Math 231b
Lecture 07

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7. Lecture 7: Stiefel-Whitney classes of projective spaces

Our next goal is to apply Stiefel-Whitney classes to prove the following important result by Stiefel.

7.1. **Division algebras and projective spaces.**

Theorem 7.1. *Suppose that there is a structure of a division algebra on* $\mathbb{R}^n$*. Then the projective space* $\mathbb{P}^{n-1}$ *is parallelizable. In particular,* n *must be a power of* 2*.*

Remark 7.2. In fact, we know that there is the much stronger result that a division algebra structure exists on $\mathbb{R}^n$ if and only if $n = 1, 2, 4, 8$. But to prove this final result we need stronger techniques. So for a moment let's be modest and see how the methods we know so far lead to a proof of this algebraic result.

7.2. **Stiefel-Whitney classes of projective spaces.**

Example 7.3. Stiefel-Whitney classes are not fine enough to decide if the tangent bundle of a sphere is trivial or not. For the tangent bundle of a sphere is stably trivial, hence $w(S^n) = w(\tau_{S^n}) = 1$.

Lemma 7.4. *The total Stiefel-Whitney class of the canonical bundle* γ_n^1 *over* $\mathbb{P}^n$ *is given by*

$$w(\gamma_n^1) = 1 + a$$

where a *denotes the nonzero element of* $H^1(\mathbb{P}^n; \mathbb{Z}/2)$*.*

Proof. The standard inclusion $j \colon \mathbb{P}^1 \to \mathbb{P}^n$ is clearly covered by a bundle map from γ_1^1 to γ_n^1. Therefore

$$j^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0.$$

Hence $w_1(\gamma_n^1)$ cannot be zero, hence it must be equal to a. Since γ_n^1 is a line bundle, the first axiom for Stiefel-Whitney classes tells us that the higher classes must be zero. □

Example 7.5. The canonical line bundle γ_n^1 over $\mathbb{P}^n$ is contained as a sub-bundle in the trivial bundle ϵ^{n+1}. Let $\gamma^\perp$ denote the orthogonal complement of γ_n^1 in ϵ^{n+1}. The total space $E(\gamma^\perp)$ consists of all pairs

$$(\{\pm x\}, v) \in \mathbb{P}^n \times \mathbb{R}^{n+1}$$

with v orthogonal to x. Claim:

$$w(\gamma^\perp) = 1 + a + a^2 + \ldots + a^n.$$

For: Since $\gamma_n^1 \oplus \gamma^\perp$ is trivial we have

$$w(\gamma^\perp) = \bar{w}(\gamma_n^1) = (1+a)^{-1} = 1 + a + a^2 + \ldots + a^n.$$

In particular, we see that it is possible that all of the n Stiefel-Whitney classes of an $\mathbb{R}^n$-bundle can be non-zero.

Lemma 7.6. *The tangent bundle τ of $\mathbb{P}^n$ is isomorphic to* $\mathrm{Hom}(\gamma_n^1,\gamma^\perp)$.

Proof. Let L be a line through the origin in $\mathbb{R}^{n+1}$, intersecting S^n in the points $\pm x$, and let $L^\perp \subset \mathbb{R}^{n+1}$ be the complementary n-plane. Let $f\colon S^n \to \mathbb{P}^n$ denote the canonical map $f(x) = \{\pm x\}$. Note that the two tangent vectors (x,v) and $(-x,-v)$ in DS^n both have the same image under the map

$$Df\colon DS^n \to D\mathbb{P}^n$$

which is induced by f. Thus the tangent manifold $D\mathbb{P}^n$ can be identified with the set of pairs $\{(x,v),(-x,-v)\}$ satisfying

$$x \cdot x = 1,\ v \cdot v = 0.$$

But each such pair determines, and is determined by, a linear mapping

$$\ell\colon L \to L^\perp,$$

where

$$\ell(x) = v.$$

Thus the tangent space of $\mathbb{P}^n$ at $\{\pm x\}$ is canonically isomorphic to the vector space $\mathrm{Hom}(L,L^\perp)$. It follows that the tangent vector bundle $\tau = \tau_{\mathbb{P}^n}$ is isomorphic to the bundle $\mathrm{Hom}(\gamma_n^1,\gamma^\perp)$. □

Let us compute the total Stiefel-Whitney class $w(\mathbb{P}^n)$. We cannot use the previous formula for τ, since we do not a formula that relates the Stiefel-Whitney classes of $\mathrm{Hom}(\gamma_n^1,\gamma^\perp)$, γ_n^1, and $\gamma^\perp$. Instead we do the following.

