DIFFERENTIAL TOPOLOGY
LECTURE NOTES

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Contents

Preface

These are the lecture notes accompanying a one-semester course. The aim of the course is to pick up the readers with a background in multi-variable calculus and linear algebra and to take them on a direct path to the fascinating world of differential topology.

In order to make the ideas and techniques as accessible as possible the arguments are explained in great detail. Thus instead of aiming for the most elegant and shortest argument we often take a longer walk and pick up every flower along the way by hand. We hope that the amount of detail will make it easier for a relatively unexperienced mathematician to witness and understand what is happening and to appreciate how some relatively straight-forward ideas lead to exciting and deep results.

This has the consequence that these notes are not brief and do not just summarise the main ideas and theorems in differential topology. For a brief and comprehensive account we recommend the excellent books by Milnor [13] and Guillemin–Pollack [5] on which these notes are based on.

The chapters are accompanied by a list of exercises. We highly recommend to work through all the exercises. At the end of the book there are suggestions for solutions to all exercises. For some it may be tempting to glimpse at the solutions before trying to solve the problems, other will rather try on their own first. We think that the readers may decide for themselves how they prefer to learn new mathematics.

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1. Introduction

- **Geometry vs topology**

Classical geometers were interested in measuring angles and distances. For example, two things are the 'same' – more mathematically speaking they are **congruent** – in **classical geometry**[1] if you can transform one into the other by moving or rotating them. A first variation to allow flexibility, is **projective geometry**: Two things are considered the same if they are both **views** of the same object. For example, an ellipse and a circle can be projectively equivalent: for one can look like the other when you look at them from the right perspective. In **topology**, we take this idea a big step further and consider two objects the same if we can **continuously transform** one into the other. For example, a triangle is equivalent to a circle and both are equivalent to a square. In **differential topology**, the part we study in this class, we only allow **smooth transformations**. Then square and circle are **different**, because a square has vertices which are not smooth points while the circle does not. As a slogan we may say that differential topology is the study of properties that do not change under smooth transformations. Smoothness of an object is something we check locally, i.e., by looking at every point and a small neighborhood around it. The smoothest space we know is Euclidean space. This leads to the following first working definition of what kind of objects we are going to study:

Definition 1.1 (Working definition: What is a manifold?) A manifold is a geometric object which locally, i.e., in a small neighborhood of every point, looks like $\mathbb{R}^n$.

In order to get a first idea, let us look at a fundamental example:

Example 1.2 (Spheres) Let us look at the unit circle

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

The circle is something one-dimensional. But how do we describe that precisely when we need two coordinates to describe its points. One way is that if we **zoom in at any point** it just looks like a bended **line segment**. Looking very closely it even looks almost like a straight line segment. See Figure 1.1. So, **locally**, and we will make precise what that means very soon, the circle looks like a segment of $\mathbb{R}^1$. The unit circle $\mathbb{S}^1$, more generally, the n-dimensional sphere

$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is an example of a **smooth manifold**.

[1] A **word** on **colors**: We make extensive use of colors in the text. Sometimes **NTNU orange** is used for **definitions**, while **NTNU blue** is used for **key words**. The use of the latter is quite frequent. The reason is that these notes are also used as actual printed notes during lectures where it is desirable to spot the key words of an argument easily. This might be a **distraction** for the reader for which we **apologize** in advance.

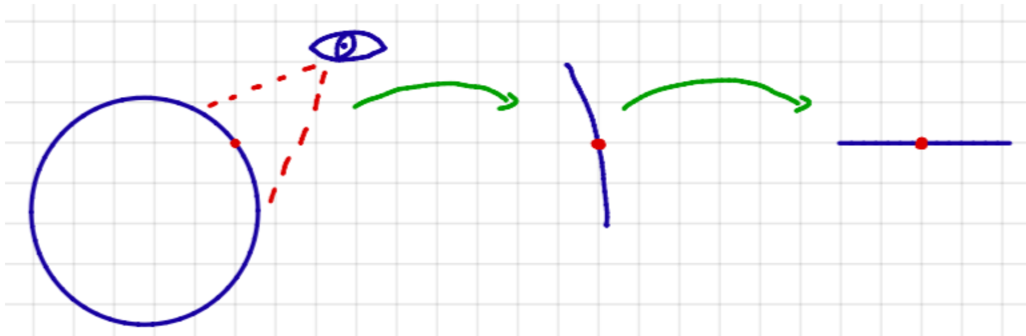

Figure 1.1: Each segment of the circle looks like an open interval in $\mathbb{R}$.

While not being precise at all, our working definition may sound at the same time quite strict. In a small neighborhood every point looks the same. How can this lead to interesting objects? We will see that there is in fact a **universe of examples** of smooth manifolds of very different kind. The point is that even though all points look pretty much the same locally, manifolds may look **very different globally**. As a first simple example consider the two-dimensional sphere and the two-dimensional torus. They look the same locally, but they are quite different globally: the torus has a whole in the middle while the sphere does not. In fact, one of the main goals in topology is to classify all types of manifolds.

- **Important idea: introduce invariants**

A key method to analyse spaces and maps, or more generally any kind of complicated object in mathematics, is to attach to them interesting **invariants**. These are usually **numbers**, groups, vector spaces, …, any sort of algebraic objects which are much easier to understand and to distinguish than the spaces and maps we started with. The name **invariant** comes from the fact that we require that the algebraic objects, for example numbers $i(X)$ and $j(f)$ we attach to each object X and each map f, **does not change** under the geometric transformations we allow. For smooth manifolds, the transformations we allow are diffeomorphisms, or, later on, smooth homotopy equivalences. For example, assume we would like to understand two manifolds X and Y. Let us assume they are defined in complicated ways so that it is not easy at all to check if they are maybe the same objects after all. If we can calculate the invariants $i(X)$ and $i(Y)$ and get $i(X) \neq i(Y)$, then X and Y could not have been the same to begin with. If X and Y were vector spaces, then we are very familiar with the idea of an invariant, namely the dimension. That is, if $\dim X \neq \dim Y$, then we know that there is no linear isomorphism between X and Y. If $\dim X = \dim Y$, then we can construct an isomorphism $X \cong Y$ by choosing bases for both X and Y. In fact, a smooth manifold has a **tangent space** at every point. The tangent space is a real vector space, and it will be one of the main tools for our study. However, tangent spaces and the dimension are not sufficient to study smooth manifolds and we will need more sophisticated tools and invariants. For example, we will develop intersection theory and will show that the smooth manifolds $\mathbb{S}^n$, the n-sphere, and $\mathbb{RP}^n$, the n-dimensional real projective space, are not diffeomorphic for $n \geq 2$. In fact, we will show that they are not even homotopy equivalent for $n \geq 2$.

- Here are some **examples of invariants** in differential topology:
  - dimension
  - Brouwer degree
  - intersection number
  - Euler characteristic
  - index of a vector field

- **Some important theorems**

After developing the theory of smooth manifolds with and without boundaries we will prove the following famous theorem: Let $\mathbb{D}^n$ be the n-dimensional unit disc

$$\mathbb{D}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \, : \, x_1^2 + \dots + x_n^2 \leq 1\}.$$

Theorem 1.3 (Brouwer Fixed Point Theorem) Every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a **fixed point**, i.e., there is an $x \in \mathbb{D}^n$ such that $f(x) = x$.

This theorem has a lot of very important applications. In particular, one way to apply it is to first transform a problem into finding a solution x_0 of an equation of the form $f(x) = x$ and to use Brouwer's theorem to show that such a solution exists. It will require a good amount of exciting work to prove the theorem.

Then we will introduce the most important invariant in differential topology: the **Brouwer degree** of a smooth map. It will turn out to be an extremely powerful tool. We will first introduce a mod 2-version of the degree and use it for example to show the following important and deep result of Hopf:

Theorem 1.4 (A Hopf invariant one primer) There is a smooth map $\mathbb{S}^3 \to \mathbb{S}^2$ which is **not** homotopic to a constant map.

The map $\mathbb{S}^3 \to \mathbb{S}^2$ of the theorem is an example of **Hopf fibration** which only exists in certain dimensions. We will use this important map as a running example throughout these notes. The way we talked about invariants so far may suggest that they only allow to show that something does not exist. However, there are also some very important situations where an equality of invariants implies that spaces are equivalent. A very famous example of such a case is the following theorem which uses the integer-valued Brouwer degree which we will study in this course after introducing orientations:

Theorem 1.5 (Hopf Degree Theorem) Let X be a compact, connected, oriented smooth k-manifold without boundary. Then two continuous maps $X \to \mathbb{S}^k$ are **homotopic** if and only if they have the **same degree**.

As a generalization of the degree we will develop mod 2-intersection theory on smooth manifolds. Surprisingly, we can use it to prove the following purely algebraic result: Let $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(a, b) \mapsto a \cdot b$, be an $\mathbb{R}$-bilinear map with no zero-divisors, i.e., $a \cdot b$ implies $a = 0$ or $b = 0$. Assume that we have $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{R}^n$. Such a map is called a **commutative division** algebra structure on $\mathbb{R}^n$. We are very familiar with commutative division algebra structures on $\mathbb{R}^n$ for $n = 1$ and $n = 2$: They are given by the field of real numbers $\mathbb{R}$ for $n = 1$ and the field of complex numbers $\mathbb{C} \cong \mathbb{R}^2$ for $n = 2$. We will show that there are no other possible cases:

Theorem 1.6 There is **no** commutative division algebra structure on $\mathbb{R}^n$ for $n > 2$.

A vector field on a smooth manifold X is a map which assigns to each point x a tangent vector to X at x. A zero of a vector field is a point to which the field assigns the zero vector. The Brouwer degree allows us to define the index of a zero which is an integer that characterises the geometry of the vector field around the zero. The sum of the indices of the zeros of a vector field a priori depends on the smooth structure of the manifold. On the other hand, the manifold X is also equipped with an integer, the **Euler characteristic** $\chi(X)$. This is a purely topological invariant which means it does not change if we transform X continuously and it does not depend on the fact that X is not just a space but a smooth manifold. One of the highlights of this course is the proof of the following famous result which relates the geometry of vector fields on a smooth manifold to its Euler characteristic. This is a first example of an **index theorem** which is part of a fascinating and very influential area in mathematics:

Theorem 1.7 (Poincaré–Hopf Index Theorem) Let X be a compact, oriented smooth manifold and let $\mathbf{v}$ be a vector field on X with only finitely many zeros. Then the sum of the indices of $\mathbf{v}$ equals the Euler characteristic of X.

Before we are able to prove these exciting results we set out to develop the basic theory of smooth manifolds in the following chapters.

2. Smooth manifolds

2.1 Topology in $\mathbb{R}^n$

2.1.1 Open and closed subsets in Euclidean space

Recall from Calculus that the norm of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is defined as the non-negative real number

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \in \mathbb{R}^{\geq 0}.$$

The norm defines a distance between two points x, y in $\mathbb{R}^n$ by taking the norm $|x - y|$ of the difference of x and y. For any n, the space $\mathbb{R}^n$ together with this norm is called **n-dimensional Euclidean space**. It is a **topological space** in the following way:

- **(Open sets in $\mathbb{R}^n$)** Here are important examples of open subsets in $\mathbb{R}^n$:
  - Let x be a point in $\mathbb{R}^n$ and $r > 0$ a real number. We define the n-dimensional **open ball** with radius r around x

    $$\mathbb{B}_r(x) = \mathbb{B}_r^n(x) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

    Note that we drop the superscript ^{n} whenever possible.
  - The **open balls** $\mathbb{B}_r(x)$ are the **prototypes of open sets** in $\mathbb{R}^n$.
  - A non-empty subset $U \subseteq \mathbb{R}^n$ is called **open** if for every point $x \in U$ there **exists** a real number $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon(x)$ is contained in U.
  - The **empty set** $\emptyset$ are defined to be **open**.
  - The whole space $\mathbb{R}^n$ is **open**.
  - A subset $Z \subseteq \mathbb{R}^n$ is called **closed** if its **complement** $\mathbb{R}^n \setminus Z$ is **open**.
  - **Arbitrary unions** of open sets are **open** and **finite intersections** of open sets are **open**.

Examples and remarks:

- Familiar examples of open sets in $\mathbb{R}$ are open intervals, e.g., the open interval $(-2, 1)$.
- The cartesian product of n open intervals (an open 'rectangle') is open in $\mathbb{R}^n$.
- Similarly, closed intervals are examples of closed sets in $\mathbb{R}$, .e.g., the closed interval $[-2, 1]$.

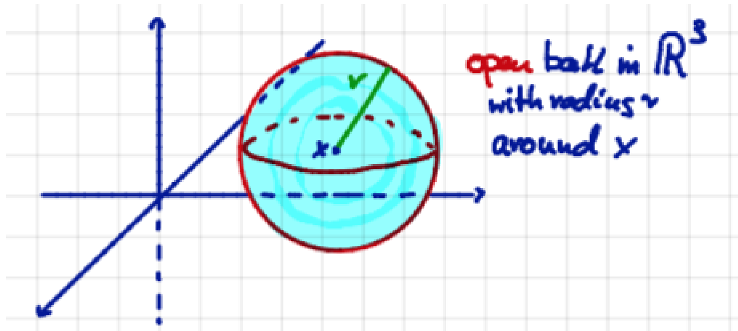

Figure 2.1: An open ball in $\mathbb{R}^3$.

- An important example of a closed set is the n-dimensional sphere $\mathbb{S}^n$ defined as

$$\mathbb{S}^n = \left\{ x \in \mathbb{R}^{n+1} \, : \, |x| = 1 \right\}.$$

- The cartesian product of n closed intervals (a closed 'rectangle') is closed in $\mathbb{R}^n$.
- The empty set $\emptyset$ and $\mathbb{R}^n$ itself are by definition both **open and closed** sets.
- Not every subset of $\mathbb{R}^n$ is open or closed. There are a lot of subsets which are neither open nor closed. For example, the interval $(0, 1]$ in $\mathbb{R}$ or the product of an open and a closed interval in $\mathbb{R}^2$.

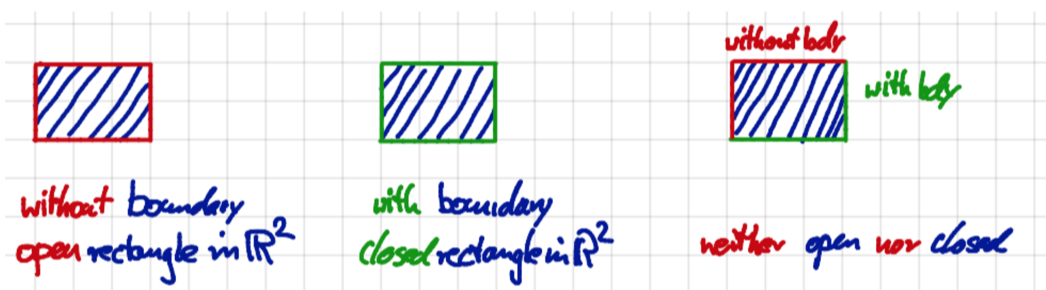

Figure 2.2: Examples of subsets in $\mathbb{R}^2$.

Definition 2.1 (Relative open sets) Let X be a subset in $\mathbb{R}^n$. Then we say that $V \subseteq X$ is **open in X** (or **relatively open**) if there is an open subset $U \subseteq \mathbb{R}^n$ which is open in $\mathbb{R}^n$ with $V = U \cap X$. **More concretely**: $V \subseteq X$ is **open in X** if and only if for every point $x \in V$ there exists a real number $\varepsilon > 0$ such that $\mathbb{B}^n_\varepsilon(x) \cap X \subseteq V$. See Figure 2.3.

In order to decide whether a subspace is open or closed it is very important to take the ambient space into account:

Remark 2.2 (Warning) It is important to note that the property of being an **open subset** very much **depends** on the bigger space we are looking at. Hence **open** always refers to being **open in** some given space. For example, a set can be open in a space $X \subset \mathbb{R}^2$, but not be open in $\mathbb{R}^2$, see Figure 2.4.

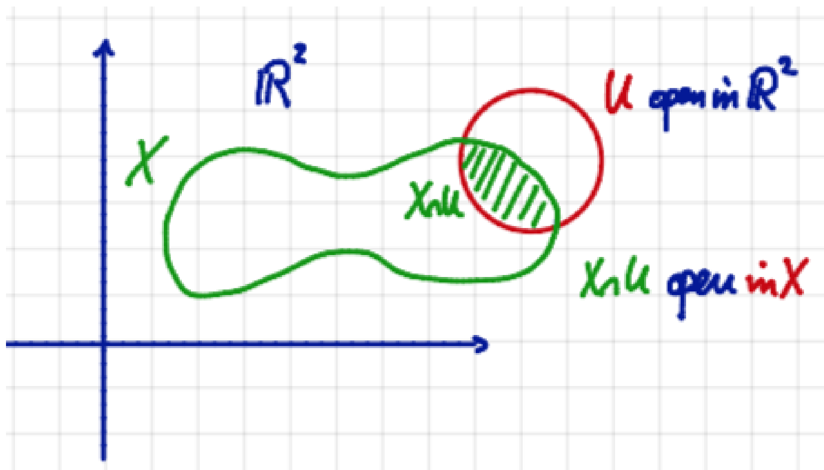

Figure 2.3: A relative open subset of X: $X \cap U$ is open in X.

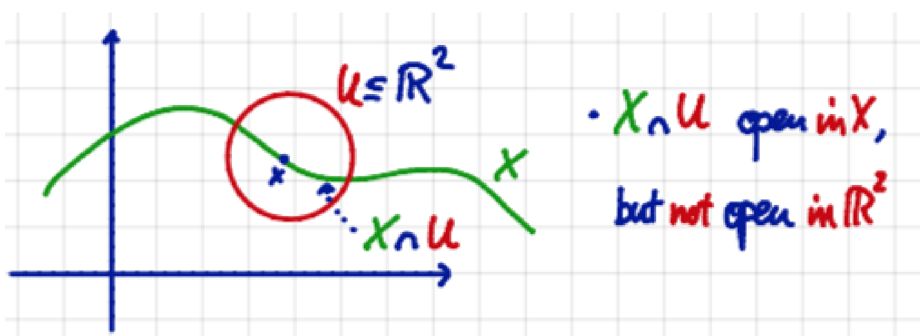

Figure 2.4: The relative open subset of $X \cap U$ is open **in** X, but is **not** open in $\mathbb{R}^2$.

Examples:

- Let $X = \mathbb{S}^2$ be the two-dimensional sphere. We consider it as a subset in $\mathbb{R}^3$ with the subspace topology. Let $x_0 \in \mathbb{S}^2$ be a point on $\mathbb{S}^2$. An example of a subset in $\mathbb{S}^2$ which is **open in** $\mathbb{S}^2$ and contains x_0 is the set

$$\mathbb{S}^2 \cap \mathbb{B}^3_{\frac{1}{2}}(x_0) \text{ with } \mathbb{B}^3_{\frac{1}{2}}(x_0) = \left\{ y \in \mathbb{R}^3 \,:\, |y| < \frac{1}{2} \right\}.$$

  In fact, every subset which is open in $\mathbb{S}^2$ and contains x_0 contains a subset if the form $\mathbb{S}^2 \cap \mathbb{B}^3_\varepsilon(x_0)$ with $\varepsilon > 0$ sufficiently small.

- **However**, the set $\mathbb{S}^2 \cap \mathbb{B}^3_{\frac{1}{2}}(x_0)$ is **not** open in $\mathbb{R}^3$. For there is no **three**-dimensional open ball $\mathbb{B}^3_\varepsilon(x_0)$ which is completely contained in $\mathbb{S}^2 \cap \mathbb{B}^3_{\frac{1}{2}}(x_0)$.

Open sets are nice for a lot of reasons. First of all, they provide us with a way to talk about things that happen **close to** a point.

Definition 2.3 (Open neighborhoods) We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of x** if there is an open subset $U \subseteq V$ with $x \in U$. If V itself is open, we call V an **open neighborhood**.

Second, a collection of open subsets in a set X, define a **topology** on X:

Definition 2.4 (Spaces) We establish the following convention for the use of the word *space*:

- A set together with a topology, i.e., a collection of open sets, is called a **topological space**.
- From now on, when we talk about a **space**, we mean a topological space, i.e., a set together with a specified topology.

Here we observe that the word **topology** is used in several ways. On the one hand, it is the name of a **whole area** in mathematics. On the other hand, it is the name for an additional **structure** on a set. We are familiar with this phenomenon: for example, the word **algebra** denotes both an area in mathematics and a structure on a set.

2.1.2 Continuous maps

The type of maps that preserve open sets, i.e., respect the topology on a set, are called **continuous** maps:

Definition 2.5 (Continuous maps: abstract definition) Continuous maps are characterized as follows:

- Let X and Y be topological spaces. A map $f : X \to Y$ is called **continuous** if, for every open subset $U \subseteq Y$, the subset $f^{-1}(U)$ is open in X.
- In the **subspace topology** in $\mathbb{R}^n$: If A is a subset of $\mathbb{R}^n$ with the subset topology, then a map $f : A \to \mathbb{R}^m$ is **continuous** if and only if, for every open subset $U \subseteq \mathbb{R}^m$, there is some open subset $V \subseteq \mathbb{R}^n$ with $f^{-1}(U) = V \cap A$ (in other words $f^{-1}(U)$ is open in A).

Remark 2.6 Just in case you have heard of categories before: topological spaces form a **category** with morphisms given by continuous maps.

We are familiar with continuous maps from Calculus. The ε-δ-characterization of continuity looks as follows:

Lemma 2.7 (Continuous maps: familiar description in $\mathbb{R}^n$) Let A be a subset in $\mathbb{R}^n$ and $a \in A$ be a point. A map $f : A \to \mathbb{R}^m$ is **continuous at a** if it satisfies the following condition: for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \text{ and } x \in A \Rightarrow |f(x) - f(a)| < \varepsilon.$$

In our new fancy notation, we can reformulate the last condition as follows: for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x \in \mathbb{B}^n_\delta(a) \cap A \Rightarrow f(x) \in \mathbb{B}^m_\varepsilon(f(a)).$$

Proof of Lemma 2.7: First, assume f satisfies ε-δ-continuity. Let $U \subseteq \mathbb{R}^m$ be an open set in $\mathbb{R}^m$. If $f^{-1}(U)$ is empty, it is open by definition. So let $a \in f^{-1}(U)$ be a point in $f^{-1}(U)$. The fact that U is open means that there is an $\varepsilon > 0$ such that $\mathbb{B}^m_\varepsilon(f(a)) \subseteq U$. Given this ε, the fact that f is continuous means that

$$\text{there is a } \delta > 0 \text{ such that } x \in \mathbb{B}^n_\delta(a) \cap A \Rightarrow f(x) \in \mathbb{B}^m_\varepsilon(f(a)).$$

But

$$f(x) \in \mathbb{B}^m_\varepsilon(f(a)) \subseteq U$$

which implies $f(x) \in U$ and hence $x \in f^{-1}(U)$. Thus, for every $x \in \mathbb{B}^n_\delta(a) \cap A$, we have $x \in f^{-1}(U)$. In other words,

$$\mathbb{B}^n_\delta(a) \cap A \subseteq f^{-1}(U).$$

Since a was an arbitrary point in $f^{-1}(U)$, this shows that $f^{-1}(U)$ is open in A.

Second, assume $f^{-1}(U)$ is open in A for every open subset $U \subseteq \mathbb{R}^m$. Given $a \in A$ and $\varepsilon > 0$, let $\mathbb{B}^m_\varepsilon(f(a)) \subset \mathbb{R}^m$ be the open ball around $f(a)$ with radius ε. Since $\mathbb{B}^m_\varepsilon(f(a))$ is open in $\mathbb{R}^m$, our assumption tells us that $f^{-1}(\mathbb{B}^m_\varepsilon(f(a)))$ is open in A. Since $a \in f^{-1}(\mathbb{B}^m_\varepsilon(f(a)))$ this means that

$$\text{there is a } \delta > 0 \text{ such that } \mathbb{B}^n_\delta(a) \cap A \subseteq f^{-1}(\mathbb{B}^m_\varepsilon(f(a))).$$

But that means

$$x \in \mathbb{B}^n_\delta(a) \cap A \Rightarrow f(x) \in \mathbb{B}^m_\varepsilon(f(a)).$$

Hence f is continuous at a. Since a was arbitrary, f is continuous. □

Next we specify the maps which have inverses in the category of topological spaces:

Definition 2.8 (Homeomorphisms) Let X and Y be topological spaces. A continuous map $f : X \to Y$ is a **homeomorphism** if it is one-to-one and onto and its inverse f^{-1} is continuous as well. Homeomorphisms preserve the topology in the sense that: if $f : X \to Y$ is a homeomorphism then $U \subset X$ is open in X if and only if $f(U) \subset Y$ is open in Y.

Examples:

- The map $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a homeomorphism.
- The map $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$ is a homeomorphism.
- However, the map $g : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ is **not** a homeomorphism, since it is neither one-to-one nor onto.

There are also more interesting examples:

- **Example:** A bijection which is **not** a homeomorphism

Example 2.9 (A bijection which is not a homeomorphism) Let

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

be the unit circle considered as a subspace of $\mathbb{R}^2$. Define a map

$$f : [0, 1) \to \mathbb{S}^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$

We know that f is bijective and continuous from calculus and trigonometry. However, the function f^{-1} is **not continuous**. For example, the image of the open subset $U = \left[0, \frac{1}{4}\right)$ under f, i.e., the subset $f(U)$, is **not open in** $\mathbb{S}^1$. Let us first remark that $U = \left[0, \frac{1}{4}\right)$ is indeed open **in** $[0, 1)$, since for examples $U = [0, 1) \cap \left(-\frac{1}{4}, \frac{1}{4}\right)$. Now we look at the point $f(0) = (1, 0) \in \mathbb{S}^1$. Since $0 \in U$, $f(0)$ is a point in $f(U)$. However, for every open ball $\mathbb{B}^2_\varepsilon((1, 0)) \subset \mathbb{R}^2$, the intersection $\mathbb{B}^2_\varepsilon((1, 0)) \cap \mathbb{S}^1$ contains points with strictly negative y-coordinate. In particular, $\mathbb{B}^2_\varepsilon((1, 0)) \cap \mathbb{S}^1$ contains points which are **not** in $f(U)$, i.e.,

$$\mathbb{B}^2_\varepsilon((1, 0)) \cap \mathbb{S}^1) \not\subset f(U).$$

Alternatively, we could observe that $f^{-1}(\mathbb{B}^2_\varepsilon((1,0)) \cap \mathbb{S}^1) \subset [0,1)$ contains points which are close to 1 in $[0,1)$ and therefore do not lie in U, i.e.,

$$f^{-1}(\mathbb{B}^2_\varepsilon((1,0)) \cap \mathbb{S}^1) \not\subset U.$$

This shows that we **cannot** find an **open** subset V of $\mathbb{R}^2$ such that

$$V \cap \mathbb{S}^1 = f(U).$$

Hence $f(U)$ is **not** open in $\mathbb{S}^1$.

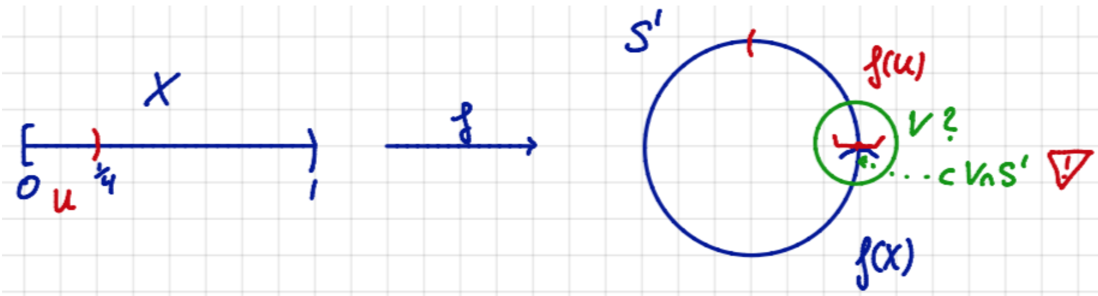

Figure 2.5: Wrapping the interval around the circle via f is a bijection, but not a homeomorphism, since $f(U)$ may **not** be open in $\mathbb{S}^1$.

2.1.3 Topological properties

One could characterize **Topology** as the study of properties which are preserved under homeomorphisms. Hence we may call a property that is preserved under homeomorphisms a **topological property**. We often refer such a topological property as a **global property**, since we cannot check that a space has it just by looking at small neighborhoods of all points. Many interesting phenomena in differential topology require an interplay of local and global properties. We will now recall some such topological properties that will play an important role for our study of smooth manifolds. We will recall here what it means for a space to be

- compact,
- connected,
- path-connected.

Compactness

Definition 2.10 (Compact space) A topological space X is called **compact** if **every** open cover $\{U_i\}_i$ of X, i.e., a collection of open subsets $U_i \subset X$ such that $X = \bigcup_i U_i$ is the union of them, has a **finite** subcover. That is, among the $\{U_i\}_i$ it is always possible to pick finitely many $U_{i_1}, \dots, U_{i_n}$ with

$$Z = U_{i_1} \cup \dots \cup U_{i_n}.$$

Recall that a subset $Z \subset \mathbb{R}^n$ is called **bounded** if there is some, possibly big, $r > 0$ such that $Z \subset B_r(0)$. For subspaces in Euclidean space we then have the following important characterization of compact subsets.

Theorem 2.11 (Theorem of Heine–Borel) A subset $Z \subset \mathbb{R}^n$ is **compact** if and only if it is **closed and bounded**.

Examples and remarks:

- **Closed balls** in $\mathbb{R}^n$ are compact.
- The n-dimensional sphere $\mathbb{S}^n$ is an important example of a **compact** space. According to the previous theorem we can show this by remembering that $\mathbb{S}^n$ is a closed subset of $\mathbb{R}^{n+1}$ and to observe that it is bounded as it is contained in, for example, $\mathbb{B}^{n+1}_2(0)$, the open ball of radius 2 around the origin.
- Theorem 2.11 tells us that open subsets in $\mathbb{R}^n$ cannot be compact. For example, **open** balls in $\mathbb{R}^n$ are never compact.
- Compactness makes a lot of things **easier**. On the one hand, it makes it easier to keep track of things, as we can cover the space with **finitely** many open sets. On the other hand, the previous theorem tells us that points cannot lie too far off, at least for subspaces in $\mathbb{R}^n$, since compact spaces in $\mathbb{R}^n$ are bounded. Hence we can think of compactness as a general condition which helps to **avoid trouble**.

Lemma 2.12 (Compact and discrete implies finite) Every **compact** and **discrete** subset S of $\mathbb{R}^n$ is **finite**.

Proof: Assume S was **not** finite. Compact subsets of $\mathbb{R}^n$ are bounded. Hence there is an $\varepsilon > 0$ such that S is contained in the n-dimensional box with edges of length ε and center 0. Divide this box into 2^n many n-dimensional boxes of equal size. The length of their edges is $\varepsilon/2$. If S was **infinite** there must be at least one small box which **still contains infinitely many points** of S. We take this box and divide it again into 2^n many n-dimensional boxes of equal size. The length of their edges is now $\varepsilon/4$. Again, if S was infinite there must be at least one of the smaller boxes which still contains infinitely many points of S. By repeating the argument, we see that we can find **an infinite sequence** of points in S which **converges**. Since S is also **closed**, any convergent infinite sequence of points in S must have **a limit in S**. Call this limit s. But then the subset $\{s\}$ would **not be open** in S, since every open subset of $\mathbb{R}^n$ containing s would also contain other points of S. Hence S would **not be discrete**. □

Connected spaces

Definition 2.13 **(Connected spaces)** Recall that a topological space X is called **connected** if X cannot be written as the union of two **nonempty disjoint** open subsets; or equivalently, if X and $\emptyset$ are the only subsets which are **both open and closed** in X.

- Familiar examples of connected spaces are intervals in $\mathbb{R}$. For example, the closed **interval $[0, 1]$ is connected**.

- If X is not connected, it has subsets $Z_\alpha \subset X$ which are both open and closed. Each such Z_α is called a **connected component** of X. Hence X can be considered as the possibly infinite union of its connected components.

- Here is a type of argument we will meet frequently: Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^n$ be two topological spaces and we would like to show that there cannot be a homeomorphism between them. First, if the numbers of connected components of X and Y are different, then there cannot be a homeomorphism $X \xrightarrow{f} Y$.

- The previous argument is often used indirectly in the following way: if X and Y have the same number connected components, we remove a suitable point $x \in X$ and count the number of connected components of $X \setminus \{x\}$ and $Y \setminus \{f(x)\}$. If these numbers are different, there cannot be a homeomorphism $X \xrightarrow{f} Y$. For if such an f exists, then the restriction $X \setminus \{x\} \xrightarrow{f} Y \setminus \{f(x)\}$ is still a homeomorphism. Continuing this or a similar process often leads to the desired conclusion.

- Another frequent application of connectedness is the following: Given a map $f : X \to S$ from a topological space X to any set S. Recall that f is called **locally constant** if for every $x \in X$ there is an open neighborhood $U_x \subset X$ such that $f_{|U_x}$ is constant.

Lemma 2.14 Let X be a **connected** space and $f : X \to S$ be **locally constant**. Then f is **constant**.

Proof: Let $s \in S$ be a value of f, i.e., $s = f(x)$ for some $x \in X$. We can write X as the **disjoint union** of the sets

$$A = \{x \in X \,:\, f(x) = s\} \text{ and } B = \{x \in X \,:\, f(x) \neq s\}.$$

Since f is **locally constant**, both A and B are **open**. For, if $a \in A$, then there is an open neighborhood $U_a \subset A$ with $f(U_a) = \{s\}$, i.e. $U_a \subset A$. Similarly, if $b \in B$, then there is an **open** neighborhood $U_b \subset X$ with $f(U_b) = \{f(b)\}$, i.e. $U_b \subset B$. But since X is **connected** and $A \neq \emptyset$, we must have $A = X$, and f is constant. $\square$

Path-connected spaces

The criterion for connectedness is **elegant to state**, but also rather **abstract**. For example, it does not tell us if we can walk, i.e., draw a line without interruptions, from one point to another, as one would intuitively expect for a connected space. This leads to a related and **more concrete** property:

Definition 2.15 (Path-connected spaces) A topological space X is called **path-connected** if for any two points $x, y \in X$ there is a **continuous** map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Path-connectedness is the **stronger property**:

Lemma 2.16 (Path-connected implies connected) If a space is path-connected, then it is also connected.

Proof: Suppose X is **path-connected**. If X was **not** connected, then there would be two **disjoint nonempty open** subsets A and B with $X = A \cup B$. Since A and B are nonempty, we can choose two points $a \in A$ and $b \in B$. Since X is **path-connected**, there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. Hence $0 \in \gamma^{-1}(A) \subset [0,1]$ and $1 \in \gamma^{-1}(B) \subset [0,1]$. Since A and B are **disjoint and open**, the subsets $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are **disjoint and open in** [0, 1]. Since $X = A \cup B$, we would have $[0,1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$ which **contradicts** the fact that $[0,1]$ is connected. Hence X must be connected. □

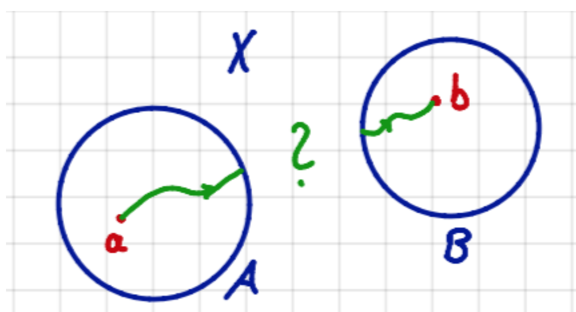

Figure 2.6: Not all points may be connected by a path.

- We will show later that smooth manifolds, however, are connected if and only if they are path-connected.

- But be aware that there are **connected spaces** which are **not path-connected**. A standard **example**, illustrated in Figure 2.7, is the subspace

$$X = \{(x, \sin(\log x)) \in \mathbb{R}^2 : x > 0\} \cup (0 \times [-1,1]) \subset \mathbb{R}^2.$$

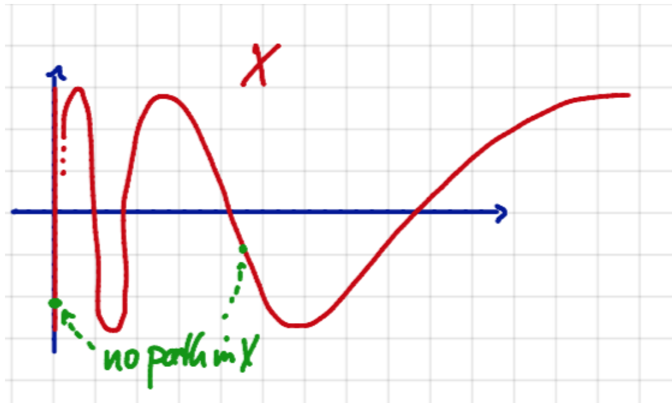

Figure 2.7: A connected but not path-connected space.

2.2 Smooth maps

2.2.1 Maps on open domains

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be **open** sets. **Recall** that a map $f : U \to V$ is called totally **differentiable at $a \in U$** if there exists a **linear map** $L_a : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - L_a(h)|}{|h|} = 0.$$

Note that if such an L_a exists, it is unique and is the best possible **linear approximation of f** at a. Moreover, if L_a exists it can be represented in the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, by the **Jacobian matrix at a**, the $m \times n$-matrix with (i, j)-entry the partial derivative $\frac{\partial f_i}{\partial x_j}(a)$. Recall that a differentiable map is in particular also continuous.

Conversely, if we know that all partial derivatives at a exist and are continuous, then f is differentiable at a. We say that f is **differentiable** if it is differentiable at every point a in U.

In differential topology we usually require that maps are not just once but **infinitely many times differentiable**. In this case, we call them **smooth**. More precisely, we define:

Definition 2.17 (Smooth maps on open subsets) Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. A map $f : U \to V$ is called **smooth** if, at every point $x \in U$, the partial derivatives of f of **all orders** exist and are continuous, i.e., all the partial derivatives $\frac{\partial^k f_i}{\partial x_{j_1} \dots \partial x_{j_k}}(a)$ exist and are continuous **for all $k \geq 1$**.

Examples:

- The familiar maps exp, cos, sin and all polynomials are smooth maps from $\mathbb{R}$ to $\mathbb{R}$.
- Let $p(x_1, x_2, \dots, x_n)$ be a **polynomial** with coefficients in $\mathbb{R}$ in the variables $x_1, x_2, \dots, x_n$. Then p induces a **smooth map** $p : \mathbb{R}^n \to \mathbb{R}$ by evaluating p on the coordinates of $\mathbb{R}^n$, i.e.,

by sending $a = (a_1, \dots, a_n)$ to $p(a_1, \dots, a_n)$. The partial derivative $\frac{\partial^k p}{\partial x_{j_1} \dots \partial x_{j_k}}(a)$ is just the partial derivative of the polynomial p. The latter always exists and is continuous.

- Let $p_1, p_2, \dots, p_m$ be **polynomials** with coefficients in $\mathbb{R}$ in the variables $x_1, x_2, \dots, x_n$. They induce a **smooth map** $P\colon \mathbb{R}^n \to \mathbb{R}^m$ by evaluating $p_1, \dots, p_m$ on the coordinates of $\mathbb{R}^n$, i.e., by sending $a = (a_1, \dots, a_n)$ to the m-tuple

$$\big(p_1(a_1, \dots, a_n), p_2(a_1, \dots, a_n), \dots, p_m(a_1, \dots, a_n)\big)\,.$$

  The partial derivative $\frac{\partial^k P_i}{\partial x_{j_1} \dots \partial x_{j_k}}(a)$ is just the partial derivative of the polynomial p_i. The latter always exists and is continuous.

- For example, the map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (x^2 - y^2, 2xy)$ is smooth.

- Another example of this type is the **smooth** map

$$F\colon \mathbb{R}^4 \to \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1x_3 + 2x_2x_4 \\ 2x_2x_3 - 2x_1x_4 \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}. \tag{2.1}$$

  To convince ourselves let us calculate some partial derivatives. For example, at a point $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ we get

$$\begin{aligned}
&\frac{\partial F_1}{\partial x_3}(a) = 2a_1, \ \frac{\partial^2 F_1}{\partial x_3 \partial x_4}(a) = 0, \\
&\frac{\partial F_2}{\partial x_1}(a) = -2a_4, \ \frac{\partial^2 F_2}{\partial x_1 \partial x_4}(a) = -2, \\
&\frac{\partial F_3}{\partial x_2}(a) = 2a_2, \ \frac{\partial^2 F_3}{\partial x_2 \partial x_2}(a) = 2, \dots
\end{aligned}$$

2.2.2 Extension to arbitrary subsets

Now we would like to extend smoothness to maps between arbitrary sets subsets of $\mathbb{R}^n$. But there is an issue we need to discuss:

In Calculus, we learned what it means for a function $f\colon (a - \varepsilon, a + \varepsilon) \to \mathbb{R}$ defined on an open interval to be differentiable at the point a. However, the definition only makes sense if there is some space on the left and right hand side of a in the interval, i.e., if $\varepsilon > 0$. For example, we cannot talk about differentiability of a function $f\colon [a, a + \varepsilon) \to \mathbb{R}$ at a. The definition requires that we can approach a from **both the left and the right** when we take the limit. That is why we required all maps to be at least defined on an **open** neighborhood of the point a.

However, there is a way out of this: Given a map $f : X \to \mathbb{R}^m$ where $X \subset \mathbb{R}^n$ is an arbitrary subset. For f to be smooth at $a \in X$, we require that f is actually just a shadow of a map which is indeed defined on an **open** ball $\mathbb{B}^n_\varepsilon(a)$ in $\mathbb{R}^n$. We will see that this simple trick makes the whole machinery work very nicely. To make things more precise, here is the definition:

Definition 2.18 (Smooth maps on arbitrary subsets) Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be **arbitrary subsets**. A map $f : X \to Y$ is called **smooth** if for each $x \in X$ there exist an open subset $U \subseteq \mathbb{R}^n$ containing x and a **smooth map** $F : U \to \mathbb{R}^m$ that coincides with f on all of $X \cap U$, i.e.,

$$F \text{ is smooth and } F_{X\cap U} = f_{X\cap U}.$$

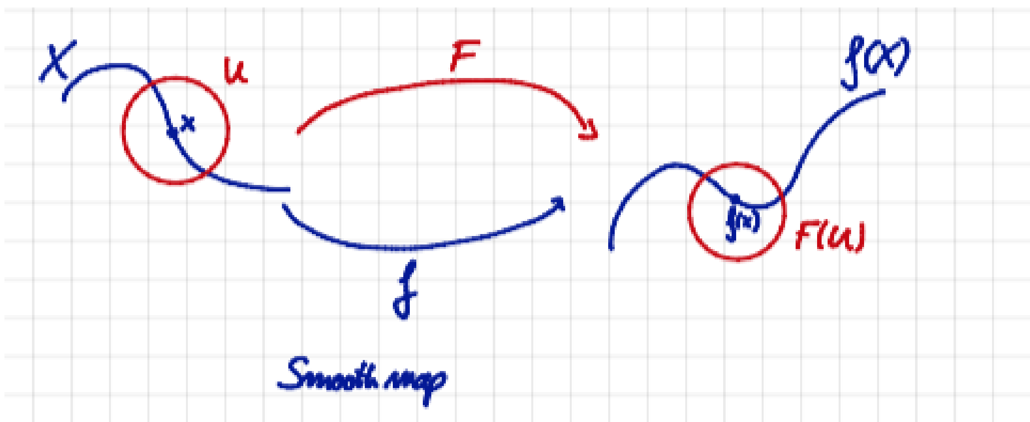

Figure 2.8: Smoothness of a map f with an arbitrary domain is defined by finding at each point a smooth map F that restrict to f on relatively open subset.

Note that **smoothness** at a point x is a **local property**, i.e., we need to check it only in a small neighborhood of x.

Examples and remarks:

- The identity map of any set X is smooth.
- If $f : X \to Y$ and $g : Y \to Z$ are smooth, then the composition $g \circ f$ is also smooth.
- The projection map

$$\pi : \mathbb{S}^1 \to \mathbb{R}, \ (x, y) \mapsto x$$

  is smooth, since it can be extended to the projection $\mathbb{R}^2 \to \mathbb{R}$ onto the first coordinate which is smooth.
- Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be the map defined by

$$f : \mathbb{S}^3 \to \mathbb{S}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1x_3 + 2x_2x_4 \\ 2x_2x_3 - 2x_1x_4 \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}.$$

This map is the restriction of the map F defined in (2.1) and hence smooth. Since F is smooth, it remains to check that if $x = (x_1, x_2, x_3, x_4)$ is in $\mathbb{S}^3 \subset \mathbb{R}^4$, then $f(x) \in \mathbb{S}^2 \subset \mathbb{R}^2$.[1]

2.2.3 Diffeomorphisms

Definition 2.19 (Diffeomorphism) A smooth map $f : X \to Y$ is called a **diffeomorphism** if f is one-to-one and onto, and its **inverse** f^{-1} is **smooth** as well. We say that X and Y are **diffeomorphic** if there exists a diffeomorphism $f : X \to Y$.

Note that every diffeomorphism is a homeomorphism, but not the other way around. Here are some examples for which it is an exercise to verify the assertions.

Examples:

- The map $g : \mathbb{R} \to \mathbb{R},\ x \to x^3 + x$ is a **diffeomorphism**.
- **However**, $f : \mathbb{R} \to \mathbb{R},\ x \to x^3$ is a homeomorphism but **not** a diffeomorphism, since the inverse map is not differentiable and therefore not smooth at the origin.
- The map $\mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$, $(x, y) \mapsto (x^2 - y^2, 2xy)$, is **not** a diffeomorphism – even though its derivative is invertible everywhere – because it is not one-to-one.
- Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the two-dimensional sphere. The map

  $$f : \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2,\ (x, y, z) \mapsto \frac{1}{1-z}(x, y)$$

  is a **diffeomorphism**. We will meet it again soon and see that it is quite useful.
- The map $f : \mathbb{S}^3 \to \mathbb{S}^2$ defined by

  $$f : \mathbb{S}^3 \to \mathbb{S}^2,\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1x_3 + 2x_2x_4 \\ 2x_2x_3 - 2x_1x_4 \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}$$

  is **not** a diffeomorphism, since it is **not one-to-one**. For example, the whole unit circle in the x_1-x_2-plane on $\mathbb{S}^3$, i.e., points on $\mathbb{S}^3$ with $x_3 = x_4 = 0$, is mapped to the north pole $(0, 0, 1)$ on $\mathbb{S}^2$. The whole unit circle in the x_3-x_4-plane on $\mathbb{S}^3$, i.e., points on $\mathbb{S}^3$ with $x_1 = x_2 = 0$, is mapped to the south pole $(0, 0, -1)$ on $\mathbb{S}^2$. In fact, we will see in Exercise 2.8 that each fiber of the Hopf fibration f is diffeomorphic to a circle on $\mathbb{S}^3$. However, as we will show later, none of these circles intersect even though they are **all linked** into each other. This is a **fascinating** and very rare phenomenon. More on

[1] We will study this map further though in different disguise in the exercises and will meet it many times in these notes. But let us remark anyway that if $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, then

$$(2x_1x_3 + 2x_2x_4)^2 + (2x_2x_3 - 2x_1x_4)^2 + (x_1^2 + x_2^2 - x_3^2 - x_4^2)^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = 1.$$

this later. For the moment, we conclude this example with the remark that after defining tangent spaces for manifolds we will see that, in fact, there **cannot exist** a diffeomorphism between $\mathbb{S}^3$ and $\mathbb{S}^2$.

Remark 2.20 (Diffeomorphic spaces are equivalent) From the point of view of differential topology, diffeomorphic spaces are equivalent, and we may (and will) consider them as copies of the same abstract space, which may happen to be differently situated in their surrounding Euclidean spaces.

2.3 Smooth manifolds

2.3.1 How to describe a space of solutions?

Many interesting spaces are given as the set of solutions of an equation of the form

$$f(x) = b$$

where $f : X \to Y$ is a map and $b \in Y$ is some specified point. It is a natural and important question: how we can **best describe** the space

$$\mathbf{S} = \{x \in X : f(x) = b\} \subset X?$$

- **(Goal)** We would like to describe the space $\mathbf{S}$ in a **simple** and **efficient** way while still expressing all its **interesting properties**.

Let us look at a **familiar situation** and consider the **linear** map

$$A : \mathbb{R}^4 \to \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_2 + 2x_4 \\ 2x_3 - 2x_1 \\ x_1 + x_2 - x_3 - x_4 \end{pmatrix}$$

ans solve the equation

$$A(\mathbf{x}) = \mathbf{0}.$$

We can use, for example, Gauss elimination to get the **set of solutions**

$$\mathbf{L} = \{\mathbf{x} \in \mathbb{R}^4 : x_2 = 0, x_3 = x_1, x_4 = -x_2\} \subset \mathbb{R}^4.$$

This is a **line** in $\mathbb{R}^4$. In particular, it is something **one-dimensional**, i.e., we can describe all the points in $\mathbf{L}$ by using just one variable, say t:

$$\mathbf{L} = \left\{ \mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = t \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We think of the variable t as a **parameter** and would like to say that t **parametrizes the set of solutions** $\mathbf{L}$. In a more formal way we have the map

$$\psi : \mathbb{R} \to \mathbf{L}, t \mapsto t \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

In fact, ψ is a **linear isomorphism**. Hence we may think of ψ as a mean to **express**

- the **one-dimensionality** of $\mathbf{L}$, and
- that $\mathbf{L}$ has a **linear structure**.

We call the map ψ a **parametrization of L**.

Now let us look at the map[2]

$$f : \mathbb{R}^4 \to \mathbb{R}^3, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1x_3 + 2x_2x_4 \\ 2x_2x_3 - 2x_1x_4 \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}.$$

This map is **not linear**, since we **multiply variables**. W have seen such maps in multivariable calculus and know how to calculate their derivatives. In fact, f is a **smooth** map, since each of its coordinates is a polynomial.The equation $f(\mathbf{x}) = \mathbf{0}$ has only a single solution – the zero vector. So let us rather determine the solutions the equation

$$f(\mathbf{x}) = \mathbf{b} =: \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

After some calculation we arrive at the **set of solutions**

$$\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^4 : x_3 = x_2, x_4 = -x_1, x_1^2 + x_2^2 = 1/2\} \subset \mathbb{R}^4.$$

This is **not** a straight line in $\mathbb{R}^4$. However, it looks like something **one-dimensional**, since knowing one of the variables, say x_1, determines x_2, x_3 and x_4, and thereby $\mathbf{x}$. Well, hold on: Let us fix a value $x_1 = t$. Then we get

$$1/2 = x_1^2 + x_2^2 = t^2 + x_2^2 \Rightarrow x_2 = \pm\sqrt{1/2 - t^2}.$$

Hence x_2 is only determined **up to a choice**. To remedy this defect, let us **restrict** our attention to points $\mathbf{x} \in \mathbf{S}$ with $x_2 \geq 0$, then $x_1 = t$ determines $\mathbf{x}$ completely. We write $\mathbf{S}_{x_2 \geq 0}$ for the set of such points.

[2]The map f is modelled on the famous **Hopf fibration** $\mathbb{S}^3 \to \mathbb{S}^2$ which is one of the very few smooth maps between spheres whose fibers are all spheres themselves. We will meet the Hopf map many times during this class.

In addition, we need to make sure that the square root is defined, i.e., we need that t only varies in the range $t \in \left[-\sqrt{1/2}, \sqrt{1/2}\right]$. Hence we can use the map

$$\tilde{\phi}_+ : \left[-\sqrt{1/2}, \sqrt{1/2}\right] \to \mathbf{S}_{x_2 \geq 0} \subset \mathbb{R}^4, t \mapsto \begin{pmatrix} t \\ \sqrt{1/2 - t^2} \\ \sqrt{1/2 - t^2} \\ -t \end{pmatrix}$$

to describe one part of the set of solutions **S**. And we check that this map is a **bijection**.

This is very similar to the **parametrization** we used to describe **L**. **However**, this map is **far from linear**. There is no way to fix this, since f was not linear in the first place. But we can check that f is differentiable at many points. In particular, it is **continuously differentiable** at all points in **S**. Hence we would like our map $\tilde{\phi}_+$ to be differentiable as well. In fact, we would like it to be **smooth**, since f is smooth at all points in **S**.

To achieve this, we need to make sure that the domain of $\tilde{\phi}_+$ is **open** and the partial derivatives are defined. Hence we replace $\tilde{\phi}_+$ with the map

$$\phi_+ : \left(-\sqrt{1/2}, \sqrt{1/2}\right) \to \mathbf{S}_{x_2 > 0} \subset \mathbb{R}^4, t \mapsto \begin{pmatrix} t \\ \sqrt{1/2 - t^2} \\ \sqrt{1/2 - t^2} \\ -t \end{pmatrix}$$

defined on an **open interval**.

The map ϕ_+ is now a **diffeomorphism**, the best we can hope for, and does a **similar job** as the parametrization ψ above: it expresses

- the **one-dimensionality** of **S** for points in $\mathbf{S}_{x_2>0}$, and
- that **S** has a **smooth structure**[3] for points in $\mathbf{S}_{x_2>0}$.

Since ϕ_+ describes only some **part** of **S**, we call the map ϕ_+ a **local parametrization of S**.

Finally, we observe that we are **missing out** on some points of **S**, in particular where $x_2 \leq 0$. Hence we **need further** local parametrizations similar to ϕ_+ to **cover** all of **S**. The **collection** of such maps will then express **S** as a **one-dimensional** and **smooth** subspace of $\mathbb{R}^4$. In fact, these maps give **S** the structure of a **one-dimensional smooth manifold**, a notion we will now define rigorously based on what we learned from this example.

2.3.2 Smooth manifolds - the definition

Let $X \subseteq \mathbb{R}^n$ be an arbitrary subset. We learned what it means for subsets in $X \subseteq \mathbb{R}^n$ to be open. One reason why open sets are useful is that they give us a way to talk about things that happen

[3] This is not yet a well-defined term. So for the moment we may think of it as saying that **S** does not have any corners or any other nasty points.

close to a point. In order to facilitate this way of thinking we are going to use the following terminology:

Definition 2.21 (Neighborhoods) We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of x** if there is an open subset $U \subseteq V$ with $x \in U$. If V itself is open, we call V an **open neighborhood**.

We will also use the following abbreviation:

Remark 2.22 (A way of speaking: Local properties) If we refer to something that happens in the neighborhood of a point $x \in X$, then we are often going to say that it happens **locally at x**. Moreover, a property of a space or a function that we only need to **test for a neighborhood of each point** is a **local property**. For example, **smoothness** of a map is a local property, since we test it in a neighborhood of each point. In contrast, there are **global properties** which are properties that describe the **whole space**.

Manifolds are now spaces that **locally look like Euclidean spaces** in the following sense:

Definition 2.23 (Smooth manifolds) Let $\mathbb{R}^N$ be some big Euclidean space.

- A subset $X \subseteq \mathbb{R}^N$ is a **k-dimensional smooth manifold** if it is **locally diffeomorphic to $\mathbb{R}^k$**. The latter means that for every point $x \in X$ there is an open subset $V \subset X$ containing x and an open subset $U \subseteq \mathbb{R}^k$ such that U and V are diffeomorphic. Note that the number k is the same for all points in X.
- Any such diffeomorphism $\phi : U \to V$ is called a **local parametrization**.
- The inverse diffeomorphism $\phi^{-1} : V \to U$ is called a **local coordinate system on V**.

The natural number N in the previous definition is not specified. We just assume that there is some $\mathbb{R}^N$ big enough to fit X into it. We are going to discuss what we can say about the minimal N later. It is actually a very interesting question.

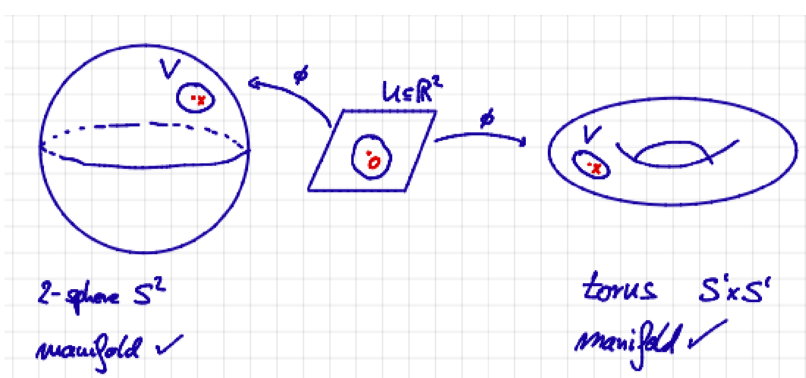

Figure 2.9: Points on $\mathbb{S}^2$ and on $\mathbb{T}^2$ have both open nighborhoods diffeomorphic to open subsets in $\mathbb{R}^2$. However, $\mathbb{S}^2$ and $\mathbb{T}^2$ have different **global** properties.

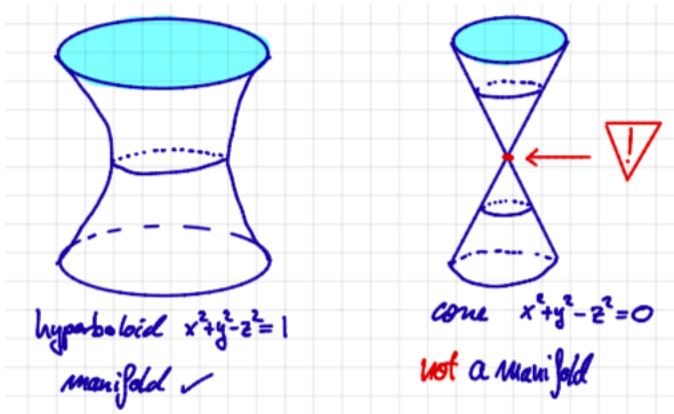

Figure 2.10: A hyperboloid is an **example** of a smooth 2-manifold. The cone, however, is **not** a manifold, since it has a point **without** an open neighborhood diffeomorphic to an open subset of $\mathbb{R}^2$. More about these two spaces in Exercise 2.5.

Remark 2.24 (Local coordinates) The set U in the definition of a local parametrization is a subset of $\mathbb{R}^k$, and it may therefore seem plausible to express a point $u \in U$ by its coordinates $u = (u_1, u_2, \ldots, u_k)$. More precisely, given a coordinate system

$$\phi^{-1} : V \to U$$

on V, we can talk about the coordinates $(\phi_1^{-1}(x), \phi_2^{-1}(x), \ldots, \phi_k^{-1}(x))$ of a point $x \in V \subset X$. Writing $u_i(x) = \phi_i^{-1}(x)$ for $i = 1, \ldots, k$, we usually drop mentioning ϕ^{-1} and just talk about the coordinates $(u_1(x), u_2(x), \ldots, u_k(x))$ of x. Hence we need to remember that the $u_1, \ldots, u_k$ are really **coordinate functions**.

Remark 2.25 (Simplified notation for parametrizations) Let $X \subseteq \mathbb{R}^N$ be k-dimensional manifold and $x \in X$ a point. Let $\phi : U \to V$ be a local parametrization around x, i.e., $U \subseteq \mathbb{R}^k$ and $V \subseteq X$ are open subsets with $x \in V$ and $\phi : U \to V$ is a diffeomorphism. Then we also **write** $\phi : U \to X$ for the **composite** $U \xrightarrow{\phi} V \hookrightarrow X$. We usually assume that ϕ is adjusted such that $\phi(0) = x$.

2.3.3 First examples

We are very well familiar with some simple examples:

- An **open** subset $U \subseteq \mathbb{R}^k$ is a k-dimensional manifold. The identity map $U \to U$ is a parametrization for all of U. For example, any k-dimensional **open ball** $\mathbb{B}_r^k(x)$ around some point $x \in \mathbb{R}^k$ is a manifold of dimension k.
- In particular, Euclidean space $\mathbb{R}^k$ is a k-dimensional manifold.
- A 0-dimensional manifold M just consists of a collection of discrete points. Given $x \in$

M, the set $\{x\} \subset M$ consisting of x alone is open in M and is diffeomorphic to the one-point set $\mathbb{R}^0 = \{0\}$.

A fundamental example that will play an important role during the whole semester is the n-dimensional sphere. We start with the one-dimensional case: the unit circle.

Example 2.26 (The unit circle) We start with $n = 1$: Let

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

be the **unit circle**. We are going to show that $\mathbb{S}^1$ is a **one-dimensional manifold**. First, suppose that (x, y) lies in the upper semicircle where $y > 0$. Then

$$\phi_1(x) = (x, \sqrt{1 - x^2})$$

maps the open interval $W = (-1, 1) \subset \mathbb{R}$ bijectively onto the upper semicircle. It is a **smooth** map $(-1, 1) \to \mathbb{R}^2$, since its partial derivatives exist and are continuous. Here it is important that we do not include the endpoints of the interval $(-1, 1)$. Its inverse is the projection map

$$\phi_1^{-1}(x, y) = x$$

which is defined on the upper semicircle. This ϕ_1^{-1} is **smooth**, since it extends to a smooth map of all of $\mathbb{R}^2$ to $\mathbb{R}^1$. Therefore, ϕ_1 is a parametrization.
A parametrization of the lower semicircle where $y < 0$ is similarly defined by

$$\phi_2(x) = (x, -\sqrt{1 - x^2}) \text{ with inverse } \phi_2^{-1}(x, y) = x.$$

These two maps give local parametrizations of $\mathbb{S}^1$ around any point except the two points $(1, 0)$ and $(-1, 0)$. To cover these points, we can use the maps

$$\phi_3(y) = (\sqrt{1 - y^2}, y) \text{ and } \phi_4(y) = (-\sqrt{1 - y^2}, y)$$

which map W to the right and left semicircles, respectively.
This shows that $\mathbb{S}^1$ is a 1-dimensional manifold.

More generally, we will show in the exercises:

Example 2.27 (n-sphere) The n-sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subset \mathbb{R}^{n+1}$$

is an n-dimensional smooth manifold.

- Another **example** is the set of solutions $\mathbf{S} \subset \mathbb{R}^4$ of the equation $f(\mathbf{x}) = \mathbf{b}$ that we have seen at the beginning of the chapter. We can check that the map ϕ_+ we defined is a local parametrization of $\mathbf{S}$. Using what we learned from the local parametrizations of $\mathbb{S}^1$ it should not be too difficult to write down the missing local parametrizations for $\mathbf{S}$. We just need to adjust for how $\mathbf{S}$ sits inside $\mathbb{R}^4$. It is a good exercise to work this out on your own. Note that we will meet the map f and the set $\mathbf{S}$ again during this course.

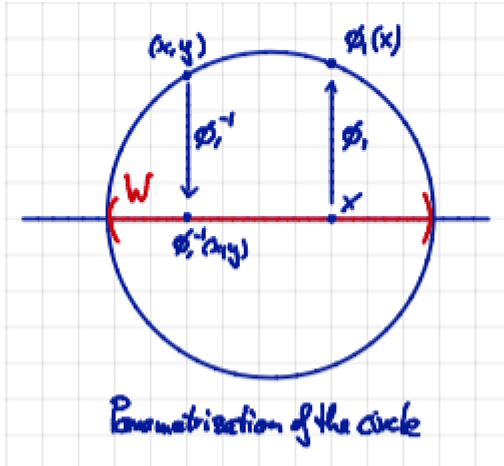

Figure 2.11: A simple parametrization of the 1-manifold $\mathbb{S}^1$.

- In fact, we will see later that many smooth manifolds arise as the **set of solutions** of a suitable equation involving a smooth function.[4]

The definition of a manifold requires parametrizations that cover the whole space. It is a natural question, what the minimal number of such maps is. The answer depends on the manifold we look at. Here is a first thought about this number for the sphere:

Remark 2.28 (Need at least two parametrizations on the sphere) Note that we have used four parametrization maps in the above example. It is an exercise to show that it is possible to cover $\mathbb{S}^1$ with only two parametrizations. But note that just **one parametrization cannot be enough**, because **$\mathbb{S}^1$ is compact**. For, if there was a diffeomorphism $\phi : \mathbb{S}^1 \to U \subset \mathbb{R}^1$ to an open subset, it would mean that U is compact contradicting the Theorem of Heine-Borel which says that the compact subsets of $\mathbb{R}^1$ are closed and bounded. This argument actually holds for the n-sphere in every dimension $n \geq 1$.

There are many different ways to choose parametrizations for a sphere. There is a very economical one which shows that two parametrizations suffice to cover the n-sphere:

Remark 2.29 (Stereographic projection) The method of **stereographic projection** yields a cover of the n-sphere with **only two parametrizations**. In Exercise 2.7 we find the formulae for the corresponding diffeomorphisms.

This is an illustration of the stereographic projection for the 2-sphere $\mathbb{S}^2$. We study the formulae of the maps involved in the exercises and will show that this actually defines a sufficient parametrization.

[4] If you cannot wait, you may want to fast-forward to Section 4.2.1 on **regular values** to have a first glimpse.

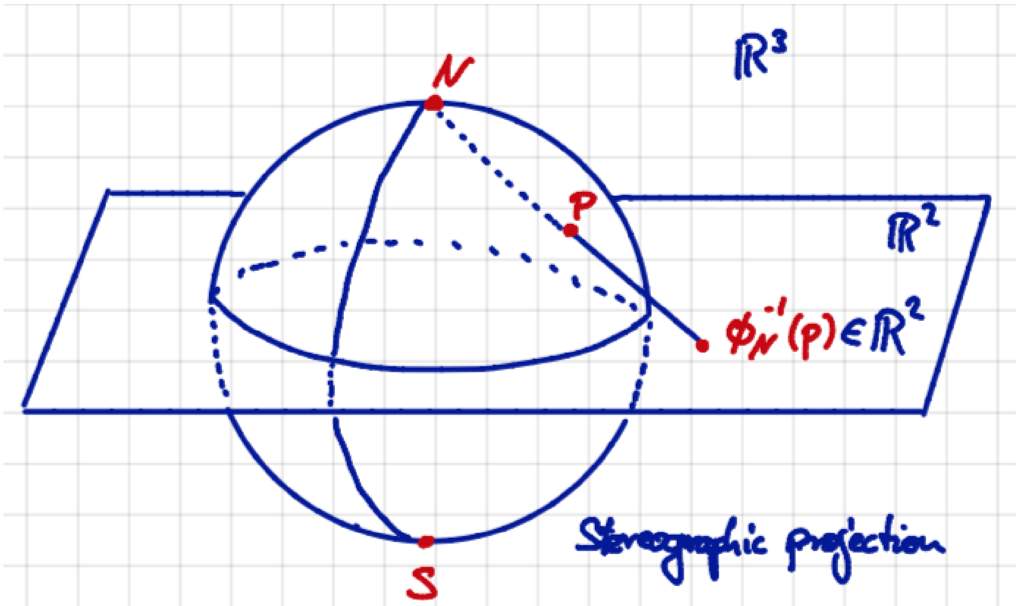

Figure 2.12: A diffeomorphism between $\mathbb{S}^n \setminus \{N\}$ and $\mathbb{R}^n$.

Definition 2.30 (Morphisms between smooth manifolds: smooth maps) Let $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ be two smooth manifolds. Then a **smooth map** between the manifolds X and Y is just a smooth map in the sense we defined previously. In fact, those are the maps which respect the smooth manifold structure on X and Y. Hence smooth maps are the **morphisms** in the category of smooth manifolds.

Manifolds have subsets. We are interested in those subsets which are manifolds on their own, possibly of lower dimension:

Definition 2.31 (Submanifolds) Let $X \subset \mathbb{R}^N$ and $Z \subset X$ be a subset considered as a topological space with relative topology induced from X and hence from $\mathbb{R}^N$. If both Z and X are manifolds — possibly of different dimensions — then Z is called a **submanifold** of X. In particular, X itself is a submanifold of $\mathbb{R}^N$. Any open subset of X is a submanifold of X.

Examples and remarks:

- We could consider the equator in $\mathbb{S}^2$ as a copy of $\mathbb{S}^1$ and hence as a submanifold.
- Similarly, we have basically two ways of considering a copy of the circle on the two-dimensional torus: once as a horizontal circle, once as a vertical circle.
- We can generalize these examples to consider copies of $\mathbb{S}^1, \mathbb{S}^2, \ldots, \mathbb{S}^{n-1}$ as submanifolds of $\mathbb{S}^n$. As in the previous cases there are different ways of how these submanifolds sit inside the bigger manifold. Understanding all possible ways of how submanifolds can sit inside a bigger manifold is actually a very interesting and useful problem to study.

We will get back to this question when we discuss **embeddings** and **intersection theory**.

□

2.3.4 Product manifolds

To find submanifolds in already existing manifolds is an important way to define and study new manifolds. But there are also other ways to produce new manifolds:

Lemma 2.32 (Creating new manifolds out of old ones) Let $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ be manifolds of dimensions k and l, respectively. Then $X \times Y \subseteq \mathbb{R}^{N+M}$ is a **manifold** of dimension $k + l$.

For, let $W \subset \mathbb{R}^k$ an open set with $\phi\colon W \to X$ a local parametrization around $x \in X$, and $U \subset \mathbb{R}^k$ an open set with $\psi : U \to Y$ a local parametrization around $y \in Y$. Then we can define the map

$$\phi \times \psi\colon W \times U \to X \times Y,\ (\phi \times \psi)(w, u) = (\phi(w), \psi(u)).$$

from the open set $W \times U \subseteq \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ to $X \times Y$. This map defines a local parametrization around (x, y). We recommend it to check this assertion as an exercise.

Example 2.33 (Torus) One way to define the **two-dimensional torus** is to think of it as the product $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Then the general statement above implies that $\mathbb{T}^2$ is a two-dimensional manifold. This is convenient. **However**, this way we consider $\mathbb{T}^2$ as a subset of $\mathbb{R}^4$, since $\mathbb{S}^1$ being a subspace of $\mathbb{R}^2$ forces us to take the product of $\mathbb{R}^2$ with itself to embed $\mathbb{T}^2$. Since we are more used to visualise the torus as a subspace in three dimensions, we will discuss a way to describe $\mathbb{T}^2$ as a subspace in $\mathbb{R}^3$ in the exercises.

□

2.3.5 A non-example

Finally, we are now going to discuss the case of a space which is **not** a manifold:

Let X denote the union of the x- and the y-axis in $\mathbb{R}^2$, in other words,

$$X = \{(x, y) \in \mathbb{R}^2 \text{ such that } xy = 0\}.$$

The **critical** point is the origin $O = (0, 0)$, as every other point on X has an open neighborhood which is diffeomorphic to an **open interval** in $\mathbb{R}$. But no point in $\mathbb{R}^d$ with $d \geq 2$ has an **open** neighborhood in $\mathbb{R}^d$ diffeomorphic to an open interval in $\mathbb{R}^1$.[5] Hence X could only be 1-dimensional.

Now let us check the point $O = (0, 0)$. **If** X was a manifold of dimension **one**, there would be an open subset $V \subseteq X$ around O **diffeomorphic to an open interval** in $\mathbb{R}^1$. By definition of open sets in a subset of $\mathbb{R}^2$, there must be an open ball $\mathbb{B}^2_\varepsilon(O)$ such that $\mathbb{B}^2_\varepsilon(O) \cap X$ is contained in V. Let I be the open interval in $\mathbb{R}$ homeomorphic to $\mathbb{B}^2_\varepsilon(O) \cap X$.

[5] We can prove this fact when we have introduced tangent spaces.

The subset $\mathbb{B}_\varepsilon(O) \cap X$ looks like a cross and contains, in particular, the points

$$P_1 = (-\varepsilon/2, 0),\ P_2 = (0, \varepsilon/2), \text{ and } P_3 = (\varepsilon/2, 0).$$

In $\mathbb{B}^2_\varepsilon(O) \cap X$, there are **paths**, i.e., continuous maps $\gamma\colon [0,1] \to \mathbb{B}^2_\varepsilon(O) \cap X$,

- γ_1 from P_2 to P_3 **not** passing through P_1.
- γ_2 from P_1 to P_3 **not** passing through P_2,
- γ_3 from P_1 to P_2 **not** passing through P_3.

But there is **no triple of distinct points with this property in the open intervall $I \subset \mathbb{R}$**. In more detail, we can argue as follows:

Since P_1, P_2, P_3 are pairwise distinct points, their images under ϕ must be pairwise distinct as well. Since $\mathbb{R}$ is a **totally ordered** set, we can order these three points. Assume first $\phi(P_1) < \phi(P_2) < \phi(P_3)$. Then the **Intermediate Value Theorem** of Calculus implies that for the path

$$\phi\circ\gamma_2\colon [0,1] \to \mathbb{B}^2_\varepsilon(O) \cap X \text{ with } \gamma_2(0) = P_1 \text{ and } \gamma_2(1) = P_3$$

there is an $s \in (0,1)$ such that $(\phi\circ\gamma_2)(s) = \phi(P_2)$. This would imply $\gamma_2(s) = P_2$ contradicting the choice of γ_2. Thus the diffeomorphism ϕ with the assumed ordering $\phi(P_1) < \phi(P_2) < \phi(P_3)$ cannot exist.

Now we can adjust and repeat this argument for any ordering of the three points $\phi(P_1)$, $\phi(P_2)$ and $\phi(P_3)$ and get contradictions to the choices of paths γ_1, γ_2 and γ_3.

Hence the homeomorphism $\phi\colon \mathbb{B}^2_\varepsilon(O) \cap X \to I$ **cannot** exist. We conclude that O does not have a neighborhood homeomorphic to an open interval in $\mathbb{R}$, and X is not a manifold.

Paths and path-connectedness - an alternative argument

In the discussion above, we used implicitly that we were looking at **path-connected** spaces. **Recall** that a topological space X is called **path-connected** if for any two points $x, y \in X$ there is a continuous map $\gamma\colon [0,1] \to X$ from the unit interval to X with $\gamma(0) = x$ and $\gamma(1) = y$. Path-connectedness is a topological property, i.e. it is preserved under homeomorphisms.

The **union of the coordinate axes** in $\mathbb{R}^2$ is an example of a **path-connected space** and every interval in $\mathbb{R}$ is path-connected. Now assume $\phi\colon \mathbb{B}^2_\varepsilon(O) \cap X \to I$ was a homeomorphism to an interval $I \subset \mathbb{R}$. Let $\phi(O) \in I$ be the image of the origin. If we remove $\phi(O)$ from I, we get **two components** of the interval. Points in the same component can be connected by a path, whereas points from different components cannot be connected to each other via a path without crossing $\phi(O)$.

If we remove O from $\mathbb{B}^2_\varepsilon(O) \cap X$ we get a space with **four components**. Again, points in the same component can be connected by a path, whereas points from different components cannot be connected to each other via a path.

We call these subsets the **path-components** of the spaces $I \setminus \{\phi(O)\}$ and $(\mathbb{B}^2_\varepsilon(O) \cap X) \setminus \{O\}$. The key observation is that if ϕ was a homeomorphism, $\phi_{|(\mathbb{B}^2_\varepsilon(O) \cap X)\setminus\{O\}}$ would still be a homeomorphism. But homeomorphic spaces need to have the same number of path-components, assuming that number is finite. This is the background for the argument we used above.

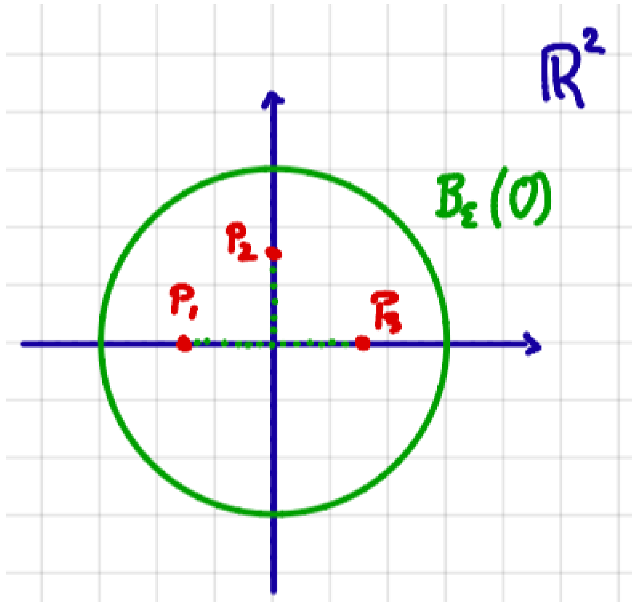

Figure 2.13: Three points on the coordinate axes, two of which can be connected without passing through the other one.

2.4 Tangent spaces and the derivative

We are now going to introduce one of the key tools to study smooth manifolds.

2.4.1 The tangent space - motivation

Let x be a point on the smooth manifold X. By definition, we can choose a local parametrization $\phi : U \to X$ around x which tells us that, at least locally at x, X is the image of $U \subset \mathbb{R}^k$ under the diffeomorphism ϕ. Images under diffeomorphism are nice, but images under linear maps are even better since the latter are **vector spaces**.

So how could we describe X, at least locally at x, via a vector space? Well, X is itself not a vector space, but what we can look for is a **linear approximation** of X at x. This is the purpose of the tangent space at x.

In order to motivate our construction we begin with a familiar situation. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $f'(x)$ be the derivative of f at x. In Calculus, we think of the derivative often as the **slope** of the **tangent line** at the **graph** of f at the point $(x, f(x))$. The graph of f, i.e., the subset $\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^2 : x \in \mathbb{R}\} \subset \mathbb{R}^2$, is an example of a smooth manifold. We have the natural map

$$\phi : \mathbb{R} \to \Gamma(f), \; x \mapsto (x, f(x)).$$

This map is smooth since f is smooth, and ϕ is injective because of the first coordinate and surjective by definition of $\Gamma(f)$. Moreover, the projection $(x, f(x)) \mapsto x$ defines a smooth inverse. Thus, ϕ is a diffeomorphism and yields us a parametrization for all points of $\Gamma(f)$.

The tangent line at the point $(x, f(x))$ is the prototype of an example of a tangent space of smooth manifold. More precisely, we prefer to consider the **parallel translate** of the tangent line to the origin, since we want the tangent space to be a **vector space**. See Figure 2.14. How can we describe the tangent line L_x passing through the origin? It is determined by its slope, i.e.,

$$L_x = \left\{ t \cdot \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} \in \mathbb{R}^2 : t \in \mathbb{R} \right\}.$$

Now we observe that $\begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$ is exactly the derivative at x of the map ϕ we defined above, i.e., $d\phi_x = \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$. Thus, the tangent line L_x is **the image** in $\mathbb{R}^2$ of the **linear map**

$$d\phi_x : \mathbb{R} \to \mathbb{R}^2.$$

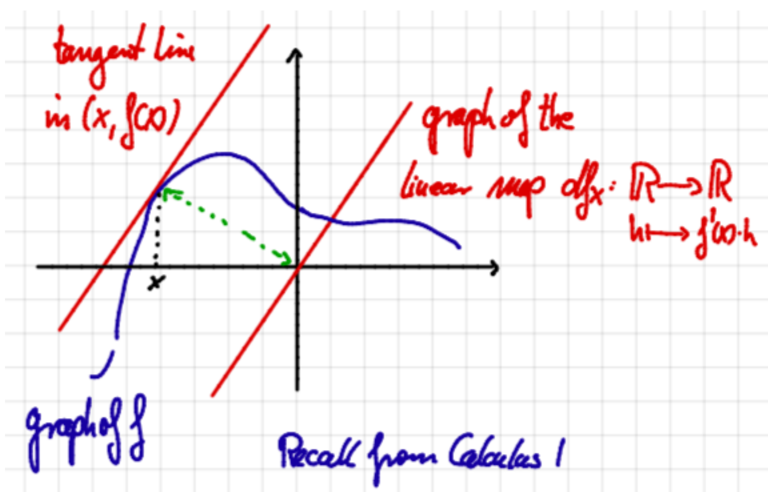

Figure 2.14: The tangent line at the graph $\Gamma(f)$ of f is the parallel translate of the tangent space of $\Gamma(f)$.

Our goal is to **generalize this observation** to an arbitrary smooth manifold by following the same recipe: the tangent space at a point should be the image of the derivative of a local parametrization. To make this precise we recall some facts about the derivative of a smooth map. Let

$$\phi : U \to V$$

where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ are open subsets. Let $u \in U$ be a point in the domain of f and $h \in \mathbb{R}^n$ be a vector in $\mathbb{R}^n$. Then the **derivative** of ϕ in the **direction** h can be defined as the limit

$$d\phi_u(h) = \lim_{t \to 0} \frac{\phi(u + th) - \phi(u)}{t}.$$

For a fixed u, the derivative is a map

$$d\phi_u : \mathbb{R}^n \to \mathbb{R}^N$$

sending a vector $h \in \mathbb{R}^n$ to the vector $d\phi_u(h) \in \mathbb{R}^N$. In Calculus we learned that this map is **$\mathbb{R}$-linear**, i.e., $d\phi_u(h + g) = d\phi_u(h) + d\phi_u(g)$ and $d\phi_u(\lambda h) = \lambda d\phi_u(h)$ for all $h, g \in \mathbb{R}^n$ and

$\lambda \in \mathbb{R}$. In particular, the derivative of ϕ is a **map on its own** which is defined on all of $\mathbb{R}^n$ even when ϕ may not be. Recall that we can calculate $d\phi_u$ in the standard bases of Euclidean spaces as the **Jacobian matrix**. Its entry in row i and column j is the partial derivative $\frac{\partial \phi_i}{\partial x_j}(u)$.

Remark 2.34 (The derivative is a linear approximation) One way to appreciate the significance of the derivative is to think of it as a simple and useful approximation to ϕ at u, i.e., knowing $\phi(u)$ and $d\phi_u$ gives us a good guess for what $\phi(u+h)$ might, namely something close to $\phi(u) + d\phi_u(h)$.

2.4.2 The tangent space - definition

Now we are ready to define tangent spaces in general:

Definition 2.35 (Tangent space) Let $X \subseteq \mathbb{R}^N$ be k-dimensional manifold and $x \in X$ a point. Let $\phi\colon U \to V$ be a local parametrization around x with $\phi(u) = x$. We define the **tangent space** of X at x to be **the image** of the **linear map** $d\phi_u \colon \mathbb{R}^k \to \mathbb{R}^N$. We denote it by $T_x(X)$. This is a **vector subspace of** $\mathbb{R}^N$.

By this definition, a **tangent vector** to $X \subseteq \mathbb{R}^N$ at x is a point $v \in \mathbb{R}^N$ that lies in the vector subspace $T_x(X) \subset \mathbb{R}^N$. However, we usually picture v **geometrically** as the **arrow pointing from x to $x + v$** in the translate $x + T_x(X)$. See Figure 2.15.

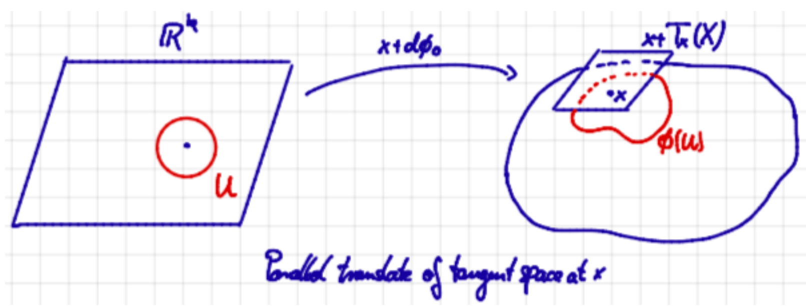

Figure 2.15: The tangent space is the isomorphic image of a $\mathbb{R}^k$ in $\mathbb{R}^N$. We visualize it as the parallel translate of this plane attached to the point of the manifold.

- **Tangent spaces are useful:** While tangent spaces may look quite boring, since they are just vector spaces, we will see very soon that they are extremely useful for understanding manifolds. Many important geometric conditions can be stated in terms of tangent spaces. The most important example for us might be transversality, a key condition for making intersection theory work. □

Lemma 2.36 (Dimension of $T_x(X)$) If X is a k-dimensional manifold, then $T_x(X)$ is a **k-dimensional vector space** over $\mathbb{R}$.

Proof: Since a local parametrization ϕ is a diffeomorphism onto its image, its derivative $d\phi_u$ is injective. Hence by definition of the vector space $T_x(X)$ of the image of $\mathbb{R}^k$ under $d\phi_u$, the dimension of $T_x(X)$ is k. □

2.4.3 Independence of choices: $T_x(X)$ is well-defined

In order to define $T_x(X)$ we made a **choice** of a parametrization ϕ. We have to check what happens if we choose a **different** parametrization.

- **Question:** Do we get the same tangent space?

We can find an answer to this question by taking another local parametrization and check whether $T_x(X)$ changes. So let $\psi : V \to X$ be **another local parametrization** around x with $\psi(0) = x$. If necessary, we shrink U and V, i.e., we replace U with $\phi^{-1}(\phi(U) \cap \psi(V)) \subset U$ and V with $\psi^{-1}(\phi(U) \cap \psi(V)) \subset V$. After doing this we can assume

$$\phi(U) = \psi(V).$$

Then the map

$$\theta := \psi^{-1} \circ \phi : U \to V$$

is a **diffeomorphism**, since it is the composite of two diffeomorphisms and the **chain rule** implies that this yields a diffeomorphism as well. By definition of θ, we have $\phi = \psi \circ \theta$. Taking derivatives yields by the chain rule

$$d\phi_0 = d\psi_0 \circ d\theta_0.$$

This implies that the image of $d\phi_0$ is contained in the image of $d\psi_0$:

$$\mathbf{d\phi_0(\mathbb{R}^k) \subseteq d\psi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N}.$$

By switching the roles of ϕ and ψ in the argument, we also get:

$$\mathbf{d\psi_0(\mathbb{R}^k) \subseteq d\phi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N}.$$

Hence $d\phi_0(\mathbb{R}^k) = d\psi_0(\mathbb{R}^k)$ in $\mathbb{R}^N$. This shows that whatever local parametrization around x we start with, the vector subspace $T_x(X) \subseteq \mathbb{R}^N$ is always the same. In mathematical terms we say that **$T_x(X)$ is well-defined**. □

2.4.4 Some examples

Example 2.37 (Tangent spaces of the unit circle) Let $p = (a, b) \in \mathbb{S}^1$ be a point with $b > 0$. A local parametrization around p with $\phi(0) = p$ is given by

$$\phi : (-\varepsilon, \varepsilon) \to \mathbb{S}^1,\ t \mapsto (t + a, \sqrt{1 - (t + a)^2})$$

for some small enough real number $\varepsilon > 0$. The derivative at t is the linear map

$$d\phi_t : \mathbb{R} \to \mathbb{R}^2,\ d\phi_t = \begin{pmatrix} 1 \\ -\frac{t+a}{\sqrt{1-(t+a)^2}} \end{pmatrix}.$$

Hence the image of $\mathbb{R}$ under $d\phi_0$ in $\mathbb{R}^2$ is the line spanned by $(-b, a)$ where we use $b = \sqrt{1-a^2}$.

We can extend this to dimension two:

Example 2.38 (Tangent spaces of the two-sphere $\mathbb{S}^2$) Let $p = (a, b, c)$ be point on $\mathbb{S}^2$ which is not the north pole. Then we can use the stereographic projection $\phi_N : \mathbb{R}^2 \to \mathbb{S}^2$ as a local parametrization.[a]

Recall that

$$\phi_N(x, y) = \frac{1}{1 + x^2 + y^2}\left(2x, 2y, x^2 + y^2 - 1\right).$$

The derivative at (x, y) is the linear map $d\phi_N : \mathbb{R}^2 \to \mathbb{R}^3$ which, with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, is given by the matrix

$$d(\phi_N)_{(x,y)} = \frac{2}{(1 + x^2 + y^2)^2}\begin{pmatrix} 1 - x^2 + y^2 & -2xy \\ -2xy & 1 + x^2 - y^2 \\ 2x & 2y \end{pmatrix}.$$

The image of $\mathbb{R}^2$ under the linear map $d(\phi_N)_{(x,y)}$ is the tangent space $T_{\phi_N(x,y)}\mathbb{S}^2$. This image is spanned by the two column vectors of the matrix $d(\phi_N)_{(x,y)}$. Let us check that we get the space we would have expected, i.e., the **plane** which is **orthogonal** to the vector $\phi_N(x, y)$:

$$\begin{aligned}
&\left(2x, 2y, x^2 + y^2 - 1\right) \cdot \begin{pmatrix} 1 - x^2 + y^2 \\ -2xy \\ 2x \end{pmatrix} \\
&= 2x(1 - x^2 + y^2) - 4xy^2 + 2x(x^2 + y^2 - 1) \\
&= 2x - 2x^3 + 2xy^2 - 4xy^2 + 2x^3 + 2xy^2 - 2x \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(2x, 2y, x^2 + y^2 - 1\right) \cdot \begin{pmatrix} -2xy \\ 1 + x^2 - y^2 \\ 2y \end{pmatrix} \\
&= -4x^2y + 2y(1 + x^2 - y^2) + 2y(x^2 + y^2 - 1) \\
&= -4x^2y + 2y + 2x^2y - 2y^3 + 2x^2y + 2y^3 - 2y \\
&= 0.
\end{aligned}$$

Hence the plane spanned by the column vectors is orthogonal to $\phi_N(x, y)$.

[a] We do not have to translate first to get $\phi_N(0) = p$. That is up to us.

- **Open subsets and tangent spaces**

Let $X \subset \mathbb{R}^N$ be a k-dimensional manifold and W be an **open** subset. Then W is also a k-dimensional manifold, since we can restrict all local parametrizations to the intersection with W (which is again open in X). In fact, for $x \in W$, let $\phi : U \to X$ be a local parametrization around x of X. We can assume $\phi(0) = x$. Then $\phi_{|W \cap U} : \phi^{-1}(W \cap U) \to W$ is a local parametrization around x of W. Since the derivative only depends on an open neighborhood around a point, we get $d\phi_0 = d(\phi_{|W\cap U})_0$. In particular, for the tangent spaces at x, we get

$$T_x(X) = d\phi_0(\mathbb{R}^k) = d(\phi_{|W\cap U})_0(\mathbb{R}^k) = T_x(W)$$

as vector subspaces of $\mathbb{R}^N$.

We summarize this discussion as a lemma:

Lemma 2.39 (Tangent spaces and open subsets) Let X be a k-dimensional manifold and W be an **open** subset. At any point $x \in W$, we have

$$T_x(X) = T_x(W).$$

A simple way to produce new manifolds is by taking products. We have already met an important example of this construction: the two-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. The tangent space of such a product behaves as nicely as we can imagine:

Lemma 2.40 (Tangent space of a product) Given two smooth manifolds $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ and points $x \in X$, $y \in Y$, then the tangent space of the product X and Y is the product of the tangent spaces, i.e.

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

Proof: This follows from the fact that we can choose neighborhoods in $X \times Y$ by taking the product of neighborhoods in X and Y, respectively. Moreover, it is easy to check that $f : X \to X'$ and $g : Y \to Y'$ are smooth maps, then the derivative of the product map is the product of the derivatives, i.e.,

$$d(f \times g)_{(x,y)} = df_x \times dg_y$$

for all $(x, y) \in X \times Y$. □

2.4.5 The induced derivative

Now we will turn to the effect of smooth maps on tangent spaces. In fact, every smooth map between manifolds induces a linear map between tangent spaces. These linear maps are very useful.

Let $f : X \to Y$ be a smooth map from a k-dimensional smooth manifold $X \subseteq \mathbb{R}^N$ to an l-dimensional smooth manifold $Y \subseteq \mathbb{R}^M$. We would like to define a map **best linear approximation of f at a point $(x, f(x))$**. For $y = f(x)$, this should result in a **linear map** of vector spaces

$$T_x(X) \to T_y(Y).$$

Suppose that $\phi : U \to X$ is a local parametrization around x with $U \subseteq \mathbb{R}^k$, and $\psi : V \to Y$ a local parametrization around y with $V \subseteq \mathbb{R}^l$. We can assume $\phi(0) = x$ and $\psi(0) = y$. Then we define a map $\theta : U \to V$ by the following commutative diagram:[6]

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, bend left, dotted, "\psi^{-1}"] \\ U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V . \arrow[u, "\psi"'] \end{tikzcd}$$

Definition 2.41 **(The derivative df_x)** Taking derivatives yields a diagram of linear maps and we **define** df_x to be the **linear map** which **makes** the diagram commutative:

$$\begin{tikzcd} T_x(X) \arrow[r, dotted, "df_x"] \arrow[d, bend left, "d(\phi^{-1})_x"] & T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "d\theta_0"'] & \mathbb{R}^l . \arrow[u, "d\psi_0"'] \end{tikzcd}$$

Since $d\phi_0$ is an isomorphism, we can **define** df_x as

$$\mathbf{df_x := d\psi_0 \circ d\theta_0 \circ d(\phi^{-1})_x}.$$

We call df_x the **derivative of f at x**.

Remark 2.42 **(Why so complicated?)** You may wonder why we need to take this detour to define df_x when we could also consider f as a map of subsets of Euclidean spaces and take the derivative of that map, since we assume $X \subset R^N$ and $Y \subset \mathbb{R}^M$ for some N and M anyway. This works nicely **if** X is an **open** subset in $\mathbb{R}^N$. For if $X \subset \mathbb{R}^N$ is **open**, we can choose ϕ as the identity map and have $T_x(X) = T_x(\mathbb{R}^N)$. Then the derivative $dF_x : T_x(X) = T_x(\mathbb{R}^N) = \mathbb{R}^N \to T_y(Y)$ is actually the derivative as a smooth map between Euclidean space.

However, **if X is not open** in $\mathbb{R}^N$ we need to work a bit harder. By definition of smoothness, there is an open subset $W \subset \mathbb{R}^N$ and a smooth map $F : W \to \mathbb{R}^M$ such that $F_{|W\cap X} = f_{|W\cap X}$. The derivative of F at x is a linear map $dF_x : \mathbb{R}^N \to \mathbb{R}^M$. But what we want is a linear map defined on $T_x(X) \subset \mathbb{R}^N$ and with image in $T_y(Y) \subset \mathbb{R}^M$. When we look at the gymnastics we do to define df_x, we see that this is exactly what we do: we restrict and adjust dF_x to the vector subspace $T_x(X) \subset \mathbb{R}^N$ such that it has image in $T_y(Y)$. Thus, in the end, the seemingly complicated definition is just the linear algebra necessary to assure that df_x has the correct domain and codomain. We will see later when we learn about regular values and transversal intersections that there is often a short cut to make our first intuition work.

[6] which means that it does not matter which way we walk around from U to Y.

2.4.6 The derivative is well-defined:

We should check that the derivative is well-defined, i.e., that df_x does **not** depend on the **choice** of local parametrizations around x and $y = f(x)$. So let $\phi' : U \to X$ and $\psi' : V' \to Y$ be another choice of local parametrizations around x and y, respectively. Again after possibly shrinking both U, U', V and V' we can assume that $\phi(U) = \phi'(U') \subseteq X$ and $\psi(V) = \psi'(V') \subseteq Y$.

Then $d\phi_0$ and $d\phi'_0$ differ by a linear isomorphism of $\mathbb{R}^k$, say α: $d\phi_0 = d\phi'_0 \circ \alpha$. Similarly, there is a linear isomorphism β of $\mathbb{R}^l$ such that $d\psi_0 = d\psi'_0 \circ \beta$. Let $\theta' : U \to V$ be defined similarly to θ, i.e., we set $\theta' = \psi'^{-1} \circ f \circ \phi'$. This gives us the following diagram in which each square commutes

$$\begin{tikzcd}
T_x(X) \arrow[rr, dotted, "df_x"] & & T_y(Y) \\
\mathbb{R}^k \arrow[u, "d\phi'_0"'] \arrow[rr, "d\theta'_0"'] & & \mathbb{R}^l \arrow[u, "d\psi'_0"] \\
\mathbb{R}^k \arrow[u, "\alpha"'] \arrow[uu, bend left=60, "d\phi_0"] \arrow[rr, "d\theta_0"'] & & \mathbb{R}^l. \arrow[u, "\beta"] \arrow[uu, bend right=60, "d\psi_0"']
\end{tikzcd}$$

The relation between $d\theta_0$ and $d\theta'_0$ is given by

$$d\theta'_0 = \beta \circ d\theta_0 \circ \alpha^{-1}.$$

Putting all relations together we get

$$\begin{aligned} d\psi'_0 \circ d\theta'_0 \circ d(\phi'^{-1})_x &= d\psi'_0 \circ (\beta \circ d\theta_0 \circ \alpha^{-1}) \circ d(\phi'^{-1})_x \\ &= (d\psi'_0 \circ \beta) \circ d\theta_0 \circ (\alpha^{-1}) \circ d(\phi'^{-1})_x) \\ &= d\psi_0 \circ d\theta_0 \circ d(\phi^{-1})_x. \end{aligned}$$

This implies the desired identity for the a priori different constructions of df_x. □

Before we look at an example, we would like to know that the new derivative satisfies a **chain rule**, since this is a very useful rule.

- **The chain rule:**

Let $g : Y \to Z$ be another smooth map. Let $\eta : W \to Z$ be a local parametrization around $z = g(y)$ with an open subset $W \subseteq \mathbb{R}^m$ and $\eta(0) = z$. Then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"'] \arrow[r, "\iota=\eta^{-1}\circ g\circ\psi"'] & W \arrow[u, "\eta"']
\end{tikzcd}$$

which gives us the commutative square

$$\begin{tikzcd} X \arrow[rr, "g\circ f"] & & Z \\ U \arrow[u, "\phi"] \arrow[rr, "\iota\circ\theta"'] & & W. \arrow[u, "\eta"'] \end{tikzcd}$$

Thus, by definition,

$$d(g\circ f)_x = d\eta_0 \circ d(\iota\circ\theta)_0 \circ d(\phi^{-1})_x.$$

The chain rule from Calculus for maps of open sets of Euclidean spaces, then gives

$$d(\iota\circ\theta)_0 = (d\iota_0)\circ(d\theta_0).$$

Thus

$$d(g\circ f)_x = (d\eta_0 \circ d\iota_0 \circ d(\psi^{-1})_y)\circ(d\psi_0 \circ d\theta_0 \circ d(\phi^{-1})_x) = dg_y \circ df_x.$$

Hence we have in fact the desired rule.

Theorem 2.43 (Chain Rule) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then

$$d(g\circ f)_x = dg_{f(x)} \circ df_x.$$

2.4.7 Example: The Hopf fibration

We conclude this section with another important example and some concrete calculations. Recall the **Hopf map** $f\colon \mathbb{S}^3 \to \mathbb{S}^2$ defined by

$$f\colon \mathbb{S}^3 \to \mathbb{S}^2, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1x_3 + 2x_2x_4 \\ 2x_2x_3 - 2x_1x_4 \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}.$$

We will now compute the derivative of f at two points concrete points in $\mathbb{S}^3$. Later we will be able to appreciate the relevance of these computations and the choice of points much better. For the moment, we consider this just as training and illustration.

First, let us pick a point which is mapped to the south pole $\mathbf{s}_2 = (0, 0, -1) \in \mathbb{S}^2$. The formula for f shows that all points $(x_1, x_2, x_3, x_4) \in \mathbb{S}^3$ with $x_1 = x_2 = 0$ are mapped to $\mathbf{s}_2$. Hence, in particular, the north pole $\mathbf{n}_3 = (0, 0, 0, 1) \in \mathbb{S}^3$ is mapped to $\mathbf{s}_2 = (0, 0, -1) \in \mathbb{S}^2$. So let us look at the point $\mathbf{n}_3$ which is sent to $\mathbf{s}_2$.

Since the formula for the stereographic projections can be quite involved when the number of variables increases, we use the local parametrizations given as the inverse of the projection onto the first coordinates. We used them in Example 2.26 for the $\mathbb{S}^1$. We choose the **open** ball $\mathbb{B}^3(\mathbf{0}_3)$ around the origin $\mathbf{0}_3$ in $\mathbb{R}^3$ of radius[7] $1/\sqrt{2}$ and use the local parametrization

$$\phi\colon U = \mathbb{B}^3_{1/\sqrt{2}}(\mathbf{0}_3) \to W_3,\ (x, y, z) \mapsto \left(x, y, z, \sqrt{1 - (x^2 + y^2 + z^2)}\right)$$

[7] The choice of the radius will become apparent in a minute when we compute $f\circ\phi$ which needs to have image in the subset of points in $\mathbb{S}^2$ with $x_3 < 0$.

where $W_3 \subset \mathbb{S}^3_{x_4>0} \subset \mathbb{S}^3$ denotes the open subset of $\mathbb{S}^3$ consisting of points with coordinate $x_4 > 0$ and $2x_1^2 + 2x_2^2 < 1$. Note that ϕ maps $\mathbf{0}_3$ to $\mathbf{n}_3$. Similarly, we choose the **open** ball $\mathbb{B}^2_1(\mathbf{0}_2)$ around the origin $\mathbf{0}_2$ in $\mathbb{R}^2$ of radius 1 and use the local parametrization

$$\psi\colon V = \mathbb{B}^2_1(\mathbf{0}_2) \to W_2,\ (x,y) \mapsto \left(x, y, -\sqrt{1-(x^2+y^2)}\right)$$

where $W_2 = \mathbb{S}^2_{x_3<0} \subset \mathbb{S}^2$ denotes the open subset of $\mathbb{S}^2$ consisting of points with coordinate $x_3 < 0$. Note that ψ maps $\mathbf{0}_2$ to $\mathbf{s}_2$.

Now we need to calculate the induced map $\theta\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that the diagram commutes

$$\begin{tikzcd}
W_3 \arrow[r, "f"] & W_2 \arrow[d, dotted, bend left, "\psi^{-1}"] \\
\mathbb{B}^3_{1/\sqrt{2}}(\mathbf{0}_3) \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & \mathbb{B}^2_1(\mathbf{0}_2). \arrow[u, "\psi"]
\end{tikzcd}$$

We calculate the effect of the maps step by step. Recall that we write $|\mathbf{x}|$ for the norm of points $\mathbf{x} \in \mathbb{R}^k$. First we apply ϕ to a point in $\mathbb{B}^3(\mathbf{0}_3)$:

$$\phi\colon \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \\ \sqrt{1-|\mathbf{x}|^2} \end{pmatrix}.$$

Next we apply the composite $f\circ\phi$ to a point in U:

$$f\circ\phi\colon \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2xz + 2y\sqrt{1-|\mathbf{x}|^2} \\ 2yz - 2x\sqrt{1-|\mathbf{x}|^2} \\ x^2+y^2-z^2-1+|\mathbf{x}|^2 \end{pmatrix}.$$

Recall that the inverse ψ^{-1} is just the projection onto the first two coordinates. Hence applying the composite $\psi^{-1}\circ\phi$ to a point in $\mathbb{R}^3$ yields:

$$\psi^{-1}\circ f\circ\phi\colon \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2xz + 2y\sqrt{1-|\mathbf{x}|^2} \\ 2yz - 2x\sqrt{1-|\mathbf{x}|^2} \end{pmatrix}.$$

This is the map $\psi^{-1}\circ f\circ\phi\colon U \to V$.[8]

Now we need to apply this result to compute the horizontal map df_{n_3} such that following diagram for tangent spaces commutes:

$$\begin{tikzcd}
T_{\mathbf{n}_3}(\mathbb{S}^3) \arrow[r, dotted, "df_{\mathbf{n}_3}"] \arrow[d, bend left, "d(\phi^{-1})_{\mathbf{n}_3}"] & T_{\mathbf{s}_2}(\mathbb{S}^2) \\
\mathbb{R}^3 \arrow[u, "d\phi_{\mathbf{0}_3}"] \arrow[r, "d\theta_{\mathbf{0}_3}"'] & \mathbb{R}^2. \arrow[u, "d\psi_{\mathbf{0}_2}"']
\end{tikzcd}$$

[8] A good reality check is that θ does map the origin $\mathbf{0}_3$ to the origin $\mathbf{0}_2$ as we claimed.

In particular, we need to calculate the derivative $d\theta_{\mathbf{0}_3} : \mathbb{R}^3 \to \mathbb{R}^2$ at the origin $\mathbf{0}_3$. We do this by first computing the Jacobian matrix J_θ of θ:

$$J_\theta = \begin{pmatrix} 2z - \frac{2xy}{\sqrt{1-|\mathbf{x}|^2}} & 2\sqrt{1-|\mathbf{x}|^2} - \frac{2y^2}{\sqrt{1-|\mathbf{x}|^2}} & 2x - \frac{2yz}{\sqrt{1-|\mathbf{x}|^2}} \\ -2\sqrt{1-|\mathbf{x}|^2} + \frac{2x^2}{\sqrt{1-|\mathbf{x}|^2}} & 2z + \frac{2xy}{\sqrt{1-|\mathbf{x}|^2}} & 2y + \frac{2xz}{\sqrt{1-|\mathbf{x}|^2}} \end{pmatrix}.$$

This looks annoyingly complicated. However, there is good news. We want to calculate the matrix representing $d\theta_{\mathbf{0}_3}$ at the origin. Hence we set $x = y = z = 0$ and see that $d\theta_{\mathbf{0}_3} : \mathbb{R}^3 \to \mathbb{R}^2$ is given by the matrix

$$d\theta_{\mathbf{0}_3} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

By our definition of $T_{\mathbf{n}_3}(\mathbb{S}^3)$ as the image of $d\phi_{\mathbf{0}_3} : \mathbb{R}^3 \to \mathbb{R}^4$. As a basis of $T_{\mathbf{n}_3}(\mathbb{S}^3)$ we can hence choose the images of the standard basis $\mathbf{e}_1^3, \mathbf{e}_2^3, \mathbf{e}_3^3$ of $\mathbb{R}^3$ under $d\phi_{\mathbf{0}_3}$. With respect this basis $d\phi_{\mathbf{0}_3}(\mathbf{e}_1^3), d\phi_{\mathbf{0}_3}(\mathbf{e}_2^3), d\phi_{\mathbf{0}_3}(\mathbf{e}_3^3)$ for $T_{\mathbf{n}_3}(\mathbb{S}^3)$ and the standard basis for $\mathbb{R}^3$, the derivative

$$d(\phi^{-1})_{\mathbf{n}_3} : T_{\mathbf{n}_3}(\mathbb{S}^3) \to \mathbb{R}^3$$

is represented by the 3×3-identity matrix.

Similarly, for $T_{\mathbf{s}_2}(\mathbb{S}^2)$ we can choose the image of the standard basis $\mathbf{e}_1^2, \mathbf{e}_2^2$ of $\mathbb{R}^2$ under $d\psi_{\mathbf{0}_2}$. With respect to the standard basis for $\mathbb{R}^2$ and the basis $d\psi_{\mathbf{0}_2}(\mathbf{e}_1^2), d\psi_{\mathbf{0}_2}(\mathbf{e}_2^2)$ for $T_{\mathbf{s}_2}(\mathbb{S}^2)$ the derivative

$$d\psi_{\mathbf{0}_2} : \mathbb{R}^2 \to T_{\mathbf{s}_2}(\mathbb{S}^2)$$

is represented by the 2×2-identity matrix.

Hence — with respect to these bases for $T_{\mathbf{n}_3}(\mathbb{S}^3)$ and $T_{\mathbf{s}_2}(\mathbb{S}^2)$ — we see that the composition

$$df_{\mathbf{n}_3} = d\psi_{\mathbf{0}_2} \circ d\theta_{\mathbf{0}_3} \circ d(\phi^{-1})_{\mathbf{n}_3}$$

is given by the matrix

$$df_{\mathbf{n}_3} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

Note that our choices of bases make it very easy to compute the matrix for $df_{\mathbf{n}_3}$. So in the end, there is not as much to compute as one might fear. To make things even more explicit we observe that $df_{\mathbf{n}_3}$ has the effect on the basis vectors:

$$\begin{aligned} d\phi_{\mathbf{0}_3}(\mathbf{e}_1^3) &\mapsto 0 \cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_1^2) - 2 \cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_2^2), \\ d\phi_{\mathbf{0}_3}(\mathbf{e}_2^3) &\mapsto 2 \cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_1^2) + 0 \cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_2^2), \text{ and} \\ d\phi_{\mathbf{0}_3}(\mathbf{e}_3^3) &\mapsto 0 \cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_1^2) + 0 \cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_2^2). \end{aligned}$$

Remark 2.44 (Something we learn from this example) Among other things we see in this example that — once we have computed θ — there is a straight-forward way to compute a matrix which describes df_x. For, in this setting, there is a **canonical choice** for the bases of $T_x(X)$ and $T_y(Y)$: the images of the standard basis vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$ under the isomorphisms $d\phi_0$ and $d\psi_0$, respectively.

Then we can **compute the matrix** which represents the linear map $df_x : T_x(X) \to T_y(Y)$ with respect to these bases just as the matrix which represents $d\theta_0$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. And we get this matrix as the **Jacobian** matrix of θ at the origin.

Second, let us apply what we just learned and practice a bit more. So let us look at a point on $\mathbb{S}^3$ which is mapped to the north pole $\mathbf{n}_2 = (0, 0, 1) \in \mathbb{S}^2$. The formula for f shows that all points $(x_1, x_2, x_3, x_4) \in S^3$ with $x_3 = x_4 = 0$ are mapped to $\mathbf{n}_2$. Hence, in particular, the point $\mathbf{q}_3 = (1, 0, 0, 0) \in S^3$ is mapped to $\mathbf{n}_2 = (0, 0, 1) \in \mathbb{S}^2$. So let us pick that $\mathbf{q}_3$ mapping to $\mathbf{n}_2$.

Much of the calculation is the same as in the previous case. However, there are some interesting changes, in particular, of some signs. Again, we will be able to appreciate this more later.

We choose the **open** ball $\mathbb{B}^3(\mathbf{0}_3)$ around the origin $\mathbf{0}_3$ in $\mathbb{R}^3$ of radius $1/\sqrt{2}$ and use the local parametrization

$$\phi : U = \mathbb{B}^3_{1/\sqrt{2}}(\mathbf{0}_3) \to W_3, \ (x, y, z) \mapsto \left(\sqrt{1 - (x^2 + y^2 + z^2)}, x, y, z\right)$$

where $W_3 \subset \mathbb{S}^3_{x_1>0} \subset \mathbb{S}^3$ denotes the open subset of $\mathbb{S}^3$ consisting of points with coordinate $x_1 > 0$ and $2x_3^2 + 2x_4^2 < 1$. We choose the **open** ball $\mathbb{B}^2_1(\mathbf{0}_2)$ around the origin $\mathbf{0}_2$ in $\mathbb{R}^2$ of radius 1 and use the local parametrization

$$\psi : V = \mathbb{B}^2_1(\mathbf{0}_2) \to W_2, \ (x, y) \mapsto \left(x, y, \sqrt{1 - (x^2 + y^2)}\right)$$

where $W_2 \subset \mathbb{S}^2_{x_3>0}$ denotes the open subset of $\mathbb{S}^2$ consisting of points with coordinate $x_3 > 0$.

We need to calculate the induced map $\theta : \mathbb{R}^3 \to \mathbb{R}^2$ such that the diagram commutes

$$\begin{tikzcd}
W_3 \arrow[rr, "f"] & & W_2 \arrow[d, dotted, bend left, "\psi^{-1}"] \\
\mathbb{B}^3_{1/\sqrt{2}}(\mathbf{0}_3) \arrow[u, "\phi"] \arrow[rr, "\theta=\psi^{-1}\circ f\circ\phi"'] & & \mathbb{B}^2_1(\mathbf{0}_2). \arrow[u, "\psi"]
\end{tikzcd}$$

We calculate the effect of the maps step by step. Recall that we write $|\mathbf{x}|$ for the norm of points $\mathbf{x} \in \mathbb{R}^k$. First we apply ϕ to a point in $\mathbb{B}^3(\mathbf{0}_3)$:

$$\phi : \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{1 - |\mathbf{x}|^2} \\ x \\ y \\ z \end{pmatrix}.$$

Next we apply the composite $f\circ\phi$:

$$f\circ\phi\colon\ \mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}2y\sqrt{1-|\mathbf{x}|^2}+2xz\\2xy-2\sqrt{1-|\mathbf{x}|^2}z\\1-|\mathbf{x}|^2+x^2-y^2-z^2\end{pmatrix}$$
$$=\begin{pmatrix}2y\sqrt{1-|\mathbf{x}|^2}+2xz\\2xy-2\sqrt{1-|\mathbf{x}|^2}z\\1-2y^2-2z^2\end{pmatrix}.$$

Applying the composite $\psi^{-1}\circ\phi$ to a point in $\mathbb{R}^3$ yields:

$$\psi^{-1}\circ f\circ\phi\colon\ \mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}2y\sqrt{1-|\mathbf{x}|^2}+2xz\\2xy-2\sqrt{1-|\mathbf{x}|^2}z\end{pmatrix}.$$

This is the map $\psi^{-1}\circ f\circ\phi\colon\ U\to V$.

Now we need to apply this result to compute the horizontal map df_{q_3} such that following diagram for tangent spaces commutes:

$$\begin{tikzcd} T_{\mathbf{q}_3}(\mathbb{S}^3) \arrow[rr, dotted, "df_{\mathbf{q}_3}"] \arrow[d, bend left, "d(\phi^{-1})_{\mathbf{q}_3}"] & & T_{\mathbf{n}_2}(\mathbb{S}^2) \\ \mathbb{R}^3 \arrow[u, "d\phi_{\mathbf{0}_3}"] \arrow[rr, "d\theta_{\mathbf{0}_3}"'] & & \mathbb{R}^2. \arrow[u, "d\psi_{\mathbf{0}_2}"'] \end{tikzcd}$$

In particular, we need to calculate the derivative $d\theta_{\mathbf{0}_3}:\ \mathbb{R}^3\to\mathbb{R}^2$ at the origin $\mathbf{0}_3$. We do this again by computing the Jacobian matrix J_θ of θ:

$$J_\theta=\begin{pmatrix}2z-\frac{2xy}{\sqrt{1-|\mathbf{x}|^2}} & 2\sqrt{1-|\mathbf{x}|^2}-\frac{2y^2}{\sqrt{1-|\mathbf{x}|^2}} & 2x-\frac{2yz}{\sqrt{1-|\mathbf{x}|^2}}\\ 2y+\frac{2xz}{\sqrt{1-|\mathbf{x}|^2}} & 2x+\frac{2yz}{\sqrt{1-|\mathbf{x}|^2}} & \frac{2z^2}{\sqrt{1-|\mathbf{x}|^2}}-2\sqrt{1-\mathbf{x}|^2}\end{pmatrix}.$$

At the origin we set $x=y=z=0$ and see that $d\theta_{\mathbf{0}_3}:\ \mathbb{R}^3\to\mathbb{R}^2$ is given by the matrix

$$d\theta_{\mathbf{0}_3}=\begin{pmatrix}0&2&0\\0&0&-2\end{pmatrix}$$

with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

Now we make our standard choice of bases of $T_{\mathbf{q}_3}(\mathbb{S}^3)$, i.e., $d\phi_{\mathbf{0}_3}(\mathbf{e}^3_1), d\phi_{\mathbf{0}_3}(\mathbf{e}^3_2), d\phi_{\mathbf{0}_3}(\mathbf{e}^3_3)$, and of $T_{\mathbf{n}_2}(\mathbb{S}^2)$, i.e., $d\psi_{\mathbf{0}_2}(\mathbf{e}^2_1), d\psi_{\mathbf{0}_2}(\mathbf{e}^2_2)$. Then — with respect to these bases for $T_{\mathbf{q}_3}(\mathbb{S}^3)$ and $T_{\mathbf{n}_2}(\mathbb{S}^2)$ — the composition

$$df_{\mathbf{q}_3}=d\psi_{\mathbf{0}_2}\circ d\theta_{\mathbf{0}_3}\circ d(\phi^{-1})_{\mathbf{q}_3}$$

is given by the matrix

$$df_{\mathbf{q}_3}=\begin{pmatrix}0&2&0\\0&0&-2\end{pmatrix}.$$

In this case, we observe that $df_{\mathbf{q}_3}$ has the effect on the basis vectors that it sends $d\phi_{\mathbf{0}_3}(\mathbf{e}_1^3)$ to the zero vector, $d\phi_{\mathbf{0}_3}(\mathbf{e}_2^3)$ to the vector $2\cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_1^2)+0\cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_2^2)$, and $d\phi_{\mathbf{0}_3}(\mathbf{e}_3^3)$ to $0\cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_1^2)-2\cdot d\psi_{\mathbf{0}_2}(\mathbf{e}_2^2)$.

Remark 2.45 (Outlook to orientations) We will appreciate this example even more when we have learned about **orientations**. For the above computation shows that $df_{\mathbf{q}_3}$ sends a **positively** oriented basis to a **negatively** oriented basis. In other words, $df_{\mathbf{q}_3}$ reverses orientations. More on this later.

2.5 Tangent Bundle

In this section we look at yet another example of an interesting space, the tangent bundle, which is formed by the collection of tangent spaces of a given smooth manifold X. We will see that it is **itself a smooth manifold**. Later on we will learn that it tells us quite a lot about the geometry of X. Moreover, the tangent bundle will turn out to be an extremely useful tool for many constructions. See for example Section 9.6.

Advice: The reader who may not feel comfortable yet with tangent spaces and what they are may want to skip this section first and get back to it later when we use the tangent bundle.

2.5.1 Tangent Bundle - the definition

Let $X\subset\mathbb{R}^N$ be a smooth manifold. For every $x\in X$, the tangent space $T_x(X)$ to X at x is a vector subspace of $\mathbb{R}^N$. If we **let x vary**, these tangent space will in general overlap in $\mathbb{R}^N$. For example, if X is a vector space itself, they will all be equal.

Hence in order to be able to keep track of the information contained in all the different tangent spaces we need a smart device that keeps those spaces apart:

Definition 2.46 (Tangent bundle) The **tangent bundle** of X, denoted $T(X)$, is the subset of $X\times\mathbb{R}^N\subset\mathbb{R}^N\times\mathbb{R}^N$ defined by

$$T(X):=\{(x,v)\in X\times\mathbb{R}^N\,:\,v\in T_x(X)\}.$$

In particular, $T(X)$ contains a **natural copy of X** consisting of the points $(x,0)$. In the direction **perpendicular** to X_0, it contains **copies of each tangent space** $T_x(X)$ embedded as the sets

$$\{(x,v)\in T(X):\ \text{for a fixed } x\}\,.$$

There is a natural **projection map**

$$\pi:\,T(X)\to X,\ (x,v)\mapsto x.$$

Any smooth map $f : X \to Y$ induces a **global derivative map**

$$df : T(X) \to T(Y),\ (x, v) \mapsto (f(x), df_x(v)).$$

Note that, since $X \subset \mathbb{R}^N$ and $T_x(X) \subset \mathbb{R}^N$ for every x, $T(X)$ is also a subset of Euclidean space:

$$T(X) \subset \mathbb{R}^N \times \mathbb{R}^N.$$

Therefore, if $Y \subset \mathbb{R}^M$, then df maps a subset of $\mathbb{R}^{2N}$ to $\mathbb{R}^{2M}$.

Lemma 2.47 The map $df : T(X) \to T(Y)$ is smooth as a map between subsets of $\mathbb{R}^{2N}$ to $\mathbb{R}^{2M}$.

Proof: Since $f : X \to \mathbb{R}^M$ is **smooth**, it extends by definition around any point $x \in X$ to a smooth map $F : U \to \mathbb{R}^M$, where U is an **open** set of $\mathbb{R}^N$. We will now show that the derivative $dF : T(U) \to \mathbb{R}^{2M}$ also **locally extends** the derivative df: Since $U \subset \mathbb{R}^N$ is open and hence $T_u(U) = \mathbb{R}^N$ for every $u \in U$, $T(U)$ is **all of** $U \times \mathbb{R}^N$. Since $U \times \mathbb{R}^N$ is an open set in $\mathbb{R}^{2N}$, dF is a **linear** and hence **smooth** map defined on an open subset of $\mathbb{R}^{2N}$. This shows that $df : T(X) \to \mathbb{R}^{2M}$ may be **locally extended** to a smooth map on an open subset of $\mathbb{R}^{2N}$, meaning that df is smooth. □

We can also say something about the derivative of the **composition of smooth maps**: Given smooth maps $f : X \to Y$ and $g : Y \to Z$, the global derivative of the composite is equal to the composite of global derivatives:

$$d(g \circ f) = dg \circ df : T(X) \to T(Z).$$

For, the chain rule implies that, for any $(x, v) \in T(X)$,

$$\begin{aligned} d(g \circ f)(x, v) &= ((g \circ f)(x), d(g \circ f)_x(v)) \\ &= ((g(f(x)), (dg_{f(x)} \circ df_x)(v)) \\ &= dg(df(x, v)) \\ &= dg \circ df(x, v). \end{aligned}$$

Now, if $f : X \to Y$ is a **diffeomorphism**, then $df : T(X) \to T(Y)$ is a **diffeomorphism**:

Lemma 2.48 **(Tangent bundles are intrinsic)** Diffeomorphic manifolds have diffeomorphic tangent bundles. As a result, $T(X)$ is an object **intrinsically** associated to X, i.e., it does **not depend** on the ambient Euclidean space.

2.5.2 Tangent bundles are manifolds

Finally, we are going to show that $T(X)$ is in fact **itself a smooth manifold**. We will use this fact for example in the proof of Whitney's theorem.

Theorem 2.49 (Tangent bundles are manifolds) Let X be a smooth n-dimensional manifold. Then the **tangent bundle** $T(X)$ is a smooth **manifold** of dimension $2n$.

Proof: Let W be an **open** set of X. In particular, W is also a manifold, and we can consider its tangent bundle $T(W)$. Since $T_x(W) = T_x(X)$ for every $x \in W$, $T(W)$ is by definition

$$T(W) = \{(x, v) \in T(X) : x \in W\} = T(X) \cap (W \times \mathbb{R}^N) \subset T(X).$$

Since $W \times \mathbb{R}^N$ is open in $X \times \mathbb{R}^N$, $T(W)$ is **open** in $T(X)$. Now suppose that W is the **image of a local parametrization** $\phi : U \to W$, where U is an open set in $\mathbb{R}^k$. Then the **global derivative** $d\phi : T(U) \to T(W)$ is a **diffeomorphism**. But $T(U) = U \times \mathbb{R}^k$ is an open subset of $\mathbb{R}^{2k}$, so **$d\phi$ is a parametrization** of the open set $T(W)$ in $T(X)$. Since every point of $T(X)$ sits in such a neighborhood, we have proved the assertion. □

In the following sections, we are going to use the tangent bundle as a tool to construct new maps. The key will be that the tangent bundle gives us **extra space** for manoeuvring.

2.5.3 Tangent bundles are vector bundles

Tangent bundles are examples of a more general class of spaces, called smooth **vector bundles**. They can be defined on any topological space. But let us assume we have a manifold X. Roughly speaking, an n-dimensional **vector bundle** E consists of two data:

- an **assignment** of an n-dimensional **vector space** V to each point $x \in X$;
- a **rule** for how to **glue** all these vector spaces together in a **nice way**.

More precisely, an n-dimensional **vector bundle E over X** consists of a topological space E together with a continuous map $\pi : E \to X$ which satisfy the following **condition**:

- for every point $x \in X$ there is an open subset $U \subset X$ around x and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that for every $y \in X$ the map $v \mapsto (y, v)$ defines a linear isomorphism between the vector space $\mathbb{R}^n$ and $\pi^{-1}(y)$.

If it is possible to choose the open subset around x in the above condition to be **all of X**, then we call $E \to X$ a **trivial bundle**.

We can refine this definition and say that $E \to X$ is a **smooth** vector bundle if we require in addition that

- E is a smooth manifold
- $\pi : E \to X$ is a smooth map
- each of the h above is a diffeomorphism.

The tangent bundle is an important example a smooth vector bundle.

Vector bundles have a **rich and very interesting theory** and many problems can bee formulated in terms of vector bundles. For example, we will see in a later chapter that there is a nice **classifying space** for vector bundles, called the **Grassmannian**. See Section 9.5 for more about this important object. The idea of a classifying space is extremely powerful and we will not be able to appreciate it in this course. However, we encourage everybody to continue reading in this.

We conclude the detour with a famous example of a problem which can be phrased in terms of vector bundles:

Remark 2.50 (Parallelizable spheres) A manifold for which the **tangent bundle is trivial** is called **parallelizable**. Examples of manifolds which are parallelizable are $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$, whereas $\mathbb{S}^2$ is **not** parallelizable. In fact, it is a famous and deep result that $\mathbb{S}^n$ **is parallelizable** if and only if $n = 0, 1, 3$ or 7. This is a consequence of the famous and **fundamental result** on the possible **multiplicative structures on** $\mathbb{R}^n$. For the above statement follows from: Let $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a map with two-sided identity element and no zero-divisors. Then n must be either 1, 2, 4 or 8.

2.6 Exercises and more examples

2.6.1 Smooth maps and manifolds

Exercise 2.1 Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x^2 - y^2, 2xy)$.

(a) Check that f is smooth by calculating the partial derivatives.

(b) Show that the Jacobian matrix of the restriction $f_{|U}\colon U = \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$, $(x, y) \mapsto (x^2 - y^2, 2xy)$, is invertible for every point in U.

(c) Is $f_{|U}$ a diffeomorphism?

Exercise 2.2 Let $X \subseteq \mathbb{R}^N$, $Y \subseteq \mathbb{R}^M$ and $Z \subseteq \mathbb{R}^L$ be arbitrary subsets, and let $f\colon X \to Y$, $g\colon Y \to Z$ be smooth maps with $f(X) \subseteq Y$.

(a) Show the composite $g \circ f\colon X \to Z$ is smooth if g and f are smooth.

Hint: If all subsets are open, this is just the Chain Rule from Calculus.

(b) Show that if g and f are diffeomorphisms, so is $g \circ f$.

Exercise 2.3 For a real number $r > 0$, let $\mathbb{B}_r = \mathbb{B}_r^k(0) = \{x \in \mathbb{R}^k : |x| < r\}$ be the open ball around the origin with radius r in $\mathbb{R}^k$.

(a) Show that the map
$$f\colon \mathbb{B}_r \to \mathbb{R}^k,\ x \mapsto \frac{rx}{\sqrt{r^2 - |x|^2}}$$
is a diffeomorphism from $\mathbb{B}_r$ to $\mathbb{R}^k$.
Hint: Compute the inverse directly, and use the previous exercise to show smoothness.

(b) Suppose that X is a k-dimensional manifold. Show that every point in X has a neighborhood diffeomorphic to an open ball in $\mathbb{R}^k$ around the origin.

(c) Suppose that X is a k-dimensional manifold. Show that every point in X has a neighborhood diffeomorphic to all of $\mathbb{R}^k$.

Exercise 2.4 Show that every k-dimensional vector subspace V of $\mathbb{R}^N$ is a manifold diffeomorphic to $\mathbb{R}^k$ and that any linear map $V \to \mathbb{R}^m$ is smooth.

Comment: Recall that choosing a basis for V corresponds to choosing a linear isomorphism $\phi\colon \mathbb{R}^k \to V$. Expressing a vector in V in terms of this basis means to attach coordinates to this vector. Since ϕ is linear, we refer to the corresponding coordinates as linear coordinates.

Exercise 2.5 Recall the hyperboloid and the cone drawn in Figure 2.10:

(a) Prove that the subspace of $\mathbb{R}^3$, defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$.

(b) Explain why $x^2 + y^2 - z^2 = 0$ does not define a manifold.

Exercise 2.6 The torus $\mathbb{T}(a, b)$ is the set of points in $\mathbb{R}^3$ at distance b from the circle of radius a in the xy-plane, where $0 < b < a$. Prove that each $\mathbb{T}(a, b)$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$. What happens when $b = a$?

Exercise 2.7 Let $N = (0, \dots, 0, 1) \in \mathbb{S}^k$ be the 'north pole' on the k-dimensional sphere. The stereographic projection ϕ_N^{-1} from $\mathbb{S}^k \setminus \{N\}$ onto $\mathbb{R}^k$ is the map which sends a point p to the point at which the line through N and p intersects the subspace in $\mathbb{R}^{k+1}$ defined by $x_{k+1} = 0$. See Figure 2.12 for the case $k = 2$.

(a) Show that ϕ_N^{-1} is given by the formula

$$(x_1, \dots, x_{k+1}) \mapsto \frac{1}{1 - x_{k+1}}(x_1, \dots, x_k).$$

(b) Find a formula for the inverse ϕ_N of ϕ_N^{-1}, and check that both maps are smooth.

(c) Let $S = (0, \dots, 0, -1) \in \mathbb{S}^k$ be the 'south pole'. Describe the parametrization using the stereographic projection starting in S instead of N, and conclude that $\mathbb{S}^k$ is a k-dimensional manifold.

Exercise 2.8 We consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, i.e., $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$, and $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e., $\mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$. Then the *Hopf map* π is the map $\mathbb{S}^3 \to \mathbb{S}^2$ given by

$$\pi(z_0, z_1) = \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right).$$

(a) Check that this actually defines a map $\mathbb{S}^3 \to \mathbb{S}^2$.

(b) Show that $\pi(z_0, z_1) = \pi(w_0, w_1)$ if and only if there is a complex number α with $|\alpha|^2 = \alpha\bar{\alpha} = 1$ such that $(w_0, w_1) = (\alpha z_0, \alpha z_1)$.

(c) Show that, for every point $p \in \mathbb{S}^2$, the fiber $\pi^{-1}(p)$ is diffeomorphic to $\mathbb{S}^1$.

We conclude that the Hopf map π realizes $\mathbb{S}^3$ as a **disjoint** union of fibers which each look like $\mathbb{S}^1$.

2.6.2 Tangent spaces

Exercise 2.9 Let V be a vector subspace of $\mathbb{R}^N$. Show that $T_x(V) = V$ for $x \in V$.

Exercise 2.10 Determine the tangent space to the torus $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ at an arbitrary point p. Recall the description of the torus $\mathbb{T}(a, b) \subset \mathbb{R}^3$ from the previous exercise set. Can you describe the tangent space at a point in $\mathbb{T}(a, b)$?

Exercise 2.11 Determine the tangent space to the subspace of $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$ for $a > 0$.

Exercise 2.12 The graph of a map $f : X \to Y$ is the subset of $X \times Y$ defined by

$$\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in X\}.$$

Define $F : X \to \Gamma(f)$ by $F(x) = (x, f(x))$. We assume that X and Y are smooth manifolds and f is a smooth map.

(a) Show F is a diffeomorphism, and conclude that $\Gamma(f)$ is a smooth manifold.

(b) We also write F for the composite map $F : X \to X \times Y$, $x \mapsto (x, f(x))$. Show that $dF_x(v) = (v, df_x(v))$. (You can use $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$.)

(c) Show that the tangent space to $\Gamma(f)$ at the point $(x, f(x))$ is the graph of $df_x : T_x(X) \to T_{f(x)}(Y)$.

Exercise 2.13 A curve in a manifold X is a smooth map $t \mapsto c(t)$ of an open interval of $\mathbb{R}$ into X. The velocity vector of the curve c at time t_0 in $x_0 = c(t_0)$, denoted simply $\frac{dc}{dt}(t_0)$, is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $dc_{t_0} : \mathbb{R}^1 \to T_{x_0}(X)$.

(a) For $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \dots, c_k(t))$, show that

$$\frac{dc}{dt}(t_0) = dc_{t_0}(1) = (c_1'(t_0), \dots, c_k'(t_0)) \in T_{x_0}\mathbb{R}^k.$$

(b) For an arbitrary k-dimensional smooth manifold, use the above observation and local parametrizations to prove that every vector in $T_{x_0}(X)$ is the velocity vector of some curve in X.

Aside: This shows that there is a correspondence between tangent vectors at $x_0 \in X$ and velocity vectors at t_0 of curves $c : I \to X$ with $c(t_0) = x_0$. Note that two curves $c_1 : I \to X$ and $c_2 : J \to X$, with I and J open in $\mathbb{R}$, have the same velocity vector in $c_1(t_1) = x_0 = c_2(t_2)$ if $d(c_1)_{t_1}(1) = d(c_2)_{t_2}(1) \in T_{x_0}(X)$. One can show that having the same velocity vector in a point of X is an equivalence relation on the set of curves through x_0 in X. Using this relation, we have shown that there is a unique correspondence between tangent vectors at X in x and equivalence classes of smooth curves through x_0 in X.

3. The Inverse Function Theorem, immersions and embeddings

3.1 The Inverse Function Theorem and local diffeomorphisms

For understanding smooth manifolds, it can be smart to study maps between manifolds even though it sounds like making things even more difficult. But assume we know something about X and about a map $f : X \to Y$, then we might be able to say something interesting about Y. In addition, there are a lot of interesting problems which can be stated in terms of properties of maps.

So let $f : X \to Y$ be a smooth map between smooth manifolds. Remember that the derivative at $x \in X$, $df_x : T_xX \to T_{f(x)}Y$, is a **linear map between vector spaces**. We have learned that we may think of the derivative as the **best linear approximation** at a point. Since it is easier to understand linear maps, it would be nice if we could classify maps like f by the behavior of df_x (with x varying in X).

> **Question** How much does df_x tell us about the map f?

For the behaviour df_x, there are three cases which we are going to study:

- $\dim X = \dim Y$ in which case the nicest possible behaviour of f at x is that df_x an isomorphism.
- $\dim X < \dim Y$ in which case the nicest possible behaviour of f at x is that df_x one-to-one.
- $\dim X > \dim Y$ in which case the nicest possible behaviour of f at x is that df_x onto.

We are going to consider these cases separately.

- **First case:** df_x is an **isomorphism**. We begin with the nicest case when df_x is an isomorphism. This implies in particular: $\dim X = \dim Y$.

Manifolds are characterized by the way they look in a neighborhood around any point. So let us **think locally**. In the nicest case, f sends a neighborhood of a point x diffeomorphically to a neighborhood of $y = f(x)$. In this case, f is called a **local** diffeomorphism at x. More precisely, we define:

> **Definition 3.1 (Local diffeomorphism)** Let $f : X \to Y$ be a smooth map between smooth manifolds. Then f is called a **local diffeomorphism at x** if there is an **open**

subset $U \subset X$ containing x such that $f(U) \subset Y$ is **open** in Y and

$$f_{|U} : U \to f(U)$$

is a diffeomorphism. We say that f is a **local diffeomorphism** if it is a local diffeomorphism at every $x \in X$.

If f is a diffeomorphism $U \to V$ between neighborhoods U around $x \in X$ and $y = f(x) \in Y$, respectively, let f^{-1} be its smooth inverse. Then we have $f^{-1} \circ f = \mathrm{Id}_U$ and $f \circ f^{-1} = Id_V$. The chain rule implies

$$d(Id_U)_x = d(f^{-1})_y \circ df_x, \text{ and } d(Id_V)_y = df_x \circ d(f^{-1})_y.$$

But we obviously have $d(\mathrm{Id}_X) = \mathrm{Id}_{T_x(X)}$ for any manifold X and any point $x \in X$. Hence df_x is an isomorphism with inverse $d(f^{-1})_{f(x)}$.

Thus a **necessary condition** for f to be a local diffeomorphism at x is that its derivative $df_x : T_x(X) \to T_y(Y)$ is an isomorphism. It is an important result that this is actually a **sufficient condition**. In order to prove this, we recall the corresponding result for Euclidean space from Calculus:

Theorem 3.2 (The Inverse Function Theorem in Calculus) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is **continuously differentiable** in an open set containing a point $a \in \mathbb{R}^n$, and $\det df_a \neq 0$, i.e., df_a is an **invertible** linear map $\mathbb{R}^n \to \mathbb{R}^n$. Then there is an open set $V \subseteq \mathbb{R}^n$ containing a and an open set $W \subseteq \mathbb{R}^n$ containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable and for all $y \in W$ satisfies

$$d(f^{-1})_y = \left(df_{f^{-1}(y)}\right)^{-1}.$$

Remark 3.3 (It's a map not a fraction) Note that this is exactly the formula you are used to from Calculus 1 where we learned

$$(f^{-1})'(y) = (f'(f^{-1}(y)))^{-1}.$$

You may be used to this formula as $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$ from Calculus. But the fraction here is misleading, since $(f^{-1})'(y)$ is a linear map. The superscript "to the -1" really means **take the inverse map**! In dimension 1, the inverse map happens to be given by multiplication by the inverse number. But for linear maps or matrices in dimensions > 1, we cannot write the inverse as a fraction.

Theorem 3.4 (Inverse Function Theorem) Let X and Y be smooth manifolds. Suppose that $f : X \to Y$ is a smooth map whose derivative

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

at a point $x \in X$ is an isomorphism. Then f is a **local diffeomorphism at x**.

The great thing about the Inverse Function Theorem (IFT) is that it tells us that in order to check that f is a diffeomorphism in a neighborhood of a point x, we just need to check that a **single number**, the determinant of df_x, is **nonzero**.

Idea of Proof: We can assume that X and Y are subsets in $\mathbb{R}^N$ for some large N. Let $\phi: U \to X$ be a local parametrization around $x \in X$, and $\psi: W \to Y$ a local parametrization around $y = f(x) \in Y$ with $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ open and $\phi(0) = x$ and $\psi(0) = y$.[1]

We define the map $\theta: U \to W$ as in the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, bend left, dotted, "\psi^{-1}"] \\ U \arrow[u, "\phi"] \arrow[r, "\theta := \psi^{-1} \circ f \circ \phi"'] & W. \arrow[u, "\psi"'] \end{tikzcd}$$

Then recall that df_x is defined such that the following diagram commutes

$$\begin{tikzcd} T_x(X) \arrow[r, dotted, "df_x"] \arrow[d, bend left, "d(\phi^{-1})_x"] & T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "d\theta_0"'] & \mathbb{R}^l. \arrow[u, "d\psi_0"'] \end{tikzcd}$$

Our **assumption** is that df_x is an **isomorphism** which implies that **$d\theta_0$ is an isomorphism**. By the **IFT in Calculus**, this implies that

- there is an open neighborhood $V \subseteq U$ around 0 and
- there is an open neighborhood $V' \subseteq W$ around 0 such that
- $\theta_{|V}: V \to V'$ is a **diffeomorphism**.

Since ϕ and ψ are diffeomorphisms, $\phi(V) \subseteq X$ and $\psi(V') \subseteq Y$ are **open** neighborhoods of x and y, respectively. Moreover, $\phi_{|V}$ and $\psi_{|V'}$ are local parametrizations around x and y, respectively, and

$$f_{|\phi(V)}: \phi(V) \to \psi(V')$$

is a **diffeomorphism**. □

Note that this is a **local statement**, i.e., if df_x is invertible, it only tells us that f is invertible in a **neighborhood of x**. Even if df_x is invertible for every $x \in X$, one cannot conclude that $f: X \to Y$ is globally a diffeomorphism. But such an f is a **local** diffeomorphism for every point $x \in X$. We call such a map a **local diffeomorphism** (without having to refer to a point).

Example 3.5 (A global diffeomorphism) The map

$$(-\pi/2, \pi/2) \to \mathbb{R}, \; t \mapsto \tan t$$

is a global diffeomorphism.

[1] Note that the dimension has to be the same when the tangent spaces are isomorphic.

Example 3.6 (A local but not global diffeomorphism) The map

$$f: \mathbb{R}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

is a **local** diffeomorphism but **not** a **global** diffeomorphism. Let us check how this example works:

First, f is **not** a global diffeomorphism because it is **not injective**. And we have seen that f is not a homeomorphism even when we restrict it to $[0, 2\pi) \to \mathbb{S}^1$. We could also argue that $\mathbb{S}^1$ **is compact** and $\mathbb{R}$ **is not**, so there is no chance of finding a diffeomorphism between them.

However, the **Inverse Function Theorem 3.4** tells us that f is indeed a **local diffeomorphism** since df_t is an isomorphism for every $t \in \mathbb{R}$. For, let $t_0 \in \mathbb{R}$ such that $\cos(t_0) > 0$ (for other points the argument is similar, we just want to be able to choose a parametrization), and consider the local parametrization

$$\psi: W = (-1, 1) \to V \subset \mathbb{R}^2,\ y \mapsto (\sqrt{1-y^2}, y)$$

of $\mathbb{S}^1$ around $f(t_0)$ with $V = \{(x, y) \in \mathbb{S}^1 : x < 0\}$. The inverse is given by projecting onto the second coordinate: $\psi^{-1}(x, y) = y$.

Let $\epsilon > 0$ be such that both $\cos(t_0 - \epsilon) > 0$ and $\cos(t_0 + \epsilon) > 0$. We let $\phi: U = (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}$ be the local parametrization around t_0 to be the inclusion (we don't shift U to be centred around 0). Then the map $\theta: U \to W$ (see proof of the IFT) is defined as

$$\theta = \psi^{-1} \circ f \circ \phi, t \mapsto \sin t.$$

Then we get

$$d\theta_t: \mathbb{R} \to \mathbb{R},\ z \mapsto (\cos t) \cdot z.$$

at any point t in U. Since ϕ is the identity at each point it is defined, we know $d\phi_t = \mathrm{id}$. To calculate the derivative of ψ, we consider it first as a map $W \to \mathbb{R}^2$. Then we get

$$d\psi_y: \mathbb{R} \to \mathbb{R}^2,\ z \mapsto \left(-\frac{y}{\sqrt{1-y^2}}, 1\right) \cdot z.$$

Remember that the tangent space $T_{f(t_0)}\mathbb{S}^1$ is by definition the image of $d\psi_{y_0}: \mathbb{R} \to \mathbb{R}^2$ where y_0 is the point in W which maps to $f(t_0)$. Thus we have

$$T_{f(t_0)}\mathbb{S}^1 = d\psi_{t_0}(\mathbb{R}) = \mathrm{span}\{(-\sin(t_0), \cos(t_0))\} \subset \mathbb{R}^2.$$

Now we collect all this information

$$df_{t_0} = d\psi_{\sin t_0} \circ d\theta_{t_0}$$

and hence

$$\begin{aligned} df_{t_0}(z) &= \left(-\frac{\sin t_0}{\cos t_0}, 1\right)(\cos t_0) \cdot z \\ &= (-\sin t_0, \cos t_0) \cdot z. \end{aligned}$$

Summarizing we have

$$\begin{aligned} df_{t_0}: T_{t_0}\mathbb{R} &\to T_{f(t_0)}\mathbb{S}^1 \\ z &\mapsto (-\sin(t_0), \cos(t_0)) \cdot z \end{aligned}$$

which is an isomorphism. For any other point in $\mathbb{R}$, there is a similar argument.

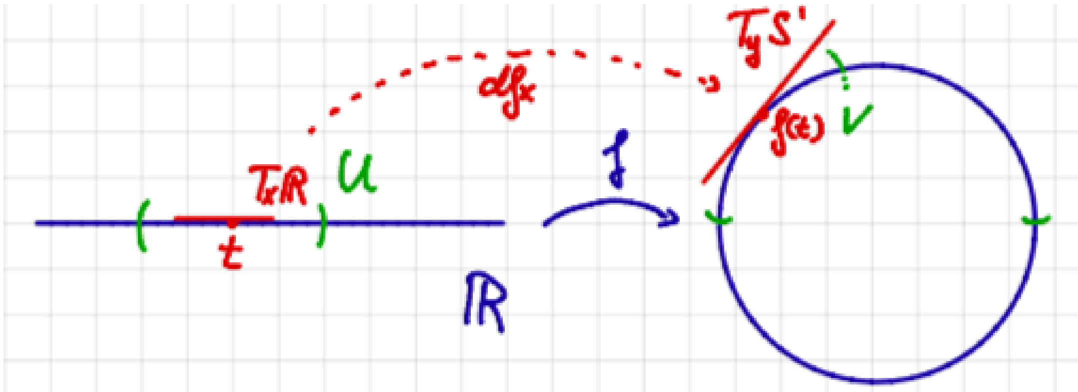

Figure 3.1: Locally, the map f is a diffeomorphism at any point. But it cannot be a global diffeomorphism, since it is not injective.

Lemma 3.7 A **bijective** local diffeomorphism is a global diffeomorphism.

Proof: See Exercise 3.2. □

Remark 3.8 **(df_x looks like the identity)** In some situations it would be nice if we could assume that the linear isomorphism df_x was the identity. This is usually not the case of course. But our freedom of choosing local parametrizations allows us to do the following. Assume that df_x is an isomorphism as in the IFT. Then we can choose local parametrizations[a] $\phi: U \to X$ and $\psi: U \to Y$ around x and $f(x)$, respectively, with the same open domain $U \subset \mathbb{R}^n$, such that the diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\mathrm{Id}_U"'] & U \arrow[u, "\psi"'] \end{tikzcd}.$$

For example, in Example 3.6 above, we would replace

- $W = (-1, 1)$ with $U = (t_0 - \varepsilon, t_0 + \varepsilon)$ and
- ψ with

$$\psi: t \mapsto (-\sqrt{1 - \sin^2 t}, \sin t) = (\cos t, \sin t).$$

[a] We are going to explain how to choose suitable parametrizations in the next sections.

3.2 Immersions and embeddings

3.2.1 Immersions

We continue our study of smooth maps between manifolds using the behaviour of their derivative. Let $f : X \to Y$ be a smooth map. We would like to understand how much do we know about f if the derivative is **injective**. Note that this is only possible if $\dim X \leq \dim Y$, so this is a silent assumption in this chapter.

Let us introduce some terminology for this case.

Definition 3.9 (Immersion) Let $f : X \to Y$ be a smooth map. If df_x is **injective**, we say that f is an **immersion at x**. If f is an immersion at every point, we say that f is an **immersion**.

Let us look at some **first examples**:

- Every **linear injective** map $L : \mathbb{R}^k \to \mathbb{R}^n$ is an immersion. This follows from the fact that L equals its own derivative dL_x at every point $x \in \mathbb{R}^k$.
- Every **local diffeomorphism** $f : X \to Y$ is also an immersion. Considering our examples for local diffeomorphisms, this also shows that an immersion f does not have to be injective itself, only its derivative df_x is injective at every x.
- Let f be the map defined by
$$f : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right).$$
We check in the exercises that this map is an **immersion**.
- Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map
$$f : (s,t) \mapsto ((1 + 2\cos s)\cos t, (1 + 2\cos s)\sin t, 2\sin s)$$
is smooth, since all its components are smooth functions. The derivative of f at a point (s,t) can be described in the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, by the matrix
$$\begin{pmatrix} -2\sin s\cos t & -(2\cos s + 1)\sin t \\ -2\sin s\sin t & (2\cos s + 1)\cos t \\ 2\cos s & 0 \end{pmatrix}.$$
We claim that this matrix has rank 2 for all (s,t). This follows from the following two observations: From the third row we get that the two columns can only be linearly dependent if $\cos s = 0$, i.e., if $s \in \left\{ n\frac{\pi}{2} | n \in \mathbb{Z} \right\}$. However, the determinant of the submatrix of the two top rows is
$$\begin{aligned} & -2\sin s(2\cos s + 1)\cos^2 t - 2\sin s(2\cos s + 1)\sin^2 t \\ = & -2\sin s(2\cos s + 1) \end{aligned}$$

which is $\neq 0$ for all $s \notin \{n\pi : n \in \mathbb{Z}\}$. Thus the two columns are always linearly independent which proves the claim. Hence f is an **immersion**. The image of f is the torus $T(1,2)$ consisting of all points in $\mathbb{R}^3$ at distance b from the circle of radius a in the xy-plane. We learn from this example, that f itself does not have to be one-to-one to be an immersion. It is about the derivative. We also observe that the image of f is a smooth manifold in $\mathbb{R}^3$.

Before we study more examples and interesting phenomena, we pause for a moment and show a technical and useful result about immersions.