Theorem 7.7. *the Whitney sum $\tau \oplus \epsilon^1$ is isomorphic the $(n+1)$-fold Whitney sum $\gamma_n^1 \oplus \gamma_n^1 \oplus \ldots \oplus \gamma_n^1$. Hence the total Stiefel-Whitney class of $\mathbb{P}^n$ is given by*

$$w(\mathbb{P}^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \ldots + \binom{n+1}{n} a^n.$$

Proof. The bundle $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is trivial since it is a line bundle with a canonical nowhere zero section. Therefore

$$\tau \oplus \epsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1).$$

But the latter is isomorphic to

$$\mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}),$$

and therefore it is isomorphic to the $(n+1)$-fold sum

$$\mathrm{Hom}(\gamma_n^1, \epsilon^1 \oplus \ldots \oplus \epsilon^1) \cong \mathrm{Hom}(\gamma_n^1, \epsilon^1) \oplus \ldots \oplus \mathrm{Hom}(\gamma_n^1, \epsilon^1).$$

But the bundle $\mathrm{Hom}(\gamma_n^1, \epsilon^1)$ is isomorphic to γ_n^1, since γ_n^1 has a Euclidean metric. This proves that

$$\tau \oplus \epsilon^1 \cong \gamma_n^1 \oplus \ldots \oplus \gamma_n^1.$$

The Whitney product formula implies that $w(\tau) = w(\tau \oplus \epsilon^1)$ is equal to

$$w(\gamma_n^1) \ldots w(\gamma_n^1) = (1+a)^{n+1}.$$

The binomial formula now completes the proof. □

Corollary 7.8. *The class $w(\mathbb{P}^n)$ is equal to 1 if and only if $n+1$ is a power of 2. Thus the only projective spaces which can be parallelizable are $\mathbb{P}^1, \mathbb{P}^3, \mathbb{P}^7, \mathbb{P}^{15}, \ldots$.*

Proof. The identity $(a+b)^2 = a^2 + b^2$ modulo 2 implies that

$$(1+a)^{2^r} = 1 + a^{2^r}.$$

Therefore if $n + 1 = 2^r$ then

$$w(\mathbb{P}^n) = (1+a)^{n+1} = 1 + a^{n+1} = 1.$$

Conversely if $n + 1 = 2^r m$ with m odd, m> 1, then

$$\begin{aligned} w(\mathbb{P}^n) &= (1+a)^{n+1} = (1+a^{2^r})^m \\ &= 1 + ma^{2^r} + \tfrac{m(m-1}{2} a^{2\cdot 2^r} + \ldots \neq 1, \end{aligned}$$

since $2^r < n+1$. □

7.3. **Proof of Stiefel's theorem.** Assume there is a bilinear product operation

$$p \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

without zero divisors.

Let $b_1, \ldots, b_n$ be the standard basis for the vector space $\mathbb{R}^n$. The correspondence

$$y \mapsto p(y,b_1)$$

defines an isomorphism of $\mathbb{R}^n$ onto itself, since p has no zero divisors. Hence the formula

$$v_i(p(y,b_1)) = p(y,b_i)$$

defines a linear transformation

$$v_i \colon \mathbb{R}^n \to \mathbb{R}^n.$$

Note that we have $v_1(x) = x$, since $v_1(p(y,b_1)) = p(y,b_1)$ by definition. Moreover, for $x \neq 0$, the vectors $v_1(x), \ldots, v_n(x)$ are linearly independent. For if there was a nontrivial relation, for some $y \in \mathbb{R}^n$ with $x = p(y,b_1)$,

$$0 = \sum_i \lambda_i v_i(x) = \sum_i \lambda_i p(y,b_i) = p(y, \sum_i \lambda_i b_i)$$

this implied

$$0 = \sum_i \lambda_i b_i$$

which implies $\lambda_i = 0$ for all i.

Now let L be a line through the origin. Each v_i defines a linear transformation

$$\bar{v}_i \colon L \to L^{\perp}$$

as follows. For $x \in L$, let $\bar{v}_i(x)$ denote the image of $v_i(x)$ under the orthogonal projection

$$\mathbb{R}^n \to L^{\perp}.$$

Since $v_1(x) = x$, we have $\bar{v}_1 = 0$. But the $\bar{v}_2, \ldots, \bar{v}_n$ are everywhere linearly independent, since the $v_2, \ldots, v_n$ are everywhere linearly independent. Hence the $v_2, \ldots, v_n$ give rise to $n-1$ linearly independent sections of the bundle

$$\mathrm{Hom}(\gamma_n^1, \gamma^{\perp}).$$

Since this bundle is isomorphic the tangent bundle $\tau_{\mathbb{P}^{n-1}}$ of $\mathbb{P}^{n-1}$, we see that $\tau_{\mathbb{P}^{n-1}}$ is trivial. This completes the proof of Theorem 7.1.