Towards the local immersion theorem

We have all seen many injective maps before. In fact, **among all** the **injective** maps $\mathbb{R}^k \to \mathbb{R}^n$ with $k \leq n$ there is a **simplest one**, namely the map

$$\iota_n^k : \mathbb{R}^k \hookrightarrow \mathbb{R}^n, \ (x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$$

which sends the coordinates of a point $\mathbf{x} \in \mathbb{R}^k$ to the first k coordinates in $\mathbb{R}^n$ and adds 0 at the remaining $n-k$ positions. This corresponds to the inclusion $\mathbb{R}^k \times \{\mathbf{0}\} \subset \mathbb{R}^n$ where $\mathbf{0}$ denotes the $(n-k)$-tuple of zeros. The map ι_n^k is called the **canonical immersion** from $\mathbb{R}^k$ into $\mathbb{R}^n$. The **matrix** which represents ι_n^k with respect to the standard bases of $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively, is the $n \times k$-matrix J_n^k which has the $k \times k$-identity matrix in the first k rows and only zeros in the remaining $n-k$ rows at the bottom. Not all injective maps $\mathbb{R}^k \to \mathbb{R}^n$ are as simple as ι_n^k. However, we will see very soon that it does not get that worse either, at least locally.

To motivate the next theorem let us start with a **linear injective** map $L : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$. Say we do not like transformations between spaces of different dimensions. The only one we think is OK is the canonical one $\iota_{k,n}$. And say we do like isomorphisms of $\mathbb{R}^n$ to itself. Then we could write L as the composition

$$\mathbb{R}^k \xrightarrow{\iota_n^k} \mathbb{R}^n \xrightarrow{\alpha} \mathbb{R}^n$$

where α is a linear isomorphism determined by sending the first k standard basis vectors $\mathbf{e}_1^n, \dots, \mathbf{e}_k^n$ of $\mathbb{R}^n$ to the images $\mathbf{v}_1 = L(\mathbf{e}_1^k), \dots, \mathbf{v}_k = L(\mathbf{e}_k^k)$ in $\mathbb{R}^n$ of the standard basis vectors of $\mathbb{R}^k$ and by sending the remaining basis vectors $\mathbf{e}_{k+1}^n, \dots, \mathbf{e}_n^n$ of $\mathbb{R}^n$ to basis vectors $\mathbf{w}_{k+1}, \dots, \mathbf{w}_n$ of the orthogonal complement of the image of L in $\mathbb{R}^n$. Since L is injective, this complement has dimension $n-k$.

So what happened is that we placed a copy of $\mathbb{R}^k$ into $\mathbb{R}^n$ as the k-dimensional plane containing the origin via ι_n^k and then we move this plane inside $\mathbb{R}^n$ to the position of $L(\mathbb{R}^k)$.

Actually, we are used to such a manoeuvre. For what α does is **changing the basis** of $\mathbb{R}^n$ from the standard basis to the new basis which consists of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_{k+1}, \dots, \mathbf{w}_n$. **With respect to this new basis** for $\mathbb{R}^n$ and the standard basis for $\mathbb{R}^k$, the **matrix** which represents L is the $n \times k$-matrix $\mathbb{S}_n^k$ described above.

We can translate this idea to immersions between smooth manifolds. In other words — up to diffeomorphisms — the canonical immersion is **locally** the only immersion:

Theorem 3.10 **(Local Immersion Theorem)** Let X and Y be smooth manifolds of dimensions k and n, respectively, with $k \leq n$. Suppose that $f: X \to Y$ is an **immersion at x**, and write $y = f(x)$. Then there exist **local coordinates** around x and y such that

$$f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0).$$

More precisely, we can **choose** local parametrizations $\phi: U \to X$ around x and $\psi: W \to Y$ around y such that in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\text{canonical immersion}", "\theta=\psi^{-1}\circ f\circ\phi"'] & W \arrow[u, "\psi"'] \end{tikzcd}$$

the map $\theta: U \to W$ is the **canonical immersion** restricted to U.

Proof of the Local Immersion Theorem 3.10: We start by choosing any local parametrization $\phi: U \to X$ with $\phi(0) = x$ and $\psi: W \to Y$ with $\psi(0) = y$:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & W. \arrow[u, "\psi"'] \end{tikzcd}$$

The plan is to manipulate ϕ and ψ such that θ becomes the canonical immersion.

By the **assumption**, $d\theta_0: \mathbb{R}^k \to \mathbb{R}^n$ is **injective**. Hence, after choosing a suitable basis for $\mathbb{R}^n$, we can assume that $d\theta_0$ is given by the $n \times k$-matrix J_n^k which has the $k \times k$-identity matrix in the first k rows and only zeros in the remaining $n-k$ rows. Now we define a new map

$$\Theta: U \times \mathbb{R}^{n-k} \to \mathbb{R}^n, \text{ by } \Theta(x, z) = \theta(x) + (0, z).$$

It is related to θ by the commutative diagram

$$\begin{tikzcd} U \arrow[rr, "\text{canonical}", "\text{immersion}"'] \arrow[dr, "\theta"'] & & U \times \mathbb{R}^{n-k} \arrow[dl, "\Theta"] \\ & \mathbb{R}^n. & \end{tikzcd}$$

Since θ is a local diffeomorphism at 0, we can choose U and V small enough such that θ sends open sets to open sets. By the assumption on $d\theta_0$ and the construction of Θ, $d\Theta_0$ is represented by the $n \times n$-**identity matrix**. By the **Inverse Function Theorem**, this implies that Θ is a **local diffeomorphism** of $\mathbb{R}^n$ to itself at 0. Hence we can find an open subset $V \subset U \times \mathbb{R}^{n-k}$ and $W' \subset \mathbb{R}^n$ such that $\Theta_{|V}$ is a diffeomorphism. After possibly shrinking U to an open subset U'

we get the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ U' \arrow[u, "\phi"] \arrow[rr, "\theta"] \arrow[drr, bend right, "\text{canonical immersion}"'] & & W \arrow[u, "\psi"'] \\ & & W' \arrow[u, "\Theta"'] \end{tikzcd}$$

Since ψ and Θ are local diffeomorphisms at 0, so is the composition $\psi \circ \Theta$. Hence we can use **$\psi \circ \Theta$ as a local parametrization around y**.

Thus, we have constructed the desired commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U' \arrow[u, "\phi"] \arrow[r, "\text{canonical}", "\text{immersion}"'] & W' \arrow[u, "\psi \circ \Theta"'] \end{tikzcd}$$

which finishes the proof. □

Remark 3.11 We observe from the proof that to be an immersion is a **local condition**, i.e., if $f: X \to Y$ is an immersion at x, then it is also an immersion for **all points in a neighborhood** of x. For, the local parametrization $\phi: U \to X$ of the proof also parametrizes any point in the image of ϕ. This is an **open** subset around x because ϕ is a diffeomorphism onto its image. Hence in order to say more about f we need to add some global topological properties to the local differential data. Recall that for a local diffeomorphism to be a global one, it suffices to require to be one-to-one and onto. We will now study what kind of additional property we wish to impose on immersions to behave nicely.

3.2.2 The image of an immersion

Even though an immersion does not have to be one-to-one itself, the injectivity of the derivative does not leave much room for different phenomena in the fibers. So let us have a closer look at the image of an immersion $f: X \to Y$, i.e., the subset $f(X) \subset Y$.

Question We can ask at least these **two questions**:

- Is $f(X)$ always a submanifold?[a]
- Is f a diffeomorphism onto its image?

[a] Recall that a subset $Z \subset X$ of a smooth manifold X is called a **submanifold** if and only if Z is itself a smooth manifold, possibly of lower dimension.

First observations:

- Recall the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ in the above examples for which the image is the two-dimensional torus. This provides an example where $f(X)$ is a submanifold. However, f is **not** a diffeomorphism onto its image, since f is **not one-to-one**.
- Note also that we can, in principle, answer the second question **independently** of the first. For we can check if f has an inverse which is defined on the subset $f(X) \subset Y$ and check if this map is smooth. **However**, we will see below that **if** f is a diffeomorphism onto $f(X)$, then this will provide the subspace $f(X) \subset$ with the structure of a smooth manifold and hence this will be a submanifold in Y.

We have seen that $f(X)$ often is a manifold. But there are also examples where this is **not** the case. We will have a look at them now:

Example 3.12 (Figure eight as twist) We just look at Figure 3.2 without making the formula for f explicit. We see here that $f(X)$ is **not** a manifold as a subset of $\mathbb{R}^2$, since the intersection point of the two branches does not have a local parametrization. We also see that f is **not** a diffeomorphism onto its image, since f is **not one-to-one**.

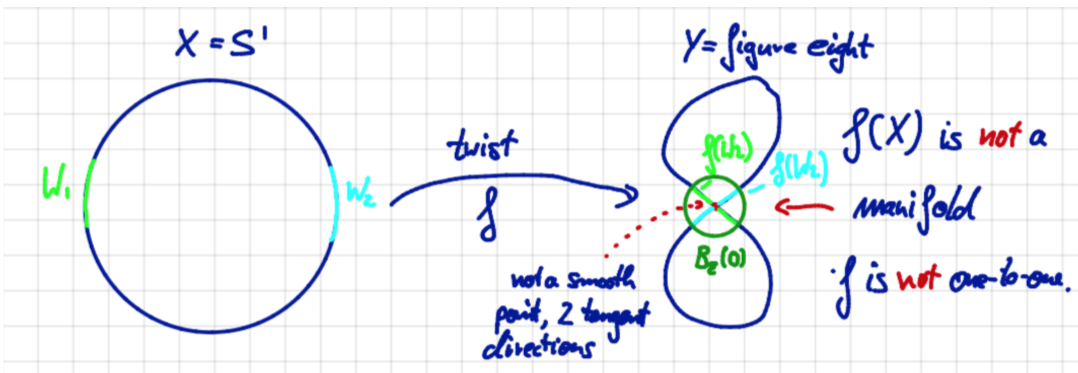

Figure 3.2: The map f twists the circle once. It is an immersion, but it is not injective. The image of f, considered as a subspace of $\mathbb{R}^2$, is **not** a submanifold.

Example 3.13 (Figure eight immersion as wrap) So let us modify the map to make it one-to-one. Consider the map

$$f : (-\pi, \pi) \to \mathbb{R}^2, \ t \mapsto (\sin 2t, \sin t).$$

The image of f is called a **lemniscate**, the locus of points (x, y) satisfying $x^2 = 4y^2(1 - y^2)$. See Figure 3.3.

We can check that f is **smooth**, **one-to-one** and **an immersion**. For the latter note that df_t can be represented by the 2×1-matrix

$$J_f(t) = \begin{pmatrix} 2\cos 2t \\ \cos t \end{pmatrix}$$

which is never zero for $t \in (-\pi, \pi)$ and hence, as a linear map between one-dimensional vector spaces, df_t is an **isomorphism** for all t.

Nevertheless, f is still **not** a diffeomorphism onto its image. In fact, we see that f is **not** a homeomorphism onto its image $f(X)$. Moreover, the image $f(X)$ is the same as above and **not** a manifold.

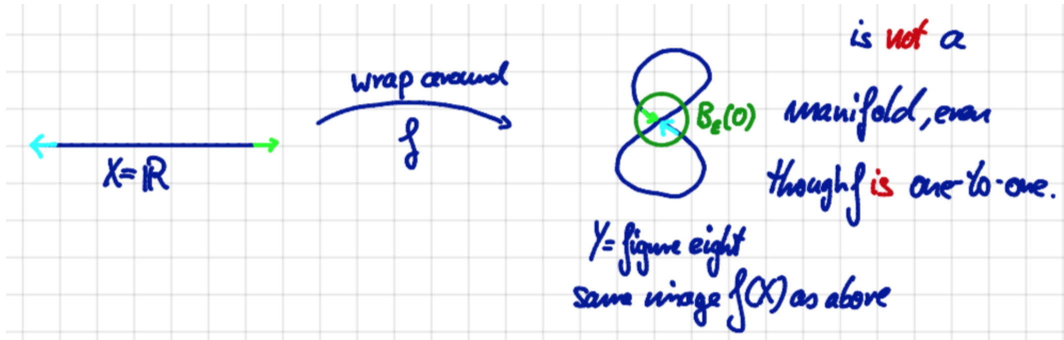

Figure 3.3: The map f wraps $\mathbb{R}$ along the figure eight. It is an immersion and one-to-one. **However**, the image of f, considered as a **subspace** of $\mathbb{R}^2$, is still **not** a submanifold. Note that, even though f is a local diffeomorphism and bijective onto its image, this is not a contradiction to Lemma 3.7, since the image of f, i.e., the figure eight, is **not** a manifold in $\mathbb{R}^2$.

3.2.3 A curve on the torus

Let us look at a classical example of another case of a map which is a one-to-one immersion, but **not** a homeomorphism — and hence not a diffeomorphism — onto its image. We will not, however, show that $f(X)$ is not a submanifold.

Let $g\colon \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C}$ be the local diffeomorphism $t \mapsto e^{2\pi i t}$. We define

$$G\colon \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1 =: \mathbb{T}^2 \subset \mathbb{C}^2,\ G(s,t) = (g(s), g(t))$$

The map G is a local diffeomorphism from the plane onto the two-dimensional torus $\mathbb{T}^2$.

We define the map γ by

$$\gamma\colon \mathbb{R} \to \mathbb{T}^2,\ \gamma(t) = (g(t), g(\alpha t)).$$

where α is an **irrational** number.

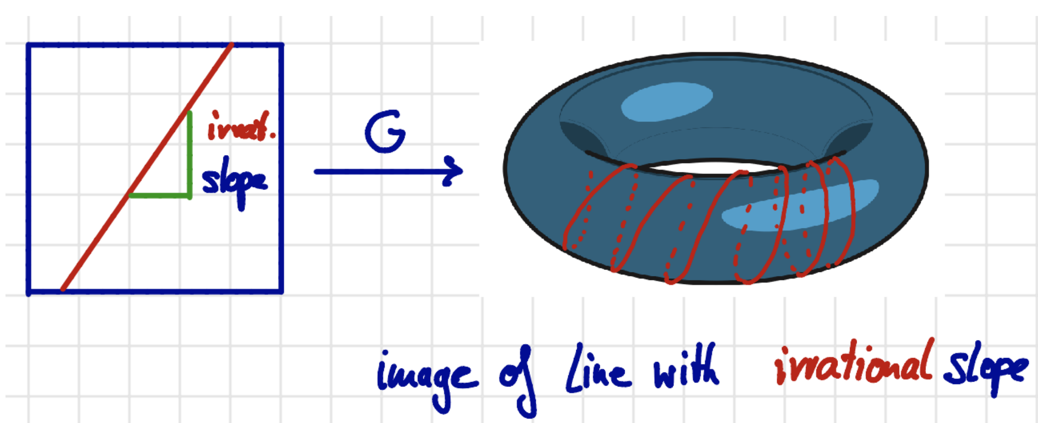

Figure 3.4: The map γ wraps a line around the torus. If the slope is **irrational**, then the image will never meet itself and is dense on $\mathbb{T}^2$.

Example 3.14 (Image of a line with irrational slope) The map γ is an **immersion** because $d\gamma_t$ **is nonzero for every** t, and, as above, a **nonzero** linear map from a one-dimensional vector space to another is automatically **injective**. Moreover, γ itself is **injective**, since $\gamma(t_1) = \gamma(t_2)$ implies

$$\begin{aligned} & g(t_1) = g(t_2) \text{ and } g(\alpha t_1) = g(\alpha t_2) \\ \Rightarrow\ & e^{2\pi i t_1} = e^{2\pi i t_2} \text{ and } e^{2\pi i \alpha t_1} = e^{2\pi i \alpha t_2} \\ \Rightarrow\ & t_1 - t_2 \in \mathbb{Z} \text{ and } \alpha(t_1 - t_2) \in \mathbb{Z}. \end{aligned}$$

Since α **is irrational**, this implies $t_1 = t_2$. One can show that the image of γ is a **dense** subset in $\mathbb{T}^2$.

However, γ is **not** a **diffeomorphism onto its image**, since it is **not** even a **homeomorphism**:
For, look at the set $\gamma(\mathbb{Z}) = \{\gamma(n) : n \in \mathbb{Z}\}$. By **Dirichlet's Approximation Theorem**, for every $\varepsilon > 0$, there are **integers** n and m such that

$$|\alpha n - m| < \epsilon.$$

Since the line segment between two points $e^{2\pi i t_1}$ and $e^{2\pi i t_2}$ on the unit circle is shorter than the circular arc of length $|t_1 - t_2|$, we have

$$|e^{2\pi i \alpha n} - e^{2\pi i m}| \leq 2\pi|\alpha n - m| < 2\pi\varepsilon.$$

Therefore, with coordinates in $\mathbb{C}^2$, we get

$$\begin{aligned} & |\gamma(n) - \gamma(0)| \\ =\ & |(e^{2\pi i n}, e^{2\pi i \alpha n}) - (e^0, e^0)| = \left|\left(1, e^{2\pi i \alpha n}\right) - (1,1)\right| \\ =\ & |e^{2\pi i \alpha n} - e^{2\pi i m}| \leq 2\pi|\alpha n - m| < 2\pi\varepsilon. \end{aligned}$$

Thus, there is a **sequence of integers** such that $\gamma(n)$ **converges** to $\gamma(0)$, i.e., $\gamma(0)$ is a **limit point** in $\gamma(\mathbb{Z})$. The image of a **convergent** sequence under a **continuous** map is again a **convergent** sequence.[a] Hence **if** γ^{-1} was continuous, then $0 = \gamma^{-1}(\gamma(0))$ had to be a limit point as well. However, $\mathbb{Z}$ does **not** have any **limit points** in $\mathbb{R}$. Hence γ is **not** a homeomorphism onto its image.

We can also show that $\gamma(\mathbb{R})$ with the subspace topology in $\mathbb{T}^2$ is **not** a manifold. We leave this as an exercise for the moment.

[a] This is actually an alternative way to define what continuity means.

3.3 Embeddings

3.3.1 Embeddings

We have seen that both our questions we asked earlier may have negative answers. Let us now focus on the situation when they do have a positive answer. We give this case a name:

Definition 3.15 (Embedding) An immersion $f: X \to Y$ is a called an **embedding** if $f(X) \subset Y$ considered with the subspace topology is a **manifold** and f is a **diffeomorphsim** onto its image.

Note that an embedding must be **one-to-one**, since it is a diffeomorphism onto its image.

Now let $f: X \to Y$ be an immersion which is also one-to-one. Let us try to show what f is an embedding to see which additional assumption we have to make:

- By the **Local Immersion Theorem 3.10**, we can choose local parametrizations $\phi: U \to W \subset X$ around x and $\psi: V \to W' \subset Y$ around $y = f(x)$ such that the induced map $\theta: U \to V$ is the canonical immersion. This provides the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] & & f(X) \\
W \arrow[u, hook, "\text{open } \checkmark"] \arrow[rr, "f_{|W} \cong"] & & f(W) \arrow[u, hook', "\text{open ?}"'] \\
U \arrow[u, "\phi", "\cong"'] \arrow[urr, "f_{|W} \circ \phi", "\cong"'] \arrow[r, "\text{canonical}", "\text{immersion}"'] & V \cap \psi^{-1}(f(W)). \arrow[ur, "\psi_{|V \cap \psi^{-1}(f(W))}"'] &
\end{tikzcd}$$

  Since the bottom horizontal map is the **canonical immersion**, the restriction $f_{|W}$ is a diffeomorphism. As ϕ is one as well, we see that the composite $f_{|W} \circ \phi: U \to f(W)$ is a diffeomorphism. Thus $f(W)$ is diffeomorphic to an open subset in $\mathbb{R}^n$.

- Since we can do this for every point $y \in f(X)$, we **would like** to say that the collection of diffeomorphisms $f_{|W} \circ \phi$ for varying ϕ provide local parametrizations for $f(X)$.

- **However**, we do **not know** that $f(W)$ is **open** in $f(X)$. This does not follow from the given assumptions on f.

- Let us **assume** for a moment that we were lucky and $f(W)$ was **open** for each open subset $W \subset X$. Then we could conclude that f is a diffeomorphism onto its image. For then f would be a bijective and a local diffeomorphism, since for every point $x \in X$ there would be an open subset $x \in W \subset X$ and an open subset $f(W) \subset f(X)$ such that $f_{|W}$ is a diffeomorphism $W \to f(W)$. A **bijective local diffeomorphism** is a **diffeomorphism** by Lemma 3.7 and Exercise 3.2.

- Thus we can **conclude**: **If** $f(W)$ is **open** for each open subset $W \subset X$, **then** f is a **diffeomorphism** onto its image.

3.3.2 The embedding theorem

We learn from this discussion that we need to find conditions which ensure that $f(W)$ is open in $f(X)$. This is the case if f is an **open map**, i.e., if f sends every open subset in X to an

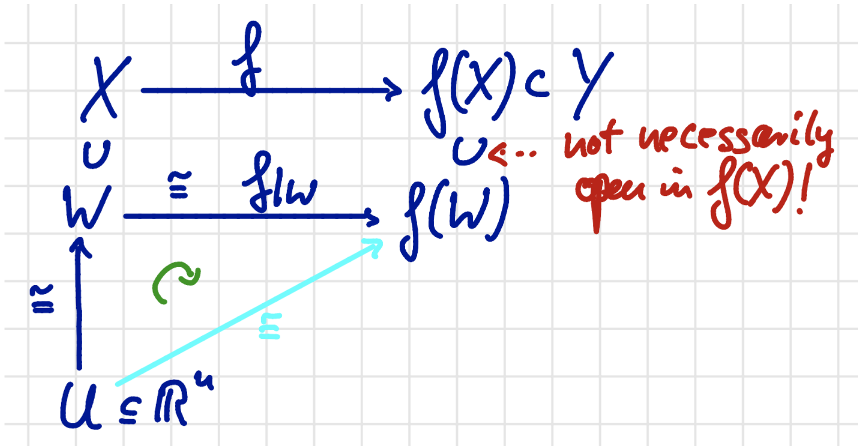

Figure 3.5: One might think that $f(W)$ provides a local parametrization, but it is not necessarily open in $f(X)$. Hence it is **not**guaranteed that $f(X)$, considered as a **subspace** of Y, is a manifold.

open subset in Y. Equivalently, we could require that f is a closed map, i.e., f sends every closed subset in X to a closed subset in Y. Note that this fits well into what we have seen in the last two examples. For, there f was neither closed nor open and $f : X \to f(X)$ failed to be a homeomorphism. A condition that is often easier to test is the following: Recall that for a general continuous map, the **image** of any compact set is compact. However, the **preimage** of a compact subset is, in general, **not** compact.

Definition 3.16 (Proper maps) A map $f : X \to Y$ between topological spaces is said to be **proper** if the **preimage** of every **compact** subset is a **compact** subset.

Being proper turns out to be a sufficient **global** topological constraint for our purposes:

Theorem 3.17 (Embedding Theorem) Let $f : X \to Y$ be a one-to-one immersion which satisfies one of the following conditions:

- f is an **open** map, or
- f is a **closed** map, or
- f is a **proper** map.

Then f is an **embedding**.

Before we prove the theorem, let us assume for a moment that X is **compact**. Then **every** continuous map $f : X \to Y$ is **proper**. This follows from the fact that closed subsets of compact sets are compact. Hence we can deduce from the theorem the following important special case:

Corollary 3.18 (Compact domain) For a **compact** smooth manifold X, **every** one-to-one immersion $f : X \to Y$ is an **embedding**.

Proof of the Embedding Theorem 3.17:

We know from our previous arguments that $f : X \to f(X)$ is a diffeomorphism if f is **open**. Since a bijective continuous map is open if and only if it is closed, $f : X \to f(X)$ is a diffeomorphism if f is **closed**. Now assume that f is **proper**. We are going to prove that this implies that f is a **closed map** and thereby prove the theorem. Actually, we show the following general statement, formulated in Lemma 3.19. Proving the lemma will finish the proof of Theorem 3.17. □

Lemma 3.19 A continuous bijective **proper** map $f : X \to Y$ is a **homeomorphism**.

Proof of Lemma 3.19:[2] We have $Y = f(X)$. Let Z be a closed subset in X. Let $y \in Y \setminus f(Z)$. We are going to show that this is an **open** subset of $f(X)$ which then implies that $f(Z)$ is **closed** in Y. Since Y is a subspace in Euclidean space, Y is **locally compact** which means that we can find a **compact neighborhood** K of y in Y, i.e., there is an **open** subset U and a **compact** subset K such that $y \in U \subset K$. Then $f^{-1}(K)$ is compact, since f is **proper**. Hence $f^{-1}(K) \cap Z$ is compact, since it is a closed subset in $f^{-1}(K) \subset \mathbb{R}^N$. Hence $f(f^{-1}(K) \cap Z) = K \cap f(Z)$ is compact, since the continuous image of a compact set is always compact. Since K is a subspace in Euclidean space and thereby **Hausdorff**, this implies that $K \cap f(Z)$ is **closed** in K and hence also closed in Y. Thus, since U is open in Y, the subset $U \setminus (K \cap f(Z))$ is open in Y as well. Now, by choice of y, we know $y \in U \setminus (K \cap f(Z))$. This shows that y has an **open** neighborhood in $Y \setminus f(Z)$, and the latter is an **open** subset of Y. □

[2] We will use a little bit more general topology in this proof than we recalled so far. We hope that is ok.

3.4 Exercises and more examples

3.4.1 Diffeomorphisms, immersions and embeddings

Exercise 3.1 Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and $b \in \mathbb{R}^n$. Show that the mapping

$$f : \mathbb{R}^n \to \mathbb{R}^n, \ x \mapsto Ax + b$$

is a diffeomorphism of $\mathbb{R}^n$ if and only if A is invertible.

Exercise 3.2 Let $f : X \to Y$ be a smooth map. Assume that f is a local diffeomorphism, and assume that f is bijective as a map of sets. Show that f is a diffeomorphism.

Exercise 3.3 Show that the map

$$f : \mathbb{R} \to \mathbb{R}^2, \ t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2} \right)$$

is an embedding.

Exercise 3.4 Show that the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$(s,t) \mapsto ((2 + \cos(2\pi s))\cos(2\pi t), (2 + \cos(2\pi s))\sin(2\pi t), \sin(2\pi s))$$

is an immersion. Is it an embedding?

Exercise 3.5 Let $\gamma_{a,b}$ be the curve on the torus defined by

$$\gamma_{a,b} : \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1, \ t \mapsto (e^{2\pi i a t}, e^{2\pi i b t})$$

where we consider $\mathbb{S}^1$ as a subset of $\mathbb{C} \cong \mathbb{R}^2$. Let a and b are integers with $a \neq 0$. Show that $\gamma_{a,b}$ factors through an embedding $\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, i.e., find a map $g_{a,b} : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ which is an embedding such that $\gamma_{a,b}$ is the composite of a map $\mathbb{R} \to \mathbb{S}^1, t \mapsto e^{2\pi i t}$, followed by $g_{a,b}$.

Exercise 3.6 Consider the map $f : (0, 3\pi/4) \to \mathbb{R}^2, \ t \mapsto \sin(2t)(\cos t, \sin t)$.

(a) Show that f is an immersion.

(b) Let $\mathrm{Im}(f) = f((0, 3\pi/4)) \subset \mathbb{R}^2$ be the image of f considered as a subspace in $\mathbb{R}^2$. Show that $f : (0, 3\pi/4) \to \mathrm{Im}(f)$ is not a homeomorphism. (Draw a picture of the image of f.)

(c) To test your understanding answer the following questions (and give reasons for your answer):

- What is the difference between $\mathrm{Im}(f)$ and the graph $\Gamma(f)$?
- Is the map $F\colon (0, 3\pi/4) \to (0, 3\pi/4) \times \mathbb{R}^2,\ t \mapsto (t, f(t))$, an embedding?
- Would f be an embedding if it was defined on the closed interval $[0, 3\pi/4]$?
- Is the map $g\colon (0, 3\pi/4) \to \mathbb{R}^3,\ t \mapsto \sin(2t)(\cos t, \sin t, t)$ an embedding?
- Is the map $h\colon [0, 3\pi/4] \to \mathbb{R}^3,\ t \mapsto (\sin(2t)\cos t, \sin(2t)\sin t, 2t)$ an embedding?

Exercise 3.7 Let X be an n-dimensional smooth manifold, Z be a k-dimensional smooth submanifold of X, and let $z \in Z$. Show that there exists a local coordinate system $(x_1, \ldots, x_n)$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{k+1} = 0, \ldots, x_n = 0$, i.e., $Z \cap U$ is the subset of points in U for which the functions $x_{k+1}, \ldots, x_n$ all vanish.

4. Submersions and regular values

4.1 Submersions

4.1.1 Submersions

Let $f : X \to Y$ be a smooth map. We will now turn to the question how much do we know about f if the derivative is **surjective**. Note that this will require that $\dim X \leq \dim Y$, and this is a silent assumption in this chapter. We will see that this case will lead to a very useful observation about the fibers of smooth maps.

We begin with some terminology:

Definition 4.1 (Submersion) Let $f : X \to Y$ be a smooth map. If df_x is **surjective**, we say that f is a **submersion at x**. If f is a submersion at every point, we say that f is a **submersion**.

Let us look at some **first examples**:

- Every **linear surjective** map $L : \mathbb{R}^k \to \mathbb{R}^n$ is a submersion. This follows from the fact that L equals its own derivative dL_x at every point $x \in \mathbb{R}^k$.

- Every local diffeomorphism $f : X \to Y$ is also a submersion. Considering our examples for local diffeomorphisms, this also shows that a submersion f does not have to be surjective itself, only its derivative df_x is surjective at every x.

- Let f be the map defined by
$$g : \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x^2 - y^2.$$
We check in Exercise 4.2 that g is not a submersion. However, it is a submersion at every point $(x, y) \neq (0, 0)$.

- The **Hopf map** $f : \mathbb{S}^3 \to \mathbb{S}^2$ is a **submersion**. We show this in the exercises. So far, our previous calculations show that the derivative of f is surjective at the north pole $\mathbf{n}_3 = (0, 0, 0, 1) \in \mathbb{S}^3$ and at $\mathbf{q}_3 = (1, 0, 0, 0) \in \mathbb{S}^3$. For we showed that $df_{\mathbf{n}_3}$ can be represented by the matrix $\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$, while $df_{\mathbf{q}_3}$ can be represented by the matrix $\begin{pmatrix} 0 & 2 & 0 \\ & 0 & -2 \end{pmatrix}$. Both these matrices have rank two and the corresponding linear maps are surjective. Hence f is a submersion at $\mathbf{n}_3$ and $\mathbf{q}_3$. In the exercises we develop a more elegant way and generalize these computations and show that f is a submersion.

- We define the map
$$\begin{aligned} f:\ \mathbb{R}^4 &\to \mathbb{R},\\ (x_1,x_2,x_3,x_4) &\mapsto x_1 + x_2^2 + x_3^3 + x_4^4. \end{aligned}$$
The derivative df_z at a point $z = (x_1,x_2,x_3,x_4)$ is a linear map $\mathbb{R}^4 \to \mathbb{R}$ given in the standard basis by the 1×4-matrix
$$df_z = \begin{pmatrix} 1 & 2x_2 & 3x_3^2 & 4x_4^3 \end{pmatrix}.$$
Since df_z is a linear map with values in $\mathbb{R}$, it suffices to observe that df_z is **not** the **zero map** to conclude that df_z is surjective for all $z \in \mathbb{R}^4$. Hence f is a **submersion**.

- Let $M(n)$ denote the space of real $n\times n$-matrices. It is isomorphic as a vector space to $\mathbb{R}^{n^2}$, since we can write every $n\times n$-matrix as a column vector of length n^2. Hence $M(n)$ is smooth n^2-dimensional manifold. Let $GL(n)$ denote the group of invertible $n\times n$-matrices with group operation given by matrix multiplication. This is an open subset and hence a submanifold of $M(n)$, also of dimension n^2. The determinant map
$$\det:\ M(n) \to \mathbb{R}$$
is a submersion at every matrix $A \in GL(n)$. We will discuss this claim later in this section and in Exercise 4.9.

As for immersions, we can — at least locally — make submersions look as simple as possible. We will discuss this useful observation next.

4.1.2 The Local Submersion Theorem

The possibly simplest **surjective** map $\mathbb{R}^n \to \mathbb{R}^m$ with $n \geq m$ is the map
$$\sigma_m^n:\ \mathbb{R}^n \to \mathbb{R}^m,\ (x_1,\dots,x_m,x_{m+1},\dots,x_n) \mapsto (x_1,\dots,x_m)$$
which sends the coordinates of a point $\mathbf{x} \in \mathbb{R}^n$ to the point in $\mathbb{R}^m$ with the first m coordinates and forgets the remaining $n-m$ ones. The map σ_m^n is called the **canonical submersion** from $\mathbb{R}^n$ onto $\mathbb{R}^m$. The matrix which represents σ_m^n with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, is the $m\times n$-matrix S_m^n which has the $m\times m$-identity matrix in the first m columns and only zeros in the remaining $n-m$ columns on the right hand side.

Starting with a **linear surjective** map $L:\ \mathbb{R}^n \to \mathbb{R}^m$ we can split L into a composition of the form
$$\mathbb{R}^n \xrightarrow{\alpha} \mathbb{R}^n \xrightarrow{\sigma_m^n} \mathbb{R}^m$$
where α is a linear isomorphism of $\mathbb{R}^n$ as follows: Since L is surjective, the kernel of L has dimension $n-m$, and the quotient vector space space $\mathbb{R}^n/\mathrm{Ker}\, L$ has dimension m. Let $\mathbf{v}_1,\dots,\mathbf{v}_m$ be vectors in $\mathbb{R}^n$ which map to a basis of $\mathbb{R}^n/\mathrm{Ker}\, L$ and such that $L(\mathbf{v}_1) = \mathbf{e}_1^n,\dots,L(\mathbf{v}_m) = \mathbf{e}_m^n$ in $\mathbb{R}^n$. Let $\mathbf{w}_1,\dots,\mathbf{w}_{n-m}$ be a basis of Ker L. We define α by sending the first m standard basis vectors $\mathbf{e}_1^n,\dots,\mathbf{e}_m^n$ of $\mathbb{R}^m$ to $\mathbf{v}_1,\dots,\mathbf{v}_m$ and the remaining $n-m$ basis vectors $\mathbf{e}_{m+1}^n,\dots,\mathbf{e}_n^n$ of $\mathbb{R}^n$ to $\mathbf{w}_1,\dots,\mathbf{w}_{n-m}$.

In this way we move all the action into $\mathbb{R}^n$ while the second step, where the dimension drops, always consists of one fixed map which just forgets coordinates.

Again, this is a familiar manoeuvre. For α changes the basis of $\mathbb{R}^n$ from the standard basis to a new basis which consists of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{w}_1, \ldots, \mathbf{w}_{n-m}$. **With respect to this new basis** for $\mathbb{R}^n$ and the standard basis for $\mathbb{R}^m$, the matrix which represents L is the $m \times n$-matrix S^n_m described above.

We can translate this idea to submersions between smooth manifolds. In other words — up to diffeomorphisms — the canonical submersion is **locally** the only submersion:

> **Theorem 4.2** **(Local Submersion Theorem)** Let X and Y be smooth manifolds of dimensions n and m, respectively, with $n \geq m$. Suppose that $f: X \to Y$ is an **submersion at x**, and write $y = f(x)$. Then there exist **local coordinates** around x and y such that
>
> $$f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = (x_1, \ldots, x_m).$$
>
> More precisely, we can **choose** local parametrizations $\phi: U \to X$ around x and $\psi: W \to Y$ around y such that in the commutative diagram
>
> $$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\text{canonical submersion}", "\theta=\psi^{-1}\circ f\circ\phi"'] & W \arrow[u, "\psi"'] \end{tikzcd}$$
>
> the map $\theta: U \to W$ is the **canonical submersion** from $\mathbb{R}^n$ to $\mathbb{R}^m$ restricted to the open subset U.

Proof of the Local Submersion Theorem 4.2: We start by choosing any local parametrizations $\phi: U \to X$ with $\phi(0) = x$ and $\psi: V \to Y$ with $\psi(0) = y$:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Now we are going to manipulate ϕ and ψ such that θ becomes the canonical submersion.

By the **assumption**, we know $d\theta_0: \mathbb{R}^n \to \mathbb{R}^m$ is surjective. Hence, after choosing a suitable basis for $\mathbb{R}^n$, we can assume that $d\theta_0$ is given by the $m \times n$-matrix S^n_m which has the $m \times m$-identity matrix in the first m columns and only zeros in the remaining $n - m$ columns.[1] We define a new map

$$\Theta: U \to \mathbb{R}^n, \text{ by } \Theta(\mathbf{x}) = (\theta(\mathbf{x}), x_{m+1}, \ldots, x_n)$$

[1] The only reason we make this change of bases is to make sure that the map Θ we are about to define is a local diffeomorphism which we know since $d\Theta_0$ is the identity matrix. Without the change of bases we would not be sure that $d\Theta_0$ is an isomorphism.

for a point $\mathbf{x} = (x_1, \dots, x_n)$. It is related to θ by the commutative diagram

$$\begin{tikzcd} U \arrow[rr, "\theta"] \arrow[dr, "\Theta"'] & & \mathbb{R}^m \\ & \mathbb{R}^n. \arrow[ur, "\text{canonical submersion}"'] & \end{tikzcd}$$

By the construction, the derivative $d\Theta_0$ at 0 is given by the $n \times n$-**identity matrix**. Hence Θ is a **local diffeomorphism** at 0. Thus we can find a **small enough** neighborhood U' around 0 in $\mathbb{R}^n$ such that **Θ^{-1} exists** as a diffeomorphism from $U' \subset \mathbb{R}^n$ onto some small neighborhood around 0 in U. By construction, θ equals the composition of the canonical submersion with Θ, i.e., we have $\theta \circ \Theta^{-1} = \sigma^n_m$ on U'. This gives us the commutative diagram

$$\begin{tikzcd} & X \arrow[rr, "f"] & & Y \\ & U \arrow[u, "\phi"] \arrow[rr, "\theta"] & & V. \arrow[u, "\psi"'] \\ U' \arrow[uur, bend left, "\phi\circ\Theta^{-1}"] \arrow[ur, "\Theta^{-1}"'] \arrow[urrr, bend right, "\text{canonical submersion}"'] & & & \end{tikzcd} \tag{4.1}$$

Hence it suffices to replace U with U' and ϕ with $\phi\circ\Theta^{-1}$ to get the desired commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U' \arrow[u, "\phi\circ\Theta^{-1}"] \arrow[r, "\text{canonical}", "\text{submersion}"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

which proves the theorem. □

Remark 4.3 (Local property) We observe from this proof that if $f : X \to Y$ is a submersion at x, then it is also a **submersion for all points in a neighborhood of x**. For if $d\theta_0$ is surjective, then $d\theta_u$ is surjective for all points u in a small open neighborhood of 0 in U.

We now look at an example to improve our understanding of how the Local Submersion Theorem 4.2 works.

Example 4.4 (Local Submersion Theorem for the Hopf map) Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be the **Hopf map**. defined by

$$f : \mathbb{S}^3 \to \mathbb{S}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1x_3 + 2x_2x_4 \\ 2x_2x_3 - 2x_1x_4 \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}. \tag{4.2}$$

In Section 2.4.7 we computed the derivative of f at $\mathbf{n}_3 = (0,0,0,1) \in \mathbb{S}^3$ which is mapped to $\mathbf{s}_2 = (0,0,-1) \in \mathbb{S}^2$. Let $\mathbb{B}^3(\mathbf{0}_3)$ be the open ball around the origin $\mathbf{0}_3$ in $\mathbb{R}^3$

of radius $1/\sqrt{2}$. We start with the local parametrization

$$\phi : U = \mathbb{B}^3_{1/\sqrt{2}}(\mathbf{0}_3) \to \mathbb{S}^3,\ (x,y,z) \mapsto \left(x, y, z, \sqrt{1-\left(x^2+y^2+z^2\right)}\right)$$

which maps $\mathbf{0}_3$ to $\mathbf{n}_3$. Let $\mathbb{B}^2_1(\mathbf{0}_2)$ be the open ball around the origin $\mathbf{0}_2$ in $\mathbb{R}^2$ of radius 1. We use the local parametrization

$$\psi : V = \mathbb{B}^2_1(\mathbf{0}_2) \to \mathbb{S}^2,\ (x,y) \mapsto \left(x, y\sqrt{1-\left(x^2+y^2\right)}\right)$$

which maps $\mathbf{0}_2$ to $\mathbf{s}_2$. In Section 2.4.7 we computed the induced map θ. As in the proof of Theorem 4.2 we will now replace ϕ with a suitable local parametrization $\phi' : U' \to X$ such that the new induced map θ' is the restriction of the canonical submersion σ^3_2 to U' to obtain a diagram as in (4.1). We do this by finding the inverse map $\Theta^{-1} : U' \to U$ of Θ. The inverse exists on the sufficiently small open subset $U' \subset \mathbb{R}^3$ by the argument in the proof of Theorem 4.2.

To find Θ^{-1} we observe that $\psi\circ\sigma^3_2$ has the effect

$$(\psi\circ\sigma^3_2)(x,y,z) = \psi(x,y) = \left(x, y, \sqrt{1-\left(x^2+y^2\right)}\right).$$

Now, Θ^{-1} satisfies the identity $f\circ\phi\circ\Theta^{-1} = \psi\circ\sigma^3_2$. Thus, using the description of the Hopf map f in (4.2), to determine Θ^{-1}, for a given $\left(u, v, \sqrt{1-\left(u^2+v^2\right)}\right) \in \mathbb{S}^2$, we need to find $(x,y,z) \in U$ such that

$$\begin{aligned}(f\circ\phi)(x,y,z) &= f\left(x, y, z, \sqrt{1-\left(x^2+y^2+z^2\right)}\right)\\ &= \left(u, v, \sqrt{1-\left(u^2+v^2\right)}\right).\end{aligned}$$

Hence using the description of the Hopf map f in (4.2), we need to find $(x,y,z) \in U$ such that

$$\begin{pmatrix} 2xz + 2y\sqrt{1-\left(x^2+y^2+z^2\right)} \\ 2yz - 2x\sqrt{1-\left(x^2+y^2+z^2\right)} \\ x^2+y^2-z^2-\left(1-\left(x^2+y^2+z^2\right)\right)\end{pmatrix} = \begin{pmatrix} u \\ v \\ \sqrt{1-\left(u^2+v^2\right)}\end{pmatrix}.$$

The fact that Θ is a local diffeomorphism implies that we can solve these equations to find (x,y,z) in a sufficiently small open ball U'. The map Θ^{-1} is then given by sending a point $(x_0, y_0, z_0) \in U'$ to the point (x,y,z) in U. However, the formulas for (x,y,z) in terms of u and v may be quite involved. We see in this concrete example that the point of the Local Submersion Theorem 4.2 is that we can hide all the complications in the parametrization $\phi\circ\Theta^{-1}$ and can make look f, in the local coordinate systems, like the canonical submersion.

4.2 Regular values

4.2.1 Manifolds as preimages

Suppose we have a smooth map between manifolds $f : X \to Y$ and a point $y \in Y$. We would like to understand the geometry or topology of the **fiber of f over y**, i.e., the set

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

For example, we would very much like $f^{-1}(y)$ to be a smooth manifold itself. Remember that this was a situation we started with when we defined manifolds. The fiber $f^{-1}(y)$ is the set of solutions of the equation $f(x) = y$.

> **Remark 4.5 (Be aware)** Unfortunately, in general, there is no reason for the set $f^{-1}(y)$ to be a manifold.

- **However**, life is much nicer in the world of submersions:

We now assume that $f : X \to Y$ is a **submersion at a point** $x_0 \in X$. Assume $n = \dim X \geq \dim Y = m$. Set $y = f(x_0)$ or in other words $x_0 \in f^{-1}(y)$. We choose local parametrizations $\phi : U \to V \subset X$ and $\psi : W \to W' \subset Y$ around x and y, respectively, with $\phi(0) = x_0$ and $\psi(0) = y$ and open subsets $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$. See Figure 4.1. By the Local Submersion Theorem 4.2, we can choose these parametrizations such that $\theta = \psi^{-1} \circ f \circ \phi$ becomes the **canonical submersion** $\mathbb{R}^n \to \mathbb{R}^m$ restricted to U:

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \\
V \cap f^{-1}(y) \arrow[r, hook] \arrow[dr, "u_1=\cdots=u_m=0"'] & V \arrow[u, hook, "\text{open}"] \arrow[rr, "f_{|V}"] & & W' \arrow[u, hook, "\text{open}"'] \\
 & U \arrow[u, "\phi", "\cong"'] \arrow[rr, "\theta\,=\,\text{canonical}", "\text{submersion}"'] & & W. \arrow[u, "\cong", "\psi"']
\end{tikzcd}$$

Since $\psi^{-1}(y) = 0$ and since ϕ and ψ are diffeomorphisms, we have

$$x \in V \cap f^{-1}(y) \iff \theta(\phi^{-1}(x)) = 0.$$

We now write

$$\phi^{-1}(x) =: u(x) = \big(u_1(x), u_2(x), \dots, u_n(x)\big)$$

to make the **local coordinates** in U of points in $x \in V$ explicit. With this notation, we can rewrite the above equivalence as

$$x \in V \cap f^{-1}(y) \iff \theta\left(u_1(x), u_2(x), \dots, u_n(x)\right) = 0.$$

Now, since θ is the **canonical submersion**, we can rewrite this yet again as

$$x \in V \cap f^{-1}(y) \iff u_1(x) = u_2(x) = \cdots = u_m(x) = 0.$$

Hence in terms of sets we get

$$f^{-1}(y) \cap V = \{x \in V : u_1(x) = \cdots = u_m(x) = 0\}.$$

However, on $f^{-1}(y) \cap V$, the remaining local coordinate functions $u_{m+1}(x), \ldots, u_n(x)$ do not have to satisfy any additional condition. Hence we can use them to define a **local coordinate system** in $\mathbb{R}^{n-m}$ for points $x \in f^{-1}(y) \cap V$. In fact, the subset $f^{-1}(y) \cap V \subset f^{-1}(y)$ is **open** in $f^{-1}(y)$, since V is open in X and $f^{-1}(y)$ has the **subspace** topology. Similarly, the subset $(\{0\} \times \mathbb{R}^{n-m}) \cap U$ is **open** in $\{0\} \times \mathbb{R}^{n-m}$, since U is open in $\mathbb{R}^n$. We now identify $\{0\} \times \mathbb{R}^{n-m}$ with $\mathbb{R}^{n-m}$ and write $U' \subset \mathbb{R}^{n-m}$ for the open subset corresponding to $(\{0\} \times \mathbb{R}^{n-m}) \cap U$.

Summarizing, we have shown that the restriction of ϕ^{-1} to $f^{-1}(y) \cap V$

$$f^{-1}(y) \cap V \to (\{0\} \times \mathbb{R}^{n-m}) \cap U, \ x \mapsto \big(u_{m+1}(x), \ldots, u_n(x)\big)$$

defines a **local coordinate system** for $f^{-1}(y)$ around x_0. In particular, this is a **diffeomorphism** between open subsets. We write

$$\phi' : (\{0\} \times \mathbb{R}^{n-m}) \cap U =: U' \to f^{-1}(y) \cap V$$

for the inverse diffeomorphism where $U' \subset \mathbb{R}^{n-m}$ is an **open** subset. That is ϕ' is a **local parametrization** around x_0 in $f^{-1}(y)$.

We would like this to be possible for every point in the fiber $f^{-1}(y)$. This is not always the case. So let us give the desired case a name:

Definition 4.6 **(Regular points and regular values)** Let $f : X \to Y$ be a smooth map of manifolds. A point $x \in X$ is called a **regular point** for f if $df_x : T_x(X) \to T_y(Y)$ is **surjective**, i.e., f is a submersion at x. An element $y \in Y$ is called a **regular value** for f if all points $x \in f^{-1}(y)$ are regular, i.e., $y \in Y$ is a regular value if $df_x : T_x(X) \to T_y(Y)$ is **surjective** at **every** point $x \in X$ such that $f(x) = y$.

The above argument shows the following very important result:

Theorem 4.7 **(Preimage Theorem)** Let $f : X \to Y$ be a smooth map of manifolds. If y is a **regular value** for $f : X \to Y$, then the **fiber** $f^{-1}(y)$ over y is a **submanifold** of X, with $\dim f^{-1}(y) = \dim X - \dim Y$.

Remark 4.8 The above Theorem 4.7 is actually one of the main tools to show that a space is a smooth manifold. It is an important **mile stone** on our journey and we should always have it in mind whenever we are asked to show that a space has the structure of a smooth manifold.

Before we look at some examples, we determine the tangent spaces of the submanifold given as the preimage of a regular value:

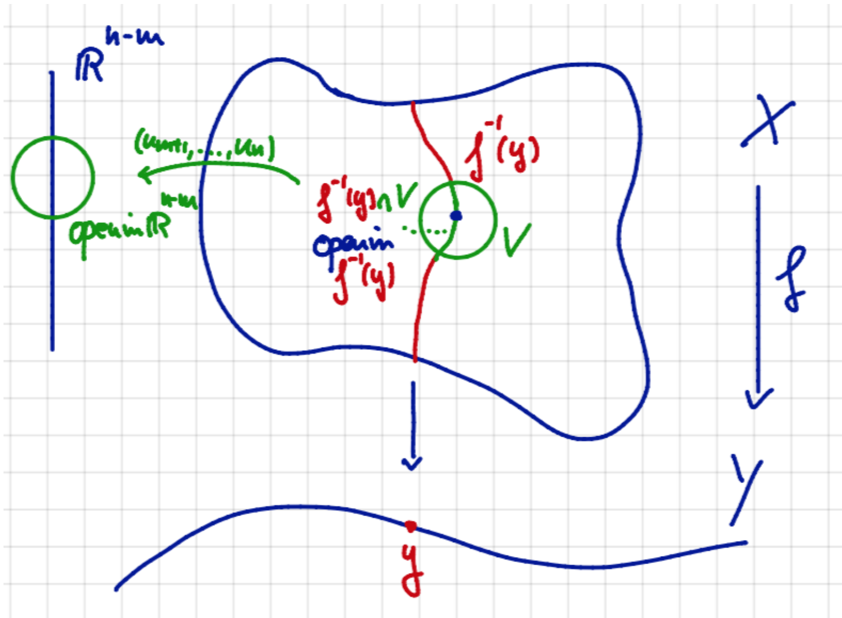

Figure 4.1: At a regular point we can use the additional coordinates to define a local coordinate system of the fiber.

Lemma 4.9 (Tangent space of a regular fiber) Let $f: X \to Y$ be a smooth map of manifolds, and let y be a **regular value**. Let $Z = f^{-1}(y)$ be the **fiber** $y \in Y$. Then the tangent space $T_x(Z)$ at a point $x \in Z$ is the **kernel of the derivative**

$$df_x : T_x(X) \to T_y(Y),$$

i.e., $T_x(Z) = \text{Ker}\,(df_x)$.

Proof: Since $f(Z) = \{y\}$, f is **constant on Z**. Therefore, df_x vanishes on the subspace $T_x(Z) \subset T_x(X)$. Hence df_x sends all of $T_x(Z)$ to zero. More concretely, using the notation of the argument to prove Theorem 4.7, we have the following commutative diagram:

$$\begin{tikzcd} T_x(Z) \arrow[r, hook] & T_x(X) \arrow[r, "df_x"] & T_y(Y) \\ \mathbb{R}^{n-m} \arrow[r, hook] \arrow[u, "d\phi_0'"] & \mathbb{R}^n \arrow[r, "d\theta_0 = \sigma^n_m"] \arrow[u, "d\phi_0"] & \mathbb{R}^m. \arrow[u, "d\psi_0"'] \end{tikzcd}$$

Since σ^n_m is the canonical submersion, the composition of the lower horizontal map is zero. Since $T_x(Z)$ is defined as the image of $d\phi_0'$ and since the diagram commutes, this shows that df_x sends all of $T_x(Z)$ to 0. This proves $T_x(Z) \subseteq \text{Ker}\, df_x$.

On the other hand, since y is a **regular value**, df_x is surjective. Hence the dimension of the kernel of df_x is $\dim T_x(X) - \dim T_y(Y) = \dim X - \dim Y = \dim Z$. This shows that $T_x(Z)$ is a **subspace** of the kernel of df_x of the **same dimension** as $\text{Ker}\, df_x$. Thus $T_x(Z) = \text{Ker}\, df_x$. □

4.2.2 First examples

- Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be the map

$$x = (x_1, \ldots, x_{n+1}) \mapsto |x|^2 = x_1^2 + \cdots + x_{n+1}^2.$$

The derivative df_x at the point $x = (x_1, \ldots, x_{n+1})$ is the linear map given by the matrix $(2x_1 \ldots 2x_{n+1})$ expressed in the standard basis. Thus $df_x : \mathbb{R}^{n+1} \to \mathbb{R}$ is surjective unless $f(x) = 0$, so every nonzero real number is a regular value of f. In particular, we get again that the sphere $\mathbb{S}^n = f^{-1}(1)$ is an n-dimensional manifold.

- We will show in the exercises that $1 \in \mathbb{R}$ is a regular value of the determinant function $\det : GL(n) \to \mathbb{R}$. This will show that the subgroup $SL(n)$ of matrices with determinant one is a smooth manifold. It is is called the **special linear group**. More on such groups later.

- Recall the map

$$f : \mathbb{R}^4 \to \mathbb{R},$$
$$(x_1, x_2, x_3, x_4) \mapsto x_1 + x_2^2 + x_3^3 + x_4^4.$$

We showed above that f is a submersion. In particular, this implies that 0 is a regular value for f. Hence the preimage

$$Z = f^{-1}(0) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2^2 + x_3^3 + x_4^4 = 0\}$$

is a **submanifold** in $\mathbb{R}^4$. Its dimension is $\dim \mathbb{R}^4 - \dim \mathbb{R} = 3$.

- Let $p_1, \ldots, p_n$ be polynomials with real coefficients in variables $t_1, \ldots, t_m$. We can consider the collection $(p_1, \ldots, p_n)$ as a smooth map $\mathbf{p} : \mathbb{R}^m \to \mathbb{R}^n$. Then the **set of common zeros** of $p_1, \ldots, p_n$ is a smooth submanifold of $\mathbb{R}^m$ if $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ is a **regular value** of p. This is the case if the Jacobian matrix $J_\mathbf{s}$ at a point $\mathbf{s} = (s_1, \ldots, s_m)$, i.e., the matrix with (i, j)-entry the partial derivative $\frac{\partial p_i}{\partial t_j}(\mathbf{s})$, has **maximal rank** whenever $\mathbf{p}(\mathbf{s}) = \mathbf{0}$. For example, if $n = m$, this requires that $\det J_\mathbf{s} \neq 0$. While, if $n = 1$ and $m > 1$, then this only requires that the partial derivatives do not vanish simultaneously. Such a set of zeros of polynomials is often referred to as an **algebraic variety**. See also Exercise 4.8. □

Remark 4.10 (Algebraic Geometry in a nutshell) The study of the zeroes of polynomials is the central theme in Algebraic Geometry. This is a classical and fascinating part of pure mathematics. In the past three decades, deep and fascinating connections between algebraic geometry and homotopy theory have been developed. This is the field of Motivic Homotopy Theory.

Since $f^{-1}(y)$ can be very complicated if y is not regular, the values which are **not** regular get the following name:

Definition 4.11 **(Critical values)** Let $f: X \to Y$ be a smooth map of manifolds. A point $x \in X$ is called a **critical point** for f if df_x is **not** surjective, i.e., if x is not a regular point. An element $y \in Y$ is called a **critical value** for f if it is **not** a regular value for f.

- For **example**, we will show in Exercise 4.9 that the **only critical** value of the determinant map $\det: M(n) \to \mathbb{R}$ is 0.

Remark 4.12 All values y which are **not** in the image of f also are regular values for f. For, if $f^{-1}(y)$ is the **empty set**, then there is no condition to be satisfied.

Depending on the relative dimensions of X and Y we can now say how regular values can arise.

Remark 4.13 **(Summary for regular values)** Suppose $f: X \to Y$ is a smooth map of manifolds. Then **y being a regular value for f** has the following meaning:

- if $\dim X > \dim Y$: y is a regular value if and only if f is a **submersion** at each point $x \in f^{-1}(y)$;
- if $\dim X = \dim Y$: y is a regular value if and only if f is a **local diffeomorphism** at each point $x \in f^{-1}(y)$;
- if $\dim X < \dim Y$: y is a regular value if and only if y is **not** in the **image** of f; for, all values in the image are critical (df_x cannot be surjective when $\dim T_x(X) < \dim T_{f(x)}(Y)$).

4.2.3 More examples: Matrix subgroups are manifolds

A very important application of the Preimage Theorem is that we can use it to show that **various matrix groups are smooth manifolds**. Let $M(n)$ again denote the space of real $n \times n$-matrices and $GL(n)$ denote the group of invertible matrices in $M(n)$.

Let $O(n)$ be the subgroup of matrices A of $GL(n)$ which satisfy $AA^t = I$ where A^t denotes the transpose of A and I is the $n \times n$-identity matrix. Note that $O(n)$ is the subgroup of invertible matrices which preserve the scalar product of vectors. In particular, matrices in $O(n)$ preserve distances and angles in $\mathbb{R}^n$.

Our goal is to show:

Theorem 4.14 **(Orthogonal matrices form a smooth manifold)** The group $O(n)$ of orthogonal matrices is a **smooth manifold** of dimension $n(n-1)/2$.

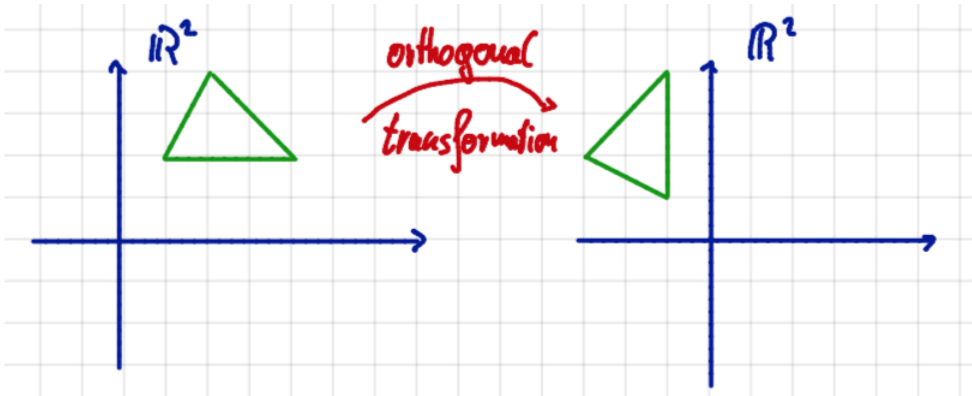

Figure 4.2: Orthogonal transformations preserve the geometry of the plane.

Proof: First, we note that AA^t is a symmetric matrix:

$$(AA^t)^t = (A^t)^t A^t = AA^t.$$

The subspace $S(n)$ of symmetric matrices is a smooth submanifold of $M(n)$ diffeomorphic to $\mathbb{R}^k$ with $k = n(n+1)/2$. The dimension is determined by the fact that the entries below the diagonal are determined by the entries above the diagonal such that there are $n(n+1)/2$ free choices. We consider the map

$$f : M(n) \to S(n), \; A \mapsto AA^t.$$

This map is smooth, since multiplication of matrices is smooth and taking transposes is smooth as well as we just reshuffle the entries in the matrix. Now we observe $O(n) = f^{-1}(I)$. Hence, in order to show that $O(n)$ is a smooth manifold, we just need to show that **I is a regular value for f**. In order to check that I is a regular value, we need to show that derivative of f at a matrix A,

$$df_A : T_A(M(n)) \to T_{f(A)}(S(n)),$$

is **surjective for all $A \in O(n)$**. Since $M(n)$ and $S(n)$ are vector spaces, we have

$$T_A(M(n)) = M(n) \text{ and } T_{f(A)}(S(n)) = S(n).$$

So let us compute the derivative of f at a matrix A:

$$\begin{aligned} df_A(B) &= \lim_{s\to 0} \frac{f(A+sB) - f(A)}{s} = \lim_{s\to 0} \frac{(A+sB)(A+sB)^t - AA^t}{s} \\ &= \lim_{s\to 0} \frac{(A+sB)(A^t+sB^t) - AA^t}{s} = \lim_{s\to 0} \frac{AA^t + sBA^t + sAB^t + s^2BB^t - AA^t}{s} \\ &= \lim_{s\to 0} \frac{sBA^t + sAB^t + s^2BB^t}{s} = \lim_{s\to 0} BA^t + AB^t + sBB^t \\ &= AB^t + BA^t. \end{aligned}$$

Hence, given a matrix $C \in S(n)$, we need to show that there is a matrix $B \in M(n)$ with $df_A(B) = BA^t + AB^t = C$. Since **$C$ is symmetric**, we have

$$C = \frac{1}{2}(2C) = \frac{1}{2}(C + C) = \frac{1}{2}\left(C + C^t\right).$$

Now we set $B = \frac{1}{2}CA$ and compute, using $AA^t = I$:

$$\begin{aligned} df_A(B) &= \left(\frac{1}{2}CA\right)A^t + A\left(\frac{1}{2}CA\right)^t \\ &= \frac{1}{2}CAA^t + \frac{1}{2}AA^tC^t \\ &= \frac{1}{2}C + \frac{1}{2}C^t \\ &= C. \end{aligned}$$

Thus, I is a **regular value**, and $O(n)$ is a submanifold of $M(n)$. We can also calculate the dimension of $O(n)$:

$$\dim O(n) = \dim M(n) - \dim S(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}. \qquad \square$$

We end this section with an outlook to an exciting field which has ramifications to almost all areas of mathematics:

Remark 4.15 (Lie groups) The manifold $O(n)$ is an example of a **very important class of smooth manifolds**. For, $O(n)$ is both a smooth manifold and a group such that the group operations are smooth. For both the multiplication map

$$O(n) \times O(n) \to O(n),\ (A, B) \mapsto AB$$

and the map of forming the inverse

$$O(n) \to O(n),\ A \mapsto A^{-1}$$

are smooth. For the latter note $A^{-1} = A^t$ for $A \in O(n)$, though taking inverses is also smooth for other matrix groups.

In general, a group which is also a **manifold** such that the **group operations are smooth** is called a **Lie group**. Lie groups are **extremely interesting and important** and have a rich and exciting theory. For example, the tangent space at a Lie group at the identity element is a Lie algebra, a vector space with a certain additional operation. Such Lie algebras can be classified completely. Lie groups and Lie algebras play an important role in many different areas of mathematics and physics. We will take a closer look, though by now way sufficient for their very rich theory, at Lie groups in chapter 5.

4.3 The Stack of Records Theorem

We will now take a closer look at a specific and important situation for regular values. This will be **useful** later at several occasions.

We assume in this section that $f : X \to Y$ is a smooth map with $\dim X = \dim Y$ and X **compact**. Our goal is to understand the fibers $f^{-1}(y)$ of regular values. We do this in a series of observations:

- Let $y \in Y$ be a **regular value** for f such that $f^{-1}(y)$ is not empty. Let x be a point in $f^{-1}(y)$. Since y is a regular value, x is a regular point, i.e., df_x is surjective and hence an isomorphism as $\dim X = \dim Y$. Hence f is a **local diffeomorphism at** x. Let $V \subset X$ and $U \subset Y$ be open neighborhoods around x and y, respectively, such that $\mathbf{f_{|V}} : V \to U$ is a **diffeomorphism**.

- If x' is **another point** in $f^{-1}(y)$ with $x \neq x'$. Then $df_{x'}$ is an isomorphism as well, and we can choose an open neighborhood $V' \subset X$ around x' such that $f_{|V'} : V' \to U'$ is a diffeomorphism onto an open subset $U' \subset Y$ containing y.

Lemma 4.16 The subsets V and V' are **disjoint**.

Proof: If $V \cap V' \neq \emptyset$, then f restricts to a diffeomorphism from the open subset $V \cap V'$ onto $U \cap U'$. Since $y \in U \cap U'$ and $f(x) = y = f(x')$, this would imply $x = x' \in V \cap V'$, since x and x' are the **unique** points in V and V', respectively, which map to y. So if $x \neq x'$, we must have $V \cap V' = \emptyset$. □

- Hence all the points in $f^{-1}(y)$ lie in **pairwise disjoint open** subsets of X. We conclude that $f^{-1}(y)$ is **discrete**, i.e., it is a space in which **every** subset is open. Since the subset $\{y\}$ is **closed** in Y, the fiber $f^{-1}(y)$ is a **closed** subset of X. Since X is **compact** and since closed subsets in compact spaces are compact, this implies that $f^{-1}(y)$ **is compact** as well. Hence $f^{-1}(y)$ is a **compact and discrete** subset of Euclidean space. We have shown previously in Lemma 2.12 that this implies that $f^{-1}(y)$ is a **finite set**.

- Let $f^{-1}(y) = \{x_1, \dots, x_n\}$. We can thus pick pairwise **disjoint open** subsets $W_1, \dots, W_n$ in X with $x_i \in W_i$ and **open** subsets $U_1, \dots, U_n$ in Y, each containing y, such that f maps W_i diffeomorphically onto the **open** subset U_i. We define the set

$$U := (U_1 \cap \cdots \cap U_n) \setminus f(X \setminus (W_1 \cup \ldots \cup W_n)).$$

Lemma 4.17 The subset U is **open** in Y.

Proof: First, the **finite** intersection $U_1 \cap \cdots \cap U_n$ is **open** in Y. Second, the set $W := W_1 \cup \ldots \cup W_n$ is **open** in X and its complement $X - W$ in X is **closed**. Since X is **compact**, $X \setminus W$ is compact and hence $f(X \setminus W)$ is compact. Since Y is **Hausdorff**, or since Y is a subspace in $\mathbb{R}^N$ in which compact subsets are closed by Theorem 2.11, this implies that $f(X \setminus W)$ is a **closed** subset in Y. Hence the subset $f(X \setminus W)$ is closed in $U_1 \cap \cdots \cap U_n$. This shows that U is open. □

- We also know $y \in U$, since $y \in U_1 \cap \cdots \cap U_n$ and $y \notin f(X \setminus W)$ by our choices of the U_i's and W_i's.

- We set $V_i := W_i \cap f^{-1}(U)$. Then $f_{|V_i} : V_i \to U$ is a diffeomorphism and maps x_i to y.

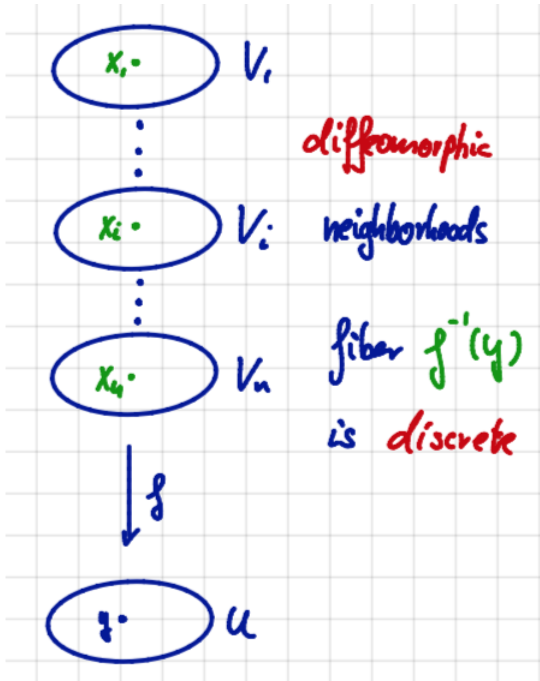

Figure 4.3: Every point y in Y has an open neighborhood U such that $f^{-1}(U)$ consists of disjoint copies of open subsets V_i in X which are diffeomorphic to U. In three-dimensional space this may be drawn as several old school records stacked on top of each other. One key consequence is that the fiber over y is a finite and discrete set.

- Thus the inverse image $f^{-1}(U)$ is a **disjoint** union of **open** subsets $V_1, \dots, V_n$ with $x_i \in V_i$ and the restriction of f to any of the V_i is a **diffeomorphism onto** U.

Hence we have shown the following very useful result, illustrated in Figure 4.3:

Theorem 4.18 (Stack of Records Theorem) Suppose $\dim X = \dim Y$, $f : X \to Y$ is a smooth map and X is **compact**. Let $y \in Y$ be a **regular value** for f such that $f^{-1}(y) \neq \emptyset$. Then:

- the set $f^{-1}(y)$ is a **discrete finite** subset $\{x_1, \dots, x_n\}$ of X, and
- we can choose an open neighborhood $U \subset Y$ around y such that $f^{-1}(U) \subset X$ is the disjoint union $V_1 \cup \dots \cup V_n$ of open subsets of X with $x_i \in V_i$ and the restriction of f to any of the V_i is a **diffeomorphism onto** U.

The above situation is a particular instance of an important phenomenon:

Remark 4.19 (Covering spaces) If in addition to the assumptions of the theorem all values in Y are regular, then $X \to Y$ is an example of a **covering**. In Topology, a continuous map $f : X \to Y$ is an **unramified** covering if every point in Y has an open neighborhood U such that $f^{-1}(U)$ is the disjoint union of open sets V_i such that f maps each V_i homeomorphically onto U. Coverings play an important role in topology and homotopy theory.

Given a regular value y, the fiber $f^{-1}(y)$ is either empty or **finite**. Hence it makes sense to talk about the **number** of elements in the set $f^{-1}(y)$. We denote this number by $\#f^{-1}(y)$. We can then make the following observation:

Lemma 4.20 **(Locally constant fiber)** The function $y \mapsto \#f^{-1}(y)$ from the **set of regular points** for f to the integers is **locally constant**, i.e., for every regular value y_0 there is an open neighborhood $U \subset Y$ of y_0 such that $\#f^{-1}(y) = \#f^{-1}(y_0)$ for all $y \in U$.

Proof: We split the proof into considering the cases $f^{-1}(y_0) \neq \emptyset$ and $f^{-1}(y_0) = \emptyset$ separately.:

- First we assume $f^{-1}(y_0) \neq \emptyset$. By the **Stack of Records Theorem 4.18** we can choose the open neighborhood U of y_0 in Y such that $f^{-1}(U) = V_1 \cup \cdots \cup V_n$ is the pairwise disjoint union of open neighborhoods V_i of x_i which are all mapped diffeomorphically onto the open subset U. Thus, for every point $y \in U$, there is **exactly one** point in V_i which maps to y. And these are the only points in X which map onto y by the choice of U. Thus
$$\#f^{-1}(y) = \#f^{-1}(y_0) \text{ for all } y \in U$$
and the function $y \mapsto \#f^{-1}(y)$ is constant on the **open** subset U.

- Now assume $f^{-1}(y_0) = \emptyset$. Since X is **compact**, $f(X)$ is compact and hence closed in Y. Thus $Y \setminus f(X)$ is **open** in Y. Since by our assumption $y_0 \in Y \setminus f(X)$, the subset $U := Y \setminus f(X)$ is therefore an open neighborhood around y_0 such that
$$\#f^{-1}(y) = \#f^{-1}(y_0) \text{ for all } y \in U. \qquad \square$$

Along the way we have also collected the pieces for the proof of the following observation which we now state for future reference and prove in detail:

Lemma 4.21 The set R of **regular values** for f is **open** in Y.

Proof: Let $x_0 \in X$ be a regular point for f, i.e., df_{x_0} is **surjective**. Since we assume $\dim X = \dim Y$, this actually implies that df_{x_0} is an **isomorphism**. By the Inverse Function Theorem 3.4, there is an open neighborhood W around x_0 such that df_x is an isomorphism for all $x \in W$. Hence the subset of **regular points** is **open** in X. Consequently, its complement in X, the set of critical points C is a **closed** subset in X. Since X is **compact**, this implies that C is a **compact** subset. Since f is continuous, $f(C)$ is a **compact** subset in Y. Since Y is **Hausdorff**, or since Y is a subspace in $\mathbb{R}^N$ in which compact subsets are closed by Theorem 2.11, this implies that $f(C)$ is **closed** in Y. The set $f(C)$ is the set of critical values of f and its complement is the set R of regular values. Hence R is **open** in Y. $\square$

4.4 Milnor's proof of the Fundamental Theorem of Algebra

As an application of regular values we will now study Milnor's proof of the Fundamental Theorem of Algebra [13, page 8–9]:

Theorem 4.22 (Fundamental Theorem of Algebra) Every non-constant complex polynomial has a zero. More precisely, let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $n \geq 1$, $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Then there is a $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. As a consequence, $P(z)$ must have **exactly n zeros** when we count them with multiplicities.

Before we look at the details of the proof let us outline the argument:

- **(Outline of the proof)** Let us first summarize the **idea of the proof**:
  - We will show that $P : \mathbb{C} \to \mathbb{C}$ is **surjective**. So 0 is a value of P.
  - We do this by **replacing P** with a smooth map $f : \mathbb{S}^2 \to \mathbb{S}^2$ such that

    $$f \text{ is onto} \iff P \text{ is onto.}$$

    The **key** feature is that $\mathbb{S}^2$ is **compact** while $\mathbb{C}$ is not!
  - We show that the **assumption** that f is **not surjective** leads to a **contradiction** as follows:
    - The **Stack of Records Theorem** implies that the function $y \mapsto \# f^{-1}(y)$ is **constant** on the set of regular values on $\mathbb{S}^2$. Here we use that P is a polynomial such that dP_z has only **finitely many critical points**.
    - If f was **not surjective**, $\# f^{-1}(y) = 0$ for **all regular values**.
    - Then f would have to be **constant**, since $\mathbb{S}^2 \setminus \{\text{critical values}\}$ is **connected**.
    - **But** f is **not constant**, since P is **not constant**.

The same argument does **not** work for **real** polynomials, at least not for polynomials of degree at least two:

Remark 4.23 ($\mathbb{R}$ is too one-dimensional) If we tried the above strategy for a **real** polynomial $P : \mathbb{R} \to \mathbb{R}$, we would replace P with a smooth map $f : \mathbb{S}^1 \to \mathbb{S}^1$. Then f is smooth defined on a compact domain and f has only finitely many critical values. But the conclusion that the function $y \mapsto \# f^{-1}(y)$ is **constant** on the set of regular values on $\mathbb{S}^1$ would **not work** anymore. For, after removing **at least two** different points from $\mathbb{S}^1$ we get a space which is **not connected**. And for a polynomial of degree at least two, we cannot exclude the possibility that f may have two critical values. See also Remark 4.27.

4.4.1 The proof of Theorem 4.22

We are going to identify the complex numbers $\mathbb{C}$ with the points in real plane $\mathbb{R}^2$, but we keep in mind that $\mathbb{C}$ is a field, i.e., we can multiply and form inverses for points in $\mathbb{C}$. To prove the theorem we need to extend the map $P : \mathbb{C} \to \mathbb{C}$ to a map on a **compact** space. Recall that $\mathbb{S}^2$ is a compact subspace of $\mathbb{R}^3$ and that we can relate $\mathbb{S}^2$ and the real plane $\mathbb{R}^2$ via **stereographic projection**, recall Figure 2.12. The formulae for the projection from the north pole $N = (0, 0, 1) \in \mathbb{S}^2$ are

$$\phi_N^{-1} : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2, (x_1, x_2, x_3) \mapsto \frac{1}{1 - x_3}(x_1, x_2) \text{ and}$$
$$\phi_N : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{N\}, (x_1, x_2) \mapsto \frac{1}{1 + |x|^2}\left(2x_1, 2x_2, |x|^2 - 1\right).$$

The formulae for the projection from the south pole $S = (0, 0, -1) \in \mathbb{S}^2$:

$$\phi_S^{-1} : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2, (x_1, x_2, x_3) \mapsto \frac{1}{1 + x_3}(x_1, x_2) \text{ and}$$
$$\phi_S : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{S\}, (x_1, x_2) \mapsto \frac{1}{1 + |x|^2}\left(2x_1, 2x_2, 1 - |x|^2\right).$$

Considering our polynomial P as a map from $\mathbb{C}$ to $\mathbb{C}$ **we define** a new map

$$f : \mathbb{S}^2 \to \mathbb{S}^2, \quad \begin{cases} f(x) := \phi_N \circ P \circ \phi_N^{-1}(x) & \text{for all } x \in \mathbb{S}^2 \setminus \{N\} \\ f(N) := N & \text{for } x = N. \end{cases} \tag{4.3}$$

Lemma 4.24 (First claim: Smoothness) The map $f : \mathbb{S}^2 \to \mathbb{S}^2$ is smooth.

Proof: Since ϕ_N and ϕ_N^{-1} are smooth and polynomials are smooth as well, it is clear that f is smooth at all points which are not the north pole. It remains to show that it is also **smooth in a neighborhood of N**. In order to do this we use the projection from the south pole and define a map

$$Q : \mathbb{C} \to \mathbb{C} \text{ by } Q := \phi_S^{-1} \circ f \circ \phi_S.$$

Note that $Q(0) = 0$, since $\phi_S(0) = N$, $f(N) = N$ and $\phi_S^{-1}(N) = 0$. To compute Q for other values we need to calculate the composite $\phi_N^{-1} \circ \phi_S$. For a point $x = (x_1, x_2) \neq (0, 0)$ we get

$$\begin{aligned} \phi_N^{-1} \circ \phi_S(x_1, x_2) &= \phi_N^{-1}\left(\frac{1}{1 + |x|^2}(2x_1, 2x_2, 1 - |x|^2)\right) = \frac{1}{1 - \frac{1-|x|^2}{1+|x|^2}}\left(\frac{2x_1}{1 + |x|^2}, \frac{2x_2}{1 + |x|^2}\right) \\ &= \frac{1 + |x|^2}{2|x|^2}\left(\frac{2x_1}{1 + |x|^2}, \frac{2x_2}{1 + |x|^2}\right) = \frac{1}{|x|^2}(x_1, x_2). \end{aligned}$$

Remembering complex conjugation $z \mapsto \bar{z}$ on $\mathbb{C}$ and $|z|^2 = z\bar{z}$, we can rewrite this as:

$$\phi_N^{-1} \circ \phi_S(z) = \frac{z}{|z|^2} = 1/\bar{z} \text{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Similarly, we get

$$\phi_S^{-1} \circ \phi_N(z) = \frac{z}{|z|^2} = 1/\bar{z} \text{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Thus, for $z \neq 0$, we get

$$\begin{aligned} Q(z) &= \phi_S^{-1} \circ \phi_N \circ P \circ \phi_N^{-1} \circ \phi_S(z) = \phi_S^{-1} \circ \phi_N(P(1/\bar{z})) \\ &= \phi_S^{-1} \circ \phi_N(a_n \bar{z}^{-n} + a_{n-1} \bar{z}^{-(n-1)} + \cdots + a_1 \bar{z}^{-1} + a_0) \\ &= 1/\left(\bar{a}_n z^{-n} + \bar{a}_{n-1} z^{-(n-1)} + \cdots + \bar{a}_1 z^{-1} + \bar{a}_0\right) \\ &= z^n/\left(\bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_1 z^{n-1} + \bar{a}_0 z^n\right). \end{aligned}$$

Note that $z = 0$ is not a zero of $D(z) := \bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_1 z^{n-1} + \bar{a}_0 z^n$, since $\bar{a}_n \neq 0$. Thus in small a neighborhood of $z = 0$ in $\mathbb{C}$, the **denominator** in the expression for $Q(z)$ is **bounded below**, i.e., there is a small $\varepsilon > 0$ and a $c > 0$ such that $|D(z)| \geq c$ for all $z \in \mathbb{B}^2_\varepsilon(0) \subset \mathbb{C}$. Since $Q(z) = z^n/D(z)$, this shows that Q is smooth at $z = 0$. See also Remark 4.25 below.

This implies that Q is smooth in a small **open neighborhood** of 0. Since ϕ_S and ϕ_S^{-1} are diffeomorphisms and since ϕ_S sends an open neighborhood of N in $\mathbb{S}^2$ to an open neighborhood of 0 in $\mathbb{C}$, this implies that

$$f = \phi_S^{-1} \circ Q \circ \phi_S$$

is **smooth** in an open neighborhood of N. □

Remark 4.25 (Derivative of Q) We can also calculate the derivative of Q at 0. Since we checked above that $Q(0) = 0$, we get

$$\begin{aligned} dQ_0 &= \lim_{h \to 0} \frac{Q(h) - Q(0)}{h} = \lim_{h \to 0} \frac{h^n/\left(\bar{a}_n + \bar{a}_{n-1} h + \cdots + \bar{a}_0 h^n\right) - 0}{h} \\ &= \lim_{h \to 0} \frac{h^{n-1}}{\bar{a}_n + h(\bar{a}_{n-1} + \cdots + \bar{a}_0 h^{n-1})} \\ &= \begin{cases} 1/\bar{a}_n & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases} \end{aligned}$$

- Now we use the fact that P is a complex **polynomial**. This will give us a **simple** way to **calculate its derivative**. For the derivative of the real function $P : \mathbb{R}^2 \to \mathbb{R}^2$ is the same as the underlying derivative of $P : \mathbb{C} \to \mathbb{C}$ as a complex function.[2] **More importantly**, the derivative of a polynomial has at most **finitely** many zeros. Recall that a point $x \in \mathbb{S}^2$ is called a critical point for f if the derivative df_x fails to be surjective.

Lemma 4.26 (Second claim: Finitely many critical points) The map $f : \mathbb{S}^2 \to \mathbb{S}^2$ has **at most n** many critical points. In particular, f has only **finitely** many **critical points**.

[2] We also have the additional information that the derivative is $\mathbb{C}$-linear, not just $\mathbb{R}$-linear.

Proof: Since ϕ_N and ϕ_N^{-1} are **diffeomorphisms**, the only points that might be critical for f are the points where P **fails** to be a local diffeomorphism, and possibly N. But the derivative of P, as a **linear** map

$$dP_z : \mathbb{C} \to \mathbb{C},\ w \mapsto dP_z \cdot w,$$

is given by

$$dP_z = P'(z) = \sum_{j=1}^{n} j a_j z^{j-1}$$

Hence dP_z fails to be an isomorphism, only if it is the **zero map**, i.e., $dP_z = 0$. However, as a function of z, dP_z is a **polynomial** of degree $n-1$ and it is **not** identically zero, since $a_n \neq 0$. This shows that there are **at most** $n-1$ complex numbers z such that $dP_z = 0$. Hence there are only **finitely many** z where dP_z is **not** an isomorphism. □

Remark 4.27 The computation of the derivative of Q at 0 in Remark 4.25 and the chain rule imply that N is, in fact, a critical point of f when $n \geq 2$. Hence, for a polynomial P of degree two, f has, in fact, two critical points since $P'(z)$ is of degree one and therefore has a zero. This observation is relevant in case we would like to apply the argument to a real polynomial P.

- Lemma 4.26 implies that f has **at most** n critical values. Thus, the set R of **regular values** for f is $\mathbb{S}^2$ with only **finitely** many points **removed**. This shows that R is **connected**.
- Now we use that fact that $\mathbb{S}^2$ is **compact**: The map f satisfies the assumption of the Stack of Records Theorem 4.18, and we can apply Lemma 4.20 to see that the function

$$R = \mathbb{S}^2 \setminus \{\text{critical values}\} \longrightarrow \mathbb{Z},\ y \mapsto \# f^{-1}(y),$$

  is **locally constant**. Since it is defined on a **connected** space, it must be **constant** by Lemma 2.14.

Lemma 4.28 (Third claim: Infinitely many values) The map f has **infinitely** many different values.

Proof: **Assume** f had only **finitely** many **different** values $y_1, \dots, y_k \in \mathbb{S}^2$. Then we could write $\mathbb{S}^2$ as the union $\mathbb{S}^2 = f^{-1}(y_1) \cup \cdots \cup f^{-1}(y_k)$. For $y_i \neq y_j$, we have $f^{-1}(y_i) \cap f^{-1}(y_j) = \emptyset$. Each $f^{-1}(y_i)$ is a closed subset of $\mathbb{S}^2$, since $\{y_i\}$ is a closed subset and f is continuous. Moreover, each $f^{-1}(y_i)$ is also **open** since the complement is a **finite** union of closed subsets and therefore closed. Thus, we could write $\mathbb{S}^2$ as the union of **non-empty**, **open**, and **closed** subsets. Since $\mathbb{S}^2$ is **connected**, this is implies $k = 1$. However, this would mean that f was **constant**. However, P is **not** constant, and ϕ_N and ϕ_N^{-1} are diffeomorphisms. Thus f is **not constant**, and our initial assumption had to be wrong. □

This enables us to show:

Lemma 4.29 (Fourth claim: Surjectivity) The smooth map f is **surjective**.

Proof: Assume there is a $y_0 \in \mathbb{S}^2$ with $f^{-1}(y_0) = \emptyset$, i.e., $\# f^{-1}(y_0) = 0$. Then y_0 is a **regular value** for f by definition. Since the function $y \mapsto \# f^{-1}(y)$ is **constant** on the set of **regular values**, it would have to be **zero** for **every** regular value. Hence $\# f^{-1}(y) \neq 0$ **only** for **critical** values y. But that would mean that f had only **finitely many values**, as f has only finitely many critical values. This **contradicts** the previous claim. Thus f must be surjective. $\square$

- **Conclusion:** In particular, $f^{-1}(S) \neq \emptyset$ for the **south pole** S on $\mathbb{S}^2$. Hence there must be at least one point $x \in \mathbb{S}^2$ with $f(x) = S$. Since ϕ_N is a diffeomorphism and $\phi_N(0) = S$, x must satisfy $P(\phi_N^{-1}(x)) = 0$. Hence $z := \phi_N^{-1}(x) \in \mathbb{C}$ is a zero of P. $\square$

4.5 Exercises and more examples

4.5.1 Submersions and regular values

Exercise 4.1 Let $f : X \to Y$ be a submersion and U an open subset of X. Show that $f(U)$ is open in Y. In other words, submersions are open maps.

Exercise 4.2 We define the map g by

$$g : \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x^2 - y^2.$$

Determine the set of regular values of g, and determine the set of critical values of g. Is g a submersion?

Exercise 4.3 Recall that a space Y is called *connected* if Y cannot be written as the union of two nonempty disjoint open subsets; or equivalently, if Y and $\emptyset$ are the only subsets which are both open and closed in Y.

(a) Show that if X is compact and Y is connected, then every nontrivial submersion $f : X \to Y$ is surjective.

(b) Show that there exist no submersions from compact manifolds to $\mathbb{R}^n$ for any n.

Exercise 4.4 Show that the orthogonal group $O(n)$ is compact.
Hint: Show that if $A = (a_{ij})$ lies in $O(n)$, then for each i, $\sum_j a_{ij}^2 = 1$.

Exercise 4.5 Show that the tangent space to $O(n)$ at the identity matrix I is the vector space of *skew symmetric* $n \times n$-matrices, i.e., matrices B satisfying $B^t = -B$.

Exercise 4.6 Let $M(2)$ denote the set of all real 2×2-matrices.

(a) Show that the determinant function is a submersion on the open submanifold of nonzero 2×2-matrices $M(2) \setminus \{0\}$.

(b) Conclude that the set R_1 of all 2×2-matrices of rank 1 is a three-dimensional submanifold of $\mathbb{R}^4 = M(2)$.

Exercise 4.7 In this exercise we prove Euler's identity for homogeneous polynomials:
Let $P(x_1, \dots, x_k)$ be a homogeneous polynomial of degree m in k variables, i.e.,

$$P(tx_1, \dots, tx_k) = t^k P(x_1, \dots, x_k) \text{ for all } t.$$

Show Euler's identity

$$\sum_i x_i \partial P/\partial x_i = mP. \tag{4.4}$$

Hint: Define a new function Q by

$$Q(x_1, \dots, x_k, t) := P(tx_1, \dots, tx_k) - t^m P(x_1, \dots, x_k)$$

and compute its derivative.

Exercise 4.8 In this exercise we show that the fibers of homogeneous polynomials form manifolds.

Let $P(x_1, \dots, x_k)$ be a homogeneous polynomial of degree m in k variables We consider P as a map

$$\mathbb{R}^k \to \mathbb{R},\ (x_1, \dots, x_k) \mapsto P(x_1, \dots, x_k).$$

(a) Show that 0 is the only critical value of P. Conclude that $P^{-1}(a)$ is a $k-1$-dimensional submanifold of $\mathbb{R}^k$ for all $a \neq 0$.

Hint: You may want to use Euler's identity for homogeneous polynomials.

(b) For two positive real numbers $a, b > 0$, show that $P^{-1}(a)$ is diffeomorphic to $P^{-1}(b)$. Similarly, For two negative real numbers $a, b < 0$, show that $P^{-1}(a)$ is diffeomorphic to $P^{-1}(b)$.

Exercise 4.9 The set $SL(n)$ of $n \times n$- matrices with determinant $+1$ form a subgroup of $GL(n)$. In this exercise we show that $SL(n)$ is a smooth manifold of dimension $n^2 - 1$ and compute its tangent space.

(a) Show that 0 is the only critical value of $\det : M(n) \to \mathbb{R}$.

Hint: You may either want to think of det as a homogeneous polynomial given by Leibniz' formula

$$\det(A) = \sum_\sigma \left(\operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \right) \tag{4.5}$$

where the sum runs over all permutations of the set $\{1, \dots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. Then use a previous exercise.

Or you use the formula $\det A = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det A_{ij}$ where A_{ij} denotes the $(n-1) \times (n-1)$-matrix defined by removing the ith row and jth column from A.

(b) Conclude that $SL(n)$ is a submanifold of $M(n)$ of dimension $n^2 - 1$.

(c) Show that the tangent space to $SL(n)$ at the identity matrix consists of all matrices with trace equal to zero.

Hint: Recall that we proved: If $Z = f^{-1}(y) \subseteq X$ is a submanifold defined by a regular value y of a smooth map $f : X \to Y$, then $T_x(Z) = \operatorname{Ker}(df_x) \subseteq T_x(X)$. You may also want to use Leibniz' formula for det.

Exercise 4.10 Recall the *Hopf map* that we have seen previously: We consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, i.e., $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$, and $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e., $\mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$. Then the *Hopf map* π is the map $\mathbb{S}^3 \to \mathbb{S}^2$ given by

$$\pi(z_0, z_1) = \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right).$$

(a) First we consider π as a map $\tilde{\pi} : \mathbb{R}^4 \cong \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ using the same formula as for π, i.e., $\pi = \tilde{\pi}_{|\mathbb{S}^3}$. Compute the derivative of $\tilde{\pi}$ at a point $q = (z_0, z_1)$.

(b) Let $q \in \mathbb{S}^3$. Explain why the restriction of $d\tilde{\pi}_q$ to $T_q\mathbb{S}^3$ has image contained in $T_{\pi(q)}\mathbb{S}^2$.

(c) Consider the points $a = (0, 0, 1)$ and $b = (0, 1, 0)$ on $\mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$. Determine the fibers $\pi^{-1}(a)$ and $\pi^{-1}(b)$ and show that a and b are regular values for π.

(d) Now show that actually each point in $\mathbb{S}^2$ is a regular value for π.

5. A brief excursion to Lie groups

Very important examples of smooth manifolds are given by Lie groups. We have seen first examples in a previous section and will now discuss them further taking the group structure into account. The study of Lie groups and their associated Lie algebras is a fascinating subject in mathematics and we highly recommend to read more about them in other books. Here we will only give a brief introduction considering Lie groups as examples of smooth manifolds.

5.1 Lie groups - the definition

Definition 5.1 (Lie groups) A **Lie group** is a group G which is also a smooth manifold such that the two maps

$$\mu\colon G \times G \to G,\ (g,h) \mapsto g \cdot h =: gh$$

and

$$\iota\colon G \to G,\ g \mapsto g^{-1}$$

corresponding to the two group operations of multiplication and taking inverses, respectively, are both smooth.

In fact, we can **summarize** the condition that μ and ι are smooth by requiring that

$$G \times G \to G,\ (g,h) \mapsto gh^{-1}$$

is **smooth**.

Translations and tangent spaces

- If G is a Lie group, then any element $g \in G$ defines maps

  $$L_g \text{ and } R_g\colon G \to G,$$

  called **left translation** and **right translation**, respectively, by

  $$L_g(h) = gh \text{ and } R_g(h) = hg.$$

  Since L_g can be expressed as the composition of smooth maps

  $$G \xrightarrow{i_g} G \times G \xrightarrow{\mu} G,$$

with $i_g(h) = (g, h)$, it follows that L_g is smooth. It is actually a **diffeomorphism of G**, because $L_{g^{-1}}$ is a smooth inverse for it. Similarly, $R_g : G \to G$ is a diffeomorphism.

- In fact, many of the important properties of Lie groups follow from the fact that we can systematically map any point to any other via a canonical global diffeomorphism given by translation by a suitable element in G. This **translation** makes the study of Lie groups much **more accessible** compared to arbitrary smooth manifolds. In particular, we can move an open neighborhood around any point in G to make it an open neighborhood of the identity element. Hence, in a Lie group, we basically only need to study neighborhoods of the identity element.

- This observation has important consequence for the **tangent spaces** of Lie groups. In fact, the translation property of Lie groups implies that the tangent space to a Lie group G at any matrix in G is isomorphic to tangent space to G at the identity element. It is a vector space with an additional structure, a **Lie bracket**, and is an example of a **Lie algebra**. The classification of Lie algebras and thereby Lie groups is a highlight in the history of mathematics. □

Here are some simple **first examples** of Lie groups:

- The real numbers $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie groups under addition, because the coordinates of $x - y$ are linear and therefore smooth functions of (x, y).

- Similarly, $\mathbb{C}$ and $\mathbb{C}^n$ are Lie groups under addition.

- Any finite group with the discrete topology is a (compact) Lie group.

- Suppose G is a Lie group and $H \subseteq G$ is an open subgroup, i.e., a subgroup which is also an open subspace. Then H is a Lie group as well.

- The set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers is a 1-dimensional Lie group under multiplication. The subset $\mathbb{R}^{>0}$ of positive real numbers is an open subgroup, and is thus itself a 1-dimensional Lie group — still with multiplication as the group operation.

- The set $\mathbb{C}^*$ of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication.

- The unit circle $\mathbb{S}^1 \subset \mathbb{C}^*$ is a Lie group under the operations induced by multiplication of complex numbers.

- A finite product of k copies of $\mathbb{S}^1$ is a Lie group. We denote it by $\mathbb{T}^k$. In particular, the 2-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is a Lie group.

- More generally, the product of Lie groups is again a Lie group.

We will see more examples below. But before, we introduce the notion of maps between Lie groups which respect the Lie group structure.

Definition 5.2 (Lie group homomorphisms) If G and H are Lie groups, a **Lie group homomorphism** from G to H is a smooth map $F : G \to H$ that is also a group homomorphism. It is called a **Lie group isomorphism** if it is also a diffeomorphism, which implies that it has an inverse that is also a Lie group homomorphism. In this case, we say that G and H are isomorphic Lie groups.

Here are some examples of Lie group homomorphisms:

- The inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{C}$ is a Lie group homomorphism.

- Considering $\mathbb{R}$ as a Lie group under addition, and $\mathbb{R}^*$ as a Lie group under multiplication, the map
$$\exp : \mathbb{R} \to \mathbb{R}^*, \ t \mapsto e^t$$
is smooth, and is a Lie group homomorphism, since $e^{s+t} = e^s e^t$. The image of exp is the open subgroup $\mathbb{R}^{>0}$ consisting of positive real numbers. In fact, $\exp : \mathbb{R} \to \mathbb{R}^{>0}$ is a Lie group isomorphism with inverse $\log : \mathbb{R}^{>0} \to \mathbb{R}$.

- Similarly, $\exp : \mathbb{C} \to \mathbb{C}^*$ given by $\exp(z) = e^z$ is a Lie group homomorphism. It is surjective but not injective, because its kernel consists of the complex numbers of the form $2\pi i k$, where k is an integer.

- The map
$$\epsilon : \mathbb{R} \to \mathbb{S}^1, \ t \mapsto e^{2\pi i t}$$
is a Lie group homomorphism whose kernel is the set $\mathbb{Z}$ of integers.

- Similarly, the map
$$\epsilon^n : \mathbb{R}^n \to \mathbb{T}^n, \ (t_1, \dots, t_n) \mapsto (e^{2\pi i t_1}, \dots, e^{2\pi i t_n})$$
is a Lie group homomorphism whose kernel is $\mathbb{Z}^n$.

- If G is a Lie group and $g \in G$, **conjugation by g** is the map $C_g : G \to G$ given by $C_g(h) = ghg^{-1}$. Because group multiplication and inversion are smooth, C_g is smooth and it is a group homomorphism:

$$C_g(hh') = ghh'g^{-1} = (ghg^{-1})(gh'g^{-1}) = C_g(h)C_g(h').$$

In fact, it is a **Lie group isomorphism**, because it has $C_{g^{-1}}$ as an inverse. A subgroup $H \subseteq G$ is said to be **normal** if $C_g(H) = H$ for every $g \in G$.

Lie group homomorphisms behave much nicer in many respects than arbitrary smooth maps between manifolds. For example, the rank of the derivative is constant:

Theorem 5.3 (Constant Rank Theorem) Let $f : G \to H$ be a Lie group homomorphism. Then, as a linear map, the derivative df_g has the same rank for all $g \in G$.

Proof: Let e_G and e_H denote the identity elements in G and H, respectively. Suppose g_0 is an arbitrary element of G. We will show that df_{g_0} has the same rank as df_e. The fact that f is a homomorphism means that for all $g \in G$,

$$f(L_{g_0}(g)) = f(g_0 g) = f(g_0)f(g) = L_{f(g_0)}(f(g));$$

or in other words, $f \circ L_{g_0} = L_{f(g_0)} \circ f$. Taking differentials of both sides at the identity and using the chain rule yields

$$df_{g_0} \circ d(L_{g_0})_{e_G} = d(L_{f(g_0)})_{e_H} \circ df_{e_G}.$$

Recall that left multiplication by any element of a Lie group is a diffeomorphism, so both $d(L_{g_0})_{e_G}$ and $d(L_{f(g_0)})_{e_H}$ are isomorphisms. Because composing with an isomorphism does not change the rank of a linear map, it follows that df_{g_0} and df_{e_G} have the same rank. □

Remark 5.4 (Lie group isomorphisms revisited) Using the constant rank theorem, one can now show that every **bijective** Lie group homomorphism $f : G \to H$ is automatically a Lie group **isomorphism**. This is yet another point which makes Lie groups special among all smooth manifolds. Here is a **hint** why this could be true: We will learn soon about **Sard's theorem** which will tell us that the subspace of regular values of a smooth map is dense in the codomain. In particular, there is at least one **regular value**, say $h_0 \in$H, for our Lie group homomorphism f. Since f is bijective, h_0 must be in the image of f. Hence, at the unique point $g_0 \in G$ with $f(g_0) = h_0$, we know that df_{g_0} is surjective. But then df_g must be an **isomorphism**, since otherwise the Local Submersion Theorem would imply that f looked like the canonical submersion and would have nontrivial kernel. Hence f would **not** be bijective. According to the previous theorem, this implies that df_g is an isomorphism for all $g \in G$. Hence f is a bijective local diffeomorphism everywhere. Bijective local diffeomorphisms are global diffeomorphisms. Since the map is a Lie group homomorphism, it is a Lie group isomorphism.

5.2 Examples of Lie groups

Now let us study some more interesting examples:

- **The General Linear Group**

The general linear group

$$GL(n) = \{A \in M(n) : \det A \neq 0\}$$

of all invertible $n \times n$-matrices with entries in $\mathbb{R}$, is a smooth manifold of dimension n^2, since it is an **open** subset of $M(n) \cong \mathbb{R}^{n^2}$. To check that it is open, look at its complement

$$M(n) \setminus GL(n) = \{A \in M(n) : \det A = 0\} = \det^{-1}(0).$$

Since $\det : M(n) \to \mathbb{R}$ is continuous, being a polynomial in the entries of the matrix, and since $\{0\}$ is a closed subset of $\mathbb{R}$, $\det^{-1}(0)$ is closed in $M(n)$.

We claim that $GL(n)$ is a **Lie group**: To show this we need to check that multiplication and taking inverses are smooth operations. Given two matrices A and B in $GL(n)$, the entry in position (i, j) in AB is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Hence $(AB)_{ij}$ is a polynomial in the coordinates of A and B. Thus matrix multiplication

$$\mu : GL(n) \times GL(n) \to GL(n)$$

is a smooth map.

Recall that the (i, j)-minor of a matrix A is the determinant of the submatrix A_{ij} of A obtained by deleting the ith row and the jth column of A. By **Cramer's rule** from linear algebra, the (i, j)-entry of A^{-1} is

$$(A^{-1})_{ij} = \frac{1}{\det A} \cdot (-1)^{i+j} ((j, i)\text{-minor of } A),$$

which is a **smooth function** of the a_{ij}'s provided $\det A \neq 0$, i.e., the map

$$M(n) \to \mathbb{R}, \ A \mapsto (A^{-1})_{ij}$$

is smooth because it depends smoothly on the entries of A. Therefore, the map of taking inverses

$$\iota : GL(n) \to GL(n)$$

is also smooth. □

Remark 5.5 **($GL(n)$ exists over many bases)** In fact, we can matrices with entries in any ring K. We denote the corresponding matrix groups by $M(n, K), GL(n, K), \ldots$.

Since $K = \mathbb{R}$ is the most important case for us, we omit mentioning the base when it is clear that we work over $\mathbb{R}$.

Another very important case is $K = \mathbb{C}$. The complex general linear group $GL(n, \mathbb{C})$ is also a Lie group. It is a group under matrix multiplication, and it is an open submanifold of $M(n, \mathbb{C})$ and thus a $2n^2$-dimensional smooth manifold. It is a Lie group, since matrix products and inverses are smooth functions of the real and imaginary parts of the matrix entries.

Note that the determinant is a Lie group homomorphism for both $\mathbb{R}$ and $\mathbb{C}$:

$$\det : GL(n, \mathbb{R}) \to \mathbb{R}^* \text{ and } \det : GL(n, \mathbb{C}) \to \mathbb{C}^*.$$

- **The Special Linear Group:** $SL(n) = \{A \in M(n) : \det A = 1\}$

In terms of **geometry**, note that $SL(n)$ consists of all transformations of $\mathbb{R}^n$ into itself which preserve **volumes** and **orientations**.We have shown in the exercises that $SL(n) = \det^{-1}(1)$ is a **smooth manifold** of dimension $n^2 - 1$. Since it is a subset of the Lie group $GL(n)$ with the operation inherited from the one of $GL(n)$, $SL(n)$ is also a Lie group. In the exercises we calculate the tangent space of $SL(n)$ at the identity to be the subspace in $M(n)$ of all matrices with trace zero. □

- **The Special Orthogonal Group**

Recall that the orthogonal group $O(n)$ is defined as the subset of matrices A in $M(n)$ such $AA^T = I$. This equation implies, in particular, that every $A \in O(n)$ is invertible with $A^{-1} = A^T$. Hence the determinant of an $A \in O(n)$ must satisfy $(\det A)^2 = 1$, i.e., $\det A = \pm 1$. Thus, $O(n)$ splits into two disjoint parts, the subset of matrices with determinant $+1$ and the subset of matrices with determinant -1.

If A and B have determinant -1, then their product AB has determinant $+1$. Hence the subset of matrices with determinant -1 is not closed under multiplication and therefore **not** a subgroup of $O(n)$. But the other part is a Lie subgroup of $O(n)$ and is called the **Special Orthogonal Group** denoted by $SO(n)$

$$SO(n) = \{A \in O(n) : \det A = 1\} \subset O(n).$$

The subgroup $SO(n)$ is a Lie group □

- **Unitary and Special Unitary Groups**

The **unitary group** $U(n)$ is defined to be

$$U(n) := \{A \in GL(n, \mathbb{C}) : \bar{A}^T A = I\},$$

where $\bar{A}$ denotes the complex conjugate of A, the matrix obtained from A by conjugating every entry of A. A similar argument as for $O(n)$ shows that $U(n)$ is a submanifold of $GL(n, \mathbb{C})$ and that $\dim U(n) = n^2$.

The **special unitary group** $SU(n)$ is defined to be the subgroup of $U(n)$ of matrices of determinant 1. □

Remark 5.6 (Spin groups) There are other important examples of Lie groups which, in general, do not arise as closed subgroups of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. For example, the nth **Spin group** $\mathrm{Spin}(n)$ is the n-dimensional Lie group which is a double cover of $SO(n)$. The latter means that $\mathrm{Spin}(n)$ is equipped with a smooth surjective map $\pi: \mathrm{Spin}(n) \to SO(n)$ such that each point in $SO(n)$ has an open neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open subsets in $\mathrm{Spin}(n)$ each of which is mapped diffeomorphically onto U by π. The map π is part of a short exact sequence of groups

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to SO(n) \to 1.$$

Spin groups can be constructed for example via Clifford algebras. However, there are some exceptional isomorphisms in low dimensions which we can write down:

$$\begin{aligned}
&\mathrm{Spin}(1) \cong O(1),\\
&\mathrm{Spin}(2) \cong SO(2),\\
&\mathrm{Spin}(3) \cong SU(2),\\
&\mathrm{Spin}(4) \cong SU(2) \times SU(2),\\
&\mathrm{Spin}(6) \cong SU(4).
\end{aligned}$$

5.3 Topology of Lie groups

For some important Lie groups, we will now study the topological properties we singled out in the beginning: compactness, connectedness and path-connectedness.

We showed in the exercises that $O(n)$ is **compact**. As a closed subset , $SO(n)$ is compact we well. Similarly, $U(n)$ and $SU(n)$ are compact. The general linear group $GL(n)$, however, is not compact as an open subset of $M(n)$.

Moreover, note that both $SO(n)$ and its complement are both open and closed in $O(n)$. They are the **two connected components of $O(n)$**. In particular, there is no continuous path in $O(n)$ from a matrix with determinant $+1$ to one with determinant -1. In fact, there is no such path in $GL(n)$:

Lemma 5.7 The **real** general linear group is **not** connected.

Proof: Let γ be a path in $GL(n)$, i.e. a continuous map

$$\gamma: [0, 1] \to GL(n).$$

Since γ and det are continuous, so is their composite

$$\det \circ \gamma : [0, 1] \xrightarrow{\gamma} GL(n) \xrightarrow{\det} \mathbb{R}.$$

Hence if $\det(\gamma(0)) > 0$ and $\det(\gamma(1)) < 0$, then the **Intermediate Value Theorem** from Calculus implies that there must be a real number $t_0 \in (0, 1)$ such that $\det(\gamma(t_0)) = 0 \notin GL(n)$. Hence γ would have to leave $GL(n)$. □

Thus also $GL(n)$ has two connected components, one of which is an open subgroup consisting to all matrices A with $\det A > 0$. The other one is just an open subset consisting to all matrices A with $\det A < 0$.

Lemma 5.8 The **complex** general linear group $GL(n, \mathbb{C})$ is **path-connected**.

We see the difference between $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ most clearly for the case $n = 1$: $GL(1, \mathbb{R}) = \mathbb{R}^*$ is not path-connected, since we cannot cross 0; whereas $GL(1, \mathbb{C}) = \mathbb{C}^*$ is path-connected, since we can just walk around 0 in the plane.

Proof of Lemma 5.8: More generally, to show that $GL(n, \mathbb{C})$ is path-connected, it suffices to show that there is a path from any matrix $A \in GL(n, \mathbb{C})$ to the identity matrix $I \in GL(n, \mathbb{C})$. Therefore, we define first the function

$$P : \mathbb{C} \to \mathbb{C}, \ z \mapsto \det(A + z(I - A)).$$

Then we have $P(0) = \det A \neq 0$ and $P(1) = \det I = 1 \neq 0$. Since P is a polynomial of degree n, it has only **finitely** many zeroes. We also note that neither $0 \in \mathbb{C}$ nor $1 \in \mathbb{C}$ are zeroes of P. Since $\mathbb{C} \setminus \{\text{finitely many points}\}$ is **path-connected**, we can find a path $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = 0$, $\gamma(1) = 1$ and which avoids the zeroes of P, i.e.,

$$P(\gamma(t)) \neq 0 \text{ for all } t.$$

Then the continuous map

$$\Gamma : [0, 1] \to GL(n, \mathbb{C}), \ t \mapsto A + \gamma(t)(I - A)$$

is the desired path from A to I. □

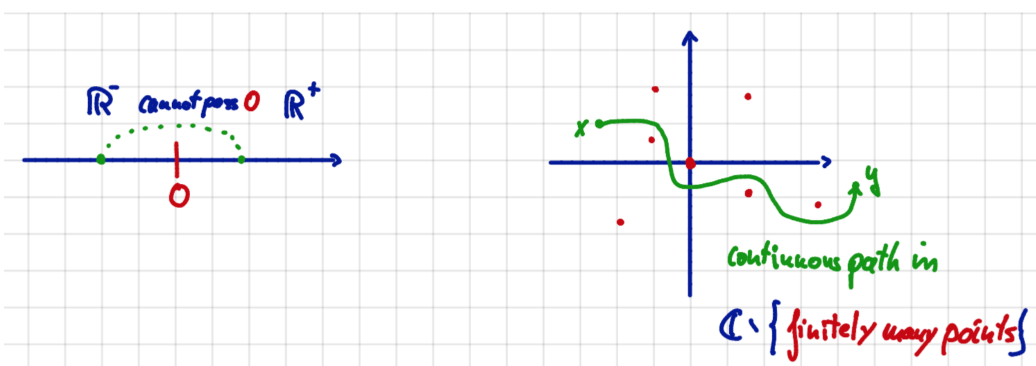

Figure 5.1: Two points in $GL(n, \mathbb{R})$ may not be connected. In $GL(n, \mathbb{C})$, however, we can always find a path between two matrices. This reduces via the determinant function to the fact that the plane remains path-connected after removing finitely many points.

Remark 5.9 The fact that $\mathrm{GL}(n, \mathbb{C})$ is connected while $GL(n, \mathbb{R})$ is not plays a crucial role for orientations of vector spaces, vector bundles, manifolds etc. For, every complex vector space, complex vector bundle, complex manifold, etc has a **natural orientation**. We will get back to this later.

Open neighborhoods of the identity:

Recall that if G is a group and $S \subset G$ is a subset, the **subgroup generated by S** is the smallest subgroup containing S, i.e., the intersection of all subgroups containing S. One can check that the subgroup generated by S is equal to the set of all elements of G that can be expressed as finite products of elements of S and their inverses.

Lemma 5.10 (Neighborhoods of the identity) Suppose G is a Lie group, and $W \subset G$ is any neighborhood of the identity. Then

- W generates an open subgroup of G.
- If G is **connected**, then W generates G. In particular, an open subgroup in a connected Lie group must be equal to the whole group.

Proof:

- Let $W \subset G$ be any neighborhood of the identity, and let H be the subgroup generated by W. To simplify notation, if A and B are subsets of G, we write

  $$AB := \{ab : a \in A, b \in B\}, \text{ and } A^{-1} := \{a^{-1} : a \in A\}.$$

  For each positive integer k, let W_k denote the set of all elements of G that can be expressed as products of k or fewer elements of $W \cup W^{-1}$. As mentioned above, H is the union of all the sets W_k as k ranges over the positive integers.

  Now, W^{-1} is open because it is the image of W under the inversion map, which is a diffeomorphism. Thus, $W_1 = W \cup W^{-1}$ is open, and, for each $k > 1$, we have

  $$W_k = W_1 W_{k-1} = \cup_{g \in W_1} L_g(W_{k-1}).$$

  Because each L_g is a diffeomorphism, it follows by induction that each W_k is open, and thus H is open as a union of open subsets.
- Assume G is connected. We just showed that H is an open subgroup of G. It is an exercise to show that an open subgroup in a connected Lie group is equal to the whole group. □

5.4 Lie subgroups

In the previous paragraph we talked about subgroups of a Lie group. But we did not discuss how the subgroup structure relates to the structure as a smooth manifold. Actually, this is a subtle and interesting point that illustrates the importance of the distinction between immersions and embeddings once again. So here is the definition of a Lie subgroup:

Definition 5.11 (Lie subgroups) A **Lie subgroup** of a Lie group G is an abstract **subgroup** $H \subseteq G$ such that

- there exists a smooth manifold X and an **immersion** $f : X \to G$ such that $H = \mathrm{Im}(f) \subseteq G$ is the image of f, and
- the group operations on H are smooth, in the sense that the compositions

$$X \times X \xrightarrow{f \times f} G \times G \xrightarrow{\mu} G, \text{ and}$$
$$X \xrightarrow{f} G \xrightarrow{\iota} G$$

are smooth.

- **Note:** It is important to note that, in the above definition, we do **not** require H to have the subspace topology induced by being a subset in G. Instead we can think of the map f to define a topology on H in the sense that a subset $U \subset H$ is open if and only if $f^{-1}(U)$ is open in X. □

Let us have a **closer look** at this rather complicated definition:

- An **abstract subgroup** simply means a subgroup in the algebraic sense. The group operations on the subgroup H are the restrictions of the multiplication map μ and the inverse map ι from G to H.
- If H were defined to be a submanifold of G, then the multiplication map $H \times H \to H$ and similarly the inverse map $H \to H$ would automatically be smooth, and the definition would be much shorter. However, since a Lie subgroup is defined to be an **immersed submanifold**, it is necessary to impose the last condition.
- If H is in fact also a **submanifold**, then H is a Lie subgroup as the following result shows.

Lemma 5.12 (Embedded Lie subgroups) If H is an abstract subgroup and a submanifold of a Lie group G, then it is a Lie subgroup of G. In this case, the inclusion map $H \hookrightarrow G$ is an embedding, and we call H an **embedded subgroup**.

Proof: Since H is a subgroup, multiplication and taking inverses in H are just the restrictions of multiplication and taking inverses in G and both have image in H. Since H is a submanifold we can take $X = H$ in the above definition, the restrictions of smooth maps to H are again smooth. □

Here are some examples of embedded subgroups:

- The subgroups $SL(n)$ and $O(n)$ of $GL(n)$ are both submanifolds, and therefore embedded Lie subgroups.

- One easily verifies that

$$\mathbb{C}^* = \mathrm{GL}(1,\mathbb{C}) \hookrightarrow GL(2,\mathbb{R}),\ z = x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

  is an embedding.

- More generally, this map induces an embedding

$$GL(n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{R})$$

  by replacing each entry $z = x + iy$ in $A \in GL(n,\mathbb{C})$ by the block $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$:

$$\begin{pmatrix} x_{11} + iy_{11} & \dots & x_{1n} + iy_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} + iy_{n1} & \dots & x_{nn} + iy_{nn} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & -y_{11} & \dots & x_{1n} & -y_{1n} \\ y_{11} & x_{11} & & y_{1n} & x_{1n} \\ \vdots & & \ddots & & \vdots \\ x_{n1} & -y_{n1} & \dots & x_{nn} & -y_{nn} \\ y_{n1} & x_{n1} & & y_{nn} & x_{nn} \end{pmatrix}.$$

  This way, $GL(n,\mathbb{C})$ is an embedded Lie subgroup of $GL(2n,\mathbb{R})$.

Now let us get back to understanding the definition of a Lie subgroup. The **subtle differences** of immersed and embedded subgroups can be illustrated by a familiar example:

Example 5.13 (An immersed but not embedded Lie subgroup) Recall from Section 3.2.3 the maps $g : \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C},\ t \mapsto e^{it}$, and

$$G : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2,\ G(s,t) = (g(s), g(t))$$

The map G is a local diffeomorphism from the plane onto the torus $\mathbb{T}^2$. Given a real number α, we defined the map γ_α by

$$\gamma_\alpha : \mathbb{R} \to \mathbb{T}^2,\ \gamma(t) = (g(t), g(\alpha \cdot t)).$$

We learned that γ_α is always an **immersion**, but its image is **not** a submanifold of $\mathbb{T}^2$ ifα is an **irrational** number. However, when α is rational, then $\gamma_\alpha(\mathbb{R})$ is a submanifold of T^2. Recall Figure 3.4. After checking that $\gamma_\alpha(\mathbb{R})$ is an abstract subgroup, we see that $\gamma_\alpha(\mathbb{R})$ is in fact a **Lie subgroup** of $\mathbb{T}^2$ for **every** real number α.

For an explanation of why a Lie subgroup is defined in such a complicated way, we refer to a fact we will only be able to appreciate when we learn more about Lie theory:

Remark 5.14 (Why so complicated?) A fundamental theorem in Lie group theory asserts the existence of a **one-to-one correspondence** between the connected Lie subgroups of a Lie group G and the Lie subalgebras of its Lie algebra $\mathfrak{g}$, i.e., tangent space at the identity with its Lie bracket:

$$\{\text{connected Lie subgroups in } G\} \overset{1-1}{\longleftrightarrow} \{\text{Lie subalgebras in } \mathfrak{g}\}.$$

In the previous example, the Lie algebra of $\mathbb{T}^2$ has $\mathbb{R}^2$ as the underlying vector space, and the one-dimensional Lie subalgebras are all the lines through the origin with addition as group operation. Such a line is determined by its slope α. Hence **every** α should correspond to a **Lie subgroup** $\gamma_\alpha(\mathbb{R})$ in $\mathbb{T}^2$.

However, **if** a Lie subgroup had been defined as a subgroup that is also a **submanifold**, then one would have to **exclude** all the lines with **irrational slopes** as Lie subgroups of the torus. In this case it would **not** be possible to have a one-to-one correspondence between the connected subgroups of a Lie group and the Lie subalgebras of its Lie algebra. But this correspondence is extremely useful in Lie theory.

The following theorem is a very useful fact which we state here without proof. See for example [11, Theorem 7.21].

Theorem 5.15 **(Closed Subgroup Theorem)** Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Then H is closed in G if and only if it is an embedded Lie subgroup.

5.5 Exercises and more examples

5.5.1 Lie groups

Exercise 5.1 **(a)** Show that $SO(2)$ is diffeomorphic to $\mathbb{S}^1$.

(b) Show that $SU(2)$ is diffeomorphic to $\mathbb{S}^3$.

Exercise 5.2 **(a)** Show that a local diffeomorphism $f : X \to Y$ which is bijective is a diffeomorphism.

(b) Show that a local diffeomorphism $f : X \to Y$ which is one-to-one is a diffeomorphism of X onto an open subset of Y.

(c) Show that a bijective smooth map $f : X \to Y$ of constant rank is a diffeomorphism.

Comment: You can assume that f is a submersion to simplify things. If you want to challenge yourself, you could only assume that X is compact. Showing that f also is a submersion in general requires the use of Baire's category theorem.

(d) Show that a bijective Lie group homomorphism is a Lie group isomorphism.

Exercise 5.3 Show that an open subgroup H, i.e., a subgroup which is also an open subset, of a connected Lie group G is equal to G.

Exercise 5.4 Let G be a Lie group and let $e \in G$ be the identity element.

(a) Let $\mu\colon G \times G \to G$ denote the multiplication map, and let $g, h \in G$. Recall that we denote by L_g the left translation in G by g, and by R_h the right translation by h. Using the identification $T_{(g,h)}(G \times G) = T_g(G) \times T_h(G)$, show that the differential of μ at (g, h)

$$d\mu_{(g,h)}\colon T_g(G) \times T_h(G) \to T_{gh}(G)$$

is given by

$$d\mu_{(g,h)}(X, Y) = d\mu_{(g,h)}(X, 0) + d\mu_{(g,h)}(0, Y) = d(R_h)_g(X) + d(L_g)_h(Y).$$

Hint: Calculate $d\mu_{(g,h)}(X, 0)$ and $d\mu_{(g,h)}(0, Y)$ separately.

(b) Let $\iota\colon G \to G$ denote the inversion map. Show that

$$d\iota_e\colon T_e(G) \to T_e(G)$$

is given by $d\iota_e(X) = -X$.

(c) Use the previous point to show that, for any $g \in G$, the derivative of ι at g is given by

$$d\iota_g\colon T_g(G) \to T_{g^{-1}}(G),\ Y \mapsto -d(R_{g^{-1}})_e(d(L_{g^{-1}})_g(Y)) \text{ for all } Y \in T_g(G).$$

Exercise 5.5 Show that for any Lie group G, the multiplication map $\mu\colon G \times G \to G$ is a submersion.

Exercise 5.6 Show that the differential of the determinant map $\det\colon GL(n, \mathbb{R}) \to \mathbb{R}$ at $A \in GL(n, \mathbb{R})$ is given by

$$d(\det)_A(B) = (\det A) \cdot (\operatorname{tr} A^{-1}B) \text{ for all } B \in M(n).$$

In particular, $d(\det)_A(AB) = (\det A) \cdot (\operatorname{tr} AB)$ for all $B \in M(n)$.

6. Transversality

6.1 Transversality and preimages

Now we would like to understand **what happens** when we do not take the preimage of just a single point, but **the preimage of a whole submanifold**.

Let X and Y be smooth manifolds and let $f : X \to Y$ be a smooth map. Assume that $Z \subset Y$ is a submanifold of Y. We would like to understand:

Question Under which conditions is the subset $f^{-1}(Z) \subseteq X$ a smooth manifold?

6.1.1 Transverse preimages of submanifolds

- Let us denote $n = \dim X$, $m = \dim Y$ and $k = \dim Z$. Let $x_0 \in X$ be a point with $f(x_0) = z_0 \in Z \subset Y$. The inclusion of subsets $Z \hookrightarrow Y$ is an immersion. Hence we can apply the **Local Immersion** Theorem 3.10.
- We can choose local parametrizations as follows:
  - $\psi : U \to V$ with $\psi(0) = f(x_0)$ and $V \subset Y$ and $U \subset \mathbb{R}^m$ open subsets,
  - and, since Z is a smooth manifold, $\phi : W \to Z \cap V$ with $\phi(0) = f(x_0)$ and $W \subset \mathbb{R}^k$ and $Z \cap V \subset Z$ open subsets, such that

$$\begin{tikzcd} Z \cap V \arrow[r, hook, "\text{inclusion}"] & V \\ W \arrow[u, "\phi"] \arrow[r, "\text{canonical}", "\text{immersion}"'] & U \arrow[u, "\psi"'] \end{tikzcd} \tag{6.1}$$

  commutes.
- The inverse map $\psi^{-1} : V \to U$ is a local coordinate system on the open neighborhood V around $z_0 \in Y$. We write $u_i : V \to \mathbb{R}$ for the ith component of ψ^{-1}, i.e., a point $y \in \psi(W)$ has the local coordinates

$$\psi^{-1}(y) = (u_1(y), \dots, u_m(y)) \in \mathbb{R}^m.$$

- Since the lower horizontal map in (6.1) is the **canonical immersion**, the points in $Z \cap V$ are exactly those points in V on which the coordinate functions $u_{k+1}, \dots, u_m$ vanish. Hence we have

$$Z \cap V = \left\{ y \in V : u_{k+1}(y) = \cdots = u_m(y) = 0 \right\}.$$

- Let us write $g\colon V \to \mathbb{R}^{m-k}$ for the map given by $y \mapsto \big(u_{k+1}(y), \dots, u_m(y)\big)$. The above relation then reads
$$Z \cap V = g^{-1}(0).$$

- Now we take the preimage of $Z \cap V$ along f and get
$$f^{-1}(Z \cap V) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0). \tag{6.2}$$

- Since f is **continuous** and V is **open** in Y, the subset
$$f^{-1}(Z \cap V) = f^{-1}(Z) \cap f^{-1}(V)$$
is an **open** subset of $f^{-1}(Z)$ containing x_0.[1]

- Now if $f^{-1}(Z \cap V)$ is a manifold, then x_0 has an open neighborhood in $f^{-1}(Z)$ which is diffeomorphic to an open subset in $\mathbb{R}^{m-k}$. Hence, **if** every point $x \in f^{-1}(Z)$ has such an **open** neighborhood, **then** $f^{-1}(Z)$ is a **smooth manifold**.

- And we do have a **criterion** that guarantees that this is the case. For, according to (6.2), $f^{-1}(Z \cap V)$ is a manifold if 0 is a **regular value** for the smooth map $g \circ f$.

- Thus we **would like** to have that $g \circ f$ is a **submersion** at **every** point
$$x \in f^{-1}(Z \cap V) = (g \circ f)^{-1}(0).$$

- One way to check this is to show that 0 is a **regular value** of the **composite** $g \circ f$. The **chain rule** tells us
$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

- Thus, the map
$$\begin{aligned} & d(g \circ f)_x \colon T_x(X) \to \mathbb{R}^{m-k} \text{ is } \textbf{surjective} \\ \iff & dg_{f(x)} \text{ maps the image of } df_x \text{ onto } \mathbb{R}^{m-k}. \end{aligned}$$

- The smooth map $g\colon V \to \mathbb{R}^{m-k}$ is the **composite** $V \xrightarrow{\psi^{-1}} \mathbb{R}^m \to \mathbb{R}^{m-k}$ of ψ^{-1} with the **projection** onto the last $m-k$ coordinates. Since ψ^{-1} is a local diffeomorphism, its derivative is an isomorphism. This implies that the **derivative** of g at $f(x)$,
$$dg_{f(x)} \colon T_{f(x)}(V) = T_{f(x)}(Y) \to \mathbb{R}^{m-k},$$
is a **surjective** linear map on the **whole** tangent space to Y at $f(x)$ for all x such that $g(f(x)) = 0$.

- By Lemma 4.9, which was a consequence of the Preimage Theorem 4.7, the **kernel** of $dg_{f(x)}$ is the subspace $T_z(Z)$. Thus $dg_{f(x)}$ induces an **isomorphism**
$$d\bar{g}_{f(x)} \colon T_{f(x)}(Y)/T_{f(x)}(Z) \xrightarrow{\cong} \mathbb{R}^{m-k}.$$

[1] Here we equip as always $f^{-1}(Z) \subset X$ has the **subspace** topology induced from X.

- This means that $(dg_{f(x)})_{|\mathrm{Im}\,(df_x)}$ can only be surjective if $\mathrm{Im}\,(df_x)$ generates the quotient space $T_{f(x)}(Y)/T_{f(x)}(Z)$. In other words, $(dg_{f(x)})_{|\mathrm{Im}\,(df_x)}$ can only be **surjective if** $\mathrm{Im}\,(df_x)$ and $T_{f(x)}(Z)$ **together span** all of $T_{f(x)}(Y)$.

- We conclude that $g \circ f$ **is a submersion** at $x \in f^{-1}(Z \cap V)$ **if and only if**

$$\mathrm{Im}\,(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y).$$

We give this condition a name:

Definition 6.1 (Transversality) Let $f : X \to Y$ be a smooth map and $Z \subseteq Y$ a submanifold. Then f is said to be **transverse to Z**, denoted $f \pitchfork Z$, if

$$\mathrm{Im}\,(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$$

at **every** point $x \in f^{-1}(Z)$ in the preimage of Z.

The above discussion then provides the proof for the following fundamental result:

Theorem 6.2 (Transverse intersections yield submanifolds) Let $f : X \to Y$ be a smooth map, and let $Z \subseteq Y$ be a submanifold. Assume that f is **transverse** to Z. Then $f^{-1}(Z)$ is a **submanifold of X**, and the **codimension** of $f^{-1}(Z)$ in X equals the codimension of Z in Y, i.e., we have

$$\dim f^{-1}(Z) = \dim X - (\dim Y - \dim Z).$$

- Note that **regularity** is a **special case** of transversality: If $Z = \{y\}$ consists of a single point, then $T_y(Z)$ is a trivial vector space consisting just of the zero vector. Hence, in this case, **transversality** means $\mathrm{Im}\, df_x = T_y(Y)$ at all points with $f(x) = y$, i.e., y is a **regular value** of f.

- Note that **transversality** tells us something about **how the image** of f and Z **meet** in Y. We will give further geometric intuition for transversality soon.

Let us look at **some simple examples** of transversality and non-transversality. We consider $Y = \mathbb{R}^2$ with the submanifold **Z being the x-axis**. Then

- The map $f : \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (0, t)$ is **transverse** to Z, with $f^{-1}(Z) = \{(0, 0)\}$.

- The map $f : \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t, t^2)$, however, is **not transverse** to Z, with $f^{-1}(Z) = \{(0, 0)\}$. See Figure 6.1.

- The map $f : \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t, t^2 - 1)$ is **transverse** to Z, with $f^{-1}(Z) = \{(-1, 0), (1, 0)\}$.

- The map $f : \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t, \cos t - 1)$ is **not transverse** to Z, with $f^{-1}(Z) = \{(0, 0)\}$. See Figure 6.2.

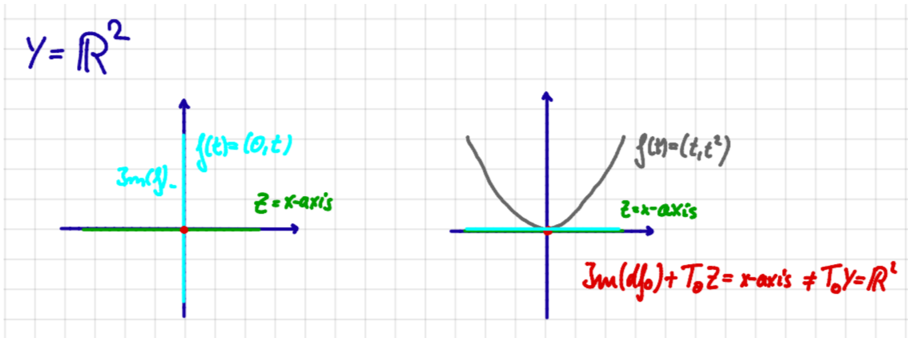

Figure 6.1: On the left, f is transverse to the x-axis, since $\mathrm{Im}\,(df_0)$ and T_0Z span all of $T_0Y = \mathbb{R}^2$. On the right, however, f is **not** transverse to Z, since both $\mathrm{Im}\,(df_0)$ and T_0Z span the same one-dimensional subspace.

Figure 6.2: We begin to see a pattern in the plane: f is transverse to the x-axis unless the x-axis is tangent to the graph of f.

- More generally, let $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ be a polynomial with real coefficients. We can consider $p(t)$ as a smooth map $\mathbb{R} \to \mathbb{R}$. The map $f : \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t, p(t))$ is **transverse** to the x-axis Z **if and only if** all zeros of p are simple, i.e., if and only if we can write $p(t)$ as a product

$$p(t) = (t - r_1) \cdot (t - r_2) \cdot \ldots \cdot (t - r_n)$$

with $r_i \neq r_j$ if $i \neq j$.

The reason is that the **derivative** of $p\colon \mathbb{R} \to \mathbb{R}$ at t_0 is given by **multiplication** by $p'(t_0)$, i.e.,

$$dp_{t_0}\colon \mathbb{R}^1 \to \mathbb{R}^1,\ s \mapsto p'(t_0) \cdot s.$$

This map is nontrivial and hence **surjective** if and only if $p'(t_0) \neq 0$, i.e., if and only if t_0 is not a zero of p'. In other words, $dp'(t_0)$ is **trivial** only if t_0 is a **multiple** zero of p.

Now the tangent space of Z at a point $f(t_0) \in \mathbb{R}^2 = Y$ is the x-axis $T_{f(t_0)}Z = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$, and the derivative df_{t_0} of f at t_0 is the map

$$df_{t_0}\colon \mathbb{R} \to \mathbb{R}^2,\ s \mapsto s \cdot \begin{pmatrix} 1 \\ p'(t_0) \end{pmatrix}.$$

Now we observe that the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ p'(t_0) \end{pmatrix}$ are **linearly independent** in $\mathbb{R}^2$ if and only if $p'(t_0) \neq 0$. Hence $T_{f(t_0)}Z$ and $\mathrm{Im}\, d_{f(t_0)}$ span $T_{f(t_0)}Y$ if and only if $p'(t_0) \neq 0$. For transversality of f and Z, we need to check that $T_{f(t_0)}Z + \mathrm{Im}\, d_{f(t_0)} = T_{f(t_0)}Y$ for all $t \in f^{-1}(Z)$, i.e., for **all zeros** of p. □

6.1.2 Tangent space of a transverse preimages

We return to the general situation: X and Y are smooth manifolds, $Z \subset Y$ is a submanifold, and $f : X \to Y$ is a smooth map. We assume now that f meets Z transversally. Hence $f^{-1}(Z)$ is a manifold, and we would like to have a formula for the tangent space of $f^{-1}(Z)$ at a given point:

Theorem 6.3 **(Tangent space of preimage)** Let $f : X \to Y$ be a smooth map and Z be a submanifold in Y. Assume that f is **transverse** to Z. Then $T_x(f^{-1}(Z))$ is the preimage of $T_{f(x)}(Z)$ under the linear map

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

As a formula:

$$T_x(f^{-1}(Z)) = (df_x)^{-1}(T_{f(x)}(Z)).$$

Proof: Let $x \in f^{-1}(Z)$ and $z := f(x)$. We can choose suitable local parametrizations and find an open subset $V \subset Y$ and a smooth map $g : V \to \mathbb{R}^{m-k}$ such that $Z \cap V = g^{-1}(0)$. Since tangent spaces are determined locally, the tangent spaces $T_z(Z \cap V)$ and $T_z(Z)$ are equal as subspaces of $T_z(Y)$. Since 0 is a regular value for g, we know that the tangent space of Z at z is given by

$$T_z(Z) = \mathrm{Ker}\, dg_z \subset T_z(Y) \tag{6.3}$$

where $dg_z : T_z(Y) \to \mathbb{R}^{m-k}$. Now we use that 0 is also a regular value for the composite $g \circ f$. Again, since tangent spaces are determined locally, we get

$$\begin{aligned} T_x(f^{-1}(Z)) &= \mathrm{Ker}\,(d(g \circ f)_x) \\ &= \mathrm{Ker}\,(dg_{f(x)} \circ df_x) \text{ (by the Chain Rule)} \\ &= (df_x)^{-1}(\mathrm{Ker}\,(dg_{f(x)})) \\ &= (df_x)^{-1}(T_{f(x)}(Z)) \text{ (by (6.3) above).} \end{aligned}$$

This finishes the proof. □

6.2 Transverse intersections

We will now study the most important case, at least for us, of transversality: the intersection of two submanifolds X and Z in the same given bigger manifold Y. This becomes, in fact, a special case of our previous studies by letting f to be the **inclusion map** $i : X \hookrightarrow Y$. For to say a point $x \in X$ belongs to the **intersection** $X \cap Z$ is equivalent to say that x belongs to the preimage $i^{-1}(Z)$.

We would like the intersection $X \cap Z$ to be a submanifold in Y. Transversality is the crucial condition to ensure this is the case:

The derivative $di_x : T_x(X) \to T_x(Y)$ is the inclusion map of $T_x(X)$ into $T_x(Y)$ as a subspace. Hence we get $i \pitchfork Z$ **if and only if**, for every $x \in X \cap Z$,

$$T_x(X) + T_x(Z) = T_x(Y).$$

Note that this equation is **symmetric** in X and Z.

Definition 6.4 (Transverse submanifolds) Two submanifolds $X \subset Y$ and $Z \subset Y$ are **transverse**, denoted $X \pitchfork Z$, if $T_x(X) + T_x(Z) = T_x(Y)$ for all $x \in X \cap Z$.

- **Warning:** For equation $T_x(X) + T_x(Z) = T_x(Y)$ to be true, it is **not sufficient** that $\dim T_x(X) + \dim T_x(Z) = \dim T_x(Y)$. The two subspaces together must **span all of $T_x(Y)$**. A simple example if the intersection of Z with itself, i.e., $X = Z$, with $\dim Z = \frac{1}{2} \dim Y$. Then we have $\dim T_z(X) + \dim T_z(Z) = \dim T_z(Y)$, but the tangent spaces $T_z(Z)$ and $T_z(X)$ are equal and do not span $T_z(Y)$.

- **However**, we will see later a way to make self-intersections interesting. This will be made possible by Thom's transversality theorem and invariance under homotopy. More on this later in the chapter on **intersection theory**.

6.2.1 Intersection of submanifolds

The theorem on transversality for this special case says:

Theorem 6.5 (Intersection of transverse submanifolds) The **intersection** of two **transverse** submanifolds X and Z of Y is a **submanifold** of Y. Moreover, the codimensions in Y satisfy

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z.$$

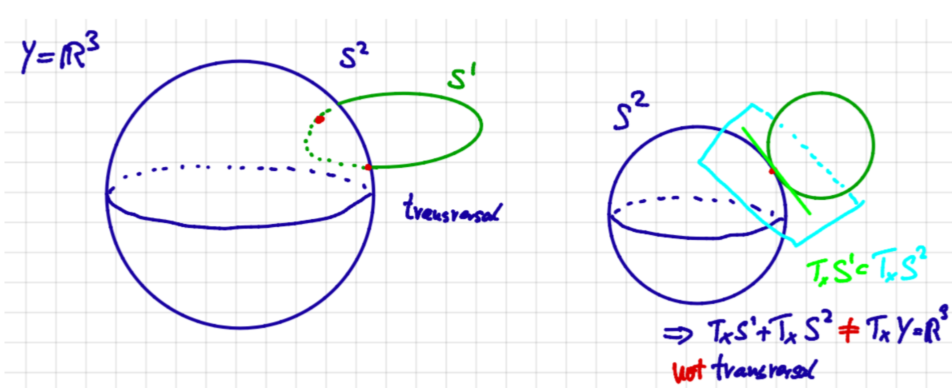

Figure 6.3: The intersection of a sphere and a circle is transverse unless the there is only one intersection point.

- The additivity of codimensions follows from the codimension formula of the previous theorem:

$$\begin{aligned} \operatorname{codim} i^{-1}(Z) \text{ in } X &= \operatorname{codim} Z \text{ in } Y \\ \Rightarrow \dim X - \dim X \cap Z &= \dim Y - \dim Z \\ \Rightarrow \dim Y - \dim X \cap Z &= (\dim Y - \dim Z) + (\dim Y - \dim X) \\ \Rightarrow \operatorname{codim} X \cap Z &= \operatorname{codim} Z + \operatorname{codim} X. \end{aligned}$$

Remark 6.6 (Intersect as little as possible) We have just learned that two manifolds intersect transversally if their tangent spaces together span the whole ambient space. A different way to think of transversality is: Two manifolds intersect transversally if they intersect as little as possible at every point. And we **measure the degree of intersection in terms of tangent spaces**: If two submanifolds intersect, then they intersect transversally if the **intersection** of their tangent spaces in the ambient tangent space **is minimal**.

- Note that the **converse** of the theorem is **not true**. We have seen a simple example above: the submanifolds $X = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ and $Z = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ do **not** intersect transversally at 0 in $Y = \mathbb{R}^2$, but their intersection $X \cap Z = \{0\}$ is a zero-dimensional manifold. More generally, when $X = Z$ are the same submanifold, then $X \cap Z$ is not transverse but $X \cap Z = X = Z$ is a smooth manifold. **However**, there do, of course, exist intersections which are not transverse and where the intersection is **not** a manifold. See Example 6.11.

- It is useful to note that any smooth map $f : X \to Y$ whose image does **not** meet a submanifold Z of Y, i.e., $f^{-1}(Z) = \emptyset$, is **transverse** to Z for trivial reasons. For in this case there is **no condition** to be satisfied. In particular, two submanifolds which do **not intersect at all** are **transverse**.

- On the other hand, if $f : X \to Y$ is a **submersion**, then f is **transverse** to **every** submanifold Z of Y since then $\operatorname{Im}(df_x) = T_{f(x)}(Y)$ for every x.

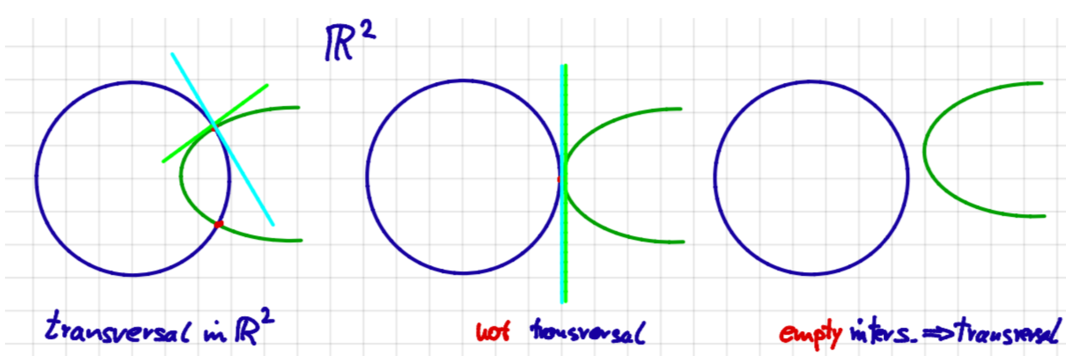

Figure 6.4: The intersection of two circles in the plane is transverse unless the there is exactly one intersection point.

Remark 6.7 (The ambient space matters!) Transversality of X and Z also **depends on the ambient space** Y. For example, the two coordinate axes intersect transversally in $\mathbb{R}^2$, but not when considered to be submanifolds of $\mathbb{R}^3$. In general, if the dimensions of X and Z do not add up to at least the dimension of Y, then they can only **intersect transversally by not intersecting at all**. For example, if X and Z are **curves in** $\mathbb{R}^3$, then $X \pitchfork Y$ if and only if $X \cap Y = \emptyset$.

By applying the formula we got for the tangent spaces of $f^{-1}(Z)$ to f being the inclusion map $X \subset Y$ we get the following useful result:

Theorem 6.8 (Tangent space of intersections) Let $X \subset Y$ and $Z \subset Y$ be submanifolds such that $X \pitchfork Z$ in Y. Then the **tangent space** to $X \cap Z$ is the **intersection of the tangent spaces**, i.e.,

$$T_x(X \cap Z) = T_x(X) \cap T_x(Z) \text{ for all } x \in X \cap Z.$$

6.2.2 Examples

Let us have a look at some examples:

Example 6.9 (A primer to Brieskorn manifolds) Recall the map

$$f : \mathbb{R}^4 \to \mathbb{R},$$
$$(x_1, x_2, x_3, x_4) \mapsto x_1 + x_2^2 + x_3^3 + x_4^4.$$

We showed before that f is a submersion and hence that the preimage

$$Z = f^{-1}(0) = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,:\, x_1 + x_2^2 + x_3^3 + x_4^4 = 0\right\}$$

is a **manifold** in $\mathbb{R}^4$.

Now we show Z and $\mathbb{S}^3$ **intersect transversally** in $\mathbb{R}^4$ where $\mathbb{S}^3 \subset \mathbb{R}^4$ denotes the three-dimensional sphere. This will imply that $Z \cap \mathbb{S}^3$ is a smooth manifold.

To do this we need to check that $T_z(Z) + T_z(\mathbb{S}^3) = T_z(\mathbb{R}^4) = \mathbb{R}^4$ for all $z \in Z \cap \mathbb{S}^3$. Since $T_z(Z)$ and $T_z(\mathbb{S}^3)$ are both three-dimensional subspaces of $\mathbb{R}^4$, it suffices to show that, for every $z \in Z \cap \mathbb{S}^3$, there is **at least one** vector $\mathbf{v}$ in $T_z(Z)$ which is **not** contained in $T_z(\mathbb{S}^3)$.

The tangent space to Z in a point $z \in Z$ is the subspace in $\mathbb{R}^4$ given by the kernel of the derivative df_z which in the standard bases is given by the 1×4-matrix

$$df_z = \begin{pmatrix} 1 & 2x_2 & 3x_3^2 & 4x_4^3 \end{pmatrix}.$$

Let $z = (x_1, x_2, x_3, x_4)$ be any point in $Z \cap \mathbb{S}^3$. Then the vector

$$\mathbf{v} := \begin{pmatrix} 12x_1 \\ 6x_2 \\ 4x_3 \\ 2x_4 \end{pmatrix}$$

lies in $T_z(Z)$, since

$$\begin{aligned} df_z(\mathbf{v}) &= \begin{pmatrix} 1 & 2x_2 & 3x_3^2 & 4x_4^3 \end{pmatrix} \cdot \begin{pmatrix} 12x_1 \\ 6x_2 \\ 4x_3 \\ 3x_4 \end{pmatrix} \\ &= 12x_1 + 12x_2^2 + 12x_3^3 + 12x_4^3 \\ &= 12f(z) \\ &= 0. \end{aligned}$$

But $\mathbf{v}$ is **not** an element in $T_z(\mathbb{S}^3)$. For, recall that $T_z(\mathbb{S}^3)$ is the subspace in $\mathbb{R}^4$ which is **orthogonal** to the vector z, i.e.,

$$T_z(\mathbb{S}^3) = \{w \in \mathbb{R}^4 \,:\, z \perp w = 0\}.$$

We can check orthogonality via the scalar product in $\mathbb{R}^4$:

$$z \perp w \iff z \cdot w = 0.$$

For $\mathbf{v}$ we calculate

$$z \cdot \mathbf{v} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \cdot \begin{pmatrix} 12x_1 \\ 6x_2 \\ 4x_3 \\ 3x_4 \end{pmatrix} = 12x_1^2 + 6x_2^2 + 4x_3^2 + 3x_4^2 > 0.$$

Thus $\mathbf{v}$ is **not** an element in $T_z(\mathbb{S}^3)$. Hence Z and $\mathbb{S}^3$ meet transversally in $\mathbb{R}^4$.

By the theorem, the codimension of $Z \cap \mathbb{S}^3$ in $\mathbb{S}^3$ equals the codimension of Z in $\mathbb{R}^4$. Thus $\dim Z \cap \mathbb{S}^3 = 2$.

Example 6.10 (Hyperboloid meets a sphere) In $Y = \mathbb{R}^3$, we consider the two submanifolds

$$X = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 - z^2 = 1\}$$

and the sphere

$$Z_a = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = a\}$$

with $a > 0$. See Figure 6.5. We would like to understand **for which a do these two submanifolds intersect transversally** in Y.

Therefore, we need to **determine the tangent spaces** of X and Z_a at points where they intersect. We observe that $X = f^{-1}(1)$ for the map

$$f : \mathbb{R}^3 \to \mathbb{R}, \ (x, y, z) \mapsto x^2 + y^2 - z^2.$$

Expressed as a matrix in the standard bases the derivative df_p at a point $p = (x_p, y_p, z_p)$ has the form

$$df_p = (2x_p \; 2y_p \; -2z_p) \colon \mathbb{R}^3 \to \mathbb{R}.$$

This map is surjective for all $p \in \mathbb{R}^3 \setminus \{0\}$. Hence **1 is a regular value** of f and the **tangent space to X** at p is the kernel of df_p. For $p = (x_p, y_p, z_p) \in X$ with $z_p \neq 0$, this is

$$T_p(X) = \mathrm{Ker}\,(df_p) = \mathrm{span}\{(z_p, 0, x_p), (0, z_p, y_p)\} \subset \mathbb{R}^3.$$

For all $p = (x_p, y_p, 0) \in X$, this is

$$T_p(X) = \mathrm{Ker}\,(df_p) = \mathrm{span}\{(-y_p, x_p, 0), (0, 0, 1)\} \subset \mathbb{R}^3.$$

Similarly, we observe that $Z_a = g^{-1}(a)$ for the map

$$g : \mathbb{R}^3 \to \mathbb{R},\ (x, y, z) \mapsto x^2 + y^2 + z^2.$$

Expressed as a matrix in the standard bases the derivative dg_p at a point $p = (x_p, y_p, z_p)$ has the form

$$dg_p = (2x_p\ 2y_p\ 2z_p) : \mathbb{R}^3 \to \mathbb{R}.$$

This map is surjective for all $p \in \mathbb{R}^3 \setminus \{0\}$. Hence **$a > 0$ is a regular value** of g and the **tangent space to Z** at p is the kernel of dg_p. For $p = (x_p, y_p, z_p) \in Z_a$ with $z_p \neq 0$, this is

$$T_p(Z_a) = \mathrm{Ker}\,(dg_p) = \mathrm{span}\{(-z_p, 0, x_p), (0, -z_p, y_p)\} \subset \mathbb{R}^3.$$

For all $p = (x_p, y_p, 0) \in Z_a$, this is

$$T_p(Z_a) = \mathrm{Ker}\,(dg_p) = \mathrm{span}\{(-y_p, x_p, 0), (0, 0, 1)\} \subset \mathbb{R}^3.$$

Now **X and Z intersect in the points** $p = (x, y, z)$ which satisfy

$$x^2 + y^2 - z^2 - 1 = 0 = x^2 + y^2 + z^2 - a.$$

Subtracting both equations yields the condition

$$2z^2 = a - 1. \tag{6.4}$$

This gives us **three cases for the intersection $X \cap Z_a$**:

- **If $a < 1$**, then X and Z_a **do not intersect**, i.e., $X \cap Z_a = \emptyset$, since there is no $z \in \mathbb{R}$ which satisfies condition (6.4).
- **If $a = 1$**, then we have $z = 0$ and X and Z_1 **intersect in the circle** with radius 1 in the xy-plane in $\mathbb{R}^3$ with the origin as center, i.e.,

$$X \cap Z_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}.$$

- **If $a > 1$**, then X and Z_a **intersect in two disjoint circles** which lie in the planes parallel to the xy-plane in $\mathbb{R}^3$ with z-coordinate $z = \pm\sqrt{(a-1)/2}$:

$$X \cap Z_a = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = \frac{a+1}{2} \text{ and } z = \pm\sqrt{(a-1)/2}\right\}.$$

Now we **check transversality**:[a]

- **if** $a < 1$, then the **intersection is empty** and therefore **transversal**.
- **if** $a = 1$, then we see $T_p(X) = T_p(Z_1)$ from the above descriptions of the tangent spaces, since $z_p = 0$ for all points in $X \cap Z_1$. In particular, the tangent spaces **span the same plane** in $\mathbb{R}^3$, and **not all of** $\mathbb{R}^3$, at every $p \in X \cap Z_1$. Thus the intersection is **not transversal**.
- **if** $a > 1$, let $p = (x_p, y_p, z_p) \in X \cap Z_a$. In this case, we have $z_p \neq 0$. Then $T_p(X)$ **and** $T_p(Z_a)$ **together span all of** $\mathbb{R}^3$: For any point $p = (x_p, y_p, z_p) \in X \cap Z_a$, x_p and y_p cannot both be zero. **If** $x_p \neq 0$, then the vector $(-z_p, 0, x_p) \in T_p(Z_a)$ is **not a linear combination** of the vectors $(z_p, 0, x_p)$ and $(0, z_p, y_p)$ which span $T_p(X)$. And **if** $y_p \neq 0$, then the vector $(0, -z_p, y_p) \in T_p(Z_a)$ is **not a linear combination** of the vectors $(z_p, 0, x_p)$ and $(0, z_p, y_p)$. Since $T_p(X)$ is 2-dimensional, this shows

$$T_p(X) + T_p(Z_a) = \mathbb{R}^3 \text{ at every } p \in X \cap Z_a.$$

Thus the **intersection is transverse**.

[a] Recall $T_p(\mathbb{R}^3) = \mathbb{R}^3$ at every p

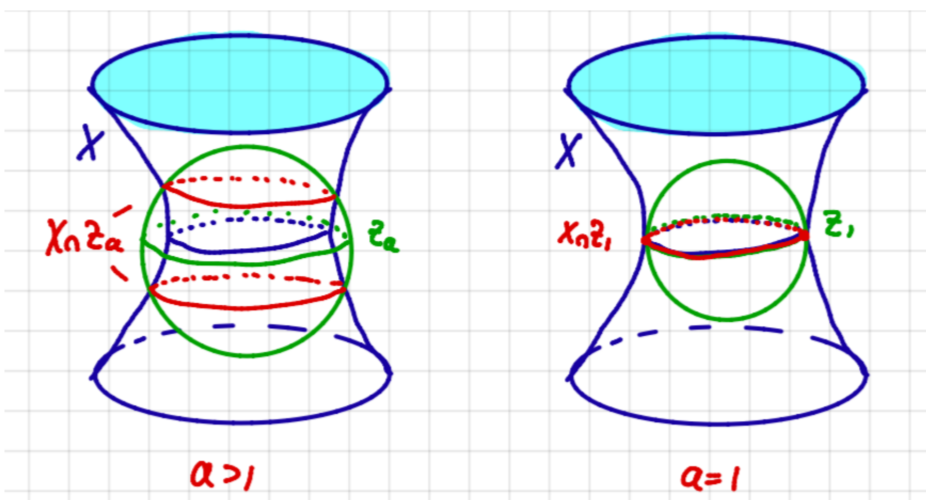

Figure 6.5: The intersection is transverse if $a > 1$, and not transverse if $a = 1$. In both cases, however, the intersection is a smooth manifold.

Here is an example of an intersection which is **not transverse** and where the **intersection** is **not** a manifold:

Example 6.11 (A non-transverse intersection) Let $Y = \mathbb{R}^3$ and let Z be the **hyperplane** defined by

$$Z = \{(x, y, z) \in \mathbb{R}^3 \,:\, x = 1\}$$

and let X be the **hyperboloid** defined by

$$X = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 - z^2 = 1\}.$$

The **intersection** of X and Z is given by the points satisfying $x = 1$ and $x^2 + y^2 - z^2 = 1$, i.e., all points such that $x = 1$ and $y^2 = z^2$. This means

$$X \cap Z = \{(x, y, z) \in \mathbb{R}^3 \,:\, x = 1,\ y = \pm z\}.$$

We have seen in Section 2.3.5 that a space consisting of **two lines crossing each other is not a manifold**. The **intersection point**, i.e., the point $p = (1, 0, 0)$, does **not** have a **neighborhood** in $X \cap Z$ which is **diffeomorphic** to an open subset in **Euclidean space**. Thus $X \cap Z$ is **not a manifold**. See Figure 6.6.

As a **reality check**, let us look at the **tangent spaces** to X and Z at p: Since Z is a parallel translate of a vector subspace of $\mathbb{R}^3$, we see that $T_p(Z)$ is the yz-plane in $\mathbb{R}^3$ (all points with $x = 0$). The tangent space to X was calculated in the previous example (and in an exercise). At $p = (1, 0, 0)$, $T_p(X)$ is the vector subspace in $\mathbb{R}^3$ spanned by the vectors $(0, 1, 0)$ and $(0, 0, 1)$. In other words, $T_p(X)$ is the xy-plane in $\mathbb{R}^3$. Thus $T_p(Z)$ and $T_p(X)$ do **not span** $T_p(Y) = \mathbb{R}^3$. The problem here is that Z is'the tangent plane to X at p.

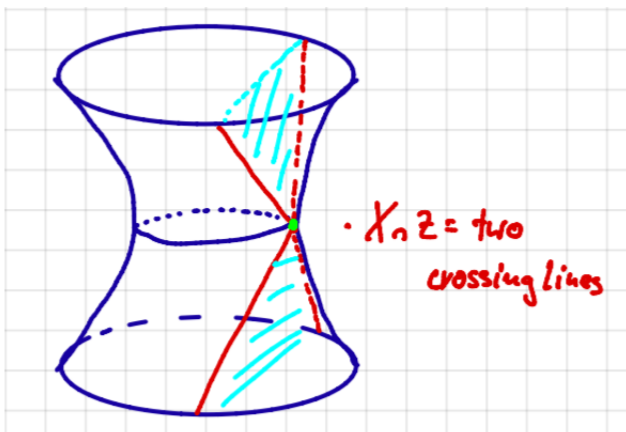

Figure 6.6: The intersection is not transverse and $X \cap Z$ is not a manifold, since we get two axes that cross each other.

Famous examples of transverse intersections are provided by **Brieskorn manifolds**:

Remark 6.12 (Exotic Spheres) Consider the following intersections in $\mathbb{C}^5 \setminus \{0\}$:

$$\begin{aligned} \mathbb{S}^7_k =& \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \\ &\cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}. \end{aligned}$$

In Exercise 6.6, we show that this is a **transverse intersection**. One can show that, for each value $k = 1, \ldots, 28$, the space S^7_k is a smooth manifold which is **homeomorphic** to the seven-sphere $\mathbb{S}^7$. But **none of these** manifolds are **diffeomorphic**.

These are so called **exotic 7-spheres**. They were constructed by **Brieskorn** and represent each of the 28 diffeomorphism classes on the space $\mathbb{S}^7$. That such exotic 7-spheres is a famous and groundbreaking result of **Milnor**. Milnor's work started an **amazing story** about the diffeomorphic structures on spheres which culminated in the solution of the **Kervaire Invariant One Problem** by **Hill, Hopkins and Ravenel** in 2009.

6.3 Exercises and more examples

6.3.1 Transversality

Exercise 6.1 As a first test, answer the following questions:

(a) Let $z = (a, b) \in \mathbb{S}^1 \subseteq \mathbb{R}^2$ and let $N_z = \{(a, y) : y \in \mathbb{R}\}$ be the vertical line intersecting the circle at z. When is $\mathbb{S}^1 \subseteq \mathbb{R}^2$ transverse to $N_z \subseteq \mathbb{R}^2$?

(b) Which of the following linear spaces intersect transversally?

- The xy-plane and the z-axis.
- The xy-plane and the plane spanned by $\{(3, 2, 0), (0, 4, -1)\}$.
- The plane spanned by $\{(1, 0, 0), (2, 1, 0)\}$ and the y-axis in $\mathbb{R}^3$.
- $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (The answer depends on k, l, and n.)
- $V \times \{0\}$ and the diagonal in $V \times V$, for a real vector space V.
- The spaces of symmetric ($A^t = A$) and skew symmetric ($A^t = -A$) matrices in $M(n)$.

(c) Do $SL(n)$ and $O(n)$ meet transversally in $M(n)$?

Exercise 6.2 Recall the maps

$$f : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2} \right) \text{ and } g : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^2$$

from previous exercise sets. Is the set $\mathrm{Im}\,(f)$, the image of f in $\mathbb{R}^2$, a manifold? Is the set $(g \circ f)^{-1}(1)$ a manifold?

Exercise 6.3 Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps between manifolds, and let $W \subset Z$ be a submanifold. Assume that g is transversal to W. Show:

$$f \pitchfork g^{-1}(W) \text{ if and only if } (g \circ f) \pitchfork W.$$

Exercise 6.4 Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A : V \to V$, consider the graph $\Gamma(A) = \{(v, Av) : v \in V\}$. Show that $\Gamma(A) \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of A.

Exercise 6.5 Let $f : X \to X$ be a map, and let x be a fixed point of f, i.e., $f(x) = x$. If $+1$ is not an eigenvalue of $df_x : T_x(X) \to T_x(X)$, then x is called a *Lefschetz fixed point* of f. The map f is called a *Lefschetz map* if all its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

Hint: Show that the intersection of the graph of f and the diagonal of X is a 0-

dimensional submanifold of $X \times X$.

Exercise 6.6 Consider the following intersections in $\mathbb{C}^5 \setminus \{0\}$:

$$S_k^7 = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

Prove that S_k^7 is a 7-dimensional manifold by showing that the intersection is transverse in $\mathbb{C}^5 \setminus \{0\}$.

Hint: At some point you may want to show that, at a point $z = (z_1, \dots, z_5)$, the vector $w := \left(\frac{m}{2} z_1, \frac{m}{2} z_2, \frac{m}{2} z_3, \frac{m}{3} z_4, \frac{m}{6k-1} z_5\right)$, with $m := 2 \cdot 3 \cdot (6k-1)$, lies in one of the tangent spaces but not in the other.

7. Sard's theorem and Morse functions

7.1 The Theorem of Brown and Sard

In the previous sections we have seen how useful regular values are. This motivates to ask:

Question Let $f : X \to Y$ be a smooth map. How many regular values are there in Y?

Recall that a $y \in Y$ which is **not** a regular value for f is called a **critical value**. Hence we may ask the equivalent question: How many critical values are there?

We have seen an answer to the above question in one situation:

- In Milnor's proof of the Fundamental Theorem of Algebra, we showed that the smooth map in question had only **finitely** many critical values. Actually, we showed that the set of critical points of the map $\mathbb{S}^2 \to \mathbb{S}^2$ we defined using the given polynomial was finite.
- We might **hope** that the set of critical values is always finite.
- However, **finiteness** is **too good** to be true when we allow X and Y to be arbitrary smooth manifolds.

So what is the correct **analog of finiteness** for our situation? Here is our first answer:

Theorem 7.1 (Sard's Theorem) Let X and Y be smooth manifolds and let $f : X \to Y$ be a smooth map. Then the set of **regular values** of f is a **dense** subset of Y, i.e., **every** open subset of Y contains a regular value.

- This theorem is a **key result** in differential topology, and we will apply it many situations. For example, it is crucial for **Thom's Transversality Theorem 13.25** which will make **intersection theory** work. See Section 14.1.

We will now reformulate and simplify the theorem. To do this we are going to use the following terminology:

Definition 7.2 (The interior of a set) Let X be a topological space, and S a subset of X. Then the **interior** of S, denoted $\mathrm{int}(S)$, is the **union of all open subsets of X contained in S**. By definition, the interior of any S is an **open** subset of X. In fact, it is the **largest open subset of X which is contained in S**. See Figure 7.1.

- If $S \subset \mathbb{R}^N$ then $\mathrm{int}(S)$ is the set of all points $s \in S$ such that there is a small open ball centred at x which is contained in S.
- If U is an open subset of X then $\mathrm{int}(U) = U$. In particular, if $X \subset \mathbb{R}^N$ is open then $\mathrm{int}(X) = X$. In general, however, $\mathrm{int}(S)$ is a proper subset of X.

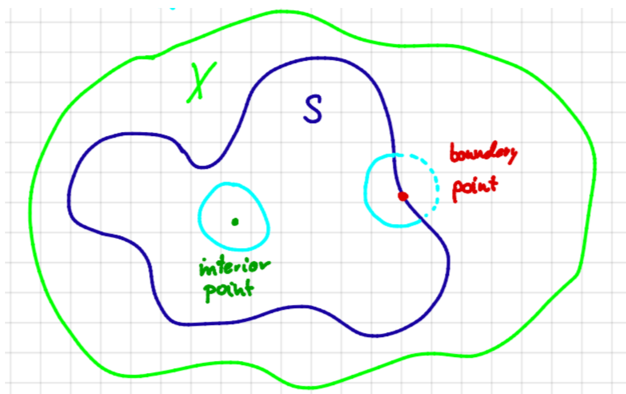

Figure 7.1: The interior of a subset S in the topological space X is the union of all open subsets of X which are contained in S.

We recall the following **key facts** from general topology:

- A subset $A \subset X$ has **empty interior** if and only if its complement $X \setminus A$ is **dense** in X.
- The **countable intersection** of open dense subsets is a dense subset in $\mathbb{R}^p$.
- Equivalently, the **countable union** of compact subsets with empty interior in $\mathbb{R}^p$ is a subset with **empty interior**.
- The analogous statements hold in every smooth p-dimensional manifold.[1]

We also introduce some more **notation**:

- For a smooth map $f : X \to Y$ we denote by C_f the set of critical points of f, by D_f the set of critical values, and by R_f the set of regular values.
- Note that it follows from the Local Submersion Theorem 4.2 that C_f is a **closed** subset of X.

We will show that Sard's theorem is a consequence of the following result:

Theorem 7.3 (Euclidean case) Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a smooth map. Then the set of **regular values** of f is a **dense** subset of $\mathbb{R}^p$. Equivalently, the set of critical values of f is a set with empty interior.

[1] This means that $\mathbb{R}^p$ and smooth manifolds in general are Baire spaces.

Before we start proving Theorem 7.3 we show it implies Sard's Theorem.

Proof that Theorem 7.3 implies Sard's Theorem 7.1: First assume $Y = \mathbb{R}^p$. Let $x \in X$ be a point and let $\phi : \mathbb{R}^n \to X$ be a local parametrization.[2] We set $W := \phi(\mathbb{R}^n)$ and $K = \phi(\bar{\mathbb{B}}^n_1(0))$, where $\bar{\mathbb{B}}^n_1(0)$ is the **closed** unit ball in $\mathbb{R}^n$. In particular, K is a **compact** subspace in X. By Theorem 7.3, the set of **critical values** of $f \circ \phi$

$$D_{f\circ\phi} = f(C_f \cap W)$$

has **empty interior**. Thus, since C_f is **closed** and hence $C_f \cap K$ is compact, $f(C_f \cap K)$ is a **compact** subset of $\mathbb{R}^p$ with **empty interior**. Since X can be covered by **countably** many neighborhoods of the form K, $D_f = f(C_f)$ is a **countable union** of compact subsets with empty interior. This implies that D_f has **empty interior**. Thus, the complement $Y \setminus D_f$, i.e., the set of regular values of f, is a **dense** subset.

Second, for an arbitrary smooth p-dimensional manifold Y, we can find **countably** many local parametrizations ψ_i of Y such that the **union** of the **compact** sets $K_i := \psi_i(\bar{\mathbb{B}}^p_1(0))$ covers Y. By the **first case**, the intersection of the D_f with any of these sets K_i, is a **compact** subset with **empty interior**. Hence D_f is a **countable** union of **compact** subsets with **empty interior** and has itself empty interior. □

Proof of Theorem 7.3: The proof will proceed in two steps: First we prove the result for a smooth **function** $\mathbb{R}^n \to \mathbb{R}$. Second we use this to reduce the case $\mathbb{R}^p$ to $\mathbb{R}^{p-1}$ and will conclude by **induction**.

- The case $p = 1$, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function.

If $n = 0$, there is nothing to prove. Now assume that the assertion is true for $n - 1$. Let $C_i \subset \mathbb{R}^n$ be the **closed** set of points where **all partial derivatives** of f of order $\leq i$ vanish. We then have $C_f = C_1$ and $C_1 \supset C_2 \supset C_3 \supset \cdots$, and hence

$$C_f = (C_1 \setminus C_2) \cup \cdots \cup (C_{n-1} \setminus C_n) \cup C_n.$$

This impllies

$$D_f = f(C_1 \setminus C_2) \cup \cdots \cup f(C_{n-1} \setminus C_n) \cup f(C_n).$$

Hence the theorem follows for n if we can show that each of the sets $f(C_i \setminus C_{i+1})$ for all $i \geq 1$ and $f(C_i)$ for $i \geq n$ is a **countable union** of closed subsets with **empty interior**. This will be shown in the following two lemmas:

> **Lemma 7.4** The set $f(C_i \setminus C_{i+1})$ is a **countable union** of closed subsets with **empty interior** for all $i \geq 1$.

Proof: We **claim** the following: For each $u \in C_i \setminus C_{i+1}$ there is a compact neighborhood K disjoint with C_{i+1} and an $(n-1)$-dimensional submanifold $Z \subset \mathbb{R}^n$ such that $C_i \cap K \subset Z$. Then every point of $C_i \cap K$ is **critical** for $f_{|Z}$, since it is critical for f. By our **induction hypothesis**,

[2] Note that we can choose the domain of ϕ to be all of $\mathbb{R}^n$, by stretching an open ball $\mathbb{B}^n_r(0)$ to all of $\mathbb{R}^n$ via a diffeomorphism.

$f(C_i \cap K)$ is then a **closed** subset in $\mathbb{R}$ **without interior** points. Since $C_i \setminus C_{i+1}$ can be covered by **countably** many arbitrarily small compact neighborhoods K, this will prove the lemma.

Now we **prove the claim**. Since $u \in C_i \setminus C_{i+1}$, there is some i-th order partial derivative of f whose first order partial derivatives do not all vanish at u. We denote this partial derivative by g. Then u is a **regular point** for g. By definition of C_i, we know $g(u) = 0$. By the **Local Submersion Theorem 4.2**, we can then find an **open** neighborhood U of u such that 0 is a **regular value** for $g_{|U}$. Hence, by the **Preimage Theorem 4.7**, $g^{-1}(0) \cap U$ is an $(n-1)$-dimensional **submanifold** in U. Moreover, since $C_i \subseteq g^{-1}(0)$ by definition of g and C_i, we can set $Z := g^{-1}(0) \cap U$ and choose a sufficiently **small compact** neighborhood K of u in U. By definition of g, we also know that K is **disjoint** with C_{i+1}. □

Lemma 7.5 The set $f(C_i)$ is a **countable union** of closed subsets with empty interior for $i \geq n$.

Proof: By **Taylor's Theorem**, we have

$$f(x+u) = f(x) + \partial_u f(x) + \cdots + \frac{1}{i!}\partial_u^i f(x) + \frac{1}{(i+1)!}\partial_u^{i+1} f(x+\lambda u)$$

for any two points x and u in $\mathbb{R}^n$, where $0 \leq \lambda \leq 1$ is a real number and ∂_u is the differential operator

$$\partial_u = u_1 \partial/\partial x_1 + \cdots + u_n \partial/\partial x_n.$$

Thus, if $x \in C_i$, then by definition of C_i:

$$f(x+u) - f(x) = \frac{1}{(i+1)!}\partial_u^{i+1} f(x+\lambda u).$$

If in addition x and $y := x+u$ lie in a **convex** set K, then $x+\lambda u$ is also in K, and we get the **inequality**

$$|f(y) - f(x)|_{\max} \leq c|y-x|_{\max}^{i+1}$$

where $|w|_{\max} = \max\{|w_1|, \ldots, |w_n|\}$ and c is a constant **depending** on K and f only.

Now we let K be the **unit cube** in $\mathbb{R}^n$ and consider the **subdivision** of K into k^n subcubes with sides of length $1/k$. Let K' be one of these subcubes and suppose $x \in C_i \cap K$ and $y \in K'$. Then

$$|y-x|_{\max} \leq 1/k.$$

This implies that $f(C_i \cap K')$ is contained in an **interval** of length c/k^{i+1}.

Thus $f(C_i \cap K)$ is contained in a **union** of k^n intervals of joint length

$$k^n c/k^{i+1} \leq c/k$$

where we use that $i \geq n$ by our assumption. Since k is any positive integer, this length can be **arbitrarily small**. Hence the set $f(C_i \cap K)$ must have empty interior. Finally, $\mathbb{R}^n$ and therefore C_i is contained in a **countable** union of unit cubes K. This proves the lemma. □

- The case $p \geq 2$, i.e., a smooth map $f : \mathbb{R}^n \mathbb{R}^p$.

We assume the assertion holds for every smooth map $f : \mathbb{R}^n \to \mathbb{R}^{p-1}$.

Let $O \subset \mathbb{R}^p$ be an **nonempty open** subset of $\mathbb{R}^p$. We will show that f has regular values in O. This implies that the set of regular values for f is dense in $\mathbb{R}^p$.

If O contains points which are **not** in $f(\mathbb{R}^n)$, then these points are **regular values** for f, and we are done.

Hence assume $O \subseteq f(\mathbb{R}^n)$. Let $\pi : \mathbb{R}^p \to \mathbb{R}^{p-1}$ be the **projection** onto the first $p-1$ coordinates. Then $\pi(O)$ is **open** in $\mathbb{R}^{p-1}$, since π is a submersion and hence an open map.[3] By our **induction hypothesis**, the map $\pi \circ f : \mathbb{R}^n \to \mathbb{R}^{p-1}$ has a **regular value** $y' \in \pi(O)$. In other words, f is **transverse** to the **line** $Y' = \pi^{-1}(y')$ in $\mathbb{R}^p$.

Now let $f' : X' \to Y'$ be the induced map. Since $O \subseteq f(\mathbb{R}^n)$, the open set O **meets** Y', i.e., $Y' \cap O \neq \emptyset$. Since Y' is diffeomorphic to $\mathbb{R}$, f' has a **regular value** $y'' \in Y' \cap O$ by the **case** $p = 1$.

Thus, we **have shown**

- f is **transverse** to Y' in $\mathbb{R}^p$, and
- $f' : f^{-1}(Y') \to Y'$ has a **regular value** y'' in $Y' \cap O$.

By the **chain rule** and what we learned about **transversality**, this implies that $y'' \in Y' \cap O$ is a **regular value for** f. This finishes the proof of Theorem 7.3. □

- At the end we used a fact we proved in the exercises: If $f : X \to \mathbb{R}^p$ is transverse to a submanifold $Y' \subset \mathbb{R}^p$ and the induced map $f' : f^{-1}(Y') \to Y'$ is transverse to $Y'' \subset Y'$, then f is transverse to Y''.
- Holm therefore thinks of the argument for the general case as **splitting off** a transverse component of lower dimension to use induction.

To conclude this section, we remark that Sard's Theorem is often formulated as follows:

Theorem 7.6 (Sard's Theorem revisited) Let $f : X \to Y$ be a smooth map of manifolds with $\dim Y = p$. Define the set C to be

$$C = \{x \in X : \operatorname{rank}(df_x) < p\}.$$

Then the subset $f(C) \subset Y$ of **critical values** has **measure zero in Y**.

Before we recall some basic measure theory, we observe how the different versions of the theorem are related:

- This result is stronger than the previous version, and we invite the reader to investigate the relationship between measure zero sets and sets with empty interior. However, the

[3] We proved this fact in the exercises.

version we proved suffices for our purposes. For, we will apply the theorem for knowing that any small open neighborhood of a point contains a regular value.

- **(Measure zero in a not-measure zero box)** A rectangular solid in $\mathbb{R}^n$ is just a cartesian product of n intervals in $\mathbb{R}^n$, and its volume is the product of the lengths of the n intervals. An **arbitrary set A in $\mathbb{R}^n$** is said to have **(Lebesgue) measure zero** if, for every $\varepsilon > 0$, there exists a **countable** collection $\{S_1, S_2, \ldots\}$ of rectangular solids in $\mathbb{R}^n$, such that A is contained in the union of the S_i, and

$$\sum_{i=1}^{\infty} \operatorname{vol}(S_i) < \varepsilon.$$

Then in a **manifold X**, an arbitrary **subset** $C \subset X$ has **measure zero** if, for **every local parametrization** ϕ of X, the preimage $\phi^{-1}(C)$ has **measure zero in Euclidean space**.

Note that measures and volumes depend on the ambient space!

- An **example** of a **measure zero** subset is given by the set of **rational numbers** in $\mathbb{R}$.

- Hence for measure theorists, almost every real number is irrational. This example illustrates that something that happens almost never, can still happen often enough to be noticed.

- By definition, no nonempty rectangular solid in $\mathbb{R}^n$ has measure zero. Hence it cannot be contained in a set of measure zero.

- Now, every nonempty **open** subset of $\mathbb{R}^n$ contains some nonempty rectangular solid. Thus, no nonempty open subset of $\mathbb{R}^n$ has measure zero.

- Hence, **no nonempty open** subset of a manifold Y has **measure zero**. In other words, no set of measure zero in a manifold Y can contain a nonempty open subset of Y.

7.2 Morse Functions

We will now study a very interesting **application of Sard's Theorem**. We understand the local behaviour of smooth maps at regular points by the Local Submersion Theorem 4.2. But what about the **local behaviour at critical points**? In fact, it is often at critical points that the **interesting stuff happens**. For example, it is often at critical points that the topology of a manifold can change.

For **example**, let $f : X \to \mathbb{R}$ be a smooth function. If X is **compact**, then we know that f must have a **maximum** and a **minimum**. At a point $x \in X$ where $f(x)$ is either a maximal or a minimal value, f does not change in any direction in X. In other words, the **derivative df_x must vanish** (recall $df_x(h)$ is a measure for the change of f in direction h). Hence x is a **critical point** in our terminology.

7.2.1 Height function on the torus

A **standard example** is given by the **height function on a torus**, see Figure 7.2:

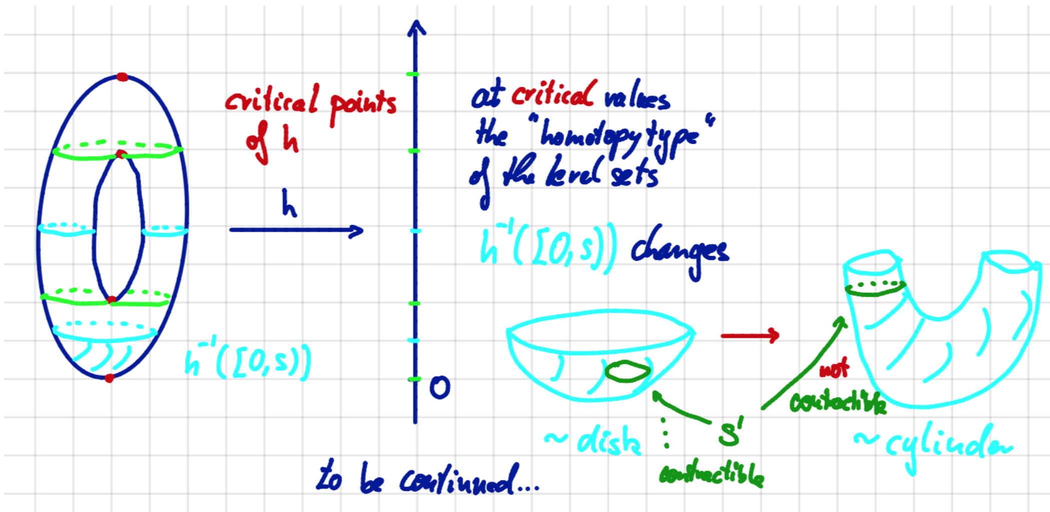

Figure 7.2: The critical points of the height function yield a decomposition of the torus.

Change of homotopy types: We observe in this example that the **homotopy type** of the fiber can change at critical points. You may have noticed that have not defined what the term **homotopy type** means. Roughly speaking, it is equivalence class of a space under the relation of homotopy equivalence (which we have not defined either). For example, all contractible spaces have the same homotopy type (with some assumptions in place). Anyway, we are not going to fix this lack of precise definitions now. Instead, we **look at** what happens with the **fibers on the torus** in this concrete example:

- For $s < 0$, the preimage $h^{-1}([0,s))$ under the height functions is empty.
- The **first critical** value is $s = 0$. For sufficiently small $s > 0$, the **preimage** $h^{-1}([0,s))$ **has changed** and looks like a **two-dimensional disk**, just a bit punched in at the center. In particular, the preimage is **contractible** in this range.
- But when we pass the **next critical value**, the light green dot on the vertical number line, another **significant change** happens. For after it, the preimage $h^{-1}([0,s))$ looks like a bent **cylinder**. On this cylinder, there are loops which are **not homotopic to a constant map**, e.g., the dark green circle. Thus, above the critical value, the preimage $h^{-1}([0,s))$ is **not contractible** anymore. Hence the **homotopy type** of the preimage has **changed** at a critical value.
- The **next change** of the homotopy type happens when we pass the **next critical value**. The preimage $h^{-1}([0,s])$ of the **closed** interval $[0,s]$ becomes homeomorphic to a **compact surface of genus one**, i.e., it has one hole, with a **circle as boundary**.
- Finally, after passing the last critical value, the preimage is the whole torus. The torus has the homotopy type of a **compact surface with genus one**, i.e., still one hole, but **without boundary**. In total, we see that **a lot of interesting stuff happens** at the critical values.

7.2.2 Morse functions on Euclidean space

We now study smooth **functions**, i.e., smooth maps to $\mathbb{R}$. We want to understand **how critical points look like locally**.

Let us look a smooth function $f : \mathbb{R}^k \to \mathbb{R}$. **Locally** around a point $c \in \mathbb{R}^k$, we can describe f by

$$f(x) = f(c) + \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(c) \cdot (x_i - c_i) + \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 f}{\partial x_i \partial x_j}(c) \cdot (x_i - c_i)(x_j - c_j) + o(|x|^3).$$

If c is a **critical point**, then by definition

$$df_c = \left(\frac{\partial f}{\partial x_1}(c), \dots, \frac{\partial f}{\partial x_k}(c) \right) = 0$$

(otherwise df_c was surjective as a linear map $\mathbb{R}^k \to \mathbb{R}$). Hence the best possible approximation for the local behavior of f at c is the **Hessian matrix** of the second partial derivatives. Critical points where the **Hessian matrix is invertible** is the **best** we can hope for.

Definition 7.7 (Non-degenerate critical points and Morse functions) For a smooth function $f : \mathbb{R}^k \to \mathbb{R}$, a point $c \in \mathbb{R}^k$ where df_c vanishes, but the **Hessian** matrix $H(f)_c = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(c) \right)$ is **invertible** at c, is called a **non-degenerate critical point**. A smooth function $f : \mathbb{R}^k \to \mathbb{R}$ for which all critical points are non-degenerate is called a **Morse function**.

- Non-degenerate critical points are much easier to study than arbitrary critical points, since they are **isolated from the other critical points**, i.e., there is an open neighborhood which does not contain any other critical points. Hence Morse functions are **easier to understand** than arbitrary smooth functions.

We check that **non-degenerate** critical points are **isolated**: We define a map

$$g : \mathbb{R}^k \to \mathbb{R}^k \text{ by the formula } g = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k} \right). \tag{7.1}$$

$$\text{Then: } df_x = 0 \iff g(x) = 0.$$

Moreover, the matrix representing the derivative dg_x is the Hessian of f at x. So if **x is non-degenerate**, then not only is $g(x) = 0$, but **g maps a neighborhood of x diffeomorphically onto a neighborhood of 0** as well. In particular, g is injective in that neighborhood of x. Thus g can be zero at no other points in this neighborhood, and f has no other critical point in this neighborhood. □

Another reason to be interested in Morse functions is the fact that there are a lot of them.

Theorem 7.8 (Morse functions on $\mathbb{R}^k$ are generic) Let $f : U \to \mathbb{R}$ be a smooth function defined on some open $U \subseteq \mathbb{R}^k$ and $a \in \mathbb{R}^k$, define

$$f_a(x) = f(x) + a \cdot x.$$

Then, **for almost all** $a \in \mathbb{R}^k$, f_a is a **Morse function**.

Proof: We us again the function g from (7.1).The derivative of f_a at a point $p \in U$ then satisfies

$$(df_a)_p = \left(\frac{\partial f_a}{\partial x_1}(p), \dots, \frac{\partial f_a}{\partial x_k}(p)\right) = g(p) + a.$$

Hence the **critical points of f_a** are the points $p \in U$ with $g(p) + a = 0$. Moreover, the **Hessian of f_a** at p is the matrix dg_p, i.e.

$$H(f_a)_p = H(f)_p = dg_p.$$

Hence

$$\begin{aligned} f_a \text{ is Morse} &\iff \det(H(f_a)_p) \neq 0 \text{ at all critical points } p \\ &\iff \det(dg_p) \neq 0 \text{ at all } p \text{ with } g(p) + a = 0 \\ &\iff -a \text{ is a regular value of } g. \end{aligned}$$

By **Sard's Theorem**, $-a$ is a regular value of g for almost all $a \in \mathbb{R}^k$. Therefore almost every f_a is a Morse function. □

7.2.3 Morse functions on manifolds

Now we would like to transport the concept of non-degenerate critical points to **manifolds**.

Lemma 7.9 (Independence of choice) So let X be a smooth manifold. Suppose that $f : X \to \mathbb{R}$ has a **critical point at x** and that $\phi : U \to X$ is a local parametrization with $\phi(0) = x$. Then

$$d(f \circ \phi)_0 = df_x \circ d\phi_0$$

and hence 0 is a critical point for the function $f \circ \phi$. We call x a **non-degenerate critical point for f** if 0 is a non-degenerate critical point for $f \circ \phi$.

Independence of choice:

Since we made a choice of a local parametrization for this definition, we need to make sure that the criterion is independent of the choice.

So let $\psi : V \to X$ be another local parametrization with $\psi(0) = x$. We define $\theta := \psi^{-1} \circ \phi : U \to V$. Since θ is a diffeomorphism, the **critical points** of $f \circ \phi$ and $f \circ \psi \circ \theta$ are the **same**.

Assuming that x is a critical point of f, i.e., $df_x = 0$, the chain rule implies for the two Hessian matrices at 0:

$$H(f\circ\phi)_0 = (d\theta_0)^t H(f\circ\psi)_0 d\theta_0.$$

Since $d\theta_0$ is invertible, we see

$$H(f\circ\phi)_0 \text{ is invertible} \iff H(f\circ\psi)_0 \text{ is invertible.} \qquad \square$$

7.2.4 Morse Lemma

An important result on Morse functions is that they can be described in some sort of **canonical form**. It extends our understanding of the **local behavior of smooth maps** and is a **key result** in this section:

Lemma 7.10 (Morse Lemma) Let X be a smooth manifold and $f : X \to \mathbb{R}$ be a smooth function. Suppose that $a \in X$ is a **non-degenerate critical point** of f. Then **there is a local parametrization** $\phi : U \to X$ with $\phi(0) = a$ and a local coordinate system $\phi^{-1} = (x_1, \ldots, x_k)$ around a such that

$$f(x) = f(a) - x_1^2 - \ldots - x_s^2 + x_{s+1}^2 + \ldots + x_k^2$$

for all $x \in \phi(U)$ where s is the **number of negative eigenvalues** of the Hessian of f at a.

For the proof of Lemma 7.10 we follow [12, Part I, §2.1]. We are going to use the following general observation:

Lemma 7.11 Let $f : \mathbb{B}^k \to \mathbb{R}$ be a smooth function defined on an open ball around the origin in $\mathbb{R}^k$ with $f(0) = 0$. Then we have

$$f(x_1, \ldots, x_k) = \sum_{i=1}^k x_i g_i(x_1, \ldots x_k)$$

for some suitable smooth functions $g_i : \mathbb{B}^k \to \mathbb{R}$ with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof: We have

$$\begin{aligned} f(x_1, \ldots, x_k) &= \int_0^1 \frac{df(tx_1, \ldots, tx_k)}{dt} dt \\ &= \int_0^1 \sum_{i=1}^k \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_k) \cdot x_i dt. \end{aligned}$$

Hence we may set $g_i(x_1, \ldots x_k) := \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_k) \cdot x_i dt$. $\square$

Proof of Lemma 7.10: Without loss of generality, we can assume that a is the origin and that $f(a) = f(0) = 0$. By Lemma 7.11 we can write

$$f(x_1, \dots, x_k) = \sum_{j=1}^{k} x_j g_j(x_1, \dots x_k)$$

in a small neighborhood of 0. Since 0 is a critical point, we must have

$$g_j(0) = \frac{\partial f}{\partial x_j}(0) = 0.$$

Hence we can apply Lemma 7.11 to each g_j and get

$$g_j(x_1, \dots, x_k) = \sum_{j=1}^{k} x_j h_{ij}(x_1, \dots x_k)$$

for suitable smooth functions h_{ij} defined on the small neighborhood around the origin. Combining these expressions we get

$$f(x_1, \dots, x_k) = \sum_{i,j=1}^{k} x_i x_j h_{ij}(x_1, \dots x_k)$$

We can assume that $h_{ij} = h_{ji}$ since we can replace h_{ij} with $\bar{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$ and then get $\bar{h}_{ij} = \bar{h}_{ji}$ and $f = \sum x_i x_j \bar{h}_{ij}$. By construction we then get

$$h_{ij}(0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

Thus the matrix H with entry $h_{ij}(0)$ in position (i, j) is the **Hessian** matrix of f, computed in the local coordinate system defined by ϕ. Since 0 is a nondegenerate critical point, the Hessian is invertible.

Now we have to show that we can choose a coordinate system such that f has the simple form of the lemma. To do so, we write $H := H(f \circ \phi))_0$ for the Hessian matrix and $\mathbf{x} = (x_1, \dots, x_k)$. Then we have locally

$$f = \mathbf{x}^t H \mathbf{x}.$$

Since the Hessian matrix H is **real symmetric**, there is an **orthonormal** matrix P such that $P^t H P$ is a diagonal matrix. Note that multiplying with P^t, or P, corresponds to a **change of basis** $\mathbb{R}^k \to \mathbb{R}^k$. We write $\mathbf{y} = P^t \mathbf{x}$ for the coordinates with respect to the **new basis**. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of H, where we order them, and hence the columns in P, such that the first s λ_i are negative and the remaining $k - s$ are positive. Note that H does not have eigenvalue 0, since it is invertible by the assumption on a being a **nondegenerate** critical point. Then we get

$$f = \mathbf{x}^t P P^t H P P^t \mathbf{x} = \mathbf{y}^t P^t H P \mathbf{y} = \sum_{i=1}^{k} \lambda_i y_i^2.$$

Finally, we make the change of basis $\mathbb{R}^k \to \mathbb{R}^k$ given by rescaling

$$z_i = \sqrt{|\lambda_i|} y_i \text{ for each } i = 1, \dots, k.$$

In the **new local coordinate system** $\mathbf{z}$ we have

$$f = -z_1^2 - \dots - z_s^2 + z_{s+1}^2 + \dots + z_k^2. \quad \square$$

Example 7.12 Let us look at the example of the height function on the **torus**. To **make the notation compatible** with the above theorem, we denote the hight function by f. This is just a projection onto the vertical coordinate of the points on the torus. At the point p on the torus with $f(p) = 0$, we can choose local coordinates x, y and write

$$f(x, y) = f(p) + x^2 + y^2 = x^2 + y^2.$$

At the next two critical values, we can choose local coordinates x, y write f as

$$f(x, y) = \text{constant } + x^2 - y^2.$$

At the final critical value, we can choose local coordinates x, y such that

$$f(x, y) = \text{constant } - x^2 - y^2.$$

7.2.5 Morse functions are generic

We are now going to discuss some situations where the Morse Lemma 7.10 is useful. We will see another important application in Section 18.6 when we discuss the Poincaré–Hopf Index Theorem 18.16. First, we can generalize the fact that **almost all** functions are Morse to the level of **manifolds**: Suppose $X \subset \mathbb{R}^N$, and let $x_1, \dots, x_N \in \mathbb{R}^N$ be the usual coordinate functions on $\mathbb{R}^N$. If $f : X \to \mathbb{R}$ is a smooth function on X and $a = (a_1, \dots, a_N)$ is an N-tuple of numbers, we define again a new function $f_a : X \to \mathbb{R}$ by

$$f_a := f + a_1 x_1 + \cdots + a_N x_N.$$

Theorem 7.13 (Morse functions on manifolds are generic) For every smooth function $f : X \to \mathbb{R}$ and for **almost every** $a \in \mathbb{R}^N$, f_a is a **Morse function** on X, i.e., all its critical points are nondegenerate.

Proof: We would like to use the above result for $U \subset \mathbb{R}^k$ open. Since $X \subset \mathbb{R}^N$ is in general **not open** (in fact, it is never open if $\dim X < N$), the **strategy is to cover X** by open subsets and then try to lift the k-dimensional result to open sets in $\mathbb{R}^N$.

So let x be any point in X. First we are going to choose a suitable local coordinate system around x. Let $v_1, \dots, v_k \in \mathbb{R}^N$ be a basis of $T_x(X)$ (for $k = \dim X$). Then the matrix $[v_1 \cdots v_k]$, having the v_i's as columns, has rank k. Hence it has k linearly independent rows, say $i_1, \dots, i_k$. Let $\pi : \mathbb{R}^N \to \mathbb{R}^k$ be projection defined by $(x_1, \dots, x_N) \mapsto (x_{i_1}, \dots, x_{i_k})$ where the $x_1, \dots, x_N$ denote the standard coordinates on $\mathbb{R}^N$. Then

$$(d\pi_x)_{|T_x(X)} : T_x(X) \to \mathbb{R}^k \text{ is an } \textbf{isomorphism}$$

by construction. Hence, by the **Inverse Function Theorem**,

$$\pi_{|X} : X \to \mathbb{R}^k \text{ is a } \textbf{local diffeomorphism}.$$

Hence we can take the k-tuple of functions $(x_{i_1}, \dots, x_{i_k}) : X \to \mathbb{R}^k$ to define a **local coordinate system around x**.

Therefore we can cover X with open subsets $U_\alpha \subseteq \mathbb{R}^N$ such that on each U_α **some k-tuple** of the functions $x_1, \ldots, x_N$ on $\mathbb{R}^N$ form a coordinate system. Moreover, by some general nonsense on the topology of Euclidean space, it is always possible to choose a **countable** subfamily of the U_α's. Hence we may assume there are only **countably many U_α**.

Let $S \subset \mathbb{R}^N$ be the subset of a such that f_a is **not Morse**. Since the **countable union** of sets with measure zero has **measure zero**, it suffices to show that **for each U_α** the set S_α of a's such that $f_a : U_\alpha \to \mathbb{R}$ is not Morse, has **measure zero**.

So let us look at one of the U_α's. We want to show that S_α has measure zero in $\mathbb{R}^N$.

To simplify the notation, assume $x_1, \ldots, x_k$ form a coordinate system around x on U_α. We can write any $a \in \mathbb{R}^N$ as $a = (b, c)$, where b denotes the **first k** coordinates and c denotes the **last $N - k$** coordinates. Around a given point x, we can thus write

$$f_a(x) = f(x) + c \cdot (x_{k+1}, \ldots, x_N) + b \cdot (x_1, \ldots, x_k).$$

The function $x \mapsto f(x) + c \cdot (x_{k+1}, \ldots, x_N)$ is smooth. Hence we can apply our previous result on genericity of Morse functions on open subsets in $\mathbb{R}^k$ to this function and get that **f_a is a Morse function for almost every $b \in \mathbb{R}^k$**.

Thus, for a fixed c, the subset of all $b \in \mathbb{R}^k$ where f_a is not Morse, has measure zero in $\mathbb{R}^k$. Hence $S_\alpha \cap (\mathbb{R}^k \times \{0\})$ has measure zero in $\mathbb{R}^N$. It is a classical result in Measure Theory, called **Fubini's Theorem**, which then implies that the set S_α of all $a = (b, c)$ where a does not yield a Morse function has measure zero in $\mathbb{R}^N$. Hence f_a is a Morse function for almost every a. □

Finally, we can also show that being a Morse function is a stable property. In order to prove stability, we start with a little lemma:

Lemma 7.14 Let f be a smooth function on an open set $U \subset \mathbb{R}^k$. For each $x \in U$, let $H(f)_x$ be the Hessian matrix of f at x. Then f is a Morse function if and only if

$$(\det(H(f)_x))^2 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}(x) \right)^2 > 0 \text{ for all } x \in U. \tag{7.2}$$

Proof: A point x is regular if $df_x = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x)) \neq 0$, and x is a nondegenerate critical point if $df_x = 0$ and $\det(H(f)_x) \neq 0$. Hence f is Morse if and only if (7.2) is satisfied. □

Lemma 7.15 Suppose that f_t is a **homotopic family** of functions on $\mathbb{R}^k$. If f_0 is a **Morse function** on some **open** subset $U \subset \mathbb{R}^k$ containing a **compact** set $K \subset \mathbb{R}^k$, then every f_t for t sufficiently small is a **Morse function**.

Proof: We define the map

$$F : U \times [0,1] \to \mathbb{R}, \; (x,t) \mapsto (\det(H(f_t)_x))^2 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}(x) \right)^2.$$

Since f is smooth, F depends smoothly on both variables. By Lemma 7.14 and the assumption, we know $F(x, 0) > 0$ for all $x \in U \times \{0\}$. Since $K \subset U$ is compact, F has a minimum on $K \times \{0\}$, i.e. there is a $\delta > 0$ such that $F(x, 0) \geq 2\delta$ for all $x \in K$. Since F is continuous, there an open neighborhood $W \subset U \times [0, 1]$ containing $K \times \{0\}$ such that $F(x, t) > \delta$ for all $(x, t) \in W$. In fact, we can cover $K \times \{0\}$ by open subsets $W_i \subset U \times [0, 1]$ such that $F(x, t) > \delta$ for all $(x, t) \in W_i$. Each such open subset W_i has the form $V_i \times [0, \epsilon_i)$ for some open $V_i \subset U$ and $\epsilon_i > 0$. Since K is compact, **finitely many** such open W_i suffice to cover $K \times \{0\}$. Let ϵ be the **minimum** of the finitely many ϵ_i. Then we have $F(x, t) > \delta$ for all $(x, t) \in K \times [0, \epsilon)$. Since F is continuous, for any fixed $t \in [0, \epsilon)$, there is again an open subset $V\mathbb{R}^k$ containing K such that $F(x, t) > 0$ for all $(x, t) \in V \times \{t\}$. Thus f_t is Morse in a neighborhood of K for all sufficiently small t. □

Finally, we can prove stability of Morse functions.

Theorem 7.16 (Stability of Morse functions) Let X be a **compact** smooth manifold, let $f_0 : X \to \mathbb{R}$ be a smooth function and f_t be a **homotopy** of f_0. If f_0 is **Morse**, then there is an $\varepsilon > 0$ such that **f_t is a Morse function** for all $t \in [0, \varepsilon)$.

Proof: For $x \in X$, let $\phi_x : U_x \to X$ be a local parametrization around x. Then $f_0 \circ \phi_x$ is a Morse function on U. Since $\{0\}$ is a compact subset of U, Lemma 7.15 implies that there is an open subset $V_x \subset U_x$ containing $\{0\}$ and an $\varepsilon(x) > 0$ such that f_t is Morse on V_x for all $t \in [0, \varepsilon(x))$. The images $\phi_x(V_x)$ are open in X and cover X. Since X is compact, finitely many suffice to cover X, say

$$X = \phi_{x_1}(V_{x_1}) \cup \cdots \cup \phi_{x_n}(V_{x_n})$$

Then we can set $\varepsilon := $ minimum of $\varepsilon(x_1), \ldots, \varepsilon(x_n)$. Then $f_t : X \to \mathbb{R}$ is a Morse function for all $t \in [0, \varepsilon)$. □

8. Smooth Homotopy

In this chapter we are going to introduce one of the most important concepts in topology: homotopies between maps. The idea of studying objects **up to homotopy** has turned out be extremely successful in many areas in mathematics. One motivation for allowing maps to vary up to homotopy is that it is far too complicated to try to classify and understand **all** maps between manifolds. We will soon define **invariants**, i.e., numbers that we attach to smooth manifolds and the maps between them. These numbers do not change if we modify a map by a homotopy. In fact, we will see that this behavior makes the numbers all the more useful. See also Remark 8.5 below.

8.1 Smooth homotopies and bump functions

8.1.1 Homotopies

We begin with a fundamental definition:

Definition 8.1 **(Homotopy)** Let X and Y be two topological spaces and let $I = [0,1]$ denote the unit interval in $\mathbb{R}$. We say that two continuous maps f_0 and f_1 from X to Y are **homotopic**, denoted $f_0 \sim f_1$, if there exists a continuous map

$$F\colon X \times [0,1] \to Y$$

such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. The map F is called a **homotopy** between f_0 and f_1. We also write $f_t(x)$ for $F(x,t)$. In other words, a homotopy is a **family** of continuous functions f_t which **continuously interpolates** between f_0 and f_1.

Here are some first **examples**:

Example 8.2 Consider $f_0\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x,0)$ and $f_1\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, \sin x)$ with homotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^2$, $(x,t) \mapsto (x, t\sin x)$. See Figure 8.1.

Example 8.3 Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a loop, i.e., a continuous map where start and end points agree: $\gamma(0) = \gamma(1)$. Then γ is **homotopic** to the **constant** map $[0,1] \to \{0\} \subset \mathbb{R}^2$. See Figure 8.2. In fact, this is true when we replace $\mathbb{R}^2$ with any $\mathbb{R}^k$, since $\mathbb{R}^k$ is **contractible** as we will show in the exercises.

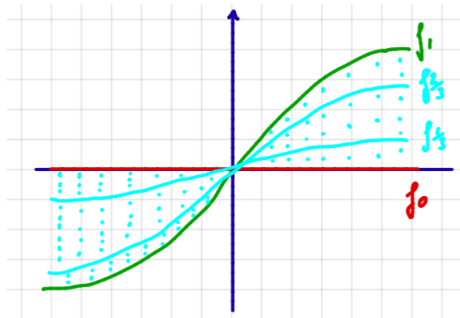

Figure 8.1: A family of maps which interpolates between a constant map and the sine function.

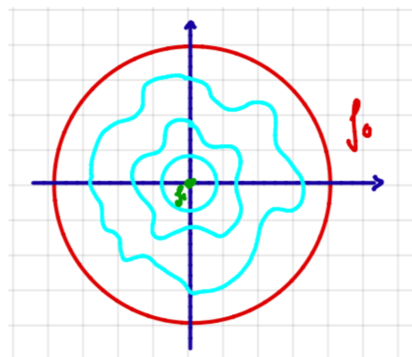

Figure 8.2: Every loop in $\mathbb{R}^2$ can be shrunk to a point, since $\mathbb{R}^2$ is contractible.

Example 8.4 In Exercise 8.4 we will show that the **antipodal map** on the k-sphere $\mathbb{S}^k \to \mathbb{S}^k$, $x \mapsto -x$, which sends a point to the point on the other side of the sphere, is **homotopic to the identity** on $\mathbb{S}^k$ if k is **odd**. See Figure 8.3. If k is **even**, however, the antipodal map is **not** homotopic to the identity. To prove this claim will require more sophisticated tools that we will develop in this course. See Exercise 16.4.

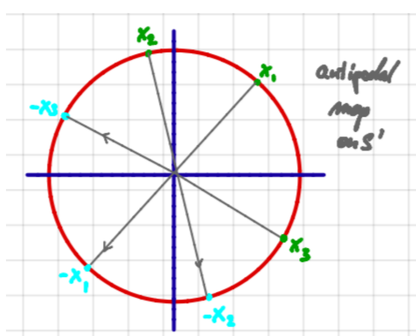

Figure 8.3: The antipodal map on the sphere may or may not be homotopic to the identity map. This is a subtle phenomenon which we will study in more detail later.

Remark 8.5 (The homotopy category) Homotopy is one of the most important concepts in topology. In fact, a lot of properties in topology are **invariant under homotopy**. Therefore, they can be studied by considering maps only **up to homotopy**. This leads to the construction of the **homotopy category of spaces** in which **morphisms are homotopy classes** of maps, i.e., continuous maps f and g represent the same morphism if and only if f and g are homotopic. All the functors that assign an algebraic object to a topological space are, in fact, homotopy-invariant, and hence they descend to the homotopy category. Passing to the homotopy category is a **very powerful idea** which has had tremendous influence in many areas of mathematics. We will, however, not be able to fully appreciate the homotopy category in this course. You will meet homotopies also in algebraic topology, homological algebra, model categories, ∞-categories, the theory of motives in algebraic geometry, new foundations of logic and set theory and in many other areas of mathematics.

8.1.2 Smooth homotopies

For the study of smooth manifolds, it is desirable to strengthen our assumptions on what a homotopy is and to require it to be a **smooth family** of maps. We first present the necessary definition and will then see that it actually is not such a strong restriction after all.

Definition 8.6 **(Smooth homotopy)** Let X and Y be smooth manifolds. We say that two **smooth** maps f_0 and f_1 from X to Y are **smoothly homotopic**, denoted $f_0 \sim f_1$, if there exists a **smooth** map $F: X \times [0,1] \to Y$ such that

$$F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x).$$

F is called a **smooth homotopy** between f_0 and f_1. We also write $f_t(x)$ for $F(x,t)$. In other words, a homotopy is a **family** of smooth functions f_t which **smoothly interpolates** between f_0 and f_1.

- Recall that **smoothness** of F in the second variable means that we can extend F to a smooth map $X \times (-\varepsilon, 1+\varepsilon) \to Y$ for some small $\varepsilon > 0$.

To allow only smooth homotopies might seem like a strong restriction. However, it turns out that we can always approximate a continuous map by a smooth map within its homotopy class:

Theorem 8.7 **(Whitney Approximation Theorem)** Let X and Y be smooth manifolds.

- Let $g: X \to Y$ be a **continuous** map. Then g is **homotopic** to a **smooth** map $f: X \to Y$. Moreover, if g was already smooth on a closed subset $Z \subset X$, then one can choose f such that $f_{|A} = g_{|A}$.
- Let $g_0, g_1: X \to Y$ be two smooth maps which are **homotopic**. Then they are **smoothly homotopic**.

We will prove this important result in Section 13.2. See Theorem 13.20 in particular. The key tool we will use for the proof are tubular neighborhoods that we will introduce and study in

Section 13.1. For the moment, however, we rather move on and show that smooth homotopy is an equivalence relation and will then see an application of homotopies.

8.1.3 Bump functions

In order to show that being smoothly homotopic defines an equivalence relation we need to introduce a new tool: **smooth bump functions**. Such functions will turn out to be extremely useful in many applications later.

Definition 8.8 (Bump function) A **bump function** on $\mathbb{R}^n$ is a **smooth** function $\mathbb{R}^n \to \mathbb{R}$ which takes the value 1 on some neighborhood of the origin and the value 0 outside some larger neighborhood of the origin.

We will now show that such functions exist:

Lemma 8.9 (Bump functions exist) For every pair of real numbers $0 < a < b$, there is a **smooth** function $\varphi\colon \mathbb{R}^n \to [0,1] \subset \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \leq a \\ 0 & \text{for } |x| \geq b. \end{cases}$$

In other words, $\varphi(x)$ is equal 1 **on the closed ball** with radius a around the origin, is 0 **outside the open ball** with radius b, and between 0 and 1 on the intermediate points.

Proof: We start with the function

$$f\colon \mathbb{R} \to \mathbb{R},\; f(t) = \begin{cases} e^{-1/t^2} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0. \end{cases}$$

We observe that **f is smooth**: We only need to think about $t \geq 0$. The **ith derivative** has the form e^{-1/t^2} times a rational polynomial. Such a **product is differentiable** and

$$\lim_{t\to 0} f^{(i)}(t) = 0$$

since e^{-1/t^2} **goes faster** to 0 than any rational polynomial can go to $\pm\infty$. The key feature of the map f is that it provides a **smooth transition** from 0 to positive values.

For given real numbers $a < b$, we then define the function

$$g(t) := \frac{f(t-a)}{f(t-a) + f(b-t)}.$$

The function g is well-defined and therefore **smooth** since $f(t-a) + f(b-t) > 0$ for all t. Moreover, $g(t) = 0$ if and only if $t \leq a$ and $g(t) = 1$ if and only if $f(b-t) = 0$, i.e., if and only if $t \geq b$. In other words, g is a **smooth** function such that

$$\begin{cases} g(t) = 0 & \text{for } t \leq a \\ 0 < g(t) < 1 & \text{for } a < t < b \\ g(t) = 1 & \text{for } t \geq b. \end{cases}$$

Finally, we can also define the **bump function** φ by setting

$$\varphi : \mathbb{R}^n \to \mathbb{R}, \ \varphi(x) := 1 - g\left(|x|\right). \qquad \square$$

8.1.4 Being smoothly homotopic is an equivalence relation

Now we are ready to improve the following important fact:

Lemma 8.10 (Smooth homotopy is an equivalence relation) Let X and Y be smooth manifolds. Smooth homotopy defines an **equivalence relation** on the set of smooth maps from X to Y. The equivalence class to which a mapping belongs to is called its **homotopy class**.

Proof: We need to check that $\sim$ is reflexive, symmetric, and transitive:

- **Reflexivity:** This is clear as every map is homotopic to itself via the homotopy $f_t = f$ for all t.
- **Symmetry:** Suppose $f \sim g$ and let F be a homotopy. Then the map defined by $(x,t) \mapsto F(x, 1-t)$ is a homotopy from g to f. Hence $g \sim f$ as well.
- **Transitivity:** Suppose $f \sim g$ and $g \sim h$, and let F be a homotopy from f to g and G be a homotopy from g to h. We would like to compose F and G to get a homotopy from f to h. Since we require our homotopies to be **smooth**, we need to make sure that the **transition** from F to G is **smooth**.

  In order to this, we need to manipulate F and G a bit. And here we are **lucky** that we have **smooth bump functions** at our disposal. So let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

  $$\varphi(t) = \begin{cases} 0 & t \leq 1/4 \\ 1 & t \geq 3/4 \end{cases}$$

  and define new homotopies $\tilde{F}$ from f to g and $\tilde{H}$ from g to h by

  $$\tilde{F}(x,t) := F(x, \varphi(t)) \text{ and } \tilde{G}(x,t) := G(x, \varphi(t)).$$

  Now we can define the map

  $$H : X \times [0,1] \to Y, \ H(x,t) = \begin{cases} \tilde{F}(x, 2t) & t \in [0, 1/2] \\ \tilde{G}(x, 2t-1) & t \in [1/2, 1]. \end{cases}$$

  This map is **well-defined** and **smooth**, since $\tilde{F}(x, 2t) = g(x)$ $t \in (3/8, 1/2]$ and $g(x) = \tilde{G}(x, 2t-1)$ for $t \in [1/2, 5/8)$. Thus H is a smooth homotopy from f to h. Hence $\sim$ is also **transitive** and an equivalence relation. $\square$

Remark 8.11 We emphasize that the role of φ in the proof of Lemma 8.10 is merely to make sure that the transition from F to G is smooth. While $t \mapsto F(x, 2t)$ and $t \mapsto G(x, 2t-1)$ are smooth in t, their concatenation, i.e., the map given by F for $t \geq 1/2$ and by G for $t \geq 1/2$, may not be smooth at $t = 1/2$. The smooth bump function φ, however, makes both $\tilde{F}(x, 2t)$ and $\tilde{G}(x, 2t-1)$ constant in t equal to $g(x)$ in an open neighborhood of $t = 1/2$.

8.2 Simply-connected spaces

We will now explore some more ideas and results related to homotopies. We will formulate parts of this section for topological spaces and not just smooth manifolds, even though we are interested in the latter. Recall that by Whitney's Approximation Theorem 13.20 we can assume that all homotopies are smooth whenever we consider maps between smooth manifolds, for example maps $\mathbb{S}^k \to \mathbb{S}^n$. Thus, when we study manifolds, we can use the techniques we developed for smooth maps to study homotopies. We begin with the following definition:

Definition 8.12 (Homotopy equivalence) Let X and Y be topological spaces. A continuous map $f : X \to Y$ is called a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_Y$. We call g a **homotopy inverse** of f. We say that X and Y are **homotopy equivalent** if there exists a homotopy equivalence $X \to Y$.

- Every **homeomorphism** is also a homotopy equivalence. However, the **converse** is **not** true. For example, recall that a topological space X is called **contractible** if its identity map is homotopic to some constant map $X \to \{x\}$ where x is any point of X. A contractible space X is homotopy equivalent to a one-point space, but it may not be homeomorphic to it. For example, $\mathbb{R}^n$ is homotopy equivalent to the one-point space $\{0\}$ consisting of the origin in $\mathbb{R}^n$. **However**, $\mathbb{R}^n$ is **not** homeomorphic to $\{0\}$ for $n \geq 1$, since any map $\mathbb{R}^n \to \{0\}$ cannot be injective.

All properties of a space which invoke a homotopy are preserved under homotopy equivalences. Being **contractible** is such a property. The following definition provides another example of such a property.

Definition 8.13 (Simply-connected) A topological space X is called **simply-connected** if X is connected and every continuous map $\mathbb{S}^1 \to X$ is homotopic to a constant map.

We study simply-connectedness also in Exercise 8.4 and Exercise 11.2. Here we focus on the following observation:

Lemma 8.14 Let X and Y be topological spaces. Assume that X and Y are **homotopy equivalent**. Then X is simply-connected if and only if Y is simply-connected.

Proof: Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. Let $H : Y \times [0,1] \to Y$ be a homotopy between $H(y,1) = f \circ g(y)$ and $H(y,0) = y$ for all $y \in Y$. Assume that X is simply-connected, and let $\gamma : \mathbb{S}^1 \to Y$ be a continuous map. The composition $g \circ \gamma : \mathbb{S}^1 \xrightarrow{\gamma} Y \xrightarrow{g} X$ is homotopic to a constant map c. Let $h : \mathbb{S}^1 \times [0,1] \to X$ be a homotopy with $h(s,1) = g(\gamma(s))$ and $h(s,0) = c$. Then we can define a homotopy $\Gamma : \mathbb{S}^1 \times [0,1] \to Y$ by

$$\Gamma(s,t) = \begin{cases} H(\gamma(s), 2t) & \text{if } 0 \le t \le 1/2 \\ f(h(s, 2(1-t))) & \text{if } 1/2 \le t \le 1. \end{cases}$$

We need to check that this map is well-defined. For $t = 1/2$ we compute

$$H(\gamma(s), 1) = f \circ g(\gamma(s)) = f(g(\gamma(s)) = f(h(s,1)).$$

Moreover, we have

$$\Gamma(s,0) = \gamma(s) \text{ and } \Gamma(s,1) = f(c) \text{ for all } s \in \mathbb{S}^1.$$

Thus Γ is a homotopy between γ and the constant map with value $f(c) \in Y$. This shows that Y is simply-connected. By applying the argument with the roles of X and Y reversed, we get the assertion. □

Remark 8.15 We note that Lemma 8.14 helps us classifying spaces up to homotopy equivalence. There is the bucket of spaces which are simply-connected and the one of those which are not simply-connected.

For every $n \ge 2$, the n-dimensional sphere $\mathbb{S}^n$ is an example of a simply-connected space:

Theorem 8.16 **($\mathbb{S}^n$ is simply-connected for $n \ge 2$)** For $n \ge 2$, the n-dimensional sphere $\mathbb{S}^n$ is simply-connected.

Remark 8.17 In Exercise 8.2 we use Sard's Theorem 7.1 to show that every **smooth** map $X \to \mathbb{S}^n$ from a smooth manifold X of dimension k is homotopic to a constant map whenever k is **strictly less** than n. Since every continuous map $\mathbb{S}^1 \to \mathbb{S}^n$ is homotopic to a smooth map $\mathbb{S}^1 \to \mathbb{S}^n$ by **Whitney's Approximation Theorem 13.20**, this implies the assertion of Theorem 8.16. Since we have not yet proven Theorem 13.20, we give an alternative proof that does not make use of smooth maps.

Proof of Theorem 8.16: Let $p \in \mathbb{S}^n$ be any point. Stereographic projection defines a diffeomorphism $\mathbb{S}^n \setminus \{p\} \cong \mathbb{R}^n$. In $\mathbb{R}^n$ every continuous map is homotopic to a constant map. Hence the assertion follows from the following Lemma 8.18. □

Lemma 8.18 **(Paths avoiding a point)** Let $n \ge 2$ and $p \in \mathbb{S}^n$. Every path γ in $\mathbb{S}^n$ with $\gamma(0) \ne p \ne \gamma(1)$ is homotopic to a path in $\mathbb{S}^n \setminus \{p\}$.

Proof of Lemma 8.18: Let $U = \mathbb{S}^n \setminus \{p\}$ and $V = \mathbb{S}^n \setminus \{-p\}$. Then $\mathbb{S}^n$ is the union the two **open** subsets U and V. Both U and V are diffeomorphic to $\mathbb{R}^n$ via stereographic projection. By **Lebesgue's number lemma**[1], we can find a subdivision $b_0 < \cdots < b_m$ of the unit interval $[0, 1]$ such that, for each i, the set $\gamma([b_{i-1}, b_i])$ is contained either in U or in V. If there is an index i such that $\gamma(b_i) \notin U \cap V$, i.e., if $\gamma(b_i) \in \{p, -p\}$, then we modify the subdivision as follows: Each of the sets $\gamma([b_{i-1}, b_i])$ and $\gamma([b_i, b_{i+1}])$ is either contained in U or in V. If $\gamma([b_i]) \in U$, then both of these sets must lie in U. Then we may **delete** b_i from our subdivision to obtain a new subdivision $c_0 < \ldots < c_{m-1}$ that still satisfies the condition that $\gamma([c_{i-1}, c_i])$ is contained either in U or in V for each i. If $\gamma([b_i]) \in V$ we proceed similarly. After a **finite** number of steps, this leads to a subdivision $a_0 < \cdots < a_n$ of $[0, 1]$ such that $\gamma([a_{i-1}, a_i])$ is contained either in U or in V and $\gamma(a_i) \in U \cap V$ for each i.

For each i, we choose a path $\gamma_i : [a_{i-1}, a_i] \to \mathbb{S}^n$ as follows: If $\gamma([a_{i-1}, a_i]) \subset U$, we set $\gamma_i = \gamma|_{[a_{i-1},a_i]}$. If $\gamma([a_{i-1}, a_i]) \subset V$, then we let γ_i be a path which is **homotopic** to $\gamma|_{[a_{i-1},a_i]}$ which **avoids** p with $\gamma_i(a_{i-1}) = \gamma(a_{i-1})$ and $\gamma_i(a_i) = \gamma(a_i)$. We can find such a path, since V is **homeomorphic** to $\mathbb{R}^n$. Composing the γ_i's as paths yields a path $[0, 1] \to \mathbb{S}^n$ which is homotopic to γ with the same start- and endpoint and which avoids p. □

However, not all spaces are simply-connected. The following example shows that $\mathbb{S}^1$ is **not** simply-connected. We will see further examples later.

Example 8.19 (The circle is not simply-connected) The **constant** map $f : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$, $p \mapsto (1, 0)$, is **not** homotopic to the map $g : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$, $p \mapsto p$. We are **not yet able** to prove this claim. It often turns out that showing that a homotopy **cannot** exist is much harder than to find a homotopy. We will later develop techniques and invariants that will allow us to handle such problems. In particular, **degrees** and **winding numbers** will help us. There are also other ways to prove the claim. For example, we will show the claim in Exercise 11.2 using Brouwer's fixed point theorem which we will discuss and prove in Section 11.4.

8.3 Homotopy groups

An important and elegant tool to study a space up to homotopy is provided by homotopy groups that we will now define.

Definition 8.20 (Homotopy groups) Let X be a topological space and let $x \in X$ be a point. Let $p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the point with coordinates $(1, 0, \ldots, 0)$. For $n \geq 1$, the set $\pi_n(X, x)$ is defined as the set of equivalence classes of continuous maps $\mathbb{S}^n \to X$ which send p to x modulo homotopy. One can show that $\pi_n(X, x)$ is, in fact, a **group** which is even **abelian** when $n \geq 2$ and does not depend on the choice of the points p and x. We note that the neutral, or **null element**, of $\pi_n(X, x)$ is the class of the **constant map** which sends all of $\mathbb{S}^n$ to $\{x\}$. Even though we will not discuss the construction of the group operation in this course, we will refer to $\pi_n(X, x)$ as the nth **homotopy group** of

[1] A fact from general topology: Let (X, d) be a compact metric space, and let $\mathcal{A}$ be an open cover of X. Then there is a number $\lambda > 0$ such that for every $x \in X$, there is a $U \in \mathcal{A}$ with $B_\lambda(x) \subset U$.

X. For $n = 1$, $\pi_1(X, x) = 0$ is called the **fundamental group** of X.

One may think of $\pi_n(X)$ as an attempt to understand X by studying how we can map a given space, i.e., the sphere, into X. Seen this way the role of $\mathbb{S}^n$ is as some kind of **test space**. However, computing $\pi_n(X)$ may be extremely difficult. In particular, the structure of the homotopy groups of spheres, i.e., the structure of $\pi_k(\mathbb{S}^n)$ is in general unknown. Many of the tools in differential topology have been invented with the goal to compute $\pi_k(\mathbb{S}^n)$. There are, however, many known cases as well.

Example 8.21 The fundamental group of a **simply-connected** space is the trivial group. In particular, Theorem 8.16 implies that the fundamental group of $\mathbb{S}^n$, for all $n \geq 2$, is trivial, i.e., $\pi_1(\mathbb{S}^n) = \{0\}$. On the other hand, Example 8.19 indicates that $\pi_1(\mathbb{S}^1, p)$ is not trivial. In fact, Hopf's Theorem 17.1 shows that $\pi_n(\mathbb{S}^n)$ is isomorphic to the integers for all $n \geq 1$. To prove this result will require a considerable amount of effort and motivates many of the forthcoming constructions. See also Section 16.3.

We will show in Exercise 8.2 that $\pi_k(X)$ is **trivial** for all smooth manifolds of dimension **strictly larger** than k. In particular, we know $\pi_k(\mathbb{S}^n) = 0$ for $k < n$. A much more interesting question is whether $\pi_k(\mathbb{S}^n)$ is trivial or not for $k > n$. For $n = 1$, one can show that $\pi_k(\mathbb{S}^1)$ is the trivial group for all $k \geq 2$. For $n = 2$, we may then ask:

Question Is $\pi_3(\mathbb{S}^2)$ trivial or not?

In fact, we know of an interesting map $f : \mathbb{S}^3 \to \mathbb{S}^2$, namely the **Hopf map**. Hence we may ask the **concrete question** whether the Hopf map is homotopic to a constant map. In Section 12.4.3 we will show that f is **not** homotopic to a constant map. This implies that $\pi_3(\mathbb{S}^2)$ is **not** trivial. This is a famous result due to Heinz Hopf [7] and is one of the highlights of early stages of differential topology. To show this result will require the development of important **invariants**, in particular the Brouwer degree, linking numbers and the Hopf invariant. We should therefore keep this question in mind as a motivation for the forthcoming chapters.

8.4 Stable properties

We will now study the following question:

Question Assume that f_0 has a given property, for example let us say that f_0 is a submersion. If there is a smooth homotopy between f_0 and f_1, do we know that f_1 also is a submersion? In other words, we would **like to know** which properties of maps **are** or are **not invariant** under homotopy.

In fact, **many of the properties** we have studied so far are **not invariant**, i.e., if f_0 has a property P and f_t is a homotopy from f_0 to f_1, then it is often not true that f_1 has property P. For **example**, we could start with an **embedding** f_0 and end up with a **constant map** f_1.

So let us ask a **more modest question**: given f_0 has property P, is there always a **small** $\varepsilon > 0$ such that f_t **has property P for all** $t \in [0, \varepsilon)$? For **example**, if f_0 is an **embedding** there is always a small $\varepsilon > 0$ such that f_t remains an embedding for $0 \le t < \varepsilon$. We will say that being an embedding is **stable**:

Definition 8.22 (Stable properties) A property P of smooth maps is called a **stable** provided that whenever $f_0 : X \to Y$ possesses the property and $f_t : X \to Y$ is a smooth **homotopy** of f_0 then, for some $\varepsilon > 0$, each f_t with $t < \varepsilon$ also possesses property P.

- We also call the maps which have a stable property, a **stable class**. **Examples** are the classes of **embeddings**, **local diffeomorphisms**, **submersions**,... as we will learn soon.
- Note that stability is a very **natural condition** to ask for: For real-world measurements, only stable properties are interesting, since any **tiny perturbation** of the data would make an unstable property appear or disappear.

In order to get a better idea of stability let us look at some examples:

- That a smooth map $f_0 : \mathbb{R} \to \mathbb{R}^2$ **passes through** a fixed point in $\mathbb{R}^2$ is **not** a stable property. It disappears immediately. See Figure 8.4.

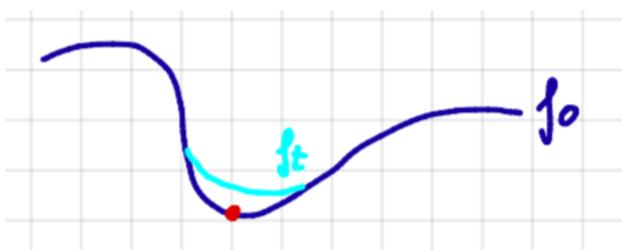

Figure 8.4: Passing through a point is not stable.

- That a smooth map $f_0 : \mathbb{R} \to \mathbb{R}^2$ **merely intersects** the x-axis is **not** a stable property. It disappears immediately. See Figure 8.5.

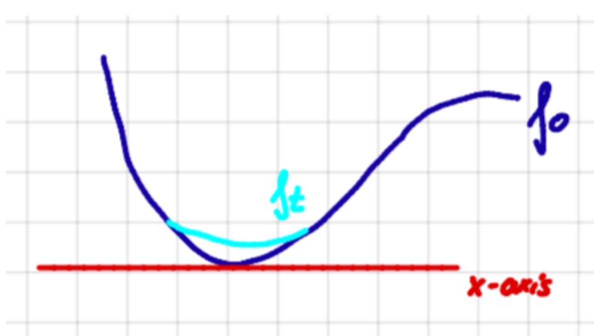

Figure 8.5: Mere intersection is not stable.

- **However**, that a smooth map $f_0 : \mathbb{R} \to \mathbb{R}^2$ intersects the x-axis **transversely** is a **stable** property. It persists after a small perturbation. See Figure 8.6. This reveals yet another very **important feature of transversality**.

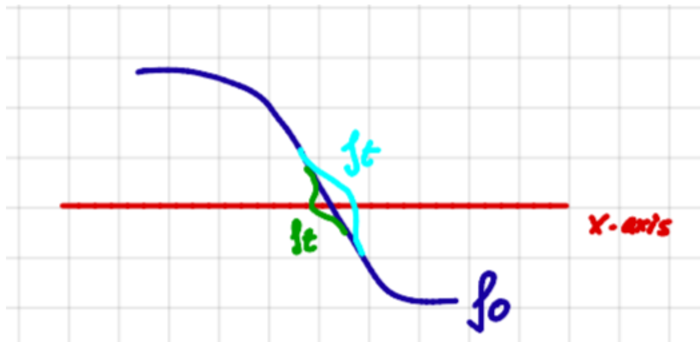

Figure 8.6: Transverse intersection, however, is indeed stable.

- That **two smooth curves** (connected 1-dimensional manifolds) **meet** in $\mathbb{R}^3$ is **not** a stable property. It disappears immediately. See Figure 8.7.

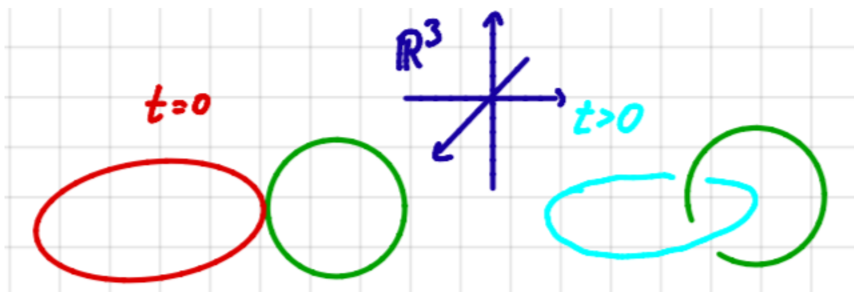

Figure 8.7: Intersection of curves in three-dimensional space is not stable.

- That a **smooth curve** and a **smooth surface** (2-dimensional manifold) **intersect transversely** in $\mathbb{R}^3$ is a **stable** property. It persists after a small perturbation. See Figure 8.8.

The following theorem tells us that the properties which turned out to be useful for us so far are all stable.

Theorem 8.23 **(Stability Theorem)** Let X and Y be smooth manifolds. We assume that X is **compact**. The following classes of smooth maps from X to Y are **stable classes**:

(a) local diffeomorphisms,

(b) immersions,

(c) submersions,

(d) maps which are transversal to any fixed **closed** submanifold $Z \subset Y$,

(e) embeddings,

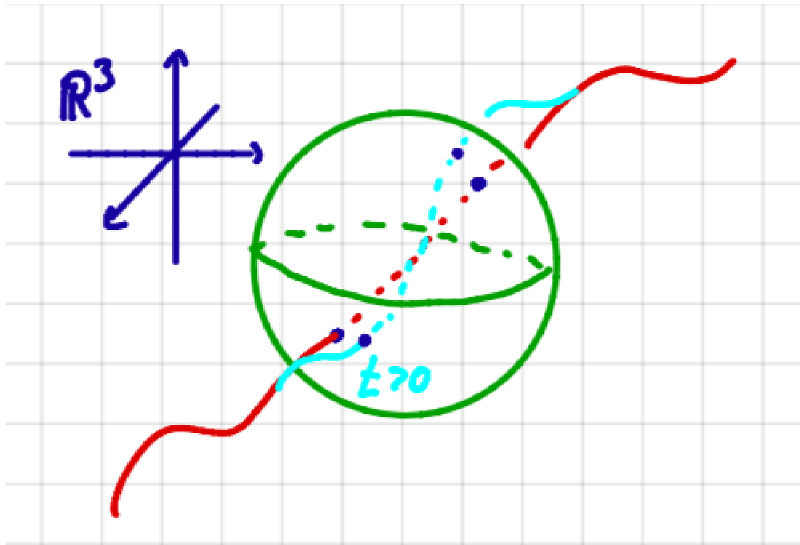

Figure 8.8: Intersection of a surface and a curve in three-dimensional space is not stable. The dimensions have to add up.

(f) diffeomorphisms.

Note that the assumption that X is **compact** is crucial and not just made for convenience. The next example will show that we **cannot drop** this assumption for any of the properties in theorem:

Example 8.24 (Compactness matters) The assertion of the Stability Theorem 8.23 **fails** to be true in general when X is **not compact**. For a simple **example**, let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\rho(s) = 1$ for $|s| < 1$ and $\rho(s) = 0$ for $|s| > 2$. Then we define

$$f_t : \mathbb{R} \to \mathbb{R}, \ f_t(x) = x\rho(tx).$$

For $t = 0$, $f_0(x) = x$ for all x, i.e., $f_0 = \mathrm{Id}$. Hence f_0 is a **local diffeomorphism**, an **immersion**, a **submersion**, an **embedding**, a **diffeomorphism** and **transversal** to every submanifold of $\mathbb{R}$.

However, for **any fixed** $t > 0$, we have $|tx| > 2$ when $x > 2/|t|$. Hence, for this fixed t, $f_t(x) = 0$ **for all** $x > 2/|t|$.

Thus f_t is **neither** a local diffeomorphism, an immersion, a submersion, an embedding, nor a diffeomorphism, and is not transverse to $\{0\} \subset \mathbb{R}$.

To emphasise **what is going wrong**, let us replace the domain with a **closed** interval, say $X = [a, b]$ with $b > 0$, which is a **compact** subspace of $\mathbb{R}$. Then we **can choose** $\varepsilon > 0$ which is **small enough** such that $1/\varepsilon > \max(|a|, |b|)$. This implies that x is not bigger than $1/|t|$. Then we have $f_t(x) = x$ for all x and all $t < \varepsilon$.

Proof of Theorem 8.23:

(a) First we note that **local diffeomorphisms** are just immersions in the special case when

$\dim X = \dim Y$, so (a) follows from (b).

(b) Assume $f_0 : X \to Y$ is an **immersion** and $\dim X = m$. Letf_t be a homotopy of f_0. That f_0 is an immersion means that **$d(f_0)_x$ is injective** for all $x \in X$. We need to show that there is an $\epsilon > 0$ such that **$d(f_t)_x$ is injective** for all points (x, t) in $X \times [0, \varepsilon) \subset X \times I$:

Given a point $x_0 \in X$, that $d(f_0)_{x_0}$ is injective implies that the matrix representing $d(f_0)_{x_0}$ (in local coordinates) has an $m{\times}m$-**submatrix** $A(x_0, 0)$ with **nonvanishing determinant**. Since the **determinant is continuous**, this submatrix will have **nonvanishing determinant** in an **open** neighborhood of $(x_0, 0)$ in $X \times [0, 1]$. Since X is **compact**, **finitely** many such neighborhoods suffice to **cover all of $X \times \{0\}$**. Hence there is a small $\varepsilon > 0$ (it is the **minimum** for the open intervals $[0, \varepsilon_i)$ covering $\{0\}$) such that the **intersection** of these finitely many neighborhoods **contains $X \times [0, \varepsilon)$**. Thus **$d(f_t)_x$ is injective** for all $(x, t) \in X \times [0, \varepsilon)$. This is what we needed.

(c) If f_0 is a **submersion**, almost the **same argument** works. We just need to choose an $n \times n$-submatrix of the **surjective** map $d(f_0)_x$ with $n = \dim Y$.

(d) Let $Z \subset Y$ be a **closed** submanifold, and assume that f_0 is a map which is **transversal to Z**. Then we have shown that, for every point $x \in X$, there is a smooth function g which sends a neighborhood of $f(x)$ to $0 \in \mathbb{R}^{\operatorname{codim} Z}$ and such that **$g \circ f_0$ is a submersion**. Since Z is closed in Y, **$f^{-1}(Z)$ is closed in X** and therefore **also compact**. Therefore, **by (c)**, there is an $\varepsilon > 0$ such that **$g \circ f_t$ is still a submersion** for all $t < \varepsilon$. This is means that **f_t is still transversal to Z** for all $t < \varepsilon$.

(e) Assume that f_0 is an **embedding**, and let f_t be a homotopy of f_0. Since **X is compact**, f_0 and each f_t are automatically **proper maps**. Hence we need to show that when f_0 is a one-to-one immersion, then so is f_t in a small neighborhood. We just checked that **being an immersion is stable**. Hence it remains to show that **f_t is still one-to-one if t is small enough**.

Therefor we define a smooth map

$$G : X \times I \to Y \times I, \; G(x, t) := (f_t(x), t).$$

Then **if (e) was false**, i.e., if f_t was **not one-to-one** in some small neighborhood of 0, then, for every $\varepsilon > 0$, we can find a t with $0 < t < \varepsilon$ and $x, y \in X$ such that $f_t(x) = f_t(y)$. For example, for every $\varepsilon_i = 1/i$, we could find such a t_i, x_i and y_i. Thus there is an **infinite sequence $t_i \to 0$**, and an **infinite sequence** of points $x_i \neq y_i \in X$ where **f_{t_i} fails to be injective**, i.e., such that

$$f_{t_i}(x_i) = G(x_i, t_i) = G(y_i, t_i) = f_{t_i}(y_i).$$

Since **X is compact**, we may pass to **subsequences which converge** $x_i \to x_0$ and $y_i \to y_0$. Since G is **continuous**, this implies

$$G(x_0, 0) = \lim_i G(x_i, t_i) = \lim_i G(y_i, t_i) = G(y_0, 0).$$

But $G(x_0, 0) = f_0(x_0)$ and $G(y_0, 0) = f_0(x_0)$. By assumption, **f_0 is injective**, and **hence $x_0 = y_0$**.

Now, **after choosing local coordinates**, we can express the **derivative of G** at $(x_0, 0)$ by the matrix

$$dG_{(x_0,0)} = \begin{pmatrix} & * \\ d(f_0)_{x_0} & \vdots \\ & * \\ 0 \cdots 0 & 1 \end{pmatrix}$$

where the 0s in the lowest row arise from the fact that the **first coordinates do not depend on** t, and the 1 in the bottom right hand corner is the derivative of the function $t \mapsto t$.

Since f_0 is an **immersion**, $d(f_0)_{x_0}$ has $k = \dim X$ many **independent rows**. Thus the matrix of $dG_{(x_0,0)}$ has $k+1$ **independent rows**, and hence $dG_{(x_0,0)}$ is an **injective linear map**. In other words, G is an **immersion around** $(x_0, 0)$ and hence G must be **one-to-one on some neighborhood of** $(x_0, 0)$.

But we have shown above that the sequences (x_i, t_i) and (y_i, t_i) both **converge to** $(x_0, 0)$. This means that for l**arge** i **both** (x_i, t_i) and (y_i, t_i) belong to this neighborhood. This **contradicts the injectivity of** G.

(f) Assume that $f_0 : X \to Y$ is a **diffeomorphism**. Since X is **compact**, this implies that Y **is compact as well**. Let f_t be a homotopy of f_0. We need to show that there is an $\varepsilon > 0$ such that f_t is diffeomorphism for all $0 \leq t < \varepsilon$.

Since X **is compact**, X has only **finitely many connected components**, and so does Y. Hence we can check the statement for each of these connected components separately. For, this gives us an ε_i for each component. Since there are **finitely many** components, we can just take the **minimum** of the ϵ_i's as the ϵ for all of X and Y.

Thus we may assume that X **and** Y **are connected**. By (a) and (e), we know that being a **local diffeomorphism** and being an **embedding** is a **stable** property. Thus there is a $\varepsilon > 0$ such that f_t is a local diffeomorphism and an embedding. For f_t being a diffeomorphism, it remains to show that f_t **is surjective**.

We fix a $0 < t < \varepsilon$. Since f_t is a **local diffeomorphism**, f_t is an **open map** and hence $f_t(X)$ **is open in** Y. But $f_t(X)$ is **also closed**, since it is **compact** being the image of a compact space. Since Y is **connected**, this implies $f_t(X) = Y$. □

8.5 Exercises and more examples

Exercise 8.1 A manifold X is called *contractible* if its identity map is homotopic to some constant map $X \to \{x\}$ where x is any point of X.

(a) Show that if X is contractible, then all smooth maps $Y \to X$ from an arbitrary manifold Y to X are homotopic.

(b) Conversely, show that if all maps of an arbitrary manifold Y to X are homotopic, then X is contractible.

(c) Show that $\mathbb{R}^k$ is contractible.

Exercise 8.2 Let X be a smooth manifold of dimension k. If $k < n$, show that every *smooth* map $f : X \to \mathbb{S}^n$ is homotopic to a constant map.

Hint: Use Sard's Theorem 7.1.

Exercise 8.3 Show that the antipodal map $\mathbb{S}^k \to \mathbb{S}^k$, $x \mapsto -x$, is homotopic to the identity if k is odd. (We will see later that this is not true if n is even.)

Hint: Start off with $k = 1$ by using the linear maps defined by

$$[0,1] \to M(2),\ t \mapsto \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}.$$

Exercise 8.4 Show that all contractible manifolds are simply-connected.

Note that the converse is false. As an example consider $X = \mathbb{S}^2$. The 2-sphere is simply connected, but it is not contractible. For example, the antipodal map is not homotopic to the identity. We will have to develop further techniques to be able to check this.

Exercise 8.5 Show that every connected smooth manifold X is *path-connected*, i.e., given any two points $x_0, x_1 \in X$, there exists a smooth map $f : [0,1] \to X$ with $f(0) = x_0$ and $f(1) = x_1$.

Hint: Use the fact that homotopy is an equivalence relation to show that the relation *x_0 and x_1 can be joined by a smooth curve* is an equivalence relation on X. Then show that the equivalence classes are both open and closed subsets of X.

Exercise 8.6 Let X be a smooth manifold. Let $f, g : X \to \mathbb{S}^k$ be two continuous maps such that $|f(x) - g(x)| < 2$ for all $x \in X$, where the norm is taken as elements in $\mathbb{R}^{k+1}$.

(a) Show that f and g are homotopic.

Hint: What does the assumption say geometrically about $f(x)$ and $g(x)$?

(b) Show that f and g are smooth, then they are smoothly homotopic.

Exercise 8.7 Let $X \subset \mathbb{R}^N$ be a smooth manifold. A *vector field* on X is a smooth section of $\pi : T(X) \to X$, i.e., a smooth map $\sigma : X \to T(X)$ such that $\pi \circ \sigma = \mathrm{Id}_X$. An equivalent way to describe such a section is to give a map $s : X \to \mathbb{R}^N$ such that $s(x) \in T_x(X) \subset \mathbb{R}^N$ for all x (with corresponding $\sigma(x) = (x, s(x))$). A point $x \in X$ is a zero of the vector field σ if $\sigma(x) = (x, 0)$ or equivalently $s(x) = 0$.

(a) Show that if k is odd, there exists a vector field on $\mathbb{S}^k$ having no zeros.

Hint: For $k = 1$, use $(x_1, x_2) \mapsto (-x_2, x_1)$.

(b) Prove that if $\mathbb{S}^k$ has a vector field which has no zeros, then its antipodal map $x \mapsto -x$ is homotopic to the identity.

Hint: Show that you may assume $|s(x)| = 1$ everywhere. Now contemplate about $(\cos(\pi t))x + (\sin(\pi t))s(x)$ when t varies from 0 to 1.

(c) Show that if k is even, then the antipodal map on $\mathbb{S}^k$ is homotopic to the reflection map

$$r : \mathbb{S}^k \to \mathbb{S}^k, (x_1, \dots, x_{k+1}) \mapsto (-x_1, x_2, \dots, x_{k+1}).$$

Hint: Consider also the reflections

$$r_i(x_1, \dots, x_{k+1}) = (x_1, \dots, -x_i, \dots, x_{k+1}).$$

Show that $r_i \circ r_{i+1}$ is homotopic to the identity on $\mathbb{S}^k$.

9. Abstract Smooth Manifolds

9.1 Abstract manifolds - the definition

We would like to define manifolds without referring to a given embedding into some $\mathbb{R}^N$. The key idea that should be preserved in any new definition is that a manifold is a space which **locally looks like Euclidean space**. First, we recall an important concept from topology:

Definition 9.1 (Hausdorff spaces) A topological space X is called **Hausdorff** if, for any two distinct points $x, y \in X$, there are two **disjoint** open subsets $U, V \subset X$ such that $x \in U$ and $y \in V$. In other words, in a Hausdorff space we can **separate points by open neighborhoods**.

- Every **subspace** of $\mathbb{R}^N$ with the relative topology is a **Hausdorff** space.
- **However**, there are spaces which are **not** Hausdorff. For a typical **example** of a space which is **not** Hausdorff, consider two copies of the real line $Y_1 := \mathbb{R} \times \{1\}$ and $Y_2 := \mathbb{R} \times \{2\}$ as subspaces of $\mathbb{R}^2$. On $Y_1 \cup Y_2$, we define the equivalence relation $(x, 1) \sim (x, 2)$ for all $x \neq 0$. Let X be the set of equivalence classes. In fact, X looks like the real line except that the origin is replaced with **two different copies of the origin**:

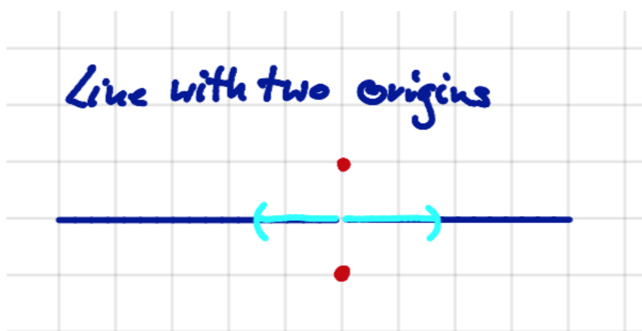

Figure 9.1: A space that looks like a line with a double-point. The topology is such that we cannot separate the two points by open subsets.

The topology on X is the **quotient topology**: a subset $W \subset X$ is **open** in X if and only if **both its preimages** in $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{2\}$ are **open**. Away from the double origin, X looks perfectly nice like a one-dimensional manifold. But every neighborhood of one of the origins has a non-empty intersection with any neighborhood of the other origin. Hence we **cannot separate** the two origins by open subsets, and X is **not Hausdorff**.

In the definition of an abstract manifold, we want to avoid such pathological spaces. There are several reasons for this choice. First of all, manifolds are characterised by how open neighborhoods of points look like, and we would like to be able to find separate neighborhoods for distinct points. That is exactly the Hausdorff property. But there are also slightly deeper reasons. For example, we would like to use the fact that a compact subset Z of a closed subset $Y \subset X$ is itself closed in X. This general conclusion requires that X is a Hausdorff space.

Definition 9.2 (Charts) Let X be a topological space. A **chart** on X is a **pair** (V, ψ) where $V \subset X$ is an open subset and $\psi : V \to U$ is a **homeomorphism** from V to an open subset $U \subset \mathbb{R}^k$.

Now we can define manifolds:

Definition 9.3 (Abstract manifolds) An **abstract smooth k-manifold** is a **Hausdorff** and **second-countable** space X, i.e., there exists a countable basis of the topology, together with a **collection of charts** (V_α, ψ_α) on X such that

- every point in X is in the domain of some chart, and
- for every pair of overlapping charts ψ_α and ψ_β, i.e.,

$$V_{\alpha\beta} := V_\alpha \cap V_\beta \neq \emptyset,$$

the **change-of-coordinates map**

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(V_{\alpha\beta}) \to \psi_\beta(V_{\alpha\beta})$$

is **smooth** as a map between open subsets of $\mathbb{R}^k$. In fact, this means that the change-of-coordinates maps are **diffeomorphisms**, since they are mutual smooth inverses to each other.

Definition 9.4 (Smooth maps between abstract manifolds) Let X be an **abstract smooth k-manifold**.

- A continuous map $f : X \to \mathbb{R}^n$ is called **smooth** if for every chart $\psi_\alpha : V_\alpha \to U_\alpha$, the composition

$$f \circ \psi_\alpha^{-1} : U_\alpha \to \mathbb{R}^n$$

is **smooth** as a map from an **open** subset of $\mathbb{R}^k$ to $\mathbb{R}^n$.

- More generally, let Y an **abstract smooth n-manifold** and $f : X \to Y$ a **continuous map**. Then **f is smooth at $x \in X$** if, for every chart $\psi^X : V \to U$ on X around x and every chart $\psi^Y : V' \to U'$ on Y around $f(x)$, the map

$$\psi^Y \circ f_{|V \cap f^{-1}(V')} \circ (\psi^X)^{-1}_{|U \cap \psi^X(V \cap f^{-1}(V'))} : U \cap \psi^X(V \cap f^{-1}(V')) \to U'$$

is a smooth map as a map from an open subset of $\mathbb{R}^k$ to an open subset of $\mathbb{R}^n$. We call f **smooth** if it is smooth at every $x \in X$.

Note that the smooth k-dimensional manifolds $X \subset \mathbb{R}^N$ we have been studying so far are

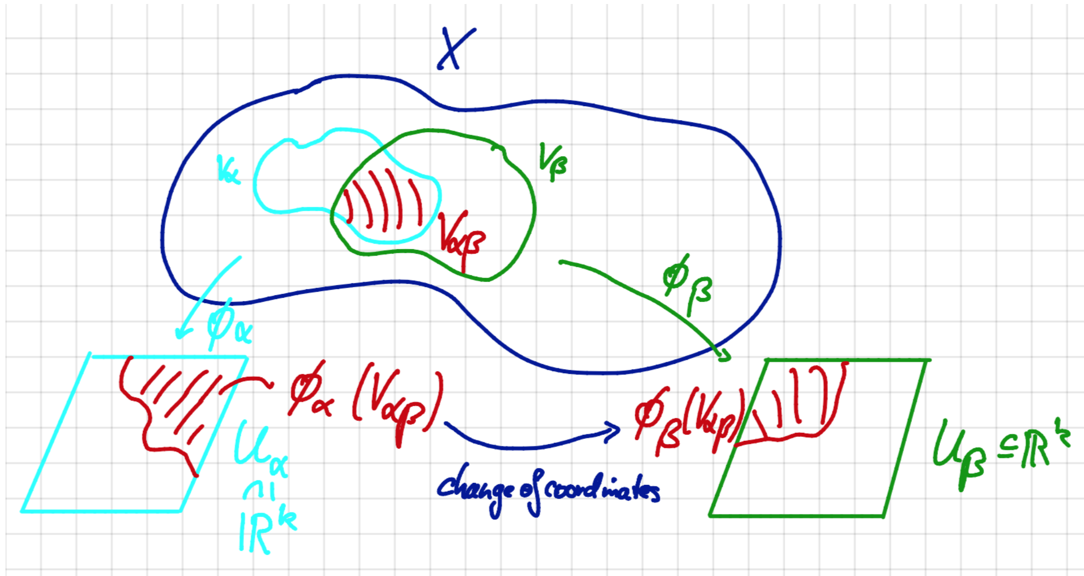

Figure 9.2: On a manifold every point is contained in the domain of a chart. If two charts overlap we can look at the composition of the chart maps and get a map between open subsets in $\mathbb{R}^k$. We call this the **change-of-coordinate** map and require this map to be **smooth** in the usual sense.

examples of abstract smooth k-manifolds:

- The **Hausdorff** property is satisfied in $\mathbb{R}^N$ and therefore also for every subspace of $\mathbb{R}^N$ with the relative subspace topology.

- Moreover, every open cover $\{U_\alpha\}$ of $\mathbb{R}^N$ has a **countable refinement**. For, we can take the collection of all open balls which are contained in some U_α, which have **rational radii**, and which are centred at points having **only rational coordinates**.

- For an open cover $\{V_\alpha\}$ of a subset $X \subset \mathbb{R}^N$, we can write $V_\alpha = U_\alpha \cap X$ for some open subsets U_α of $\mathbb{R}^N$. Then let $\{\tilde{U}_i\}$ be a **countable refinement** of $\{U_\alpha\}$ in R^N, and define $\tilde{V}_i = \tilde{U}_i \cap X$.

- The **charts** are just what we called **local coordinates** and the inverses of charts are what we called **local parametrizations**. One difference is that we required local parametrizations to be diffeomorphisms. For an abstract manifold X, we use the charts to **define what smoothness means** for a map on X. Hence **a priori** it makes only sense to talk about the smoothness of the change of coordinate maps. **A posteriori** we can then check that charts are in fact diffeomorphisms.

- Similarly for **smooth maps** between manifolds. We only know what smoothness of maps between Euclidean spaces means. Hence we need to use the charts to first translate the maps into maps between Euclidean spaces.

- In the abstract definition, we take care of the fact that the images of the charts/local parametrizations **overlap**. In fact, we use the overlap to **define** the smooth structure.

- Finally, a chosen collection of charts is called an **atlas** on the manifold. One can show that every manifold has a **maximal atlas**, i.e., the images of the charts are as **big as possible**.

Luckily, our initial definition fits nicely into this picture:

Lemma 9.5 (Smooth manifolds are also abstract manifolds) Let X be a smooth manifold according to our initial definition. Then X is also an **abstract smooth manifold**.

Proof: Subspaces in $\mathbb{R}^N$ are **Hausdorff**. By our definition of a smooth manifold, we can cover X by the open subsets associated to local parametrizations. Since $X \subset \mathbb{R}^N$, it suffices to use **countable many** such open sets to cover X. It remains to show that X has charts.

Let $x \in X$ be a point in X and let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be local parametrizations around x with open subsets $U_1, U_2 \subset \mathbb{R}^k$ and $V_1, V_2 \subset X$. The overlap of V_1 and V_2 is **not empty**, since it contains x. Moreover, $V_1 \cap V_2$ is an open subset of X. Since both ϕ_1 and ϕ_2 are **homeomorphisms**, $\phi_1^{-1}(V_1 \cap V_2) \subset U_1$ and $\phi_2^{-1}(V_1 \cap V_2) \subset U_2$ are **both open subsets** in $\mathbb{R}^k$. Since **both ϕ_1 and ϕ_2 are diffeomorphisms**, the **transition map**

$$\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(V_1 \cap V_2) \to \phi_2^{-1}(V_1 \cap V_2)$$

is a **diffeomorphism** between open subsets in $\mathbb{R}^k$. Thus our local parametrizations (or rather our local coordinate systems) do equip X with the structure of an abstract smooth manifold. □

Remark 9.6 (It all fits together) It is **nice** and important to have such an **intrinsic definition** of a manifold. However, the definition is **quite abstract indeed**. And, in fact, we are going to show that **every abstract smooth manifold can be embedded into Euclidean space** and is therefore a manifold for our previous definition. Hence all the machinery we have developed can be applied to abstract manifolds.

9.1.1 Tangent space of an abstract smooth manifold

Abstract smooth manifolds also have tangent spaces. There are several different, though equivalent, ways to define the tangent space of an abstract manifold. We will look at just one way which is closest to our intuitive and concrete approach. In addition, we have seen it in earlier exercises.

Let X be an abstract smooth manifold of dimension k and let $x \in X$ be a point. We consider the set of all **smooth curves through x** on X, i.e., the set of all smooth maps $\gamma : \mathbb{R} \to X$ with $\gamma(0) = x$. On this set we define the following **equivalence relation**:

- Let $\gamma_1 : \mathbb{R} \to X$ and $\gamma_2 : \mathbb{R} \to X$ be two smooth curves with $\gamma_1(0) = x = \gamma_2(0)$. We consider γ_1 and γ_2 to be equivalent, written $\gamma_1 \sim \gamma_2$, if **for all charts** $\psi : V \to U$ with $x \in V$ and $U \subset \mathbb{R}^k$ we have an **equality of derivatives** at 0

$$(\psi \circ \gamma_1)'(0) = (\psi \circ \gamma_2)'(0)$$

where the derivative is taken as maps $\mathbb{R} \to \mathbb{R}^k$. Hence $(\psi\circ\gamma_1)'(0) = (\psi\circ\gamma_2)'(0)$ is just an equality of vectors in $\mathbb{R}^k$.

Definition 9.7 (Abstract tangent spaces) The **tangent space** of X at x, denoted $T_x(X)$, is defined as the set of all **equivalence classes of curves** through x.

- Actually, it is not necessary that the curves γ are defined on all of $\mathbb{R}$. It suffices that there is an open subset $W \subset \mathbb{R}$ containing 0 and $\gamma : W \to X$ is a smooth map defined on W with $\gamma(0) = x$. In a more sophisticated terminology, it suffices to consider the set of **germs** of curves through x. Then we consider the set of all such curves and the equivalence relation only requires that the derivatives of the composite with any chart are equal. Anyway, we see that $T_x(X)$ only depends on the **local structure** of X at x. □

9.2 Real projective space

This is an important example which we can easily be described with the new definition of an abstract manifold, but for which it is **not obvious** at all how we can embed it into $\mathbb{R}^N$.

Definition 9.8 (Real Projective Space) The **real projective n-space $\mathbb{R}\mathrm{P}^n$** is the set of all straight lines through the origin in $\mathbb{R}^{n+1}$. As a topological space, $\mathbb{R}\mathrm{P}^n$ is the **quotient space**

$$\mathbb{R}\mathrm{P}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim$$

where the equivalence relation is given by $x \sim y$ if there is a nonzero real number λ such that $x = \lambda y$. This means that a **subset V is open in $\mathbb{R}\mathrm{P}^n$** if and only if its preimage $U = \left\{x \in \mathbb{R}^{n+1} \setminus \{0\} : [x] \in V\right\}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.

- Note that each line through the origin intersects the unit sphere in two antipodal points. Hence $\mathbb{R}\mathrm{P}^n$ can also be described as $\mathbb{S}^n/\sim$ where the equivalence relation is given by $x \sim -x$. As a quotient of $\mathbb{S}^n$, we see that $\mathbb{R}\mathrm{P}^n$ is actually **compact**. As we have seen, this is always very good to know about a space.

Theorem 9.9 ($\mathbb{R}\mathrm{P}^n$ is a smooth manifold) Real projective n-space $\mathbb{R}\mathrm{P}^n$ is an abstract **n-dimensional smooth manifold**.

Proof: If $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we write $[x]$ for its equivalence class considered as a point in $\mathbb{R}\mathrm{P}^n$. One also often writes $[x] = [x_0 : \dots : x_n]$.

For $0 \leq i \leq n$, let

$$V_i := \left\{[x] \in \mathbb{R}\mathrm{P}^n : x_i \neq 0\right\}.$$

The preimage of V_i in $\mathbb{R}^{n+1}$ is the open subset $\{x \in \mathbb{R}^{n+1} : x_i \neq 0\}$. Hence **each V_i is open** in $\mathbb{R}\mathrm{P}^n$. By varying i, this gives an **open cover** of $\mathbb{R}\mathrm{P}^n$ because **every representative** $(x_0, \dots, x_n)$

of a point $[x] \in \mathbb{R}\mathrm{P}^n$ must have at least **one coordinate** $\neq 0$ (otherwise it would be the origin which is excluded).

For each i, we have the maps $\phi_i : \mathbb{R}^n \to V_i$

$$\left(x_0, \dots, \widehat{x_i}, \dots, x_n\right) \mapsto \left[x_0 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n\right].$$

and $\phi_i^{-1} : V_i \to \mathbb{R}^n$

$$\left[x_0 : \dots : x_i : \dots : x_n\right] \mapsto \frac{1}{x_i}\left(x_0, \dots, \widehat{x_i}, \dots, x_n\right)$$

where $\widehat{x_i}$ means that x_i is **omitted**. Note that the quotient $\frac{1}{x_i}$ is well-defined on all of V_i.

Since we use a representative of an equivalence class for the definition of ϕ_i^{-1}, we **need to check** that the definition is **independent of the chosen representative**:
If $[x_0 : \dots : x_i : \dots : x_n] = [\lambda x_0 : \dots : \lambda x_i : \dots : \lambda x_n]$ for some $\lambda \neq 0$, then

$$\begin{aligned}
\phi_i^{-1}([\lambda x]) &= \frac{1}{\lambda x_i}(\lambda x_0, \dots, \lambda x_{i-1}, \lambda x_{i+1}, \dots, \lambda x_n) \\
&= \frac{1}{x_i}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \phi_k^{-1}([x]).
\end{aligned}$$

It is easy to see that ϕ_i and ϕ_i^{-1} are **mutual inverses** which are **both continuous**.

Finally, the **change-of-coordinate** maps are **smooth**: For the composite

$$\phi_i^{-1}(V_i \cap V_j) \xrightarrow{\phi_i} V_i \cap V_j \xrightarrow{\phi_j^{-1}} \phi_j^{-1}(V_i \cap V_j)$$

is just

$$\left(x_0, \dots, \widehat{x_i}, \dots, x_n\right) \mapsto \frac{1}{x_j}\left(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, \widehat{x_j}, \dots, x_n\right)$$

which is **smooth** whenever $x_j \neq 0$.[1] □

Remark 9.10 (Tangent space of $\mathbb{R}\mathrm{P}^n$) **Intuitively**, we can think of the tangent space of $\mathbb{R}\mathrm{P}^n$ at a point $[x] \in \mathbb{R}\mathrm{P}^n$ as follows: We can consider the point $[x]$ as a pair of antipodal points $(x, -x)$ in $\mathbb{S}^n$. Then a tangent vector at $[x]$ may be viewed as a **pair** of **'antipodal vectors'** $(v, -v)$ where $v \in T_x\mathbb{S}^n$ and $-v \in T_{-x}\mathbb{S}^n$. To write $-v$ may be justified by identifying both $T_x\mathbb{S}^n$ and $T_{-x}\mathbb{S}^n$ with the subspace in $\mathbb{R}^{n+1}$ which is **orthogonal** to the line L through x and $-x$:

$$T_{[x]}\mathbb{R}\mathrm{P}^n = \{(v, -v) : v \in L^\perp = T_x\mathbb{S}^n \subset \mathbb{R}^{n+1}\}. \tag{9.1}$$

In Remark 9.21, and in the arguments leading up to it, we will see that there is a more precise and canonical description of the tangent space of $\mathbb{R}\mathrm{P}^n$ at a point $[x] \in \mathbb{R}\mathrm{P}^n$ as

$$T_{[x]}\mathbb{R}\mathrm{P}^n = \mathrm{Hom}_{\mathbb{R}}(L, L^\perp) \tag{9.2}$$

[1] Our notation seems to indicate that j must be bigger than i. But this is of course not the case. Both cases $i < j$ and $i > j$ work out in the same way.

where $L \subset \mathbb{R}^{n+1}$ is the line through the origin determined by $[x]$ and $L^{\perp}$ denotes the orthogonal complement of L in $\mathbb{R}^{n+1}$.

There are many reasons why real projective space is important. One is that it comes equipped with a very useful additional structure:

Remark 9.11 **(Canonical line bundle)** We discussed vector bundles briefly in Section 2.5.3. One class of vector bundles are line bundles, i.e., each vector space at a point is one-dimensional. Real projective n-space has a **canonical line bundle**, often also called the tautological line bundle. A point in $\mathbb{R}\mathrm{P}^n$ consists of a line L in $\mathbb{R}^{n+1}$ through the origin. Considering L both as a point in $\mathbb{R}\mathrm{P}^n$ and as a one-dimensional vector space defines a **line bundle** on $\mathbb{R}\mathrm{P}^n$.

It turns out that for every sufficiently nice topological space Y with a line bundle $\mathcal{L} \to Y$, there is a continuous map $Y \to \mathbb{R}\mathrm{P}^n$ (for n large enough) such that $\mathcal{L}$ is the pullback of the canonical line bundle along this map. Up to homotopy, this map is in fact **unique**. We refer to this phenomenon as that $\mathbb{R}\mathrm{P}^n$ is a **classifying space for line bundles**. The idea is classifying spaces is very powerful and important.

Another, more geometric reason for considering projective spaces is the following:

Remark 9.12 **(Intersection at ∞)** A **plane** V in $\mathbb{R}^3$ can be described as the orthogonal complement of given vector $v \neq 0$ in $\mathbb{R}^3$:

$$V = \left\{(x_0, x_1, x_2) \in \mathbb{R}^3 : x_0 v_0 + x_1 v_1 + x_2 v_2 = 0\right\}.$$

Since multiplying the equation $x_0v_0 + x_1v_1 + x_2v_2 = 0$ with a **nonzero** real number does **not change** the set of solutions, we can consider the equivalence classes in $\mathbb{R}\mathrm{P}^2$ of the points of V. This gives us a **line $\mathcal{L}$ in $\mathbb{R}\mathrm{P}^2$**:

$$\mathcal{L} = \left\{[x_0 : x_1 : x_2] \in \mathbb{R}\mathrm{P}^2 : x_0 v_0 + x_1 v_1 + x_2 v_2 = 0\right\}.$$

In fact, **every line** in $\mathbb{R}\mathrm{P}^2$ is represented by a plane through the origin in $\mathbb{R}^3$ and is hence **determined by a nonzero** vector v in $\mathbb{R}^3$. Moreover, v and λv with $\lambda \neq 0$ determine the same line.

Now assume we are given two distinct lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbb{R}\mathrm{P}^2$ determined by two distinct vectors $v, w \neq 0$ in $\mathbb{R}^3$, i.e.,

$$\begin{aligned}
\mathcal{L}_1 &= \left\{[x_0 : x_1 : x_2] \in \mathbb{R}\mathrm{P}^2 : x_0 v_0 + x_1 v_1 + x_2 v_2 = 0\right\} \\
\mathcal{L}_2 &= \left\{[x_0 : x_1 : x_2] \in \mathbb{R}\mathrm{P}^2 : x_0 w_0 + x_1 w_1 + x_2 w_2 = 0\right\}.
\end{aligned}$$

The **orthogonal complements** of v and w, respectively, are **two planes through the origin**. Hence **they meet in a line through the origin** in $\mathbb{R}^3$ which is the set of solutions of the two linear equations defining $\mathcal{L}_1$ and $\mathcal{L}_2$ above. This is a **one-dimensional** vector subspace of $\mathbb{R}^3$ (the kernel of a 2×3-matrix). By definition of $\mathbb{R}\mathrm{P}^2$, this subspace corresponds to a **point** in $\mathbb{R}\mathrm{P}^2$. This is the **intersection point** of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbb{R}\mathrm{P}^2$. If this intersection line happens to be the z-axis, i.e., when $\mathcal{L}_1$ and $\mathcal{L}_2$ are represented by the planes given by the xz-plane and the yz-plane, then the intersection point is $[0 : 0 : 1] \in \mathbb{R}\mathrm{P}^2$. We can think of it as the **point at infinity in $\mathbb{R}\mathrm{P}^2$**.

In the **Euclidean plane** $\mathbb{R}^2$, however, it **may** very well **happen** that two lines are **parallel** and hence do **not intersect**. The idea for $\mathbb{RP}^2$ is to add a **point at infinity** which is the **intersection point** for all **parallel lines**.

9.3 Torus and Klein bottle

9.3.1 The torus as an abstract manifold

We already know the torus as an important example of a two-dimensional compact manifold. So far we have constructed it via coordinates in $\mathbb{R}^3$ or as a product $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{R}^4$. Here is another way to construct a two-dimensional torus $\mathbb{T}^2$ without referring to an ambient space:

On the product $\mathbb{R} \times \mathbb{R}$ we consider the relation

$$(x_1, y_1) \sim (x_2, y_2) \iff x_1 - x_2 \in \mathbb{Z} \text{ and } y_1 - y_2 \in \mathbb{Z}.$$

We readily verify that this is an **equivalence relation**. W equip the **quotient space** with the **quotient topology** and can check that it is just the torus

$$\frac{\mathbb{R} \times \mathbb{R}}{\sim} \cong \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2.$$

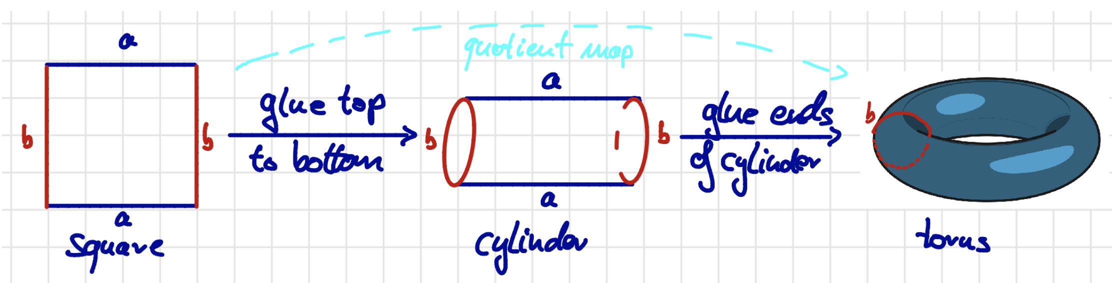

Figure 9.3: The torus $\mathbb{T}^2$ can be constructed by gluing together the opposite edges of a square.

However, let us have a closer look because we would like to use this picture to equip $\mathbb{T}$ with the structure of a manifold:

Claim: The quotient map

$$q\colon \mathbb{R} \times \mathbb{R} \to \frac{\mathbb{R} \times \mathbb{R}}{\sim}$$

is an **open map**, i.e., q sends open subsets to open subsets.

Let $U \subset \mathbb{R} \times \mathbb{R}$ be open. By definition of the quotient topology, the image $V = q(U)$ is open in $\mathbb{T}$ if and only if $q^{-1}(V) = q^{-1}(q(U))$ is open in $\mathbb{R} \times \mathbb{R}$. The latter subset is

$$q^{-1}(V) = q^{-1}(q(U)) = \bigcup_{n,m \in \mathbb{Z}} U_{(n,m)} \text{ with } U_{(n,m)} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x - n, y - m) \in U\}.$$

This is a **union of open subsets**, each **homeomorphic to** U, and hence open in $\mathbb{R} \times \mathbb{R}$. $\square$

Now we can check that $\mathbb{T} = \frac{\mathbb{R}\times\mathbb{R}}{\sim}$ is a **Hausdorff** space. Moreover, it has a countable basis for its topology. Let us just accept these facts. For we are much more interested in the **charts** that turn $\mathbb{T}^2$ into a **manifold**. In fact, defining these charts is **almost trivial**:

Let $(x, y) \in \mathbb{R} \times \mathbb{R}$ and let $[x, y]$ be its equivalence class in $\mathbb{T}$. Let $U_r = \mathbb{B}_r^2(x, y) \subset \mathbb{R} \times \mathbb{R}$ be the **open ball** of radius $r > 0$ around (x, y). Then the map

$$q_{|U_r} : U_r \to \frac{\mathbb{R} \times \mathbb{R}}{\sim}$$

is a **homeomorphism onto its image** for all $r < \frac{1}{2}$. For it is surjective and it is injective, since for any two points in U_r the difference of the x- or y-coordinates, respectively, is strictly less than $2r < 1$ by the choice of r. Moreover, the **inverse is continuous**, since we checked that q is an open map. We use $q_{|U_r}$ as a chart around the point $[x, y]$. The **change-of-coordinate** maps are of the form (or almost, we need to restrict the maps accordingly, but the notation would become too annoying, so we simplify):

$$\mathbb{B}_{r_1}(x_1, y_1) \to \mathbb{T} \to \mathbb{B}_{r_2}(x_2, y_2),\ (x, y) \mapsto (x + (x_2 - x_1), y + (y_2 - y_1)).$$

This is a **linear map** and hence **smooth**.

In summary: We have shown that the quotient space $\mathbb{T} = \frac{\mathbb{R}\times\mathbb{R}}{\sim}$ is an abstract smooth manifold. And if we ignore or accept the topological general nonsense in the beginning, this was a very simple way to show that the torus is a manifold. □

9.3.2 The Klein bottle

Another famous and slightly more complicated space is the **Klein bottle** defined as follows: This time we consider the relation on $\mathbb{R} \times \mathbb{R}$ defined by

$$(x_1, y_1) \sim (x_2, y_2) \iff x_1 + x_2 \in \mathbb{Z} \text{ and } y_1 - y_2 \in \mathbb{Z}.$$

This is again an **equivalence relation** and the **quotient map is open**.

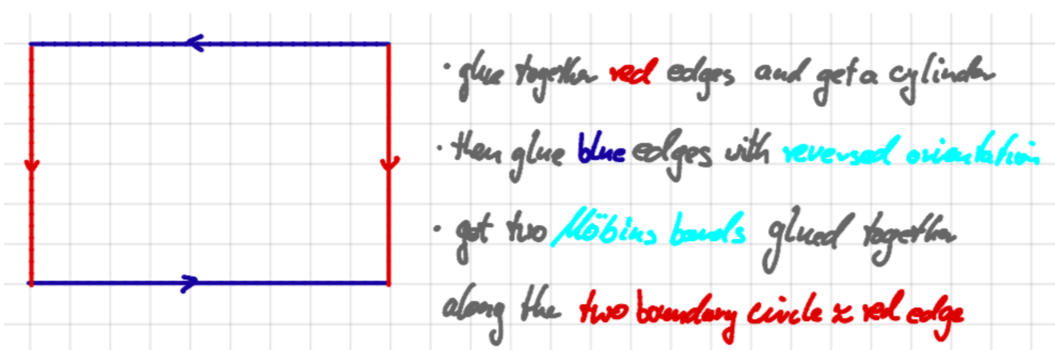

Figure 9.4: The Klein bottle can be constructed by gluing together the opposite edges of a square, but we twist one pair of the two edges.

The **quotient space** $K = \frac{\mathbb{R}\times\mathbb{R}}{\sim}$ is a Hausdorff space and is the **famous** so-called Klein bottle. It is a two-dimensional space which we **cannot embed into** $\mathbb{R}^3$. However, one can draw its shadows in three-dimensional space. We see that it may not be the best strategy to first embed K into an $\mathbb{R}^N$. However, to show that K is an **abstract smooth manifold** is not so difficult. The charts for K are as simple as for the torus: Again, let $(x, y) \in \mathbb{R} \times \mathbb{R}$ and let $[x, y]$ be its equivalence class in K. Let $U_r = \mathbb{B}^2_r(x, y) \subset \mathbb{R} \times \mathbb{R}$ be the **open ball** of radius $r > 0$ around (x, y). Then the restriction of the quotient map

$$q_{|U_r} : U_r \to K$$

is a **homeomorphism onto its image** for all $r < \frac{1}{2}$ for the same reason as for the torus. We use $q_{|U_r}$ as a chart around the point $[x, y]$. The **change-of-coordinate** maps are of the form (again we cheat a bit here):

$$\mathbb{B}_{r_1}(x_1, y_1) \to K \to \mathbb{B}_{r_2}(x_2, y_2), \ (x, y) \mapsto (-x + (x_2 + x_1), y + (y_2 - y_1)).$$

This is a **linear map** and hence **smooth**. □

Remark 9.13 In the **exercises** we study **Hopf manifolds** as another important and interesting class of examples. They play a crucial role in complex geometry, since they provide the simplest examples of compact complex manifolds which do **not** admit a **Kähler metric** and therefore cannot be embedded into complex projective space.

9.4 Stiefel manifolds

Definition 9.14 (Stiefel manifold) A **k-frame** in $\mathbb{R}^{n+k}$ is a k-tuple $[v_1, \ldots, v_k]$ of **orthonormal vectors** in $\mathbb{R}^{n+k}$. Define the **Stiefel manifold** $V_k(\mathbb{R}^{n+k})$ as the subset

$$V_k(\mathbb{R}^{n+k}) = \left\{k\text{-frames in } \mathbb{R}^{n+k}\right\} \subset \mathbb{R}^{(n+k)k}.$$

Note that we have already met a Stiefel manifold before, since $V_1(\mathbb{R}^{n+1})$ may be identified with $\mathbb{S}^n$.

The **topology** on $V_k(\mathbb{R}^{n+k})$ is given as follows: We consider $V_k(\mathbb{R}^{n+k})$ as the **subspace** of $\mathbb{S}^{n+k-1} \times \ldots \times \mathbb{S}^{n+k-1}$ of k copies of spheres $\mathbb{S}^{n+k-1}$ given by all **orthonormal** k-tuples and equip $V_k(\mathbb{R}^{n+k})$ with the subspace topology. It is a **closed subspace** since orthogonality of two vectors can be expressed by an algebraic equation. In particular, $V_k(\mathbb{R}^{n+k})$ is **compact**, since the product of spheres is compact (and closed subspaces of compact spaces are compact). In fact, we can show that $V_k(\mathbb{R}^{n+k})$ actually deserves the name **manifold**:

Theorem 9.15 (Stiefel manifolds are manifolds) The space $V_k(\mathbb{R}^{n+k})$ of k-frames is a **compact smooth manifold** of dimension $nk + \frac{k(k-1)}{2}$.

Remark 9.16 We observe that $V_k(\mathbb{R}^{n+k})$ can be identified with the quotient $O(n + k)/O(n)$. For, if $[v_1, \ldots, v_k]$ and $T \in O(n + k)$, then multiplying each v_i with T yields another k-frame $[Tv_1, \ldots, Tv_k]$. In fact, any two k-frames are connected in this way, i.e., if $[v_1, \dots, v_k]$ and $[v'_1, \dots, v'_k]$ are two k-frames then there is an $T \in O(n+k)$ such that

$$[v'_1, \dots, v'_k] = [Tv_1, \dots, Tv_k].$$

In other words, $O(n+k)$ acts **transitively** on the set of k-frames. Moreover, the stabilizer subgroup of a given frame is the subgroup isomorphic to $O(n)$ which acts nontrivially on the **orthogonal complement** of the space spanned by that frame.

Proof of Theorem 9.15: We have already proven that $V_k(\mathbb{R}^{n+k})$ is compact. It remains to show the manifold part. Note that a k-frame $[v_1, \dots, v_k]$ corresponds to an $(n+k) \times k$-matrix A with v_i as ith column. That the column vectors are **orthonormal** is equivalent to that the product of the transpose of the ith column with the jth column is 1 if $i = j$ and 0 otherwise. In a formula:

$$v_i^t \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

In other words,

$$A \text{ represents a } k\text{-frame in } \mathbb{R}^{n+k} \iff A^t A = I_k,$$

where I_k denotes the $k \times k$-identity matrix. Let $M(n+k, k)$ be the space of $(n+k) \times k$-matrices and $S(k)$ denote the space of symmetric $k \times k$-matrices. We define the map

$$f \colon M(n+k, k) \to S(k), \; A \mapsto A^t A$$

and we observe $V_k(\mathbb{R}^{n+k}) = f^{-1}(I_k)$.

Hence, in order to show that $V_k(\mathbb{R}^{n+k})$ is a smooth manifold, we need to show that I_k is a **regular value** for f. We have computed the derivative of f at a matrix $A \in V_k(\mathbb{R}^{n+k})$ in the proof of Theorem 4.14. There we showed

$$df_A(B) = A^t B + B^t A.$$

In order to check that I_k is a regular value, we need to show that

$$df_A \colon T_A(M(n+k, k)) = M(n+k, k) \to S(k) = T_{f(A)}(S(k))$$

is surjective for all $A \in V_k(\mathbb{R}^{n+k})$. Recall for the above computation that $M(n+k, k) \cong R^{(n+k)k}$ and $S(k) \cong \mathbb{R}^{k(k+1)/2}$ are vector spaces and that the tangent space of a vector space equals the vector space. For $C \in S(k)$, we set $B = \frac{1}{2}AC \in M(n+k, k)$ and get

$$df_A(B) = A^t\left(\frac{1}{2}AC\right) + \left(\frac{1}{2}AC\right)^t A = \frac{1}{2}A^t AC + \frac{1}{2}C^t A^t A = \frac{1}{2}C + \frac{1}{2}C^t = C.$$

By the **Preimage Theorem 4.7**, this shows that $V_k(\mathbb{R}^{n+k})$ is a **smooth manifold** of dimension

$$\begin{aligned} \dim V_k(\mathbb{R}^{n+k}) &= \dim M(n+k, k) - \dim S(k) \\ &= (n+k)k - \frac{k(k+1)}{2} = nk + \frac{k(k-1)}{2}. \quad \square \end{aligned}$$

9.5 Grassmannian

There is another very important space that arises from Stiefel manifolds. Any k-frame in $\mathbb{R}^{n+k}$ spans a k-dimensional linear subspace in $\mathbb{R}^{n+k}$.

Definition 9.17 (Grassmannian) The set of all k-dimensional linear subspaces in $\mathbb{R}^{n+k}$ is called the **Grassmann manifold**, or short the **Grassmannian**,

$$\mathrm{Gr}_k(\mathbb{R}^{n+k}) = \{k\text{-dimensional linear subspaces in } \mathbb{R}^{n+k}\}.$$

The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ can be identified with the quotient of the Stiefel manifold $V_k(\mathbb{R}^{n+k})$ of orthonormal sequences

$$[v_1, \ldots, v_k]$$

of vectors $v_i \in \mathbb{R}^{n+k}$, modulo the **equivalence relation** given by $[v_1', \ldots, v_k'] \sim [v_1, \ldots, v_k]$ if and only if there exists an **orthogonal** $k \times k$-matrix T such that

$$[v_1', \ldots, v_k'] = [v_1, \ldots, v_k] \cdot T,$$

In other words, $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is the **quotient** of $V_k(\mathbb{R}^{n+k})$ that we get by identifying k-frames which **span the same subspace** in $\mathbb{R}^{n+k}$:

$$q\colon V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k}),\ [v_1, \ldots, v_k] \mapsto \mathrm{span}(v_1, \ldots, v_k) \text{ in } \mathbb{R}^{n+k}.$$

We have already seen an example of a Grassmannian:

Example 9.18 (Projective spaces are Grassmannians) For $k = 1$, $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ is the **space of lines**, i.e., one-dimensional subspaces, in $\mathbb{R}^{n+1}$. Thus we have

$$\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{R}\mathrm{P}^n$$

the **real projective space** of dimension n.

It is actually **not so easy** to **embed** projective spaces and Grassmannians into Euclidean spaces. For the moment, we content ourselves with noting that $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ **can be embedded** at least in $\mathbb{R}^{2nk+1}$. However, it is an **interesting and difficult question** what the **minimal** dimension N such that we can embed them into $\mathbb{R}^N$. We will get back to this point in a general context in Section 9.6.

There are many reasons why **Grassmannians are important**. We will now sketch one of them. If it does not make complete sense to you yet, consider it as a **advert for a future adventure**.

9.5.1 The canonical bundle

We mentioned vector bundles briefly in Section 2.5. The tangent bundle on a smooth manifold is an important example. The Grassmannian is equipped with a **canonical** or **tautological** vector bundle, denoted γ_k^{n+k}. It is defined as follows: A point in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ consists of a k-plane V in $\mathbb{R}^{n+1}$. We can consider V both as a point in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ and as a k-dimensional vector space, and we can pick a vector $v \in V$. The space γ_k^{n+k} consists of the collection of such pairs, i.e.,

$$\gamma_k^{n+k} = \left\{ (V, v) : V \in \mathrm{Gr}_k(\mathbb{R}^{n+k}),\ v \in V \right\}.$$

Together with the projection $\pi: \gamma_k^{n+k} \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$ defined by sending (V, v) to V turns γ_k^{n+k} into a a **vector bundle** on $\mathrm{Gr}_k(\mathbb{R}^{n+k})$.

This vector bundle is extremely **useful**. Let $X \subset \mathbb{R}^N$ be a smooth manifold of dimension k. Let n be a natural number such that $N = n + k$. Let $x \in X$ be a point and $T_x X$ be the tangent space at x. This is a k-dimensional vector subspace of $\mathbb{R}^{n+k}$. Hence we can consider $T_x X$ as an element in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$. This defines a map

$$f: X \to \mathrm{Gr}_k(\mathbb{R}^{n+k}),\ x \mapsto T_x(X).$$

Now recall that the **tangent bundle** TX consists of pairs (x, v) with $x \in X$ and $v \in T_x X$. Sending the pair (x, v) to the pair $(T_x X, v)$ defines a map $\bar{f}: TX \to \gamma_k^{n+k}$. Assuming the results we will prove in this section, both these maps f and $\bar{f}$ are **smooth**. Moreover, they fit into a commutative diagram

$$\begin{tikzcd} TX \arrow[r, "\bar{f}"] \arrow[d] & \gamma_k^{n+k} \arrow[d] \\ X \arrow[r, "f"'] & \mathrm{Gr}_k(\mathbb{R}^{n+k}). \end{tikzcd}$$

We can show that this actually defines a morphism of vector bundles. This diagram is often called a **generalized Gauss map**. The maps f and $\bar{f}$ are very useful for studying manifolds. More importantly, this picture actually has a fascinating generalization:

Theorem 9.19 (Grassmannians classify vector bundles) Let Y be a sufficiently nice topological space Y with a k-dimensional vector bundle $E \to Y$. Then, for n large enough, there is a continuous map $f: Y \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$ such that E is the **pullback** of the canonical vector bundle along this map:

$$\begin{tikzcd} E \cong f^*\gamma \arrow[r, dotted, "\bar{f}"] \arrow[d] & \gamma \arrow[d] \\ Y \arrow[r, "f"'] & \mathrm{Gr}_k(\mathbb{R}^{n+k}). \end{tikzcd}$$

Up to **homotopy**, the map f is in fact **unique** for the **isomorphism class** of $E \to Y$. We refer to this phenomenon by saying that $\mathrm{Gr}_k(\mathbb{R}^\infty)$ is a **classifying space** for k-dimensional vector bundles. The concept of classifying spaces is **very powerful** and we strongly encourage to learn more about it in the future.

9.5.2 Grassmannians are manifolds - geometric proof

Now we show the following key fact:

Theorem 9.20 (Grassmannians are manifolds) The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is a **compact smooth manifold** of dimension $k \cdot n$.

- **Note** that we will be cheating a bit for the moment. For, we define a map

$$f\colon \mathrm{Gr}_k(\mathbb{R}^{n+k}) \to \mathbb{R}^N$$

to be **smooth** if and only **if the composite** $f \circ q\colon V_k(\mathbb{R}^{n+k}) \to \mathbb{R}^N$ is **smooth**. We will get back to this point later. Recall that in order to shorten our sentences we will often refer to a k-dimensional linear subspace as a **k-plane**.

Proof: Since the quotient map $q\colon V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k})$ is **continuous and surjective** and since $V_k(\mathbb{R}^{n+k})$ is compact, we see that $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is **compact**. It remains to show that it is a smooth manifold.

Let $V \subset \mathbb{R}^{n+k}$ be a k-dimensional linear subspace in $\mathbb{R}^{n+k}$, and let $V^\perp$ be the **orthogonal complement** of V in $\mathbb{R}^{n+k}$ (with respect to the standard inner product). Define the subspace $\mathcal{U}_V \subset \mathrm{Gr}_k(\mathbb{R}^{n+k})$ consisting of k-dimensional linear subspaces V' of $\mathbb{R}^{n+k}$ with the property that $V' \cap V^\perp = \{0\}$:

$$\mathcal{U}_V = \left\{ V' \in \mathrm{Gr}_k(\mathbb{R}^{n+k}) : V' \cap V^\perp = \{0\} \right\}.$$

Equivalently, $\mathcal{U}_V$ is the set of all k-dimensional subspaces $V' \subset \mathbb{R}^{n+k}$ which are mapped surjectively onto V by the projection $p\colon \mathbb{R}^{n+k} = V \oplus V^\perp \to V$.

We will use subsets of the form $\mathcal{U}_V$ to construct local parametrizations. To do this we need to check several things:

- **First claim:** U is an open neighborhood of V.

To show that $\mathcal{U}_V$ is **open** in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ it suffices to show that its preimage $\tilde{\mathcal{U}}_V$ under q is open in $V_k(\mathbb{R}^{n+k})$, since we defined the topology on the Grassmannian as the **quotient topology** induced by

$$q\colon V_k(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k}),\ [v_1, \dots, v_k] \mapsto \mathrm{span}(v_1, \dots, v_k) \text{ in } \mathbb{R}^{n+k}.$$

By definition of q and $\mathcal{U}_V$, the set $\tilde{\mathcal{U}}_V$ consists of all orthonormal k-frames $[v_1', \dots, v_k']$ such that

$$\mathrm{span}(v_1', \dots, v_k') \cap V^\perp = \{0\}. \tag{9.3}$$

Let $\{w_1, \dots, w_n\}$ be a basis of $W := V^\perp$. Then condition (9.3) is equivalent to saying that the set of vectors $\{v_1', \dots, v_k', w_1, \dots, w_n\}$ **forms a basis** of $\mathbb{R}^{n+k}$, since they are **linearly independent and span** an $(n+k)$-dimensional subspace of $\mathbb{R}^{n+k}$. Equivalently, the $(n+k)\times(n+k)$-matrix $M(v_1', \dots, v_k', w_1, \dots, w_n)$ with column vectors $v_1', \dots, v_k', w_1, \dots, w_n$ has nonzero determinant. Keeping the basis of W fixed, we see that $\tilde{\mathcal{U}}_V$ consists of the **preimage of the open subset** $\mathbb{R} \setminus \{0\}$ under the map

$$\det : V_k(\mathbb{R}^{n+k}) \to \mathbb{R},\ [v_1', \dots, v_k'] \mapsto \det\left(M(v_1', \dots, v_k', w_1, \dots, w_n)\right).$$

Hence $\tilde{\mathcal{U}}_V$ is an **open subset**. This proves the first claim.

Now we are going to define a map from $\mathcal{U}_V$ to **Euclidean space**. We can think of each $V' \in \mathcal{U}_V$ as the graph of a linear map $V \to W = V^\perp$ as follows:

Given $V' \in \mathcal{U}_V$, let $\mathrm{pr}^{V'}_V$ be the **orthogonal projection** of V' onto V and $\mathrm{pr}^{V'}_W$ be the **orthogonal projection** of V' onto W. Note that **since** $V' \cap V^\perp = \{0\}$ **and** $\dim V' = \dim V$, we know that $\mathrm{pr}^{V'}_V$ is an **isomorphism**. Hence we can define a map

$$\psi_V : \mathcal{U}_V \to \mathrm{Hom}_{\mathbb{R}}(V, W),\ V' \mapsto \psi_V(V') = \mathrm{pr}^W_{V'} \circ (\mathrm{pr}^{V'}_V)^{-1}$$

from $\mathcal{U}_V$ to $\mathrm{Hom}_{\mathbb{R}}(V, W)$. See Figure 9.5.

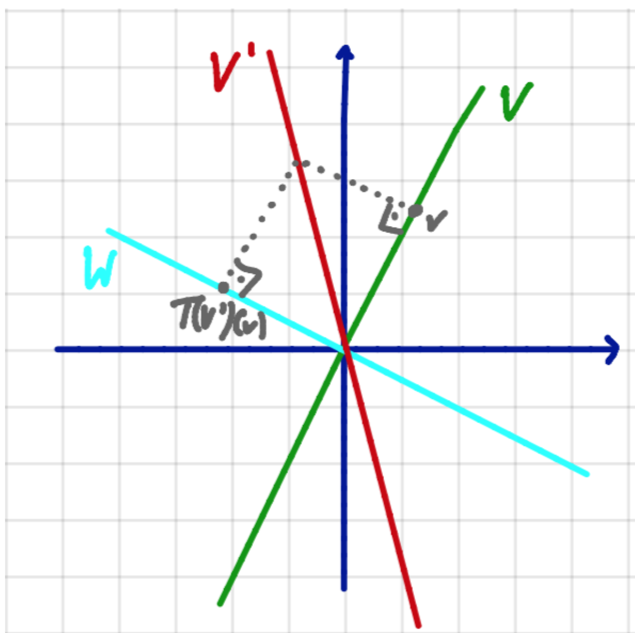

Figure 9.5: A vector v' in V' defines a linear transformation from V to W by taking orthogonal projections.

The space $\mathrm{Hom}_{\mathbb{R}}(V, W)$ is a real vector space isomorphic to $\mathbb{R}^{nk}$.

We can define an **inverse** to ψ_V as well: Given a linear map $f : V \to W$, the **graph of** f,

$$\Gamma(f) = \{v + f(v) \in V \oplus W\} \subset V \oplus W \cong \mathbb{R}^{n+k},$$

is a linear subspace of dimension k which satisfies

$$\Gamma(f) \cap W = V \cap W = \{0\}.$$

Hence $\Gamma(f) \in \mathcal{U}_V$ and we **define** a map ψ_V^{-1} by

$$\psi_V^{-1} : \mathrm{Hom}_{\mathbb{R}}(V, W) \to \mathcal{U}_V,\ f \mapsto \Gamma(f).$$

- **Second claim:** ψ_V and ψ_V^{-1} are mutual **inverses**.

First, note that $\Gamma(\psi_V(V')) = V'$ by definition of $\psi_V(V')$ by considering V' as the graph of a map $V \to W$. Now let $f : V \to W$ be a linear map. Then

$$\psi_V(\Gamma(f))(v) = \mathrm{pr}_W^{\Gamma(f)} \circ (\mathrm{pr}_V^{\Gamma(f)})^{-1}(v) = \mathrm{pr}_W^{\Gamma(f)}(v + f(v)) = f(v).$$

Hence $\psi_V(\Gamma(f)) = f$. This proves the **second claim**.

- **Third claim:** ψ_V is a **diffeomorphism**.

To prove the claim it remains to show that ψ_V and ψ_V^{-1} are **smooth**. Note that we consider $\mathrm{Hom}_{\mathbb{R}}(V, W)$ as a topological space by identifying it with $\mathbb{R}^{nk}$. That is we fix an orthonormal basis $v_1, \dots, v_k$ for V and an orthonormal basis $w_1, \dots, w_n$ for $W = V^\perp$. Then we identify linear maps $f : V \to W$ with their associated matrix A_f with respect to these bases. The entries in A_f are real numbers and we can collect them to $n \cdot k$-tuples, i.e., elements in $\mathbb{R}^{n \cdot k}$.

We show **first** that T is smooth: By definition of the topology on the Grassmannian and our definition of smoothness on $\mathrm{Gr}_k(\mathbb{R}^{n+k})$, T is smooth if and only if the composite

$$\begin{tikzcd}
\widetilde{\mathcal{U}}_V = q^{-1}(\mathcal{U}_V) \arrow[r, "q"] \arrow[dr, "\psi_V \circ q"'] & \mathcal{U}_V \arrow[d, "\psi_V"] \\
& \mathrm{Hom}_{\mathbb{R}}(V, W)
\end{tikzcd}$$

is smooth. Let $[v'_1, \dots, v'_n]$ be an orthonormal k-frame which spans V'. The entries in the matrices of the maps $\mathrm{pr}_{V'}^W$ and $(\mathrm{pr}_V^{V'})^{-1}$, respectively, expressed in the bases of V, W and V', depend smoothly on the coordinates of the v'_i. Hence matrix of the map $\psi_V(V') = \mathrm{pr}_{V'}^W \circ (\mathrm{pr}_V^{V'})^{-1}$ depends smoothly on the k-frame $[v'_1, \dots, v'_n]$ and **$\psi_V \circ q$ is smooth**.

Now let $f : V \to W$ be a linear map and let A_f be its representing matrix (in the bases we chose). Then there is a unique orthonormal basis $v'_1, \dots, v'_n$ of $\Gamma(f)$ with

$$v'_i = v_i + f(v_i) = v_i + A_f(v_i)$$

for all i. Hence the coordinates of each v'_i depend smoothly on the entries in A_f. Hence the **k-frame** $[v'_1, \dots, v'_n]$ in $V_k(\mathbb{R}^{n+k})$ **depends smoothly** on f. Since the k-frame $[v'_1, \dots, v'_n]$ spans $\Gamma(f) = \psi_V^{-1}(f)$, we can interpret this as $\Gamma(f)$ depends smoothly on f and **ψ_V^{-1} is also smooth**. This proves the **third claim**.

In total we have shown that an arbitrary point $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ has an open neighborhood $\mathcal{U}_V$ which is diffeomorphic to $\mathbb{R}^{nk}$. In other words, $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is indeed a smooth manifold of dimension $n \cdot k$. □

Remark 9.21 (Tangent space of the Grassmannian) The local coordinate chart ψ_V gives us also a way to understand the tangent space of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ at V. The map ψ_V **behaves** like a **linear map** (if we keep vectors small enough) and the image $\mathrm{Hom}_{\mathbb{R}}(V, V^\perp)$ of ψ_V is a **vector space**. Hence the tangent space of $\mathrm{Hom}_{\mathbb{R}}(V, V^\perp)$ at 0 is $\mathrm{Hom}_{\mathbb{R}}(V, V^\perp)$

itself. Therefore, we can use $d\psi_V$ to get a **linear isomorphism**

$$T_V(\mathrm{Gr}_k(\mathbb{R}^{n+k})) \cong T_0(\mathrm{Hom}_{\mathbb{R}}(V, V^{\perp})) = \mathrm{Hom}_{\mathbb{R}}(V, V^{\perp}).$$

9.5.3 Grassmannians are manifolds - linear algebraic proof

We now give **an alternative argument** using row vectors and more linear algebra to prove Theorem 9.20:

Any k-dimensional linear subspace in $\mathbb{R}^{n+k}$ is spanned by **k row vectors** in $\mathbb{R}^{n+k}$. Hence any such k-dimensional linear subspace V can be represented by a $k \times (n+k)$-matrix

$$A = \begin{pmatrix} v_{11} & \cdots & v_{1n+k} \\ \vdots & \ddots & \vdots \\ v_{k1} & \cdots & v_{kn+k} \end{pmatrix}$$

where any two such matrices A and A' represent the same element in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ if and only if there is an **invertible** matrix $k \times k$-matrix T such that

$$A' = T \cdot A.$$

Now let $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n+k\}$ be a k-tuple with $i_1 < i_2 < \cdots < i_k$. We define the subset $\mathcal{U}_I$ of $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ to be the set of all k-planes which are represented by a $k \times (n+k)$-matrix A such that the **$k \times k$-submatrix** consisting of the columns $i_1, \dots, i_k$ is **invertible**. We call this submatrix the **I-th $k \times k$-minor** of A. Recall that **elementary row operations** that we learned about in our very first encounter with linear algebra **do not change the row space**, i.e., the space spanned by the row vectors. Hence, after **performing suitable row operations**, we see that every $V \in \mathcal{U}_I$ is **represented by a unique $k \times (n+k)$-matrix** in which this I-th $k \times k$-minor is the **$k \times k$-identity matrix**. In other words, we can make the following observation:

Lemma 9.22 Every $V \in \mathcal{U}_I$ is represented by a **unique** $k \times (n+k)$-matrix $A_I(V)$ in which the jth column is the standard basis vector e_j of length k if $j \in I$ and an arbitrary vector of length k if $j \notin I$.

Example 9.23 (Representing planes by minors) For a **concrete example**, let $k = 3$ and $n + k = 5$ and $I = \{1, 3, 4\}$. Let V be the 3-plane, i.e., 3-dimensional linear subspace, spanned by the rows of the matrix

$$A = \begin{pmatrix} 3 & 6 & 0 & 1 & -18 \\ 3 & 1 & 1 & 0 & -7 \\ 5 & 6 & 1 & 1 & -21 \end{pmatrix}.$$

The **I-th minor** is the matrix

$$M = \begin{pmatrix} 3 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

consisting of the first, third and fourth column of A. This minor is an **invertible** matrix with **inverse**

$$M^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -2 & 3 \\ -2 & -3 & 3 \end{pmatrix}.$$

Then $V \in \mathcal{U}_I$ is **uniquely represented** by the matrix

$$A_I = M^{-1} \cdot A = \begin{pmatrix} 1 & 1 & 0 & 0 & -4 \\ 0 & -2 & 1 & 0 & 5 \\ 0 & 3 & 0 & 1 & -6 \end{pmatrix}$$

in which the first, third and fourth column form the 3×3-identity matrix.

Back to the **general discussion**:

Conversely, any $k \times (n+k)$-matrix with **invertible** I-th $k \times k$-minor defines a **unique** k-plane in $\mathcal{U}_I$.

Thus, for each I, the $k \cdot n$ entries in the remaining columns of any $k \times (n+k)$-matrix of this form define a **bijection** of sets

$$\psi_I : \mathcal{U}_I \xrightarrow{\cong} \mathbb{R}^{k \cdot n}.$$

Example 9.24 (Real projective space) For $k = 1$, we have seen that $\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{R}\mathrm{P}^n$. A point in $\mathbb{R}\mathrm{P}^n$ is represented by an $(n+1)$-tuple $x_1, \dots, x_{n+1}$. Recall that we denote the equivalence class by $[x] = [x_1 : \dots : x_{n+1}]$. The sets I then consist each of just **one number** $i \in \{1, \dots, n+1\}$. The set $\mathcal{U}_i$ is then equal to the subset V_i we have defined previously:

$$\mathcal{U}_i = V_i = \{[x] \in \mathbb{R}\mathrm{P}^n : x_i \neq 0\}.$$

The **representing matrix** A_i of $[x]$ is the matrix with just one row $\left(x_1/x_i, \dots, 1, \dots, x_{n+1}/x_i\right)$ and $e_1 = 1$ in column i.

Claim: For each I, the subset $\mathcal{U}_I$ is open in $\mathrm{Gr}_k(\mathbb{R}^{n+k})$.

By definition of $\mathcal{U}_I$, each k-plane in $\mathcal{U}_I$ is represented by a $k \times (n+k)$-matrix A such that I-th **$k \times k$-minor** consisting of the columns $i_1, \dots, i_k$ is **invertible**. We define $\det_I$ to be the map which sends a $k \times (n+k)$-matrix to the determinant of its I-th $k \times k$-minor. Then $\det_I$ is a map on the space of all $k \times (n+k)$-matrices with values in $\mathbb{R}$. This map is **continuous**, since the determinant of the I-th minor is a polynomial in the entries of this minor. Thus $\det_I$ depends continuously on these entries. Finally, $\mathcal{U}_I$ is a preimage of the open subset $\mathbb{R} \setminus \{0\}$ under $\det_I$. □

We also note that the subset $\psi_I(\mathcal{U}_I \cap \mathcal{U}_{I'})$ is **open** in $\mathbb{R}^{k \cdot n}$ for **every pair I, I'**. Now we can give the second proof:

Proof of Theorem 9.20: We already know that Grassmannians are compact. To show it once again anyway we could argue as follows: Let E_k be the k-plane in $\mathbb{R}^{n+k}$ spanned by the first k standard basis $e_1, \dots, e_k \in \mathbb{R}^{n+k}$ of length $n+k$. We deduce from what we learned above that we have a **surjective map**

$$q\colon O(n+k) \to \mathrm{Gr}_k(\mathbb{R}^{n+k}),\ T \mapsto T \cdot E_k.$$

Since $O(n+k)$ is compact and the **continuous image** of compact sets is **compact**, this shows that $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is **compact**.

We need to show that, for every pair I, I', the map

$$\psi_{I'}(\mathcal{U}_I \cap \mathcal{U}_{I'}) \xrightarrow{\psi_I \circ \psi_{I'}} \psi_I(\mathcal{U}_I \cap \mathcal{U}_{I'})$$

is a **smooth map** between open subsets of $\mathbb{R}^{k \cdot n}$, where we omit to denote the appropriate restrictions of ψ_I and $\psi_{I'}$ to simplify the notation. To prove this let V be a k-plane in $\mathcal{U}_I \cap \mathcal{U}_{I'}$. Let A_I be a matrix representing V with its I-th $k \times k$-minor being the identity matrix and let $A_{I'}$ be a matrix representing V with its I'th $k \times k$-minor being the identity matrix. Let $T^I_{I'}$ be the I'th $k \times k$-minor of A_I. Then we have

$$T^I_{I'} \cdot A_{I'} = A_I.$$

Since the entries of $T^I_{I'}$ vary smoothly with the entries in A_I, it follows that the entries in $A_{I'}$ also vary smoothly with the entries in A_I. Thus $\psi_I \circ \psi_{I'}$ is **smooth**. □

9.6 Embedding abstract manifolds in Euclidean space

We will now study the following fundamental question:

> **Question** Given an abstract smooth manifold X in the sense of Definition 9.3: Is it possible to **embed** X into **Euclidean space**? That is, is there a natural number N and an embedding $X \hookrightarrow \mathbb{R}^N$?

The answer to this question is **yes** which may justify our initial approach to smooth manifolds as subsets of Euclidean space. We will discuss the proof only for compact manifolds in Section 9.6.2. The non-compact requires much more familiarity with arguments in general topology and we omit this discussion.

Once we know that every smooth abstract manifold can be considered as subspace of some Euclidean space we will try to answer the following question:

> **Question** Given a k-dimensional manifold $X \subset \mathbb{R}^N$: What is the **minimal** n such that we can be sure that there is an **embedding** $X \subset \mathbb{R}^n$?

We will approach an answer to this question in two steps: In Section 9.7.1 we show that there always is an **one-to-one immersion** of X into $\mathbb{R}^{2k+1}$. If X is **compact**, this immersion will automatically be an **embedding** as we learned in Section 3.3. In Section 9.6.2 we will then show that also for **non-compact** smooth k-manifolds there is always an **embedding** into $\mathbb{R}^{2k+1}$. In fact, Whitney showed that $N = 2k$ always works. The proof is much harder, and we will not discuss it in these notes.

9.6.1 Partition of unity

In several proofs in this section we will employ an important and very useful tool. It will also be useful in several occasions later on. First we recall some terminology from general topology:

Definition 9.25 **(The closure of a subset)** Let X be a topological space and A be an arbitrary subset. The **closure of A** in X, denoted $\overline{A}$, is the **intersection of all closed subsets** in X which contain A.

We are familiar with the closure of subsets in many cases. For example, the **closure of an open ball** $\mathbb{B}_\varepsilon(0)$ in $\mathbb{R}^N$ is just the **closed ball**

$$\overline{\mathbb{B}_\varepsilon(0)} = \{x \in \mathbb{R}^N \,:\, |x| \leq \varepsilon\}.$$

We need the closure of a subset for example when we want to talk about the **support** of a function:

Definition 9.26 **(Support of a function)** Let X be a smooth manifold and $f : X \to \mathbb{R}$ be a smooth function. The **closed** subset

$$\operatorname{supp}(f) := \overline{\{x \in X \,:\, f(x) \neq 0\}}$$

is called the **support of f**.

We are now going to introduce a **fundamental tool** for studying manifolds:

Definition 9.27 **(Partition of unity)** Let X be a **smooth manifold** and let $\{U_\alpha\}$ be an **open cover**, i.e., a collection of **open** subsets in X such that $\bigcup_\alpha U_\alpha = X$. A **sequence of smooth functions** $\{\rho_i : X \to \mathbb{R}\}$ is called a **partition of unity subordinate to the open cover $\{U_\alpha\}$** if it has the following properties:

(a) $0 \leq \rho_i(x) \leq 1$ for all $x \in X$ and all i.

(b) Each $x \in X$ has a **neighborhood** on which **all but finitely many** functions ρ_i are identically **zero**.

(c) For each i, $\operatorname{supp}(\rho_i) \subset U_\alpha$ for some α.

(d) For each $x \in X$, $\sum_i \rho_i(x) = 1$. Note that according to (b), this sum is **always finite**.

The most general existence result for partitions of unity (only requiring that each ρ_i is merely continuous and not smooth) is that they **exist on every paracompact space**, i.e., spaces on which every open cover has a **locally finite refinement**. The latter means that every point has a neighborhood that intersects only finitely many sets in the cover.

We will **postpone the proof** of the existence of partitions of unity on abstract smooth manifolds to Section 9.8 . It is a rather technical proof which would diverts our attention from our main story. We will therefore first look at some applications of partitions of unity and will then show that they actually exist.

9.6.2 Embedding abstract manifolds in Euclidean space

We use partitions of unity to prove the following **important result**:

Theorem 9.28 (Embedding abstract manifolds in Euclidean space) Let X be a **compact** abstract smooth k-manifold. Then there is an **embedding** $X \hookrightarrow \mathbb{R}^N$ for some large N.

Proof: The collection of all V_α for all charts (V_α, ϕ_α) is an open cover of X. Since X is **compact**, we can cover X by the domains of a **finite number of charts** $V_1, \dots, V_n$. Let $\{\rho_i\}$ be a **partition of unity** subordinate to the open cover defined by the V_i's. For a chart $\phi_i \colon V_i \to U_i \subset \mathbb{R}^k$, we define a new map

$$f_i \colon X \to \mathbb{R}^k,\ f_i(x) = \begin{cases} \rho_i(x) \cdot \phi_i(x) & \text{for } x \in V_i \\ 0 & \text{for } x \in X \setminus \operatorname{supp}(\rho_i). \end{cases}$$

The map f_i is well-defined, since if $x \in V_i \setminus \operatorname{supp}(\rho_i)$, then both definitions agree to be 0. Moreover, f_i is **smooth**, since its **restrictions** to the two **open** subsets V_i and $X \setminus \operatorname{supp}(\rho_i)$ are **smooth**.

Remark 9.29 At this point we see why we do not use $X \setminus V_i$ in the definition of f_i because that would be a **closed** subset. We also observe that it would not work to drop the ρ_i and just use the ϕ_i, since there is no reason why ϕ_i would get closer to zero the closer we get to the boundary of V_i. Hence there are several reasons why **continuity and smoothness** would **not** be guaranteed without the ρ_i.

Now we define the map

$$G \colon X \to \mathbb{R}^n \times \mathbb{R}^{nk},\ x \mapsto (\rho_1(x), \dots, \rho_n(x), f_1(x), \dots, f_n(x)).$$

We observe that **G is continuous**, since the f_i's and the ρ_i's are continuous.

- **Claim:** G is an **injective proper** map.

Since X is **compact**, G is **proper**. Now we show that G is **injective**. So **assume** $G(x) = G(y)$. Then $\rho_i(x) = \rho_i(y)$ **for all** i by the definition of G. However, by the definition of a partition of unity, **for at least one** i, we must have $\rho_i(x) = \rho_i(y) \neq \mathbf{0}$. Thus x and y must lie in the **same** V_i, since ρ_i is supported on V_i, i.e., $\rho_i(x) \neq 0$ implies $x \in V_i$ and similarly for y. Hence, since we have $f_i(x) = f_i(y)$ and $\rho_i(x) = \rho_i(y) \neq \mathbf{0}$, we must have

$$\phi_i(x) = \phi_i(y).$$

Since ϕ_i is a **bijection**, this shows $x = y$. Thus G is **injective**.

Finally, we can show that G is an **immersion**. However, we have **not yet defined** what that means for **abstract** manifolds. While this is just an exercise in translating the definitions, we omit this discussion here. □

Remark 9.30 (All manifolds can be embedded in Euclidean space) In fact, **every abstract** k-manifold X can be **embedded in Euclidean space**. One can just **keep on going** with the above argument in the **non-compact case** and use local charts to map pieces of X into $\mathbb{R}^k$. Though when using only finitely many copies of $\mathbb{R}^k$ to accommodate infinitely many neighborhoods of X, **we loose injectivity**. The **key tool that restores injectivity** is a **partition of unity** which **evens out the troubles** caused by overlapping neighborhoods. For this to work, it is **crucial** that the topology on X has a **countable** basis. This is a technical point which we do not discuss any further because it would divert us too far from the main story.

9.7 Whitney's Theorems for smooth manifolds

9.7.1 Whitney's Immersion Theorem for smooth manifolds

As Theorem 9.28 tells us that manifolds can always be embedded into Euclidean space, we now turn our focus to the following question:

Question Assume we know that a smooth manifold X can be embedded into $\mathbb{R}^N$. For example, $X \subset \mathbb{R}^N$ could be a smooth manifold in the sense of Definition 2.23. How much can we reduce the dimension N in general?

We first consider this problem for an immersion:

Theorem 9.31 (Whitney's Immersion Theorem) Let $X \subset \mathbb{R}^N$ be a smooth k-dimensional manifold. Then X admits a **one-to-one immersion** into $\mathbb{R}^{2k+1}$.

Remark 9.32 Note that Theorem 9.34 does **not** necessarily give us the **minimal** N for an individual manifold. For example, we know that $\mathbb{S}^n$ is embedded in $\mathbb{R}^{n+1}$ for every n. In fact, Theorem 9.34 tells us that $N = 2k + 1$ **always works**. The example of real projective space shows that N cannot, in general, be reduced beyond $2k - 1$ if we require an immersion: **if** there is an **immersion** $\mathbb{R}\mathrm{P}^{2^r} \to \mathbb{R}^{2^r+k}$, then k must be **at least** $2^r - 1$. We will learn about the techniques that allow to show this in more **advanced algebraic topology** course. An excellent book to read more about this fact is [14].

Proof of Theorem 9.31: Assume that X is a smooth k-dimensional manifold which is a subset in $\mathbb{R}^N$ for some $N > 2k + 1$. In particular, we are given an **injective immersion** $X \hookrightarrow \mathbb{R}^N$. **Our goal** is to show that we can choose N to be $2k + 1$ and **still have an injective immersion**. Therefor we are going to construct a **linear projection** $\mathbb{R}^N \to \mathbb{R}^{2k+1}$ that restricts to a one-to-one immersion $X \hookrightarrow \mathbb{R}^{2k+1}$ on X. The construction will proceed by **induction**:

Whenever we are given an injective immersion $f : X \hookrightarrow \mathbb{R}^N$ with $N > 2k+1$, then **we will show** that there **exists a vector** $a \in \mathbb{R}^N$ such that the composition

$$X \xrightarrow{f} \mathbb{R}^N \xrightarrow{\pi} H := \{b \in \mathbb{R}^N : b \perp a\}$$

of f with the projection map π carrying $\mathbb{R}^N$ onto the **orthogonal complement** H of a is **still** an **injective immersion**. The complement $H = \{b \in \mathbb{R}^N : b \perp a\}$ is an $N-1$**-dimensional** vector subspace of $\mathbb{R}^N$, hence isomorphic to $\mathbb{R}^{N-1}$. Thus, after **choosing a basis** for H, we obtain an **injective immersion into** $\mathbb{R}^{N-1}$. **Continuing** this procedure yields a chain of linear maps

$$\mathbb{R}^N \to \mathbb{R}^{N-1} \to \cdots \to \mathbb{R}^{2k+1}$$

such that the composition $X \to \mathbb{R}^N \to \mathbb{R}^{2k+1}$ is still an injective immersion.

So we now **assume** that we have an **injective immersion**

$$f : X \hookrightarrow \mathbb{R}^N \text{ with } N > 2k+1.$$

We define two smooth maps

$$\begin{tikzcd} & X \times X \times \mathbb{R} \arrow[d, "h"] \\ T(X) \arrow[r, "g"'] & \mathbb{R}^N \end{tikzcd}$$

by

$$h : X \times X \times \mathbb{R} \to \mathbb{R}^N, \ (x, y, t) \mapsto t(f(x) - f(y))$$

and, using $T_y(\mathbb{R}^N) = \mathbb{R}^N$ at any $y \in \mathbb{R}^N$ and hence the derivative df_x is a smooth map $df_x : T_x(X) \to \mathbb{R}^N$, by

$$g : T(X) \to \mathbb{R}^N, \ (x, v) \mapsto df_x(v).$$

By **Sard's Theorem 7.1**, the sets of regular values R_g and R_h for g and h, respectively, are **dense** subsets in $\mathbb{R}^N$. Thus their **intersection** is **non-empty** and there exists a point in $\mathbb{R}^N$ which is a **regular value** for **both** g and h simultaneously.

Since $\dim T(X) = 2k$, $\dim(X \times X \times \mathbb{R}) = 2k+1$, but $N > 2k+1$, the **only** regular values of g and h are the points in $\mathbb{R}^N$ which are **not in the image** of g or h. Hence there **exists** a point $a \in \mathbb{R}^N$ which is **neither** in the image of g nor in the image of h. Note that, since 0 belongs to **both images**, we must have $a \neq 0$.

> **Remark 9.33 (Cannot reduce beyond $2k+1$)** Before we move on, note that we could only make this argument for $N > 2k+1$. Hence this induction argument of shrinking N further and further cannot be extended beyond $2k+1$.

Let π be the **projection** of $\mathbb{R}^N$ onto the **orthogonal complement** H of a.

- **First claim:** $\pi \circ f : X \to H$ is **injective**.

To prove the claim, suppose that $\pi \circ f(x) = \pi \circ f(y)$. Then, since π is **linear**, we have

$$\pi(f(x) - f(y)) = 0,$$

i.e.,

$$\begin{aligned} f(x) - f(y) \in \operatorname{Ker}(\pi) &= \operatorname{span}(a) \text{ in } \mathbb{R}^N \\ &= \{w \in \mathbb{R}^N : w = t \cdot a \text{ for some } t \in \mathbb{R}\}. \end{aligned}$$

Thus there is a $t \in \mathbb{R}$ with $f(x) - f(y) = ta$. If $x \neq y$ then $t \neq 0$, since f **is injective**. But then

$$a = 1/t(f(x) - f(y)) = h(x, y, 1/t)$$

which **contradicts** the choice of a **not** being in the image of h.

- **Second claim:** $\pi \circ f : X \to H$ is an **immersion**.

To prove the claim we suppose there was a nonzero vector v in $T_x(X)$ for which $d(\pi \circ f)_x = 0$. Because π is **linear**, we have $d\pi_{f(x)} = \pi$ and the chain rule yields

$$d(\pi \circ f)_x = \pi \circ df_x.$$

Thus $\pi(df_x(v)) = 0$, so $df_x(v) = ta$ for some $t \in \mathbb{R}$. Because f **is an immersion**, we must have $ta \neq 0$. Since we know $a \neq 0$, this implies $t \neq 0$. Thus, since df_x is linear,

$$a = \frac{1}{t} df_x(v) = df_x\left(\frac{1}{t}v\right) = g\left(x, \frac{1}{t}v\right)$$

which again **contradicts** the **choice** of a **not** being in the image of g. □

For **compact** manifolds, one-to-one immersions are **embeddings**. So we have just proved the embedding theorem in the compact case.

Theorem 9.34 **(Whitney's Embedding for compact manifolds)** Every **compact** smooth k-dimensional manifold $X \subset \mathbb{R}^N$ admits an **embedding** into $\mathbb{R}^{2k+1}$.

9.7.2 Whitney's Embedding Theorem

In order to extend Whitney's Theorem 9.31 to **non-compact** manifolds, we have to **modify** the immersion to **make it proper**. This is a **topological**, not a differential problem. The **key tool** to solve this problem will again be partitions of unity. In particular, they allow us to prove the following key lemma:

Lemma 9.35 **(Existence of proper functions on manifolds)** On every smooth manifold X, there is a **proper** smooth function $p : X \to \mathbb{R}$.

Proof: Let $\{U_\alpha\}$ be the collection of open subsets of X that have **compact closure**, and let $\{\rho_i\}$ be a subordinate **partition of unity**. Then

$$p(x) = \sum_{i=1}^{\infty} i\rho_i(x)$$

is a **well-defined smooth** function, since, in a neighborhood of every point, it is a **finite sum** of smooth functions.

In order to show that p is **proper**, we need to show that the **preimage** of any compact subset of $\mathbb{R}$ is **again compact**. Every compact subset $K \subset \mathbb{R}$ is contained in a **closed interval** of the form $[-j, j]$ for some large enough natural number j. Hence if we can show that $p^{-1}([-j, j])$ is compact, then $p^{-1}(K)$ is a **closed** subset of a **compact** set and therefore **also compact**.

For a given natural number j, **assume** we have $\rho_1(x) = \cdots = \rho_j(x) = 0$ for any $x \in X$. Then, by **definition of a partition of unity**, we have

$$\sum_{i=j+1}^{\infty} \rho_i(x) = 1$$

and therefore

$$p(x) \geq (j+1) \sum_{i=j+1}^{\infty} \rho_i(x) = j + 1 > j.$$

This shows

$$p^{-1}([-j, j]) \subset \bigcup_{i=1}^{j} \{x \in X : \rho_i(x) \neq 0\}.$$

Since $\{\rho_i\}$ is a partition of unity subordinate to $\{U_\alpha\}$, we have $\operatorname{supp}(\rho_i) \subset U_i$. Since U_i has **compact closure** and the **finite union** of compact sets is compact, this shows that $p^{-1}([-j, j])$ is a **closed** subset in a **compact** set and therefore it is also compact. □

Theorem 9.36 (Whitney's Embedding Theorem) Every smooth k-dimensional manifold $X \subset \mathbb{R}^N$ admits an embedding into $\mathbb{R}^{2k+1}$.

Remark 9.37 We note that the strongest general result is that $N = 2k$ suffices. But this is much harder to prove. There are, of course, many examples of smooth manifolds for which an even **lower dimension suffices**: e.g., the n-sphere $\mathbb{S}^n$ is embedded in $\mathbb{R}^{n+1}$.

Proof of Theorem 9.36: The **idea** is to **replace the injective immersion** $f : X \hookrightarrow \mathbb{R}^N$ with the map $(f, p) : X \hookrightarrow \mathbb{R}^{N+1}$ with a **proper** $p : X \to \mathbb{R}$. Then (f, p) is **still an injective immersion**, and it is **proper**, since p is proper. It remains to reduce the dimension $N + 1$. The details are a bit **more involved**:

Starting with $X \subset \mathbb{R}^N$ we have seen that we can find an **injective immersion** $f : X \to \mathbb{R}^{2k+1}$. By **composing** f with the **diffeomorphism**

$$\mathbb{R}^{2k+1} \to \mathbb{B}_1^{2k+1}(0), \; x \mapsto \frac{x}{1 + |x|^2},$$

we **can assume** that $|f(x)| < 1$ for all $x \in X$.

Let $p\colon X \to \mathbb{R}$ be a **proper** function which we know to exist by Lemma 9.35. We define a **new injective immersion**

$$F\colon X \to \mathbb{R}^{2k+2},\ x \mapsto (f(x), p(x)).$$

Since $2k+2 > 2k+1$, we can **apply the argument** from the proof of the previous theorem by Whitney and find a **nonzero** vector $a \in \mathbb{R}^{2k+2}$ such that

$$\pi \circ F\colon X \to H$$

is **still an injective immersion**, where π is the **projection** onto the **orthogonal complement** $H = \{b \in \mathbb{R}^{2k+2} : b \perp a\}$ of a in $\mathbb{R}^{2k+2}$. By **rescaling** we can assume $|a| = 1$. In other words, we can **assume** $a \in \mathbb{S}^{2k+1}$.

Since $\pi \circ F$ is an injective immersion for **almost every** $a \in \mathbb{S}^{2k+1}$, we **can assume** that a is **neither the north nor the south pole** on $\mathbb{S}^{2k+1}$. For if $\pi \circ F$ failed to be an injective immersion on these two points, it would suffice to rotate $\mathbb{S}^{2k+1}$ a bit (which is a diffeomorphism of $\mathbb{S}^{2k+1}$) in order to avoid north and south pole.

This will allow us to show that **$\pi \circ F$ is proper** as follows:

- **Claim:** Given any bound c, there exists another number d such that

$$\{x \in X : |(\pi \circ F)(x)| \leq c\} \subset \{x \in X : |p(x)| \leq d\}.$$

The **claim implies properness:** Since p is **proper**, the set

$$\{x \in X : |p(x)| \leq d\} = p^{-1}([-d, d])$$

is a **compact** subset of X. Thus the **preimage** under $\pi \circ F$ of **every closed ball** in H is a **compact** subset of X. Since every compact subset K of H is a **closed** subset of **some closed ball** in X, this shows that $(\pi \circ F)^{-1}(K)$ is a **closed** subset of a compact subset in X and **therefore also compact**.

Proof of the claim: **If** the claim was **false**, then there **exists** a c and a **sequence** of points $\{x_i\}$ in X for which

$$|(\pi \circ F)(x_i)| \leq c, \text{ but } |p(x_i)| \to \infty$$

as there would be no d bounding $|p(x_i)|$. By definition of the **projection** onto an orthogonal complement, for every $z \in \mathbb{R}^{2k+2}$, $\pi(z)$ is the one point in H for which $z - \pi(z)$ is a **multiple of a**. In particular,

$$F(x_i) - \pi(F(x_i)) \text{ is a } \textbf{multiple of a} \text{ for each } i,$$

and hence **so is** the vector

$$w_i := \frac{1}{p(x_i)} \left(F(x_i) - \pi(F(x_i))\right).$$

Let us look at **what happens** when i **tends to infinity**:

$$\frac{F(x_i)}{p(x_i)} = \left(\frac{f(x_i)}{p(x_i)}, 1\right) \longrightarrow (0, \dots, 0, 1)$$

because $|f(x_i)| < 1$ for all i and $p(x_i) \longrightarrow \infty$. We have

$$\left|\frac{\pi(F(x_i))}{p(x_i)}\right| \le \frac{c}{|p(x_i)|}.$$

Thus

$$\frac{\pi(F(x_i))}{p(x_i)} \longrightarrow 0 \Rightarrow w_i \longrightarrow (0, \dots, 0, 1).$$

But each w_i **is a multiple of** a. Hence the **limit** of the w_i must be a **multiple of** a as well. We conclude that a must be **either the north or south pole** of $\mathbb{S}^{k+1}$ which **contradicts our assumption** on a. This proves the claim and finishes the proof of the theorem. □

9.8 Existence of partitions of unity on abstract manifolds

Now we return to partitions of unity and prove that they exist on smooth manifolds. Before we can start the proof, we need some further preparation.

9.8.1 Bump functions revisited

Lemma 9.38 (Separating closed subsets) Let A and C be **disjoint closed** subsets in $\mathbb{R}^N$. Then there are **disjoint open** subsets U and V such that $A \subset U$ and $C \subset V$.

Proof: For each $a \in A$, choose an $\varepsilon_a > 0$ such that $B_{2\varepsilon_a}(a) \cap C = \emptyset$. This is possible since C is closed. Similarly, for each $c \in C$, choose an $\varepsilon_c > 0$ such that $B_{2\varepsilon_c}(c) \cap A = \emptyset$. We define

$$U := \cup_{a \in A} B_{\varepsilon_a}(a) \text{ and } V := \cup_{c \in C} B_{\varepsilon_c}(c).$$

Then U and V are open subsets with $A \subset U$ and $C \subset V$. We claim that U and V are **disjoint**.

For, if $x \in U \cap V$, then

$$x \in B_{\varepsilon_a}(a) \cap B_{\varepsilon_c}(c)$$

for some $a \in A$ and $c \in C$. By the **triangle inequality**, this implies

$$|a - c| < \varepsilon_a + \varepsilon_c.$$

But, if $\varepsilon_a \le \varepsilon_c$, then $|a - c| < 2\varepsilon_c$ and $a \in B_{2\varepsilon_c}(c)$. And, if $\varepsilon_c \le \varepsilon_a$, then $|a - c| < 2\varepsilon_a$ and $c \in B_{2\varepsilon_a}(a)$. Both cases are **impossible**. □

An **important tool** that we will need are **smooth bump functions** which we introduced in Section 8.1. Now we will need them in a slightly **more interesting form**:

Lemma 9.39 (Smooth bump functions revisited) Let $U \subset \mathbb{R}^N$ be **open** and $K \subset U$ be **compact**. Then there is a **smooth function** $\varphi : \mathbb{R}^N \to \mathbb{R}$ with $\varphi(x) = 1$ for all $x \in K$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^N \setminus C$ for some closed subset C with $K \subset C \subset U$.

Proof: By Lemma 8.9, we can construct a **smooth**, **nonincreasing** function h^r_ε as in Figure 9.6

such that

$$\begin{cases} h^r_\varepsilon(x) = 1 & |x-a| \leq r \\ 0 < h^r_\varepsilon(x) < 1 & r < |x-a| < r+\varepsilon \\ h^r_\varepsilon(x) = 0 & |x-a| \geq r+\varepsilon \end{cases}$$

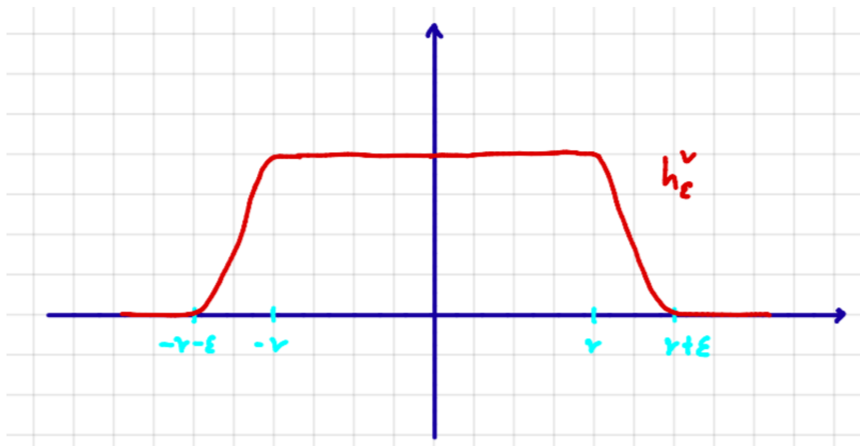

Figure 9.6: A function with a smooth transition from having constant value 1 to having constant value 0.

This gives us a smooth function $\mathbb{R}^N \to \mathbb{R}$ which has **value** 1 on the compact subset $\overline{\mathbb{B}_r(a)}$ and has **value** 0 **outside** the closed subset $\overline{\mathbb{B}_{r+\varepsilon}(a)}$. Now let $U \subset \mathbb{R}^N$ be **open** and $K \subset U$ be **compact**. For this general situation we need to **work a bit harder** and rearrange the argument as follows:

Let ψ be the function

$$\psi : \mathbb{R}^N \to \mathbb{R},\ \psi(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

We remember from Calculus that this is a smooth function. For a given $\varepsilon > 0$, we define $\psi_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ by

$$\psi_\varepsilon(x) := \frac{\psi(x/\varepsilon)}{\int_{\mathbb{R}^N} \psi(x/\varepsilon) dx}.$$

This is still a **smooth function** with $\int_{\mathbb{R}^N} \psi_\varepsilon dx = 1$ where dx denotes the standard Lebesgue measure on $\mathbb{R}^N$.

Since $\mathbb{R}^N \setminus U$ is closed and K is compact, we can choose by Lemma 9.38 a small $\varepsilon > 0$ such that, for each point $x \in K$, we have $B_{2\varepsilon}(x) \cap U = \emptyset$. Then the $V := \cup_{x \in K} B_\varepsilon(x)$ is an open set containing K with **compact closure** $\overline{V} \subset U$ contained in U.

Let χ_V be the **characteristic function** on V, i.e. the function

$$\chi_V : \mathbb{R}^N \to \mathbb{R}, \quad \begin{cases} \chi_V(x) = 1 & \text{for } x \in V \\ \chi_V(x) = 0 & \text{for } x \notin V. \end{cases}$$

The function χ_V is identically 1 on K and has **compact support** contained in U. But it is of course not smooth on $\mathbb{R}^N$, not even continuous. Hence we **need to modify it**, to make it **smooth**. The function ψ_ε, for the fixed ε, will serve as a **tool** to make χ_V **smooth**.

Then the desired smooth function φ is the **convolution** $\psi_\varepsilon * \chi_V$ of χ_V and ψ_ε:

$$\varphi : \mathbb{R}^N \to \mathbb{R}, x \mapsto \int_{\mathbb{R}^N} \psi_\varepsilon(x-y)\chi_V(y)dy.$$

Note that the integral is well-defined, since the support of χ_V, i.e., the closure of V, is **compact**.

□

9.8.2 Existence of partitions of unity

We are going to show that partitions of unity exist on manifolds **step by step** with increasing difficulty. We start with the case of **compact subspaces** in $\mathbb{R}^N$. Then we are going to transport this result to **compact smooth manifolds**. Finally, we discuss **arbitrary compact smooth k-manifolds**. There is no need to restrict to compact manifolds. In fact, partitions of unity exist on every **paracompact** topological space, i.e., on spaces where every open cover has a **locally finite refinement**. This is a class of spaces which is much larger than abstract manifolds.

- **First case:** $X \subset \mathbb{R}^N$ compact.

Let $\{U_\alpha\}$ be an open cover of X. Since X is **compact**, $\{U_\alpha\}$ has a **finite** subcover $\{U_1, \dots, U_n\}$. A partition of unity **subordinate** to the **finite subcover** is also a partition of unity subordinate to the **original cover**.

Step 1: We are going to show that we can **shrink the covering** to an open covering $\{V_1 \ \dots, V_n\}$ such that $\overline{V}_i \subset U_i$ for each i.

Consider the **closed** subset

$$A := X \setminus (U_2 \cup \dots \cup U_n)$$

of X. Since $\{U_1, \dots, U_n\}$ cover X, we know $A \subset U_1$. Since A and $X \setminus U_1$ are **closed disjoint**, we can choose an **open** subset V_1 **containing** A such that V_1 is disjoint to an open subset W which contains $X \setminus U_1$. Thus V_1 is contained in the complement $X \setminus W$. Since $X \setminus W$ is a closed subset which contains V_1, we know $\overline{V}_1 \subset X \setminus W$ since the **closure** of V_1 is the **intersection** of all closed subsets which contain V_1. Since $X \setminus U_1 \subset W$ by the choice of W, we have

$X \setminus W \subset X \setminus (X \setminus U_1) = U_1$. Thus we have $\overline{V}_1 \subset U_1$. Since V_1 contains the complement of $U_2 \cup \cdots \cup U_n$ in X, the collection $\{V_1, U_2, \ldots, U_n\}$ covers X.

Now we **proceed by induction** as follows: Given open subsets $V_1, \ldots, V_{k-1}$ such that

$$X = \{V_1, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\},$$

let A_k be the subset

$$A_k = X \setminus (V_1 \cup \cdots \cup V_{k-1}) \cup (U_{k+1} \cup \cdots \cup U_n).$$

Then A_k is a closed subset of X which is contained in the open set U_k. Choose an open subset V_k containing A_k such that $\overline{V}_k \subset U_k$. Then $\{V_1, \ldots, V_{k-1}, V_k, U_{k+1}, \ldots, U_n\}$ covers X. At the nth step of the induction **we are done**.

Step 2: Given the **open covering** $\{U_1, \ldots, U_n\}$ of X, we **use Step 1** to choose an open cover $\{V_1, \ldots, V_n\}$ of X such that $\overline{V}_i \subset U_i$ for each i. Then we **repeat this process** and choose an open cover $\{W_1, \ldots, W_n\}$ of X such that $\overline{W}_i \subset V_i$ for each i.

For each i, we choose by Lemma 9.39 a **smooth bump function**

$$\varphi_i : X \to [0,1] \text{ such that } \varphi_i(\overline{W}_i) = \{1\} \text{ and } \varphi_i(X - V_i) = \{0\}.$$

Since $\varphi_i^{-1}(\mathbb{R} \setminus \{0\}) \subset V_i$, we have

$$\operatorname{supp}(\varphi_i) \subset \overline{V}_i \subset U_i.$$

Note: Here is the point where we see why we need to **apply Step 1 twice**: If we were working with the V_i's instead of W_i's, then we would have $\operatorname{supp}(\varphi) \subset \overline{U}_i$ instead of $\operatorname{supp}(\varphi) \subset U_i$ as required for a partition subordinate to the cover $\{U_i\}$.

Since $\{W_1, \ldots, W_n\}$ covers X, we have

$$\varphi(x) := \sum_{i=1}^{n} \varphi_i(x) > 0 \text{ for all } x \in X.$$

Finally, for each i, we define

$$\rho_i(x) := \frac{\varphi_i(x)}{\varphi(x)}.$$

- **Second case:** $X \subset \mathbb{R}^N$ and $X = X_1 \cup X_2 \cup X_3 \cup \cdots$ where each X_i is **compact** and $X_i \subset \operatorname{int}(X_{i+1})$.

Let $\{U_\alpha\}$ be an **open cover** of X. For each i, we define

$$U_\alpha^i := U_\alpha \cap (X_{i+1} \setminus \operatorname{int}(X_{i-2})).$$

Then $\{U_\alpha^i\}$ is an **open cover** of $Y_i := X_i \setminus \operatorname{int}(X_{i-1})$. Since $\operatorname{int}(X_{i-1})$ is an open subset, Y_i is a closed subset of X_i and therefore Y_i is also **compact**. Then, for each i, the **first case implies** that there is a **partition of unity** φ_α^i on Y_i subordinate to the cover $\{U_\alpha^i\}$.

For each $x \in X$, there is an i such that $x \in X_i$ and hence $\varphi^j_\alpha(x) = 0$ for all $j \geq i+2$. Hence, for each $x \in X$, the sum

$$\varphi(x) := \sum_{\alpha,i} \varphi^i_\alpha(x)$$

is a finite sum in some open set containing x. For each α, we define

$$\rho^i_\alpha(x) := \frac{\varphi^i_\alpha(x)}{\varphi(x)}$$

This is a **partition of unity** subordinate to the open cover $\{U_\alpha\}$.

- **Third case:** $X \subset \mathbb{R}^N$ is open.

Define subsets

$$X_i := \{x \in X : |x| \leq i \text{ and the distance to } \mathbb{R}^N \setminus X \text{ is } \geq 1/j\}.$$

Then these subsets satisfy:

- each X_i is **compact**, since it is the intersection $X \cap \overline{B_i(0)} \cap (X \setminus (\cup_{p\in\mathbb{R}^N\setminus X} B_{1/i}(p))$ and therefore closed and bounded in $\mathbb{R}^N$;
- for each i: $X_i \subset \text{int}(X_{i+1})$;
- $X = X_1 \cup X_2 \cup \cdots$.

Hence we can **apply the second case**.

- **Fourth case:** $X \subset \mathbb{R}^N$ arbitrary.

Let $\{U_\alpha\}$ be an open cover of X. By the definition of the subspace topology on X, for each α, there is a subset $V_\alpha \subset \mathbb{R}^N$ open in $\mathbb{R}^N$ such that $U_\alpha = X \cap V_\alpha$. Let Y be the union of all the V_α in $\mathbb{R}^N$. By the **third case**, there is a partition of unity on Y subordinate to the open cover $\{V_\alpha\}$. This is also a partition of unity on X subordinate to the open cover $\{U_\alpha\}$.

- **Last case:** X is a **compact abstract** smooth k-manifold.

Let $\{V_\alpha\}$ be an open over of X. By intersecting with the domains of charts on X, we get a refinement of the cover. Hence we can assume that V_α are the **domains of charts** on X. Since X is **compact**, the domains of **finitely many** charts on X **suffice** to cover X. Let us label them $(V_1, \phi_1), \dots, (V_n, \phi_n)$. Then each $U_i = \phi_i(V_i)$ is an open subset in $\mathbb{R}^k$.

Now we can **proceed exactly as in the case of a compact subspace in $\mathbb{R}^N$** for the **finite cover** $\{U_1, \dots, U_n\}$ of the space $Y := U_1 \cup \cdots \cup U_n \subset \mathbb{R}^k$. This yields a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_1, \dots, U_n\}$. Composition of each ρ_i with ϕ_i yields a partition of unity $\{\rho_i \circ \phi_i\}$ on X subordinate to the cover $\{V_1, \dots, V_n\}$. □

9.9 Exercises and more examples

Exercise 9.1 Let X be the set of all straight lines in $\mathbb{R}^2$ (not just lines through the origin).

(a) Show that X is an abstract smooth 2-manifold by showing that we can identify X with an open subset of the real projective plane $\mathbb{RP}^2$.[a]

(b) Show that there is a bijection between X and the set of equivalence classes

$$(\mathbb{S}^1 \times \mathbb{R}) / \sim$$

where $\sim$ is the equivalence relation defined by

$$(s, x) \sim (y, t) \iff t = \pm s \text{ and } y = x.$$

[a] Here we use that open subsets of abstract smooth k-manifolds are again abstract smooth k-manifolds.

Exercise 9.2 Recall the *Hopf map* π that we have seen previously: We consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, i.e., $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, and $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e., $\mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$. Then the *Hopf map* π is the map $\mathbb{S}^3 \to \mathbb{S}^2$ given by

$$\pi(z, w) = \left(2z\bar{w}, |z|^2 - |w|^2\right).$$

In this exercise we study another way to define the Hopf map:

Let $\mathbb{CP}^1$ denote the complex projective space consisting of all complex-one-dimensional linear subspaces in $\mathbb{C}^2$. We can consider $\mathbb{CP}^1$ as the set of pairs $(z, w) \neq (0, 0)$ of complex numbers modulo the equivalence relation

$$(z_0, w_0) \sim (z_1, w_1) \iff z_1 = \lambda z_0 \text{ and } w_1 = \lambda w_0 \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

We consider $\mathbb{CP}^1$ as a topological space as the quotient $(\mathbb{C}^2 \setminus \{(0, 0)\}) / \sim$ and write $[z : w]$ for the equivalence class of (z, w).

(a) Show that $\mathbb{CP}^1$ is a two-dimensional abstract smooth manifold.

Hint: Follow the outline for $\mathbb{RP}^n$.

(b) We consider $\mathbb{S}^3$ as a subset in $\mathbb{C}^2$. Show that $\mathbb{CP}^1$ also can be defined as a quotient $\mathbb{CP}^1 = \mathbb{S}^3 / \sim_s$ by an appropriate equivalence relation $\sim_s$.

(c) We consider $\mathbb{S}^3$ as a subset in $\mathbb{C}^2$ and define the map

$$\varphi : \mathbb{S}^3 \to \mathbb{CP}^1, \; (z, w) \mapsto [z : w].$$

Find a map $h : \mathbb{CP}^1 \to \mathbb{S}^2$ such that the composition

$$\mathbb{S}^3 \xrightarrow{\varphi} \mathbb{CP}^1 \xrightarrow{h} \mathbb{S}^2$$

equals π.

Hint: Think of the stereographic projection.

(c) Let $[z_0, w_0]$ be a point in $\mathbb{CP}^1$. Describe the fiber $\varphi^{-1}([z_0 : w_0])$.

In the next exercise we let a group act on a space. If you are not familiar with group actions yet, you may want to skip this exercise or first read about group actions in a textbook of your choice.

Exercise 9.3 In this exercise we study examples of *Hopf manifolds*:

Let $0 < \lambda < 1$ be a fixed real number. We let $k \in \mathbb{Z}$ act on $\mathbb{C}^n \setminus \{0\}$ by

$$(z_1, \dots, z_n) \mapsto (\lambda^k z_1, \dots, \lambda^k z_n).$$

The Hopf manifold H^{2n}_λ is defined as the quotient

$$H^{2n}_\lambda := (\mathbb{C}^n \setminus \{0\})/\mathbb{Z},$$

i.e., we identify points z and $\lambda^k z$ for every $k \in \mathbb{Z}$.

We can show that H^{2n}_λ is a $2n$-dimensional abstract smooth manifold.[a] To focus on the important features, we will restrict our study to the cases $n = 1$ and $n = 2$. Though the general argument follows the same outline. We are also going to fix a λ and write H^{2n} for H^{2n}_λ.

(a) Show that H^2 is a 2-dimensional abstract smooth manifold.[b]

(b) Let $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\} \subset \mathbb{C}$. Show that H^2 homeomorphic to the quotient $A/\mathbb{Z}$ and deduce that H^2 is a compact space.

(c) Show that H^4 is a 4-dimensional abstract smooth manifold.[c]

Hint: It is the same argument as for $n = 1$. We are just practising a bit more.

(d) Show that there is a homeomorphism between H^4 and $\mathbb{S}^3 \times \mathbb{S}^1$. Then show that this actually a diffeomorphism by showing that the compositions with coordinate charts are smooth.[d]

Note: The Hopf surface H^4_λ is the simplest example of a compact complex manifold which cannot be equipped with a Kähler metric and cannot be holomorphically embedded into complex projective space. This follows from that fact that $H^4_\lambda \cong \mathbb{S}^3 \times \mathbb{S}^1$ implies that the first cohomology of a Hopf surface is one-dimensional, whereas every Kähler manifold must have an even-dimensional first cohomology. This is a first glimpse at the exciting theory of complex geometry.

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[a] Once we have learned more general theory, we can say that this follows from the fact that the action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ is *free and discrete*.

[b] Note that H^2 is called the *Hopf curve*, since it is usually considered as a *complex* manifold of *complex* dimension one.

[c] Note that H^4 is called the *Hopf surface*, since it is usually considered as a *complex* manifold of *complex* dimension two.

[d] In general, H^{2n} is diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. The previous point for $n = 1$ is a special case of this fact, since $A/\mathbb{Z} \cong \mathbb{S}^1 \times \mathbb{S}^1$.

Exercise 9.4 For $0 < m \le n$, the real *Milnor hypersurface* $H(m,n)$ is defined by

$$H(m,n) = \left\{ ([x_0 : \ldots : x_m], [y_0 : \ldots : y_n]) \in \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^n : \sum_{i=0}^{m} x_i y_i = 0 \right\}.$$

Show that $H(m,n) \subset \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^n$ is an abstract smooth manifold of dimension $m+n-1$.

Hint: Show that the subsets $V_{ij} = \{([x_0 : \ldots : x_m], [y_0 : \ldots : y_n]) \in H(m,n) : x_i \neq 0, y_j \neq 0\}$ are open and their union for all $i \neq j$ covers $H(m,n)$.

10. Manifolds with Boundary

10.1 Motivation: A first glimpse at intersection theory

Before we introduce the main new definition of this chapter, we look at an interesting problem we like to solve.

Consider the two smooth manifolds $\mathbb{S}^2$ and $\mathbb{R}\mathrm{P}^2$, real projective 2-space. We have seen that both are smooth manifolds of **dimension** 2, both are **connected** and **compact**. So one might wonder if there is any topological feature that distinguishes these two spaces. In other words, one might even wonder: are they homeomorphic or even diffeomorphic? The answer is **no**:

- **(A new challenge)** The manifolds $\mathbb{S}^2$ and $\mathbb{R}\mathrm{P}^2$ are **not** homotopy equivalent and therefore not diffeomorphic. But **how** can we prove this?

To attack this problem, we need a **new idea**. For example, we could study the **homotopy classes** of loops on $\mathbb{S}^2$ and $\mathbb{R}\mathrm{P}^2$, i.e., of continuous maps $\mathbb{S}^1 \to X$ for X being either $\mathbb{S}^2$ or $\mathbb{R}\mathrm{P}^2$. If these two spaces were homeomorphic, the sets of such homotopy classes of loops would be the same. But we can show that the sets we get for $\mathbb{S}^2$ and $\mathbb{R}\mathrm{P}^2$, respectively, are different and hence $\mathbb{S}^2 \not\approx \mathbb{R}\mathrm{P}^2$:

- For $\mathbb{S}^2$, every continuous map $\mathbb{S}^1 \to \mathbb{S}^2$ is **homotopic to a constant map**. In fact, $\mathbb{S}^2$ is a simply-connected space. One way to show this is to use the stereographic projection and to use that $\mathbb{R}^2$ is contractible. This requires, however, to show that every loop is homotopic to a loop which does not pass through a given point on $\mathbb{S}^2$. This is not difficult, but takes some time to write down. We skip this for the moment, but we could do it!
- For $\mathbb{R}\mathrm{P}^2$, however, there is a map $\bar{f}\colon \mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2$ that is **not** homotopic to a constant map.

Let us see how such a map $\bar{f}\colon \mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2$ could look like: We define $\bar{f}$ via the commutative diagram

$$\begin{tikzcd} {[0,2\pi]} \arrow[r, "f"] \arrow[d] & \mathbb{S}^2 \arrow[d, "\pi"] \\ \mathbb{S}^1 \arrow[r, "\bar{f}"'] & \mathbb{R}\mathrm{P}^2 = \mathbb{S}^2/\sim \end{tikzcd}$$

where π is the quotient map, the left vertical map sends t to $(\cos t, \sin t) \in \mathbb{S}^1$ and f is defined by

$$f\colon [0, 2\pi] \to \mathbb{S}^2,\ t \mapsto (\cos(t/2), \sin(t/2), 0).$$

Since $(\cos 0, \sin 0, 0) = (-1) \cdot (\cos \pi, \sin \pi, 0)$, the map $\pi \circ f$ induces a **continuous map**

$$\bar{f}\colon [0, 2\pi]/(0 \sim 2\pi) = \mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2.$$

We can even show that $\bar{f}$ is **smooth**: We show this locally for the induced map on local coordinate charts. On $\mathbb{R}P^2$ we use the charts we defined previously. Then we can check:

- For the charts $(0,\pi) \to \mathbb{S}^1$ and $(\pi, 2\pi) \to \mathbb{S}^1$ defined by $t \mapsto (\cos t, \sin t)$, $\bar{f}$ induces the map
$$t \mapsto \left(\frac{\cos(t/2)}{\sin(t/2)}, 0 \right).$$
- For the charts $(\pi/2, 3\pi/2) \to \mathbb{S}^1$ and $(3\pi/2, 5\pi/2) \to \mathbb{S}^1$ defined by $t \mapsto (\cos t, \sin t)$, $\bar{f}$ induces the map
$$t \mapsto \left(\frac{\sin(t/2)}{\cos(t/2)}, 0 \right).$$

All of these maps are **smooth** and hence $\bar{f}$ is smooth. But now we need to answer the question:

Question **How** do we show that $\bar{f}$ is **not** homotopic to a constant map?

The answer will be given in Section 14.1 and Section 14.4:

- **(Intersection Theory will do it!)** We will develop a theory of **intersections up to homotopy** that will help us solve the problem. For we will be able to show: If $g : \mathbb{S}^1 \to \mathbb{R}P^2$ is any **curve homotopic** to $\bar{f}$ so that $g \pitchfork \operatorname{Im}(\bar{f})$, then $\# g^{-1}\left(\operatorname{Im}(\bar{f})\right)$ is an **odd** number. In particular, we will have $g^{-1}\left(\operatorname{Im}(\bar{f})\right) \neq \emptyset$. However, the constant map $c : \mathbb{S}^1 \to \mathbb{R}P^2$ with image $[1:1:1]$ has **no** intersection with $\operatorname{Im}(\bar{f})$. Hence $c \pitchfork \operatorname{Im}(\bar{f})$ with $c^{-1}\left(\operatorname{Im}(\bar{f})\right) = \emptyset$. Thus $\bar{f}$ **cannot** be homotopic to c.

In order to obtain such a theory of intersections, we already have many tools available, for example, homotopy and transversality. But we still lack one **important piece of the puzzle**: manifolds **with boundary**. For allowing manifolds to have a boundary will make it possible to include, in particular, the **closed** interval $[0,1]$ and, more generally, products $X \times [0,1]$ which we use in the definition of homotopies.

10.2 Manifolds with Boundary

In order to be able to analyse a wider class of phenomena we would like to enlarge the class of manifolds. A typical example which **we would like to include** is the domain of a **homotopy** $X \times [0,1]$ for a smooth k-dimensional manifold X. The points on $X \times \{0\}$ and $X \times \{1\}$ do **not** have an **open** neighborhood which is diffeomorphic to $\mathbb{R}^k$. Another example is the **closed** unit ball in $\mathbb{R}^k$.

So far such spaces do **not qualify** as a smooth manifold. From now on, we would like to allow such spaces. This sounds like an laborious endeavour, and it is one. However, the good news is that most of the theorems we have proved so far are also valid for manifolds with boundary.

The **idea** for what a manifold with boundary should be **is the same** as before: it is a space which locally looks like some model space with boundary which we understand well. Hence we need to choose a suitable **new model space**:

Definition 10.1 (New Euclidean model) The standard model of a Euclidean space with boundary is the half-plane

$$\mathbb{H}^k = \left\{(x_1, \dots, x_k) \in \mathbb{R}^k \,:\, x_k \geq 0\right\}$$

in $\mathbb{R}^k$. The **boundary** of $\mathbb{H}^k$, denoted $\partial\mathbb{H}^k$, is given by

$$\partial\mathbb{H}^k = \left\{(x_1, \dots, x_k) \in \mathbb{R}^k \,:\, x_k = 0\right\} = \mathbb{R}^{k-1} \times \{0\} \subset \mathbb{R}^k.$$

Now a manifold with boundary is a space which locally looks like $\mathbb{H}^k$:

Definition 10.2 (Manifolds with boundary) A subset X of $\mathbb{R}^N$ is a **smooth k-dimensional manifold with boundary** if every point x of X there is an **open neighborhood** $V \subset X$ containing x and an **open** subset $U \subset \mathbb{H}^k$ together with a **diffeomorphism** $\phi : U \to V$. As before, any such a diffeomorphism is called a **local parametrization** of X.

The **boundary** of X, denoted ∂X, consists of those points that belong to the image of the boundary of $\mathbb{H}^k$ under some local parametrization. Its complement is called the **interior** of X, denoted $\mathrm{Int}(X) = X \setminus \partial X$.

- A manifold X with $\partial X = \emptyset$ is just a smooth manifold in our previous terminology. In order to make the distinction clear, if necessary, we call them also **boundaryless manifolds** or **manifolds without a boundary**.

Example 10.3 (The closed unit interval) The unit interval $[0, 1]$ is a one-dimensional smooth **manifold with boundary** and the boundary consists of the two endpoints $\{0, 1\}$. We can choose local parametrizations

$$\phi_0 : [0, 1) \to [0, 1],\ x \mapsto x,\ \text{and}\ \phi_1 : [0, 1) \to [0, 1],\ x \mapsto 1 - x$$

defined on $[0, 1)$ which is an open subset on $\mathbb{H}^1 = [0, \infty) \subset \mathbb{R}^1$. The interior is the open interval $(0, 1)$.

- **Warning:** The **interior** of $X \subset \mathbb{R}^N$ as a manifold is in general different from the interior of X as a subspace of $\mathbb{R}^N$. The interior of X as a manifold is the complement of the boundary, whereas the interior of the topological space X is the union of all its open subsets. But also every point in ∂X lies in some open neighborhood of X.

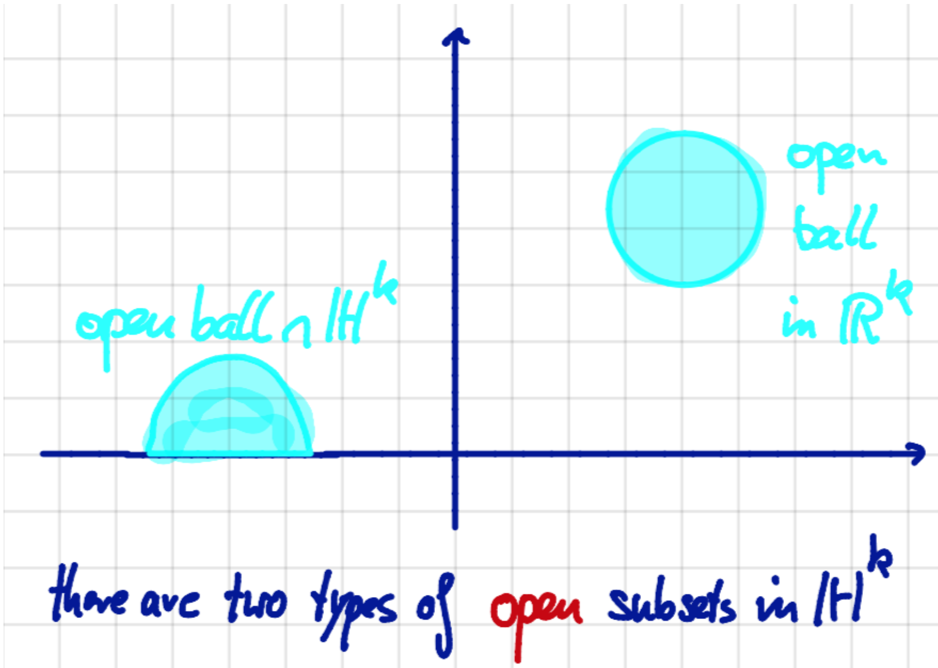


Figure 10.1: There are two types of open balls in $\mathbb{H}^k$. The balls where all points satisfy $x_k > 0$ are well-known. The balls which allow $x_k = 0$ are new.

10.2.1 The interior is well-defined

Let X be a manifold **with boundary**. We need to check that our definition of points in the interior and on the boundary is **independent of the choice** of a local parametrization.

So let $x \in X$ be a point which is in the image of a local parametrization $\phi: U \to V \subset X$ such that $U \subset \mathbb{H}^k$ is an open set of $\mathbb{H}^k$ which is contained in the interior of $\mathbb{H}^k$. Then $\mathbb{R}^k$ is an open subset of $\mathbb{R}^k$. Now assume x is also in the image of another local parametrization $\phi' : U' \to V' \subset X$. Then $x \in W := V \cap V' \subset X$, and the composition $\phi' \circ \phi^{-1} : \phi^{-1}(W) \to (\phi')^{-1}(W)$ is a diffeomorphism. Hence, after possibly shrinking U', we see that U' is also an open subset in $\mathbb{R}^k$. Thus x is being an interior point is well-defined. □

This shows in particular: if X is a manifold with boundary, then the **interior** of X, $\mathrm{Int}(X)$, is a **boundaryless manifold** of the same dimension as X.

10.2.2 The boundary is well-defined

It remains to show that being a **boundary point** is also well-defined. We show this by proving the following interesting result:

Theorem 10.4 **(Boundaries are manifolds)** If X is a k-dimensional **manifold with boundary**, then ∂X is a $(k-1)$-dimensional **manifold without boundary**.

Proof: Let $x \in X$ and let ϕ and ψ be two local parametrizations around x. After possibly shrinking the domains and codomains, we can assume that $\phi : U \to V$ and $\psi : W \to V$ are both diffeomorphisms from open sets $U \subset \mathbb{H}^k$, $W \subset \mathbb{H}^k$ to the **same open subset** $V \subset X$.

We would like to show $\phi(\partial U) = \psi(\partial W)$. For then $\partial V = \phi(\partial U)$ is independent of our choice of local parametrization and therefore well-defined. Moreover, since $\partial U = U \cap \partial \mathbb{H}^k$ is an open subset of $\mathbb{R}^{k-1}$, we would get that every point $y \in \partial X$ is contained in a local parametrization $\phi_{|\partial U} : U \cap \partial \mathbb{H}^k \to \partial X$. This will show that ∂X is a manifold of dimension $k-1$.

By our assumption on ϕ and ψ, it suffices to show $\psi(\partial W) \subset \phi(\partial U)$. The other inclusion will follow by symmetry. Hence we would like to show:

- **Claim:** $\phi^{-1}(\psi(\partial W)) \subset \partial U$.

To simplify notation, we define the map $g := \phi^{-1} \circ \psi : W \to U$. **Suppose** that the claim is **false** and there is a point $w \in \partial W$ which is mapped to an **interior point** $u = g(w)$ of U by g. Since both ϕ and ψ are diffeomorphisms, **g is a diffeomorphism** of W onto an open subset $g(W)$ of U, and the derivative $d(g^{-1})_u$ is an isomorphism. However, since $u \in \mathrm{Int}(U)$, $g(W)$ contains a neighborhood of u that is **open in $\mathbb{R}^k$**. Thus the **Inverse Function Theorem 3.4**, applied to the map g^{-1} defined on this open subset of $\mathbb{R}^k$, implies that the image of g^{-1} contains a neighborhood of w that is **open in $\mathbb{R}^k$**. This contradicts the assumption $w \in \partial W$. □

10.3 Derivatives and tangent spaces vs boundaries

Tangent spaces and derivatives are still defined in the setting of manifolds with boundary. Derivatives of smooth maps can be defined as before: Since smoothness at a point requires a functions to be defined on open neighborhood around that point, we need to be a bit more careful at boundary points:

10.3.1 Derivatives on $\mathbb{H}^k$

Definition 10.5 Suppose that g is a smooth map of an open set U of $\mathbb{H}^k$ to $\mathbb{R}^l$. If u is an interior point of U, then the derivative dg_u is defined as before. If $u \in \partial U$ is a boundary point, the smoothness of g means that it may be **extended to a smooth map** G defined in an **open** neighborhood of u in $\mathbb{R}^k$. We **define** dg_u to be the derivative $dG_u : \mathbb{R}^k \to \mathbb{R}^l$.

Lemma 10.6 This definition is **independent of the choice** of G.

Proof: Let G' be **another local extension** of g. We need to show $dG'_u = dG_u$. The equality of the two derivatives is **no problem at points in the interior** $\mathrm{Int}(U)$ of U, because then we have a small open neighborhood which is still in $\mathrm{Int}(U)$. We are going to use this and **approximate** u be a **sequence** $\{u_i\}$ of interior points $u_i \in \mathrm{Int}(U)$ which **converge to** u.

Since G and G' agree with g on $\mathrm{Int}(U)$, we have

$$dG_{u_i} = dG'_{u_i} \text{ for all } i.$$

Since the **derivative** of a smooth map at a point depends **continuously** on the point, this implies that $dG_{u_i} \to dG_u$ and $dG'_{u_i} \to dG'_u$ when $u_i \to u$ and **both limits agree**. Hence dg_u is well-defined **at boundary points as well**. □

- **Note:** It is important to observe that, at all points, dg_u is still a **linear map** of all of $\mathbb{R}^k$ to $\mathbb{R}^l$. For we have defined dg_u as the derivative dG_u of an extension G to an **open subset of $\mathbb{R}^k$**.

10.3.2 Tangent spaces

Let $X \subset \mathbb{R}^N$ be a smooth manifold with boundary, and $x \in X$. Let $\phi : U \to X$ be a local parametrization with $U \subset \mathbb{H}^k$ open. Let $u \in U$ be the point with $\phi(u) = x$. Note that we **cannot assume** $u = 0$ when x is an **interior** point. Then we have just learned that we can **form the derivative**

$$d\phi_u : \mathbb{R}^k \to \mathbb{R}^N$$

no matter what kind of point x is.

Definition 10.7 (Tangent spaces revisited) Let X be a smooth k-manifold with boundary. As before, we define the **tangent space** to X at x, denoted $T_x(X)$, to be the **image of $\mathbb{R}^k$** in $\mathbb{R}^N$ under the **linear map** $d\phi_u$.

It follows again from the **Chain Rule** that $T_x(X)$ **does not depend** as a subspace of $\mathbb{R}^N$ on the choice of ϕ.

10.3.3 Derivatives on tangent spaces

Now let $f : X \to Y$ be a smooth map between manifolds **with boundaries** with $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$. Given a point $x \in X$. Then after choosing local parametrizations $\phi : U \to X$ with $\phi(u) = x$ and $\psi : V \to Y$ with $\psi(v) = f(x)$, then we **define**

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

as the **unique linear map** which makes the following diagram **commutative**

$$\begin{tikzcd} T_x(X) \arrow[r, dotted, "df_x"] & T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_u"] \arrow[r, "d\theta_u"'] & \mathbb{R}^l \arrow[u, "d\psi_v"'] \end{tikzcd}.$$

where θ is the map $\psi^{-1} \circ f \circ \phi$ (note $v = \theta(u)$). □

10.3.4 Products and boundaries

We see that things work out nicely so far for manifolds with boundaries. But:

- **Warning:** Sometimes we do **have to be careful** when we apply our developed concepts to manifolds with boundaries. For example, the **product** of two manifolds with boundary **may not be a manifold** anymore.

Example 10.8 (Square is not a manifold) The **square** $S = [0,1] \times [0,1]$ is **not** a smooth manifold with boundary:

Suppose S was a smooth manifold with boundary. Then the corner $s = (0,0)$ had an **open** neighborhood $V \subset S$ and there is a **diffeomorphism** $f : U \to V$ to an open $U \subset \mathbb{H}^2$ such that $f(\partial V) \subset \partial \mathbb{H}^2$. After shrinking V if necessary, let $F : \tilde{V} \to \mathbb{R}^2$ be a **smooth extension** of f on a subset $V \subset \tilde{V} \subset \mathbb{R}^2$ **open** in $\mathbb{R}^2$. Then the derivative dF_s is an isomorphism, since f is a diffeomorphism. Writing $F = (F_1, F_2)$ we have $F_2(x,0) = 0 = F_2(0,y)$ for any $(x,0)$ and $(0,y)$ in U, since ∂V is mapped to $\partial \mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$. Thus taking partial derivatives yields

$$\frac{\partial F_2}{\partial x}(s) = 0, \text{ and } \frac{\partial F_2}{\partial y}(s) = 0.$$

But this implies that $dF_s(e_1)$ and $dF_s(e_2)$ both lie in $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$, where e_1 and e_2 denote the first and second standard basis vector in $\mathbb{R}^2$, respectively. In particular, $dF_s(e_1)$ and $dF_s(e_2)$ are **linearly dependent**. This **contradicts** that dF_2 is an isomorphism and therefore the existence of the diffeomorphism f.

However, if **only one** manifold has a boundary we are **ok**:

Lemma 10.9 (Products and Boundaries) The **product** of a manifold **without boundary** X and a manifold **with boundary** Y is a **manifold with boundary**. Furthermore,

$$\partial(X \times Y) = X \times \partial Y,$$

and $\dim(X \times Y) = \dim X + \dim Y$.

Proof: If $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^l$ are open, then

$$U \times V \subset \mathbb{R}^k \times \mathbb{H}^l = \mathbb{H}^{k+l}$$

is **open**. Moreover, if $\phi : U \to X$ and $\psi : V \to Y$ are local parametrizations, so is $\phi \times \psi : U \times V \to X \times Y$. □

Example 10.10 (Closed cylinder) The closed cylinder $\mathbb{S}^1 \times [0,1]$ is a manifold with boundary, see Figure 10.2, and the boundary consists of

$$\mathbb{S}^1 \times \{0\} \cup \mathbb{S}^1 \times \{1\}.$$

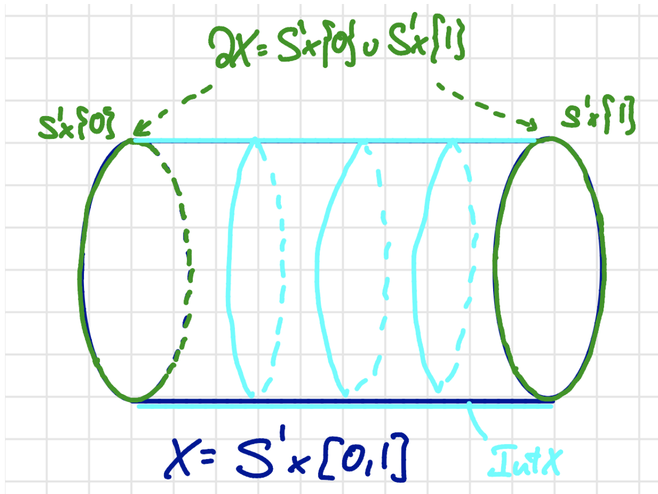


Figure 10.2: A cylinder has the two outer circles as its boundary.

Example 10.11 (Domain of a homotopy) More generally, if X is a manifold without boundary, then the **domain** $X \times [0, 1]$ of a homotopy $X \times [0, 1] \to Y$ is a **manifold with boundary**.

10.4 Regular values and transversality

One of the most **important concepts** we have studied is **transversality** of smooth maps to submanifolds. We would like to extend this to manifolds with boundary. This is possible, but **requires some care**.

10.4.1 Regular values of smooth functions

We start with the special case of regular values for functions on manifolds without boundary. This is a well-known case, but it turns out that it actually produces manifolds with boundary as follows:

Lemma 10.12 (Regular values for real-valued functions) Suppose that S is a manifold **without boundary** and that $f : S \to \mathbb{R}$ is a smooth **function with regular value** 0. Then the subset $\{s \in S : f(s) \geq 0\}$ is a manifold **with boundary**, and the boundary is $f^{-1}(0)$.

Proof: The set $\{x \in S : f(x) > 0\}$ is **open in** S, since it is the preimage of the open subset $(0, \infty) \subset \mathbb{R}$ under the continuous map f. It is therefore a **submanifold** of the same dimension as S. Hence every point in $\{x \in S : f(x) > 0\}$ has an **open neighborhood** which is diffeomorphic to an open subset of $\mathbb{R}^k$, $k = \dim S$. It **remains** to study the preimage of 0.

So suppose that s is a point with $f(s) = 0$. By assumption, 0 is a **regular value** which means that s is a regular point of f. Hence, **locally** near s, we can show as in the proof of the Local Submersion Theorem 4.2 that f is equivalent to the **canonical submersion**. But for the canonical submersion

$$f : \mathbb{H}^k \to \mathbb{R}, (x_1, \dots, x_k) \to x_k$$

the lemma just states the definition of the boundary of $\mathbb{H}^k$:

$$\partial\mathbb{H}^k = f^{-1}(0) = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_k = 0\}. \quad \square$$

An immediate consequence of this fact is:

Example 10.13 (Spheres are boundaries) Let π be the smooth function defined by

$$f : \mathbb{R}^k \to \mathbb{R}, (x_1, \dots, x_k) \mapsto 1 - \sum_i x_i^2.$$

Then 0 is a **regular value** of π, and the closed unit ball $\mathbb{D}^k$ in $\mathbb{R}^k$ can be described as

$$\mathbb{D}^k = \{x \in \mathbb{R}^k : f(x) \geq 0\}.$$

The **boundary of** $\mathbb{D}^k$ is the $(k-1)$-sphere $\mathbb{S}^{k-1} = f^{-1}(0)$.

10.4.2 Fibers over regular values with boundary

Recall that transversality is formulated as a **criterion on tangent spaces and derivatives**. We would like to formulate a similar criterion for maps between manifolds with boundary.

As we learned above, the boundary ∂X of a k-manifold with boundary X is a manifold of dimension $k-1$ without boundary. Let $x \in \partial X$ be a point on the boundary. We have $\dim T_x(\partial X) = k-1$ and $\dim T_x(X) = k$. Moreover, since ∂X is a submanifold of X, we know that

$$T_x(\partial X) \subset T_x(X)$$

is a **vector subspace** of **codimension** 1 in $T_x(X)$ (recall that the latter means the difference of dimensions equals 1).

Definition 10.14 (Notation: Restriction to boundary) For any smooth map $f : X \to Y$, we introduce the **notation**

$$\partial f = f_{|\partial X}$$

for the restriction of f to ∂X. The **derivative of** ∂f at x is just the **restriction of** df_x to the subspace $T_x(\partial X)$:

$$d(\partial f)_x = (df_x)_{|T_x(\partial X)} : T_x(\partial X) \to T_{f(x)}(Y).$$

Now let $f : X \to Y$ be a smooth map from a smooth manifold **with boundary** X to a **boundaryless** manifold Y, and let $Z \subset Y$ be a **submanifold**. We would like to know under which circumstances is $f^{-1}(Z)$ a **submanifold with boundary** of X, i.e., a subset of X which is itself a smooth manifold with boundary, such that we have **control over the boundary** in the sense that we **would like** to have

$$\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X. \tag{10.1}$$

It turns out that it is **not enough** to ask that f is transversal to Z in the previous sense, i.e., $\operatorname{Im}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$.

Example 10.15 Even for the restriction of the **canonical submersion**

$$\sigma : \mathbb{H}^2 \to \mathbb{R},\ (x_1, x_2) \mapsto x_2$$

our previous condition for transversality is **not sufficient** to guarantee that Equation 10.1 is satisfied as we will now explain:

The derivative $d\sigma_{(x_1,x_2)} : \mathbb{R}^2 \to \mathbb{R}$ is **just the projection** onto the second factor. Hence it is **surjective at every point** (x_1, x_2). In particular, 0 is a **regular value** for σ. Let $Z := \{0\}$. Then

$$\sigma^{-1}(Z) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\right\} = \partial\mathbb{H}^2.$$

Since 0 is **regular value**, we know that $\sigma^{-1}(Z)$ is a submanifold of dimension one. The **problem** is that the **boundary** does **not satisfy** the condition of Equation 10.1:

$$\partial\left(\sigma^{-1}(Z)\right) = \emptyset, \text{ whereas } \sigma^{-1}(Z) \cap \partial X = \partial\mathbb{H}^2 \neq \emptyset.$$

In order to make sure that the **boundary behaves nicely**, we need to impose an **additional** transversality **condition** on ∂f. We start again with regular values:

Theorem 10.16 (With boundary: fibers of regular values) Let g be a smooth map of a k-manifold X **with boundary** onto a **boundaryless** n-manifold Y, and suppose that $y \in Y$ is a **regular value** for **both** $f : X \to Y$ and $\partial f : \partial X \to Y$. Then the preimage $f^{-1}(y)$ is a $(k-n)$-dimensional manifold with boundary and

$$\partial\left(f^{-1}(y)\right) = f^{-1}(y) \cap \partial X.$$

- Note that $d(\partial f)_x$ is the restriction of df_x to the subspace $T_x(\partial X) \subset T_x(X)$. Hence, if $x \in \partial X$ is a **regular point** for ∂f, then it is also a regular point for df_x. However, not every point in $f^{-1}(y)$ is in ∂X, and we also need the points $x \in \operatorname{Int}(X) \cap f^{-1}(y)$ to be regular points.

Proof of Theorem 10.16: That $f^{-1}(y)$ is a manifold with boundary is a **local question**. This means that it suffices that **each point** in $f^{-1}(y)$ has an **open neighborhood** which is a **manifold with boundary**. So let $x \in X$ be a point with $f(x) = y$. After **choosing local coordinates**, we can assume that f is a map of the form

$$f : \mathbb{H}^k \to \mathbb{R}^n.$$

If x is an **interior point** in X, then $f^{-1}(y)$ is a manifold without boundary in an open neighborhood around x by the **Preimage** Theorem 4.7 for **boundaryless manifolds**.

So let us look at what happens when $x \in \partial X$. That f is smooth at x means by definition that there is an open subset $U \subset \mathbb{R}^k$ and a smooth map

$$F : U \to \mathbb{R}^n \text{ such that } F_{U \cap \mathbb{H}^k} = f_{U \cap \mathbb{H}^k}.$$

After possibly replacing U with a smaller subset, we can **assume** that F has **no critical** points in U. Then $F^{-1}(y)$ is a smooth manifold by the **Preimage** Theorem 4.7 for **boundaryless manifolds**. We need to show that

$$f^{-1}(y) = F^{-1}(y) \cap \mathbb{H}^k \text{ is a manifold with boundary.}$$

In order to show this, we set $S := F^{-1}(y)$ and let π be the **projection** to the last coordinate:

$$\pi : S \to \mathbb{R},\ (x_1, \dots, x_k) \mapsto x_k.$$

Then π is a smooth map and S is a smooth manifold with

$$S \cap \mathbb{H}^k = \{s \in S : \pi(s) \geq 0\}.$$

- **Claim:** 0 is a regular value of π.

If we can show the claim, then Lemma 10.12 implies that $S \cap \mathbb{H}^k$ is a manifold with boundary and the boundary is $\pi^{-1}(0)$.

To **prove the claim** we assume there was an $s \in S$ with both $\pi(s) = 0$, i.e., $s \in S \cap \partial \mathbb{H}^k$, and $d\pi_s = 0$. We want to show that the assumption $d\pi_s = 0$ leads to a **contradiction**. To do so, first note that π is a **linear** map, and therefore $d\pi_s = \pi$. Thus,

$$d\pi_s = \pi : T_s(S) \to \mathbb{R}$$

being **trivial**, just means that the **last coordinate** of every vector in $T_s(X)$ is 0, i.e.,

$$d\pi_s = 0 \Rightarrow T_s(S) \subset T_s(\partial \mathbb{H}^k) = \mathbb{R}^{k-1}.$$

Hence we want to show $T_s(S) \not\subset \mathbb{R}^{k-1}$. The tangent space to $S = F^{-1}(y)$ at s is the kernel of dG_s:

$$T_s(S) = T_s(F^{-1}(y)) = \operatorname{Ker}(dF_s = df_s : \mathbb{R}^k \to \mathbb{R}^n)$$

where $df_s = dF_s$ by definition of df_s. We know that $d(\partial f)_s$ is the restriction of $df_s : \mathbb{R}^k \to \mathbb{R}$ to $\mathbb{R}^{k-1}$:

$$d(\partial f)_s = (df_s)_{|\mathbb{R}^{k-1}}.$$

Thus, **if** $T_s(S) = \operatorname{Ker}(df_s) \subseteq \mathbb{R}^{k-1}$, then

$$\operatorname{Ker}(df_s) = \operatorname{Ker}(d(\partial f)_s). \tag{10.2}$$

Now, finally, we **apply the assumption** of regularity of y. Since y is a **regular** value of **both** f and ∂f, we know that **both** df_s and $d(\partial g)_s$ are **surjective**. This implies

$$\dim \operatorname{Ker}(dg_s) = k - n \text{ and } \dim \operatorname{Ker}(d(\partial g)_s) = k - 1 - n.$$

This **contradicts** Equation 10.2. Thus, we must have

$$T_s(S) = \operatorname{Ker}(df_s) \not\subset \mathbb{R}^{k-1}.$$

Hence we know $d\pi_s \neq 0$ and therefore $d\pi_s$ is **surjective**, and 0 is a **regular value**. □

10.4.3 Preimages of manifolds with boundary

We can now generalize Theorem 4.7:

Theorem 10.17 (Preimages of manifolds with boundary) Let f be a smooth map of a manifold X with boundary onto a boundaryless manifold Y, and suppose that **both** $f : X \to Y$ and $\partial f : \partial X \to Y$ are **transverse** with respect to a boundaryless submanifold Z in Y. Then the preimage $f^{-1}(Z)$ is a **manifold with boundary**

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X,$$

and the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.

Proof: The restriction of f to the boundaryless manifold $\mathrm{Int}(X)$ is transversal to Z. Hence, by the Preimage Theorem 4.7 for **boundaryless manifolds**, the intersection with the **interior** of X, i.e., $f^{-1}(Z) \cap \mathrm{Int}(X)$, is a **manifold without boundary** of correct codimension. Thus, it remains to examine $f^{-1}(Z)$ in a neighborhood of a point $x \in f^{-1}(Z) \cap \partial X$. Let $l := \operatorname{codim} Z$ in Y. As in the boundaryless case, we can choose a submersion $h : W \to \mathbb{R}^l$ defined on an open neighborhood W of $f(x)$ in Y to $\mathbb{R}^l$ such that $Z \cap W = h^{-1}(0)$. Then $h \circ f$ is defined in a neighborhood V of x in X, and $f^{-1}(Z) \cap V = (h \circ f)^{-1}(0)$.

Now let $\phi : U \to X$ be a local parametrization around x, where U is an open subset of $\mathbb{H}^k$. Then define

$$g := h \circ f \circ \phi : U \to \mathbb{R}^l.$$

Since $\phi : V \to \phi(V)$ is a **diffeomorphism**, the set

$$\begin{aligned} & f^{-1}(Z) \text{ is a manifold with boundary in a neighhorhood of } x \\ \iff & (f \circ \phi)^{-1}(Z) = g^{-1}(0) \text{ is a manifold with boundary near } u = \phi^{-1}(x) \in \partial U. \end{aligned}$$

Since **both** f and ∂f are **transverse** to Z by assumption, we know that 0 is a **regular value** of g. Hence we can apply Theorem 10.16 which finishes the proof. □

10.4.4 Sard's Theorem with boundary

Finally, also Sard's Theorem has a version with boundary.

Theorem 10.18 (Sard's Theorem with boundary) Let $f : X \to Y$ be a smooth map from a manifold X **with boundary** to a manifold Y **without** a boundary. Then the set of **regular values** for **both** f and ∂f is a dense subset in Y.

Proof: For any point $x \in \partial X$ on the **boundary** of X, we have

$$d(\partial f)_x = (df_x)_{|T_x(\partial X)} : T_x(\partial X) \to T_{f(x)}(Y).$$

This implies that, if $d(\partial f)_x$ is surjective, then df_x is surjective. Hence **if** ∂f is **regular** at x, so is f. Thus, the set of $y \in Y$ which are **regular value** of both f and ∂f is the **intersection** of the set of regular values for $f : \mathrm{Int}(X) \to Y$ and the set of regular values for $\partial f : \partial X \to Y$. Since

$\mathrm{Int}(X)$ and ∂X are manifolds **without** boundary, **both** sets of regular values are **dense** subsets of Y by Sard's Theorem 7.1 for manifolds without boundary we have shown before. Since the intersection of two dense subsets is again a dense subset, this proves the claim. $\square$

10.5 Exercises and more examples

10.5.1 Manifolds with boundary

Exercise 10.1 Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^k$ be open neighborhoods of 0. Show that there exists no diffeomorphism of V with U.

Hint: Use the Inverse Function Theorem 3.4.

Exercise 10.2 Prove that if $f: X \to Y$ is a diffeomorphism of manifolds with boundary, then ∂f maps ∂X diffeomorphically onto ∂Y.

Hint: Use the Inverse Function Theorem 3.4.

Exercise 10.3 We define the smooth maps

$$F: \mathbb{R} \times [-1/2, 1/2] \to \mathbb{R}^3, (t, s) \mapsto (\cos t, \sin t, s)$$

and

$$G: \mathbb{R} \times [-1/2, 1/2] \to \mathbb{R}^3,$$
$$(t, s) \mapsto ((1 + s\cos(t/2))\cos t, (1 + s\cos(t/2))\sin t, s\sin(t/2)).$$

We define X to be the image of F in $\mathbb{R}^3$, and $Y = G$ to be the image of G in $\mathbb{R}^3$.

(a) Show that X is a 2-dimensional manifold with boundary whose boundary is diffeomorphic to the disjoint union of two copies of the unit circle. (Convince yourself that X is a cylinder obtained by starting with a rectangular surface and then glueing two opposite edges together.)

(b) Show that Y is a 2-dimensional manifold with boundary whose boundary is diffeomorphic to just one copy of the unit circle. (Convince yourself that Y is a Möbius band obtained by starting with a rectangular surface and then glueing two opposite edges after twisting one edge once. If you do not get through all the formulae, make sure you understand the answer visually at least.)

Exercise 10.4 Suppose that X is a manifold with boundary and $x \in \partial X$. Let $\phi: U \to X$ be a local parametrization with $\phi(0) = x$, where U is an open subset of $\mathbb{H}^k$. Then $d\phi_0: \mathbb{R}^k \to T_x(X)$ is an isomorphism. Define the upper halfspace $H_x(X)$ in $T_x(X)$ to be the image of $\mathbb{H}^k$ under $d\phi_0$, $H_x(X) := d\phi_0(\mathbb{H}^k)$.

(a) Prove that $H_x(X)$ does not depend on the choice of local parametrization.

(b) Show that there are precisely two unit vectors in $T_x(X)$ that are perpendicular to $T_x(\partial X)$ and that one lies inside $H_x(X)$, the other outside. The one in $H_x(X)$ is called the inward unit normal vector to the boundary, and the other is the outward unit normal vector to the boundary. Denote the outward unit normal vector by $n(x)$.

(c) If $X \subset \mathbb{R}^N$, we consider $n(x)$ as an element in $\mathbb{R}^N$ and get a map $n: \partial X \to \mathbb{R}^N$.

Show that n is smooth.

Exercise 10.5 Let $X = \{(x, y) \in \mathbb{R}^2 : x \geq -1\}$, $Y = \mathbb{R}$ and

$$f : X \to Y,\ (x, y) \mapsto x^2 + y^2.$$

(a) What is the boundary of X? Show that 1 is a regular value of f. Is 1 a regular value of ∂f?

(b) Determine $f^{-1}(1)$, $\partial(f^{-1}(1))$ and $f^{-1}(1) \cap \partial X$. Why does the answer not contradict the assertion of the Preimage Theorem for manifolds with boundary?

11. Brouwer Fixed Point Theorem

11.1 One-Manifolds

The following theorem gives us a complete list of smooth one-dimensional manifolds. Note that in general, since every manifold is the disjoint union of its connected components, it suffices to classify **connected** manifold.

Theorem 11.1 (Classification of one-Manifolds) Let X be a connected smooth one-dimensional manifold.

(a) If X is **compact** and **without boundary**, then X is diffeomorphic to $\mathbb{S}^1$.

(b) If X is **compact with boundary**, then X is diffeomorphic to $[0, 1]$.

(c) If X is **noncompact without boundary**, then X is diffeomorphic $(0, 1)$.

(d) If X is **noncompact with boundary**, then X is diffeomorphic to either $[0, 1)$ or $(0, 1]$.

As a first consequence of the classification of one-manifolds we can deduce the following fact about the boundary of a one-manifold:

Lemma 11.2 (Boundary of one-manifolds) The boundary of a **compact** one-dimensional manifold with boundary consists of an **even** number of points.

Proof: Every compact one-manifold with boundary X is the disjoint union of **finitely many connected components**. Each component is diffeomorphic to a copy of $[0, 1]$. Hence the boundary of each component consists of **two points**. The boundary of X consists of these finitely many **pairs** of points. □

11.1.1 Some heuristics for the classification

Before we prove the theorem, we begin with the rough idea why it should be true.

(a): Assume X is a nonempty, **compact**, connected 1-manifold. Each point has a neighborhood diffeomorphic to $(-1, 1)$. By compactness, finitely many such neighborhoods $U_1, \dots, U_n$ cover X. If n was equal 1, then $X \cong (-1, 1)$. But an open interval is not compact. Thus, there must be at least two neighborhoods. Since X is connected, these **two charts must overlap**. The union of these two intervals has to be either an open interval (if they overlap on one side of each) or a circle (if they overlap on both sides). But if their **union** is an open interval, there has to be another chart, by the **compactness** of X. Since there are **only finitely** many U_i's, we must eventually arrive at the situation where the neighborhoods **intersect on both sides** and form a **circle**. Then one has to use this to **construct a diffeomorphism** to $\mathbb{S}^1$.

(b): Let X be a **compact**, connected, one-dimensional smooth manifold **with** boundary. Since X has **at least one boundary point**, there must be a neighborhood in X containing that boundary point. This neighborhood must be diffeomorphic to $[a, b)$ for some a, b. Since this interval is **not compact**, there must be another neighborhood in X. This neighborhood **either** intersects another boundary point which would yield us $X \cong [a, c]$ for some c, **or** it does not contain a boundary point. In the latter case, the union of the neighborhoods is diffeomorphic to a half-open interval $[a, d)$ which is not compact. Hence there has to be another neighborhood. Since X is compact, this process will end after finitely many steps when we eventually get that X is the union of neighborhoods which is diffeomorphic to a closed interval.

(c) and (d): When X is **not compact**, we repeat the above processes. The difference is that the **process may not terminate** and we end up with **open** or **half-open** intervals. □

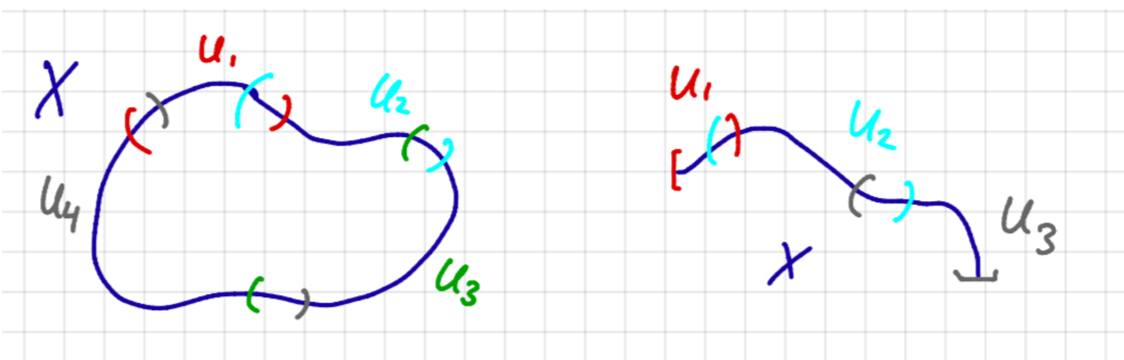


Figure 11.1: We can cover a compact one-manifold by finitely many open subsets which look like an open interval. Either we get back to the beginning and get a circle or we stop with another boundary point. This is the heuristic idea for the classification. There are some details to be straightened out though.

11.2 Proof of the classification theorem

We will follow Milnor's proof using arc-lengths given in the appendix of [13]. In this section X, will aways denote a smooth **one-manifold**.

> **Definition 11.3 (Parametrization by arc-length)** Let I be an **interval**, i.e., a **connected** subset of $\mathbb{R}$. A map $f : I \to X$ is called a **parametrization by arc-length** if f maps I **diffeomorphically** onto an **open** subset of X, and if the **velocitiy vector** $df_t(1) \in T_{f(t)}X$ has length **one** for each $t \in I$, where $df_t : \mathbb{R} = T_t I \to T_{f(t)}X$.

Note that the definition forces that I can have a boundary **only if** X has a boundary. The proof of the classification theorem relies on the following lemma:

Lemma 11.4 (Number of components) Let $f : I \to X$ and $fg : J \to X$ be two parametrizations by arc-length. Then $f(I) \cap g(J)$ has **at most** two components. If it has only **one** component, then f can be **extended** to a parametrization by arc-length of the union $f(I) \cup g(J)$. If it has **two** components, then X must be **diffeomorphic** to $\mathbb{S}^1$.

Proof: We write $U := f(I) \cap g(J) \subset X$. Note that U is **open** in X, since both $f(I)$ and $g(J)$ are open. In order to determine the number of components of U we define the map

$$\kappa : \ f^{-1}(U) \xrightarrow{f_|} U \xrightarrow{(g_|)^{-1}} g^{-1}(U)$$

where $f_|$ and $g_|$ denote the restrictions of f and g, respectively, such that the maps are defined. By definition of parametrizations by arc-length, κ is a **local diffeomorphism** with derivative equal to ± 1 and maps open subset of I diffeomorphically onto open subsets of J. In particular, it is an **open map**. We consider the **graph** of κ, denoted by $\Gamma \subset I \times J$, given by

$$\Gamma = \{(s,t) \in I \times J \ : \ f(s) = g(t)\}.$$

Since Γ is the preimage of the diagonal Δ in $X \times X$ under the map $(f,g) : \ I \times J \to X \times X$ and since Δ is closed in $X \times X$, Γ is a **closed** subset of $I \times J$. Since $d\kappa_{(s,t)} = \pm 1$, $d\kappa_{(s,t)}$ takes values in the discrete set $\{\pm 1\}$. Hence $d\kappa_{(s,t)}$ is **locally constant** and is therefore constant on each connected component of Γ. Thus, thinking of $I \times J$ as a rectangle with sides I and J, Γ consists of line segments of **slope** either $+1$ or -1. We need to understand how many such line segmens there are. Note that the image of κ equals the projection of the components of Γ to J. Now let K be a **connected component** of Γ.

- **Claim:** The endpoints of K lie on the boundary of $I \times J$.

Proof of the claim: Since Γ is closed, K is closed in $I \times J$ as well. Assume that K ends in the **interior** of $I \times J$ with endpoint a. Then the projection of K onto J has endpoint $\kappa(a)$. But since κ is a local diffeomorphism, the image of κ has to contain an **open** neighborhood around $\kappa(a)$. Hence $\kappa(a)$ **cannot** be an endpoint of a subinterval of J. This proves the claim.

Moreover, since κ is one-to-one, each of the **four edges** of the rectangle $I \times J$ can contain **at most one endpoint** of a component of Γ. Since each component has two endpoints, Γ can have **at most two components**. This proves the **first** assertion in the lemma.

Now we show the second assertion by considering the two cases:

- Assume Γ has **one** component: Then κ has constant slope and Γ is just a **linesegment** in the rectangle $I \times J$. Hence we can extend κ to a **linear** map $L : \mathbb{R} to \mathbb{R}$. Since both f, g and L are smooth and have equal derivatives, we can **glue** f and $g \circ L_{L^{-1}(J)}$ together to get an **extension**
  $$F : \ I \cup L^{-1}(J) \to f(I) \cup g(J)$$
  which is a parametrization by arc-length of the union $f(I) \cup g(J)$.

- Assume Γ has **two** components: Then all four edges are hit by the components. This implies that the **slope** of both components must be **the same**. Let us assume the slope is $+1$. The case with slope -1 is analogous. Let $a < d$ be the endpoints of I and $\gamma < \beta$ be the endpoints of J. Let $\alpha \in J$ be the starting point and $b \in I$ be the endpoint of the first component of Γ, and let $c \in I$ be the starting point and $\delta \in J$ be the endpoint of the second component. After possibly translating the interval J in $\mathbb{R}$ we can assume that $\gamma = c$ and $\delta = d$. By the previous arguments, we then have

$$a < b \leq c = \gamma < d = \delta \leq \alpha < \beta.$$

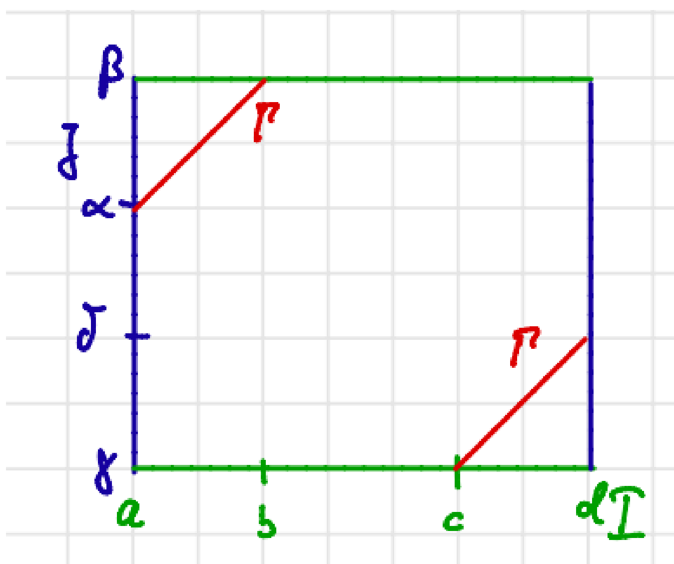


Figure 11.2: The case when Γ has two components. The slope is the same for both components, here $+1$. All four edges of the rectangle $I \times J$ are hit by Γ.

Now we can define a map $h : \mathbb{S}^1 \to X$ as follows: We write $\theta := 2\pi t/(\alpha - a)$ and define

$$h(\cos\theta, \sin\theta) = \begin{cases} f(t) & \text{for } a < t < d, \\ g(t) & \text{for } c < t < \beta. \end{cases}$$

We need to check that h is actually well-defined. First, if t varies between a and β, θ takes all values in $[0, 2\pi]$, since $\alpha < \beta$. Thus h is defined on all points of $\mathbb{S}^1$. Second, we need to check what happens on the overlaps. We have two intervals on which h is defined by two apriori different terms: for t between $c = \gamma$ and $d = \delta$, and for t between α and β. However, by definition of Γ, we have $f(c + r) = g(\gamma + r)$ for $0 \leq r \leq d - c = \delta - \gamma$. Similarly, we have $f(a + r) = g(\alpha + r)$ for $0 \leq r \leq b - a = \beta - \alpha$.

Hence h is well-defined and since f and g are smooth, h is smooth. Moreover, h is an open map, since f and g are. Thus $h(\mathbb{S}^1)$ is an **open** subset of X. Since h is continuous and $\mathbb{S}^1$ is compact, $h(\mathbb{S}^1)$ compact and hence a **closed** subset of X, as X is Hausdorff. As X is **connected**, this implies that $h(\mathbb{S}^1) = X$. Since f and g are local diffeomorphisms, so is h. Thus, h is a **diffeomorphism**. $\square$

Proof of the Classification Theorem 11.1: Let $f : I \to X$ be any parametrization by arc-length. If there exists a parametrization by arc-length $g : J \to X$ with $I \cap J \neq \emptyset$, then we can **glue** these parametrizations together to get a parametrization by arc-length $I \cup J \to X$ as described in the previous lemma. Hence after possibly **extending** f as far as possible to the left and then as far as possible to the right, we can assume that f is **maximal** in the sense that it cannot be extended over any larger interval as a parametrization by arc-length. If X is not diffeomorphic to $\mathbb{S}^1$, we will prove that f is **onto**. This will imply that f is a **diffeomorphism**, since we already know it is injective and a local diffeomorphism.

If the **open** set $f(I) \subset X$ was **not** all of X, then there would be a **limit point** x of $f(I)$ in $X \setminus f(I)$. Then we can parametrize a neighborhood of x by arc-length. The previous lemma then implies that f can be extended over a **larger** interval. This **contradicts** the assumption that f is maximal. The assertion of the theorem now follows from the comparison with the different types of intervals in $\mathbb{R}$. □

At least as interesting as the theorem are its consequences which are surprisingly rich. We will begin to study them next.

11.3 Boundaries and retractions

Now we study the surprising consequences of the classification of one-manifolds.

Definition 11.5 (Retraction) Let X be a smooth manifold and $Z \subset X$ be a submanifold. Then a **retraction** is a smooth map $g\colon X \to Z$ such that $f_{|Z}$ is the **identity**.

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}"] \arrow[d, hook] & Z \\ X \arrow[ur, "g"'] & \end{tikzcd}$$

There is an important restriction for the existence of such retractions for manifolds with boundary:

Lemma 11.6 (No retraction onto boundary) If X is any **compact** manifold with boundary, then there is **no retraction** of X onto its **boundary**.

Proof: Suppose there is such a smooth map $g\colon X \to \partial X$ such that $\partial g\colon \partial X \to \partial X$ is the identity. By **Sard's Theorem 10.18**, we can choose a regular value $z \in \partial X$ of g. Since ∂g is the identity, all values in ∂X are regular for ∂g. Hence z is regular for both g and ∂g. By the **Preimage Theorem 10.17** for manifolds **with boundary**, we know that $g^{-1}(z)$ is a submanifold of X with **boundary** given by

$$\partial\left(g^{-1}(z)\right) = g^{-1}(z) \cap \partial X. \tag{11.1}$$

Moreover, the codimension of $g^{-1}(z)$ in X equals the codimension of $\{z\}$ in ∂X, namely $\dim X - 1$ as $\{z\}$ has dimension 0. Hence $g^{-1}(z)$ is **one-dimensional**. Since it is a **closed** subset in the compact manifold X, it is also **compact**. Thus, if g is a retraction onto ∂X, then by Lemma 11.2 we must have

$$\#\partial \left(g^{-1}(z)\right) \text{ must be } \textbf{even}. \tag{11.2}$$

By definition of ∂g as the restriction of g to ∂X, we have for just **set-theoretic reasons**

$$(\partial g)^{-1}(z) = (g_{|\partial X})^{-1}(z) = (g \circ i)^{-1}(z) = i^{-1}(g(z)) = g^{-1}(z) \cap \partial X$$

where $i : \partial X \hookrightarrow X$ denotes the **inclusion** of ∂X into X. Hence, using Equation 11.1, we have

$$(\partial g)^{-1}(z) = \partial(g^{-1}(z)).$$

Since $\partial g = g_{|\partial X} = \mathrm{Id}_{\partial X}$, this implies

$$\partial \left(g^{-1}(z)\right) = (\partial g)^{-1}(z) = \{z\}.$$

In particular, this shows that $\partial \left(g^{-1}(z)\right)$ consists of an **odd** number of points. However, this **contradicts** the conclusion in Equation 11.1. Hence the retraction g cannot exist. □

11.4 Brouwer Fixed Point Theorem - smooth case

We can now prove a famous consequence of Lemma 11.6:

Theorem 11.7 (Brouwer Fixed Point Theorem for smooth maps) Let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a **smooth** map of the **closed** unit ball $\mathbb{D}^n = \{x \in \mathbb{R}^n : |x| \leq 1\} \subset \mathbb{R}^n$ into itself. Then f has a **fixed point**, i.e., there is an $x \in \mathbb{D}^n$ with $f(x) = x$.

Before we prove the theorem, let us have a look at dimension one, where the result is very familiar:

Remark 11.8 (Familiar in dimension one) Note that we have seen this theorem for $n = 1$ in Calculus 1: Every continuous map $f : [0, 1] \to [0, 1]$ has a **fixed point**. For define the function $g(x) = f(x) - x$ which is a continuous map from $[0, 1]$ to itself. We have $g(0) \geq 0$ and $g(1) \leq 0$, since $f(0) \geq 0$ and $f(1) \leq 1$. If $g(0) = 0$ or $g(1) = 1$, we are done. And if $g(0) > 0$ and $g(1) < 1$, then the **Intermediate Value Theorem** implies that there is an $x_0 \in (0, 1)$ with $g(x_0) = 0$, i.e., $f(x_0) = x_0$.

Proof of Brouwer Fixed Point Theorem 11.7:

Suppose that there exists an f **without fixed points**. We will show that such an f would allow us to construct a **retraction** $g : \mathbb{D}^n \to \partial\mathbb{D}^n$. But, since $\mathbb{D}^n$ is **compact**, we have just proved in Lemma 11.6 that such a retraction **cannot exist**.

So suppose $f(x) \neq x$ for all $x \in \mathbb{D}^n$. Then, for every $x \in \mathbb{D}^n$, the **two different** points x and $f(x)$ **determine a line**. Let $g(x)$ be the **point** where the line segment starting at $f(x)$ and passing through x **hits the boundary** $\partial\mathbb{D}^n$. This defines a map $g : \mathbb{D}^n \to \partial\mathbb{D}^n$. See Figure 11.4.

However, if $x \in \partial\mathbb{D}^n$, then $g(x) = x$ by construction of g. Hence $g : \mathbb{D}^n \to \partial\mathbb{D}^n$ is the **identity on** $\partial\mathbb{D}^n$. Thus, in order to show that g is a **retraction**, it remains to show that g is **smooth**.

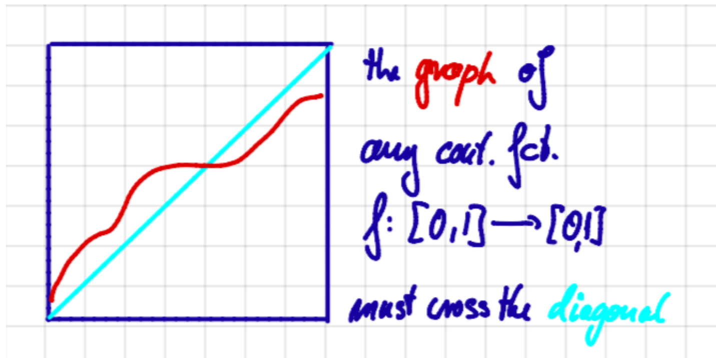


Figure 11.3: In dimension one this is the Intermediate Value Theorem. The graph has to cross the diagonal which consists of fixed points of f.

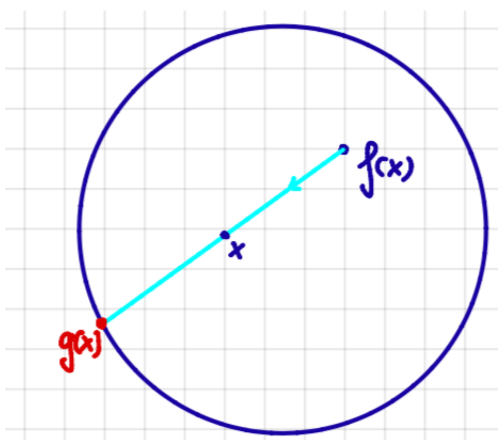


Figure 11.4: We construct g by extending the line from $f(x)$ to x until we hit the boundary. If this map existed for all x, then we would have found a retraction. That is a contradiction to Lemma 11.6.

So let us describe $g(x)$ more **explicitly**. As a point on the line from $f(x)$ to x, $g(x)$ can be written in the form

$$g(x) = x + tv, \text{ where } v := \frac{x - f(x)}{|x - f(x)|}$$

for some real number $t = t(x)$. Note that, since we assume $x \neq f(x)$, the **vector v is always defined**. In fact, it is the unit vector pointing from $f(x)$ to x. Moreover, since f is smooth, **v depends smoothly on x**. We need to calculate t and show that **t depends smoothly on x**. Since $g(x)$ is a point on **boundary** of $\mathbb{D}^n$, we know $|g(x)| = 1$, and t is determined by the equation

$$1 = |g(x)|^2 = (x + tv) \cdot (x + tv) = x \cdot x + 2tx \cdot v + t^2 v \cdot v$$

or, equivalently,

$$0 = (v \cdot v)t^2 + (2x \cdot v)t + x \cdot x - 1. \tag{11.3}$$

By definition of v, we know $v \cdot v = |v|^2 = 1$. Since v points from $f(x)$ to x, we know that t must be positive. Now we just need to find the positive solution of the quadratic Equation 11.3 for t and get

$$\begin{aligned} t &= \frac{-2x \cdot v + \sqrt{4(x \cdot v)^2 - 4(x \cdot x - 1)}}{2} \\ &= -x \cdot v + \sqrt{(x \cdot v)^2 - x \cdot x + 1} \end{aligned}$$

where $(x \cdot v)^2 - x \cdot x + 1$ is positive, since $x \cdot x = |x|^2 \leq 1$ and $(x \cdot v)^2 > 0$. Since the **scalar products** and **square roots** involved depend **smoothly** on x, we see that t depends smoothly on x. Hence g is smooth. □

Remark 11.9 (Proof in dimension one) Note that, **for $n = 1$**, in the above proof we would have constructed a map $g : [-1, 1] \to \{-1, 1\}$ which would send -1 to -1 and 1 to 1. Such a map **cannot** be **continuous**, since $[-1, 1]$ is **connected**.

11.5 Brouwer Fixed Point Theorem - continuous case

Actually, just as in the one-dimensional case, the theorem also holds for continuous maps:

Theorem 11.10 (Brouwer Fixed-Point Theorem for continuous maps) Every **continuous** map $F : \mathbb{D}^n \to \mathbb{D}^n$ has a **fixed point**.

The idea is to deduce Theorem 11.10 from Theorem 11.7 by **approximating** F by a smooth map.

Lemma 11.11 (Approximation by a smooth map) Let $F : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous map. For every $\varepsilon > 0$, there exists a **smooth** map $P : \mathbb{D}^n \to \mathbb{D}^n$ such that

$$|P(x) - F(x)| < 2\varepsilon \text{ for all } x \in \mathbb{D}^n.$$

Proof: Since $\mathbb{D}^n$ is **compact**, we can apply **Weierstrass' Approximation Theorem**, which says: Given $\varepsilon > 0$, there is a **polynomial function** $Q\colon \mathbb{D}^n \to \mathbb{R}^n$ with

$$|Q(x) - F(x)| < \varepsilon \text{ for all } x \in \mathbb{D}^n.$$

However, Q **may** send points in $\mathbb{D}^n$ to **points outside of $\mathbb{D}^n$**. In order to **remedy this defect**, we replace Q with

$$P(x) := \frac{Q(x)}{1+\varepsilon}.$$

The map P is still a polynomial and hence **smooth**. Since $|F(x)| \leq 1$, the polynomial P satisfies:

$$(1+\varepsilon)|P(x)| = |Q(x)| \leq |Q(x) - F(x)| + |F(x)| < \varepsilon + 1$$

where we apply the **triangle inequality**. Hence $|P(x)| \leq 1$ and P is a map $\mathbb{D}^n \to \mathbb{D}^n$. Moreover, we have

$$\begin{aligned}(1+\varepsilon)|P(x) - F(x)| &= |Q(x) - (1+\varepsilon)F(x)| \\ &= |Q(x) - F(x) + \varepsilon F(x)| \\ &\leq |Q(x) - F(x)| + \varepsilon|F(x)| \\ &< 2\varepsilon\end{aligned}$$

where we use that $|F(x)| \leq 1$. Since $1 + \varepsilon > 1$, this shows

$$|P(x) - F(x)| < 2\varepsilon \text{ for all } x \in \mathbb{D}^n. \quad \square \tag{11.4}$$

Now we can prove the continuous case:

Proof of Theorem 11.10: We **suppose** that $F(x) \neq x$ for all $x \in \mathbb{D}^n$. Then the continuous function

$$\mathbb{D}^n \to \mathbb{R},\ x \mapsto |F(x) - x|$$

must have a **minimum** μ since $\mathbb{D}^n$ is **compact**. Since $F(x) \neq x$ for all x, we must have $\mu > 0$.

Now, for $\varepsilon = \mu/2$, we can find, by Lemma 11.11, a smooth map $P\colon \mathbb{D}^n \to \mathbb{D}^n$ such that

$$|P(x) - F(x)| < \mu \text{ for all } x \in \mathbb{D}^n.$$

However, since $|F(x) - x| \geq \mu$ for all $x \in \mathbb{D}^n$, the **triangle inequality** yields

$$\begin{aligned}\mu \leq |F(x) - x| &= |F(x) - P(x) + P(x) - x| \\ &\leq |F(x) - P(x)| + |P(x) - x|.\end{aligned}$$

This implies that $|P(x) - x| > 0$, and therefore $P(x) \neq x$ for all $x \in \mathbb{D}^n$. Hence P is a **smooth** map from $\mathbb{D}^n$ to itself **without a fixed point**. This contradicts Theorem 11.7 and completes the proof. $\square$

11.6 Counter-example on an open ball

The assertion of Theorem 11.7 is **not true** for the **open ball**:

Let $\mathbb{B}_1^k(0) = \{x \in \mathbb{R}^k : |x| < 1\}$ be the **open** ball in $\mathbb{R}^k$. We define the map

$$\varphi : \mathbb{B}_1^k(0) \to \mathbb{R}^k, \ x \mapsto \frac{x}{\sqrt{1-|x|^2}}.$$

This is a **smooth** map with **smooth inverse**

$$\varphi^{-1} : \mathbb{R}^k \to \mathbb{B}_1^k(0), \ y \mapsto \frac{y}{\sqrt{1+|y|^2}}$$

Thus φ is a **diffeomorphism** $\mathbb{B}_1^k(0) \to \mathbb{R}^k$.

The **translation** $T : \mathbb{R}^k \to \mathbb{R}^k$, $x \mapsto x + 1$ does **not have a fixed point**. Hence the composite map

$$\varphi^{-1} \circ T \circ \varphi : \mathbb{B}_1^k(0) \to \mathbb{B}_1^k(0)$$

is a **smooth map** which does **not** have a fixed point. For, if it had a fixed point x, then

$$\varphi^{-1}(T(\varphi(x))) = x \Rightarrow T(\varphi(x)) = \varphi(x),$$

i.e., T had a fixed point, which is **not the case**.

11.7 Some interesting consequences

Brouwer's Fixed Point Theorem has many important applications. Here is one of them:

Theorem 11.12 (Brouwer Invariance of Domain) Let U be an **open** subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be a **continuous injective** map. Then $f(U)$ is also **open**.

Instead of studying the proof of this theorem, let us note a **consequence** of this result. You can read more about this story and the proof on Terence Tao's blog.

Theorem 11.13 (Topological Invariance of Dimension) **If** $n > m$, and U is a nonempty **open** subset of $\mathbb{R}^n$, then there is **no continuous injective** map from U to $\mathbb{R}^m$. In particular, since a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$ would be such a continuous injective map, $\mathbb{R}^n$ and $\mathbb{R}^m$ are **not homeomorphic** whenever $n \neq m$.

Even though it sounds like an obvious fact, this is a rather deep theorem. Note that there **exist weird things** like a continuous surjection from $\mathbb{R}^m$ to $\mathbb{R}^n$ for $m < n$ due to variants of the Peano curve construction. Hence often we have to be careful with our topological intuition.

Proof Theorem 11.12 implies Theorem 11.13: If there was such a continuous injective map from U to $\mathbb{R}^m$, then we could compose it with the embedding

$$\mathbb{R}^m \xrightarrow{\cong} (\mathbb{R}^m \times \{0\}) \subset \mathbb{R}^n.$$

Hence the composite would yield a **continuous injective** map from U into $\mathbb{R}^n$. By Theorem 11.13, the image would be both open in $\mathbb{R}^n$ and lie in the subspace $\mathbb{R}^m \times \{0\}$. But no **open** subset of $\mathbb{R}^n$ can be contained in $\mathbb{R}^m \times \{0\}$, since we must be able to fit at least a tiny **open ball** of $\mathbb{R}^n$ into that subset and there is no room for such a ball in the direction of the remaining $n - m$ coordinates. $\square$

Note that invariance of domain and dimension for **smooth** injective maps is just a consequence of the **Inverse Function Theorem**. But for maps which are **just continuous** and injective, it is **much harder** to achieve.

11.8 Exercises and more examples

11.8.1 Brouwer Fixed Point Theorem

Exercise 11.1 Prove the Theorem of *Perron–Frobenius*: Let A be an $n \times n$-matrix A with only nonnegative entries. Then A has a real nonnegative eigenvalue.

Hint: We can assume that A is invertible, otherwise 0 is an eigenvalue. Let A also denote the associated linear map of $\mathbb{R}^n$, and consider the map $v \to Av/|Av|$ restricted to $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. Show that this maps the first quadrant

$$Q = \left\{(x_l, \dots, x_n) \in \mathbb{S}^{n-1} \ : \ \text{all } x_i \geq 0\right\}$$

into itself. Now use the fact that there is a homeomorphism $\mathbb{D}^{n-1} \to Q$ to get a continuous map $\mathbb{D}^{n-1} \to \mathbb{D}^{n-1}$.

Exercise 11.2 In this exercise we use the following fact: A space X is *simply-connected* if it is path-connected and every continuous map $f : \mathbb{S}^1 \to X$ can be extended to a continuous map $F : \mathbb{D}^2 \to X$ such that $F_{|\mathbb{S}^1} = f$. In this exercise we are going to show that $X = \mathbb{R}^2 \setminus \{(0,0)\}$ is not simply-connected using Brouwer's Fixed Point Theorem 11.10.

(a) Consider the antipodal map $a : \mathbb{S}^1 \to \mathbb{S}^1$ which sends p to $-p$. Composing with the inclusion $\mathbb{S}^1 \subset X = \mathbb{R}^2 \setminus \{(0,0)\}$, we consider a as a map $f : \mathbb{S}^1 \to X$. Show that, if $X = \mathbb{R}^2 \setminus \{(0,0)\}$ was simply-connected, then this would induce a map

$$g : \mathbb{D}^2 \to X \to \mathbb{S}^1 \hookrightarrow \mathbb{D}^2.$$

(b) Still assuming $X = \mathbb{R}^2 \setminus \{(0,0)\}$ was simply-connected, show that g does not have a fixed point.

(c) Conclude that $X = \mathbb{R}^2 \setminus \{(0,0)\}$ is not simply-connected.

Exercise 11.3 Deduce from the previous point that the composed map

$$\iota : \mathbb{S}^1 \xrightarrow{\text{id}} \mathbb{S}^1 \subset \mathbb{R}^2 \setminus \{(0,0)\}, \ p \mapsto p,$$

is not homotopic to the constant map $c : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ sending all points to $(1,0)$.
Hint: Use that $\mathbb{R}^2 \setminus \{(0,0)\}$ is not contractible, since it is not simply-connected.

12. The Brouwer Degree modulo 2

12.1 The Brouwer Degree of maps modulo 2

We will now define a very important and powerful invariant for smooth maps, the degree modulo 2. Despite its rather simple definition, the degree determines a lot of the interesting properties of a map. In this chapter we define the degree with values in $\mathbb{Z}/2$, the integers modulo 2. This has the advantage that the degree is defined for a large class of maps and the definition only requires what we have learned so far. In Section 12.3 and Section 12.4 we will study important applications of the degree modulo two.

Let us first give a **rough outline** of what is about to come: Let $f : X \to Y$ be smooth map. We will always assume in this chapter that X is **compact** and without boundary. Moreover, we will assume that X and Y have the **same** dimension. The **idea** for the degree is rather simple: Pick a **regular value** y for f. By the Stack of Records Theorem 4.18, the fiber $f^{-1}(y)$ is a **finite** set. Hence we may count how many points there are in $f^{-1}(y)$. Since we also know that $\#f^{-1}(y)$ is locally constant, we may want to use the number $\#f^{-1}(y)$ to characterise f. However, there are **two caveats**. First, the number $\#f^{-1}(y)$ may not be constant for **all** regular values. Second, since describing maps and manifolds up to homotopy seems to be a much more promising task, we would like to have a number that only depends on the homotopy class of f. Luckily, we can remedy the defects of our initial idea with a little care and construct an invariant which only depends on the homotopy class of f and not on the choice of regular value. The price we pay is that we only get a number modulo two. We note, however, that there is also an integer-valued version of the degree which we will meet in chapter 16.

12.1.1 The Homotopy Lemma mod 2

The first step for the construction is to understand how the number of points in the fiber over a regular value depends on the homotopy class of a map:

Lemma 12.1 (Homotopy Lemma mod 2) Let X and Y be smooth manifolds. We assume that $\dim X = \dim Y$ and that X is **compact** and **without boundary**. Let $f, g : X \to Y$ be two smooth maps. Assume f and g are smoothly homotopic. If $y \in Y$ is a **regular value** for **both** f and g, then

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \mod \mathbf{2}.$$

Proof: Let $F : X \times [0,1] \to Y$ be a smooth homotopy between $F_0 = f$ and $F_1 = g$. **First** we suppose that y is a **regular value** for F. By the **Preimage Theorem 10.16** with boundary, this implies $F^{-1}(y)$ is a submanifold with **boundary** of $X \times [0,1]$ of dimension $\dim X \times [0,1] - \dim Y = 1$, since we assume $\dim X = \dim Y$. Moreover, the **boundary** of

$F^{-1}(y)$ is

$$\begin{aligned}\partial F^{-1}(y) = F^{-1}(y) \cap \partial(X \times [0,1]) &= \left(F_0^{-1}(y) \times \{0\}\right) \cup \left(F_1^{-1}(y) \times \{1\}\right) \\ &= \left(f^{-1}(y) \times \{0\}\right) \cup \left(g^{-1}(y) \times \{1\}\right).\end{aligned}$$

Since X is **compact**, $X \times [0,1]$ is compact. Since $\{y\}$ is a closed subset of Y, $F^{-1}(y)$ is closed in $X \times [0,1]$. This implies that $F^{-1}(y)$ is **compact**. Hence Lemma 11.2, which follows from the **classification of compact one-manifolds**, implies that $\partial F^{-1}(y)$ must consist of an **even** number of points. Thus, computing **mod** 2 we get

$$\# f^{-1}(y) + \# g^{-1}(y) \equiv 0,$$

and hence

$$\# f^{-1}(y) \equiv \# g^{-1}(y) \quad \text{mod } \mathbf{2}.$$

Now suppose that y is **not** a regular value of F (but still a regular value for f and g). By Lemma 4.20, the functions $y' \mapsto \# f^{-1}(y')$ and $y' \mapsto \# g^{-1}(y')$ are both **locally constant** on the set of regular values $y' \in Y$ of f and g, respectively. Thus there is an open neighborhood $U_1 \subset Y$ of y consisting only of regular values of f so that

$$\# f^{-1}(y') = \# f^{-1}(y) \text{ for all } y' \in U_1.$$

Similarly, there is an open neighborhood $U_2 \subset Y$ of y consisting only of regular values of g so that

$$\# g^{-1}(y') = \# g^{-1}(y) \text{ for all } y' \in U_2.$$

The intersection $U_1 \cap U_2$ is an **open** subset of Y. Hence, by Sard's Theorem 10.18, there is a regular value z of F in $U_1 \cap U_2$. By the first case we have $\# f^{-1}(z) = \# g^{-1}(z)$ modulo 2 and we get

$$\# f^{-1}(y) = \# f^{-1}(z) \equiv \# g^{-1}(z) = g^{-1}(y) \quad \text{mod } \mathbf{2}. \qquad \square$$

12.1.2 The Isotopy Lemma

In order to make to make our construction independent of the choice of a regular value, we will need a special type of homotopy which preserves more information than homotopies in general:

Definition 12.2 (Isotopy) An **isotopy** is a homotopy h_t in which **each map h_t is a diffeomorphism**. We say that two diffeomorphisms are **isotopic** if they can be joined by an isotopy. An isotopy is called **compactly supported** if the maps h_t are all equal to the identity map when restricted to the complement of some fixed compact set.

The following result will allow us to **move points** on connected manifolds via a **family of diffeomorphisms**. The fact that every map in the homotopy family is a **diffeomorphism** makes it much easier to keep track of the **orientation numbers** at preimages.

Lemma 12.3 (Isotopy Lemma) Let Y be a **connected** smooth manifold. Given any two points y and z in Y, there **exists a diffeomorphism** $h : Y \to Y$ such that

- $h(y) = z$ and
- h is **isotopic** to the **identity**.

Moreover, the isotopy can be chosen to be **compactly supported**.

We proof Lemma 12.3 and an extension to finitely many points in Section 12.2.

Using the Homotopy Lemma 12.1 and the Isotopy Lemma 12.3 we can now prove the following key result:

Theorem 12.4 (Fiber modulo 2 is well-defined) Let $f : X \to Y$ be a smooth map from a **compact** manifold X to a **connected** manifold Y and $\dim X = \dim Y$. Assume y and z are **regular values** of f. Then

$$\# f^{-1}(y) \equiv \# f^{-1}(z) \mod \mathbf{2}.$$

This common residue class $\# f^{-1}(y)$ modulo 2 only depends on the **homotopy class** of f.

Proof: Let y and z be two regular values of f. Since Y is **connected**, we can apply the **Isotopy Lemma 12.3** and find a diffeomorphism $h : Y \to Y$ which is **isotopic to the identity** and with $h(y) = z$. Since h is a diffeomorphism, z is a **regular value** of the **composite** $h \circ f$. Since h is homotopic to the identity, the composite $h \circ f$ is homotopic to f. Hence the Homotopy Lemma 12.1 implies that

$$\#(h \circ f)^{-1}(z) \equiv \# f^{-1}(z) \mod \mathbf{2}.$$

The left-hand fiber is given by

$$(h \circ f)^{-1}(z) = f^{-1}(h^{-1}(z)) = f^{-1}(y).$$

Hence we have

$$\#(h \circ f)^{-1}(z) = \# f^{-1}(y).$$

Thus we can conclude

$$\# f^{-1}(y) \equiv \# f^{-1}(z) \mod \mathbf{2}.$$

To prove the second assertion, assume that $g : X \to Y$ is a smooth map which is smoothly homotopic to f. By **Sard's Theorem 10.18**, there exists an element $y \in Y$ which is a **regular value** for **both** f and g. By the Homotopy Lemma 12.1, this implies

$$\# f^{-1}(y) \equiv \# g^{-1}(y) \mod \mathbf{2}.$$

This completes the proof of the theorem. □

12.1.3 A well-defined invariant

As a consequence of the above observations we conclude that the number $\#f^{-1}(y)$ is independent of the choice of y and only depends on the smooth homotopy class of f. Hence we make the following definition:

Definition 12.5 (Degree modulo 2) Let X be a **compact** smooth manifold X without boundary and Y a **connected** smooth manifold with $\dim X = \dim Y$. Then the **mod 2 degree of** f, denoted $\deg_2(f)$, is the mod 2 residue class of $\#f^{-1}(y)$ for any regular value y of f.

Note: The degree mod 2 is defined **only** when the range manifold Y is **connected**, the domain X is **compact**, and $\dim X = \dim Y$. Whenever we write $\deg_2$, we assume that these assumptions are satisfied. □

Now that we have the **invariant** $\deg_2$, let us contemplate on what it is good for. First of all, we note that there are **upsides and downsides** equipped to $\deg_2$:

- The first **good news** is that $\deg_2(f)$ is a **powerful invariant** as we will see soon.
- The second **good news** is that $\deg_2(f)$ is **easy to calculate**: just pick any regular value y for f and **count** preimage points
$$\deg_2(f) = \#f^{-1}(y) \mod 2.$$
- The **bad news** is that its **power is limited**. For **example**, the map
$$\mathbb{C} \to \mathbb{C},\ z \mapsto z^n,$$
which **wraps** the circle $\mathbb{S}^1$ smoothly around $\mathbb{S}^1$ n times, has mod 2 **degree zero** if n is **even**, and **degree one** if n is **odd**. For example, $\deg_2$ of the **constant** map $\mathbb{S}^1 \to \mathbb{S}^1$ is equal to $\deg_2$ of the **squaring** map $\mathbb{S}^1 \to \mathbb{S}^1$ sending $z \mapsto z^2$. Hence $\deg_2$ **cannot distinguish** between many different maps.

We now prove an important property of the degree that we will use in the following sections:

Lemma 12.6 ($\deg_2$ is additive) Let X and Z be **compact** smooth manifolds without boundary and let Y be a **connected** smooth manifold with $\dim X = \dim Z = \dim Y$. Let $f : X \sqcup Z \to Y$ be a smooth map. Then we have
$$\deg_2(f) = \deg_2(f_{|X}) + \deg_2(f_{|Z}).$$

Proof: Let y be a regular value for f. Then $\deg_2(f) = \#f^{-1}(y)$, and the number of elements in $\#f^{-1}(y)$ is the sum of the number of elements lying in X and Z, respectively. These two numbers are $\deg_2(f_{|X})$ and $\deg_2(f_{|Z})$, respectively. □

We close this section with a remark that puts our previous observation in Remark 4.23 about the necessity of the connectedness of the set of regular values for the number of points in the fiber to be constant with the findings of this section:

Remark 12.7 (No contradiction to previous observation) One may wonder why Theorem 12.4 does not contradict our observations when we discussed Milnor's proof of the FTA in Section 4.4. In Remark 4.23 we stated that it is not sufficient for Y to be connected that the number $\#f^{-1}(y)$ is constant for all regular values. For the number $\#f^{-1}(y)$ to be constant for all regular values, we actually need that $Y \setminus \{\text{regular values}\}$ is connected.

In fact, in the proof of Theorem 12.4 we compose f with the diffeomorphism h and then use that $\deg_2$ is homotopy-invariant. However, the Homotopy Lemma 12.1 only shows that $\#f^{-1}(y)$ **modulo** 2 is **constant** within the homotopy class of f. Fore example, if $\deg_2(f) = 0$, the fiber $f^{-1}(y)$ can be empty for some regular value, while it can be non-empty consisting of an even number of points for others. All we know is that the parity of $\#f^{-1}(y)$ does not change, but the number $\#f^{-1}(y)$ itself may vary.

12.1.4 Some applications of the mod 2 degree

There are many important and powerful applications of $\deg_2$. Here we just have a glimpse at a few of them.

- **Application:** Compact manifolds without boundary are **not** contractible

Let X be a compact smooth manifold without boundary of dimension at least one. The identity map on X has degree one, since $\#(\mathrm{id}^{-1}(y)) = 1$ for every $y \in X$. Thus $\deg_2(\mathrm{id}) = 1$. Now let $c : X \to X$ be a constant map with value $x_0 \in X$. Then every $x \neq x_0$ is a regular value for c, since every point not in the image is regular. Thus $\deg_2(c) = 0$. Hence we can conclude $\deg_2(\mathrm{id}) \neq \deg_2(c)$. Since $\deg_2$ is invariant under smooth homotopy, we have proven the following interesting fact:

Theorem 12.8 A **compact** manifold without boundary of dimension at least one is **not** smoothly contractible.

An immediate consequence of this result is a useful result we have proven previously. See Lemma 11.6.

Lemma 12.9 (No retraction to the boundary - revisited) There is no smooth map $f : \mathbb{D}^{n+1} \to \mathbb{S}^n$ which leaves $\mathbb{S}^n$ pointwise fix, i.e., we cannot have $f_{|\mathbb{S}^n} = \mathrm{id}_{\mathbb{S}^n}$.

Proof: If such a map $f : \mathbb{D}^{n+1} \to \mathbb{S}^n$ existed, then we could define a smooth map

$$F : \mathbb{S}^n \times [0,1] \to \mathbb{S}^n, \; F(x,t) = f(tx).$$

This map would be a smooth homotopy between a **constant** map and the **identity** map which **contradicts** Theorem 12.8. □

- **Application:** Obstruction to extending maps to boundary

Theorem 12.10 (Boundary Theorem for $\deg_2$) Let Y be a connected smooth manifold. Assume $X = \partial W$ is the **boundary** for some **compact** manifold W and that $f : X \to Y$ is a smooth map which can be **extended** to all of W, i.e., there is a smooth map $F : W \to Y$ with $F_{|\partial W} = f$. Then $\deg_2(f) = 0$.

- Note that when W is compact, then the closed subset $X = \partial W$ is also compact. Hence $\deg_2(f)$ is defined.

Proof: Let $F : W \to Y$ be an extension of f. By **Sard's Theorem 10.18**, there is an element $y \in Y$ which is a regular value for **both** f and F. By the **Preimage Theorem 10.16** with boundary, this implies $F^{-1}(y)$ is a submanifold of W of dimension $\dim W - \dim Y = 1$, since we assume $\dim X = \dim Y$ and $\dim W = \dim X + 1$. Since W is **compact** and since $\{y\}$ is a closed subset of Y, $F^{-1}(y)$ is closed in W. This implies that $F^{-1}(y)$ is **compact**. Hence Lemma 11.2 implies that $\#\partial(F^{-1}(y))$ must be **even**. Moreover, the **boundary** of $F^{-1}(y)$ is

$$\partial F^{-1}(y) = F^{-1}(y) \cap \partial W = F^{-1}(y) \cap X = (F_{|X})^{-1}(y) = f^{-1}(y).$$

Thus, computing **mod** 2 we get

$$\deg_2(f) \equiv \# f^{-1}(y) = \#\partial(F^{-1}(y)) \equiv 0. \qquad \square$$

This has a useful consequence:

Theorem 12.11 (Obstruction for extending maps) Let Y be a connected smooth manifold. Let W be a **compact** manifold with $\dim W = \dim Y + 1$ and $f : \partial W \to Y$ a smooth map. **If** $\deg_2(f) \neq 0$, then f **cannot be extended** to a smooth map $W \to Y$ on all of W.

- **Application:** Existence of zeros for complex valued functions

Suppose that $p : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map and $W \subset \mathbb{C}$ is a smooth **compact** region in the plane, i.e., a **two-dimensional compact manifold with boundary**.

Question: Is there a $z \in W$ with $p(z) = 0$?

Assume that p has **no zeros on the boundary** ∂W. Then

$$\frac{p}{|p|} : \partial W \to \mathbb{S}^1$$

is defined and smooth as a map of **compact one-manifolds**. Now **if** p has **no zeros inside** W, then $\frac{p}{|p|}$ is **defined** on all of W, i.e., the map $\frac{p}{|p|} : \partial W \to \mathbb{S}^1$ can be **extended** to a smooth map $W \to \mathbb{S}^1$ on all of W. If this is the case, we just learned that we must have $\deg_2\left(\frac{p}{|p|}\right) = 0$. See Figure 12.1.

Theorem 12.12 (Existence of zeros via $\deg_2$) **If** the mod 2 degree of $\frac{p}{|p|} : \partial W \to \mathbb{S}^1$ is **nonzero**, then the function p has a **zero inside** W. □

- **Note** that **calculating** $\deg_2\left(\frac{p}{|p|}\right)$ simply consists of **picking a point** $z \in \mathbb{S}^1$, we could think of it as a **direction**, and just **counting** the number of times we find a $w \in \partial W$ with $p(w) = z$, i.e., how often $p(w)$ points in the **chosen direction**.
- Theorem 12.12 says that this **simple procedure** tells us whether p has a **zero inside** W.
- Finally, if you have learned about Complex Analysis, then this should remind you of the **Residue Theorem** and **Cauchy's formula**.
- We look at some applications of Theorem 12.12 in the exercises, see Exercise 12.3 and Exercise 12.4.

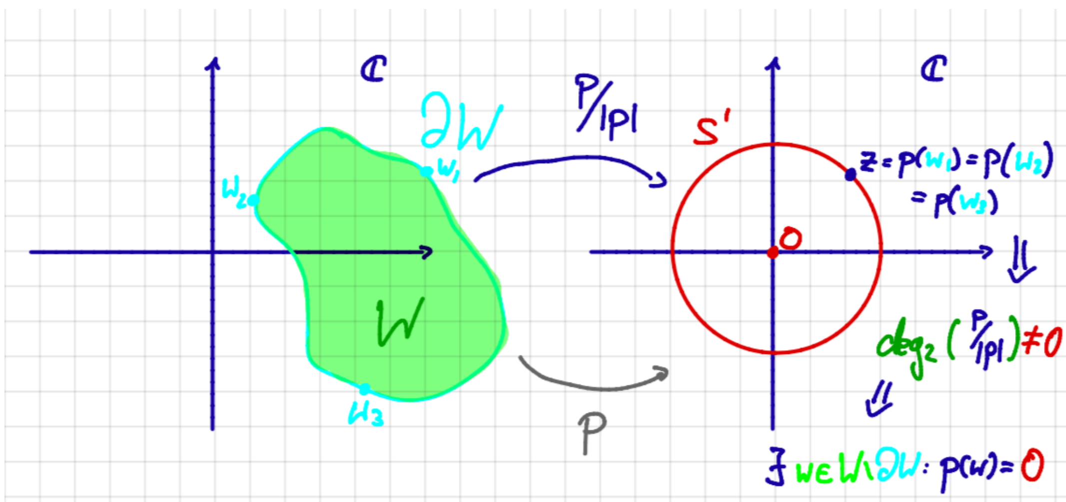


Figure 12.1: The degree is able to detect a zero of the polynomial in the region W.

12.2 Proof of the Isotopy Lemma

We now provide the proof of the Isotopy Lemma, an important result for the construction of the Brouwer degree. We will then prove a generalization of the lemma.

Proof of Lemma 12.3: For the proof, we call two points y and z **isotopic** if the statement of Lemma 12.3 holds. This defines an **equivalence relation** on the set of points in Y. The proof of Lemma 12.3 will consist in showing that each equivalence class is **open**. Since equivalence classes are disjoint, this will show that Y is the **disjoint union** of **open** subsets. Since Y is **connected**, this implies there can only be **one** equivalence class, i.e., the assertion holds for all points in Y.

To prove that the equivalence classes are **open**, we will **first** construct an isotopy h_t on **Euclidean space** $\mathbb{R}^k$ such that

- h_0 is the identity,
- each h_t is the identity outside some specified small ball around 0, and
- $h_1(0)$ is any desired point sufficiently close to 0.

Then we **transport the isotopy** to Y: Given $y \in Y$, choose a **local parametrization** $\phi\colon \mathbb{R}^k \to V \subset Y$ with $\phi(0) = y$. Let $\mathbb{B}$ be a small open ball around 0 such that h_t on $\mathbb{R}^k$ is the identity outside $\mathbb{B}$. Since ϕ is a diffeomorphism onto V, $\phi(\mathbb{B})$ is an open subset in Y contained in V. Then the map $H_t := \phi \circ h_t \circ \phi^{-1}$ extends to an isotopy on all of Y by defining H_t to be the identity outside $\phi(\mathbb{B})$. Then the existence of the isotopy H_t shows that y is **isotopic** to all points in the open subset $\phi(\mathbb{B})$ of Y. Hence the isotopy-equivalence class of y is **open**.

Now we construct the isotopy h_t on $\mathbb{R}^k$:

- We begin with the **case** $k = 1$: Given $\varepsilon > 0$, by Lemma 8.9 we can find a **smooth** function $\varphi\colon \mathbb{R} \to \mathbb{R}$ that vanishes outside $(-\varepsilon, \varepsilon)$ and equals 1 at 0. For $z \in \mathbb{R}^1$ and $t \in [0, 1]$, we define
$$h_t(x) = x + t \cdot \varphi(x) \cdot z.$$
Then h_0 is the **identity** and $h_t(x) = x$ if $x \notin (-\varepsilon, \varepsilon)$ for all t, and $h_1(0) = z$. To show that h_t is an **isotopy** we compute its **derivative** with respect to x:
$$h_t'(x) = 1 + t \cdot \varphi'(x) \cdot z.$$
Since $|\varphi'(x)|$ vanishes outside a **compact** set, it must be **bounded**. Hence as long as $|z|$ is small enough, $|t \cdot \varphi'(x) \cdot z| < 1$ for all $t \in [0, 1]$ and $x \in \mathbb{R}$. Thus $h_t'(x) > 0$ for all t and x and h_t is **strictly increasing**. By the **Inverse Function Theorem 3.2** this implies that inverse of h_t is **also smooth**. Hence h_t is a **diffeomorphism** of $\mathbb{R}^1$ for all z with $|z|$ small enough.

- Now let $k \geq 2$: We are going to use what we just learned from the case $k = 1$. Given any point p in $\mathbb{R}^k$ near the origin, we may rotate the coordinate axes so that the point lies on the first axis. Hence, writing $\mathbb{R}^k = \mathbb{R}^1 \times \mathbb{R}^{k-1}$, we can assume that p has coordinates of the form $p = (z, 0)$ with $z \in \mathbb{R}^1$.

  First we choose a function ρ as for $k = 1$. Given $\varepsilon > 0$, let again φ be a **smooth** function that vanishes outside $(-\varepsilon, \varepsilon)$ and equals 1 at 0. By Lemma 8.9 we can choose a function $\sigma\colon \mathbb{R}^{k-1} \to \mathbb{R}$ such that $\sigma(0) = 1$ and σ is zero outside some small ball of radius δ. We then define $h_t\colon \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ as follows: for $(x, y) \in \mathbb{R}^1 \times \mathbb{R}^{k-1}$ we set
$$h_t(x, y) = (x + t \cdot \sigma(y) \cdot \varphi(x) \cdot z, y)$$
  We need to check that h_t has the desired properties:

  - We have $h_t(x, y) = (x, y)$ unless $|x| < \varepsilon$, $|y| < \delta$, and $t > 0$.
  - $h_1(0, 0) = (z, 0)$ by the choice of σ and ρ.

- It remains to show that h_t is a **diffeomorphism** of $\mathbb{R}^k$ when $|z|$ is small. First, we observe that h_t is **one-to-one and onto**: We can show as above that $|t \cdot \sigma(y) \cdot \varphi'(x) \cdot z| < 1$ for all t and x if $|z|$ is small enough. Hence, for each fixed $y \in \mathbb{R}^{k-1}$, h_t restricts to a diffeomorphism of $\mathbb{R}^1 \times \{y\}$. Since h_t is the identity on $\mathbb{R}^{k-1}$, h_t must be one-to-one and onto on all of $\mathbb{R}^k$. Since h_t is smooth, it remains to check this its inverse is also smooth: The derivative of h_t at $(x, y) \in \mathbb{R}^1 \times \mathbb{R}^{k-1}$ is represented by the following matrix in the standard basis:

$$\begin{pmatrix} 1 + t \cdot \sigma(y) \cdot \varphi'(x) \cdot z & * \cdots * \\ 0 & \\ \vdots & I_{k-1} \\ 0 & \end{pmatrix}$$

where I_{k-1} is the $(k-1) \times (k-1)$-identity matrix. When $|z|$ is small enough, we can show as above that the left-hand corner entry $1 + t \cdot \sigma(y) \cdot \varphi'(x) \cdot z$ is always positive. Hence the matrix always has **strictly positive** determinant. By the **Inverse Function Theorem 3.2** this implies that the inverse of h_t is smooth. □

We can actually extend the assertion to any finite number of points. We will use this result later.

Theorem 12.13 (Isotopy Theorem) Suppose that Y is a **connected** manifold of **dimension bigger than** 1, and let $\{y_1, \dots, y_n\}$ and $\{z_1, \dots, z_n\}$ be two sets of distinct points in Y. Then there exists a **diffeomorphism** $h \colon Y \to Y$ which is **isotopic** to the **identity** with

$$h(y_1) = z_1, \dots, h(y_n) = z_n.$$

Moreover, the isotopy may be taken to be **compactly supported**.

Proof of Theorem 12.13: The proof is **by induction**. The **Isotopy Lemma 12.3** is the case $n = 1$. Now we assume the assertion being true for $n-1$. Then we have a **compactly supported isotopy** $h'_t \colon Y \setminus \{y_n, z_n\} \to Y \setminus \{y_n, z_n\}$ such that $h'_1(y_i) = z_i$ for all $i < n$ and $h'_0 = \mathrm{Id}$.

Since $\dim Y > 1$, the punctured manifold $Y \setminus \{y_n, z_n\}$ is connected. Since the isotopy h'_t has compact support, there are **open neighborhoods** around y_n and z_n in Y on which the h'_t are all equal to the identity. Hence we can extend the family h'_t to a family of **diffeomorphisms** of Y that **fix** those two points.

Now we apply the induction hypothesis again to the punctured manifold

$$Y \setminus \{y_1, \dots, y_{n-1}, z_1, \dots, z_{n-1}\} \text{ and the points } y_n, z_n.$$

Then we get a **compactly supported isotopy** h''_t with $h''_1(y_n) = z_n$ and $h''_0 = \mathrm{Id}$. By the same argument as for h'_t, we can extend h''_t to an isotopy on all of Y such that all h''_t satisfy $h''_t(y_i) = z_i$ for all $i < n$. Then

$$h_t := h''_t \circ h'_t$$

is the desired isotopy. □

12.3 Winding Numbers and the Borsuk–Ulam Theorem

Now we study an important application of the degree modulo 2 and prove a famous theorem. First we introduce a useful new invariant.

12.3.1 Winding numbers modulo 2

Let X be a **compact**, **connected** smooth manifold, and let

$$f: X \to \mathbb{R}^n$$

be a smooth map. We assume $\dim X = n-1$. Let z be a point of $\mathbb{R}^n$ **not** lying in the image $f(X)$. We would like to **understand how $f(x)$ winds around** z. To do this, we look at the unit vector

$$u(x) = \frac{f(x)-z}{|f(x)-z|}.$$

It points in the direction from z to $f(x)$ and has **length one**. With z **fixed** and x **varying**, we can consider u as a map

$$u: X \to \mathbb{S}^{n-1}.$$

We would like to know how often this **vector points in a given direction**, i.e., how often $u(x)$ has a given value. We learned in Section 12.1.3 that the degree of u is an invariant that encodes this information:

- The number $\#u^{-1}(y)$ modulo 2 is by definition $\deg_2(u)$. This number is constant for **regular values** y of the map u. To be a regular value means that $y-z$ hits $f(X)$ transversally, or in other words, the line through z and y must hit $f(X)$ transversally. See Figure 12.2.

Definition 12.14 (Winding number) We give this number a name and call it the **winding number** of f around z. We denote it by

$$W_2(f,z) := \deg_2(u).$$

Our goal is to prove the following famous result:

Theorem 12.15 (Borsuk–Ulam Theorem) Let $f: \mathbb{S}^k \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth map, and suppose that f is **odd**, i.e., f satisfies the **symmetry condition**

$$f(-x) = -f(x) \text{ for all } x \in \mathbb{S}^k. \tag{12.1}$$

Then the winding number of f equals one, i.e., $W_2(f,0) = 1$.

In other words, any map that is **anti-symmetric** around the origin must wind around the origin an **odd number of times**.

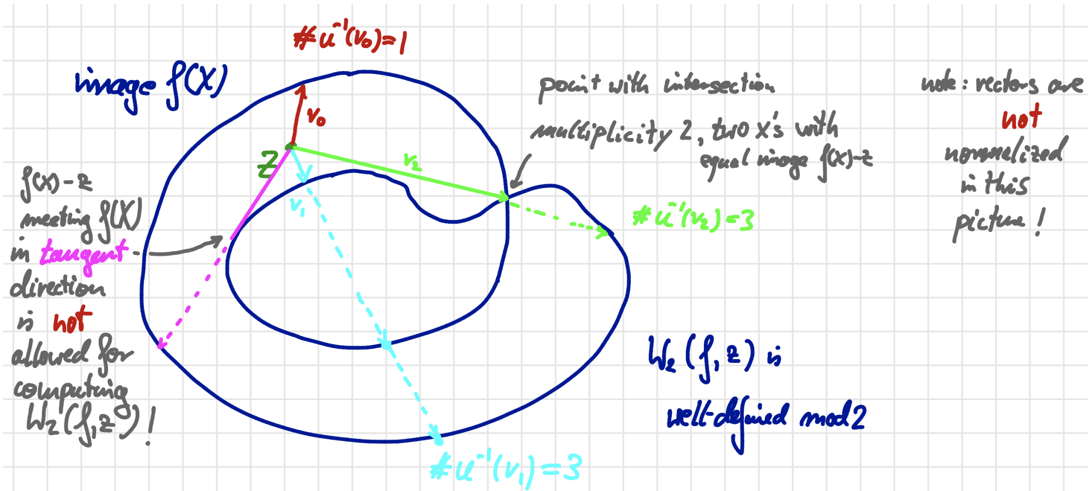


Figure 12.2: The winding number of f around z: we count the number of times the vector $f(x) - z$ points in a given direction where we neglect tangential points. Note also that some points on the graph may contribute multiple times. In the end we take our count modulo 2.

Aside: We will see below, there is a nice interpretation of this result for the **meteorologists among us**: At any given time, there are **two antipodal points** on the **Earth** that have the **same temperature and pressure**. Assuming temperature and pressure vary smoothly on the surface of the Earth. □

Before we approach the proof, we observe that the Borsuk–Ulam theorem is equivalent to the following assertion:

> **Theorem 12.16 (Equivalent formulation of the Borsuk–Ulam Theorem)** If $f : \mathbb{S}^k \to \mathbb{S}^k$ is a map which sends **antipodal** points **to antipodal** points, i.e., $f(-x) = -f(x)$, then $\deg_2(f) = 1$.

Proof of Theorem 12.15 $\iff$ Theorem 12.16: First assume Theorem 12.15 is true: Given a smooth map $f : \mathbb{S}^k \to \mathbb{S}^k$ with $f(-x) = -f(x)$, we can consider it as a map $f : \mathbb{S}^k \to \mathbb{S}^k \subset \mathbb{R}^{k+1}$. Then we have $f = f/|f|$ and therefore

$$1 = W_2(f, 0) = \deg_2(f/|f|) = \deg_2(f).$$

Now assume Theorem 12.16 is true: Given a smooth map $f : \mathbb{S}^k \to \mathbb{R}^{k+1} \setminus \{0\}$ with $f(-x) = -f(x)$, then $f/|f|$ is a well-defined smooth map $f/|f| : \mathbb{S}^k \to \mathbb{S}^k$. Hence we have

$$1 = \deg_2(f/|f|) = W_2(f, 0)$$

by definition of winding number of f around 0. □

Recall that we called real functions f with $f(-x) = -f(x)$ **odd**. Hence, as a slogan, we can remember the Borsuk–Ulam Theorem for a smooth map $f : \mathbb{S}^k \to \mathbb{S}^k$ as follows:

- **(Borsuk–Ulam Theorem in a nutshell)** If f is odd, its degree is odd.

12.3.2 Winding numbers and boundaries

In order to prove the theorem, we first need to investigate the relationship of winding numbers and boundaries:

Theorem 12.17 (Winding numbers and boundaries) Suppose that $X = \partial D$ is the **boundary** of a **compact** n-dimensional manifold D with boundary, and let $F: D \to \mathbb{R}^n$ be a smooth map **extending** $f: X \to \mathbb{R}^n$, i.e., $\partial F = f$. Suppose that z is a **regular value** of F that does **not** belong to the image of f. Then $F^{-1}(z)$ is a **finite set**, and

$$W_2(f, z) = \# F^{-1}(z) \mod 2.$$

In other words, **f winds X around z as often as F hits z**, at least modulo 2.

Proof: We prove the assertion by studying the following two cases:

- **First case:** $F^{-1}(z) = \emptyset$, i.e., $\# F^{-1}(z) = 0$.

In this case, the map

$$u: X = \partial D \to \mathbb{S}^{n-1}, \ x \mapsto \frac{f(x) - z}{|f(x) - z|}$$

can be extended to a map

$$D \to \mathbb{S}^{n-1}, \ x \mapsto \frac{F(x) - z}{|F(x) - z|}$$

since $F(x) - z$ is never 0. Hence, by the **Boundary Theorem 12.10** for $\deg_2$, we have

$$W_2(f, z) = \deg_2(u) = 0 \mod 2.$$

- **Second case:** $F^{-1}(z) \neq \emptyset$.

Since D is **compact** and of dimension n, $F^{-1}(z)$ is a zero-dimensional closed submanifold of D, and hence compact and hence a **finite set**. Suppose

$$F^{-1}(z) = \{y_1, \dots, y_m\}.$$

Then we can choose local parametrizations around each y_i in D and let $\mathbb{B}_i$ be the image of a closed ball in $\mathbb{R}^n$ around y_i. See Figure 12.3. Since z is a **regular value**, the **Stack of Records Theorem 4.18** shows that $F^{-1}(z)$ is **discrete** and **disjoint** to $X = \partial D$. Thus we **can choose the radii small enough** such that these balls satisfy

$$\mathbb{B}_i \cap \mathbb{B}_j = \emptyset \text{ and } \mathbb{B}_i \cap X = \emptyset \text{ for all } i \neq j, \text{ and } i = 1, \dots, m.$$

We define

$$f_i := F_{|\partial \mathbb{B}_i}: \partial \mathbb{B}_i \to \mathbb{R}^n.$$

to be the restriction of F to $\partial \mathbb{B}_i$.

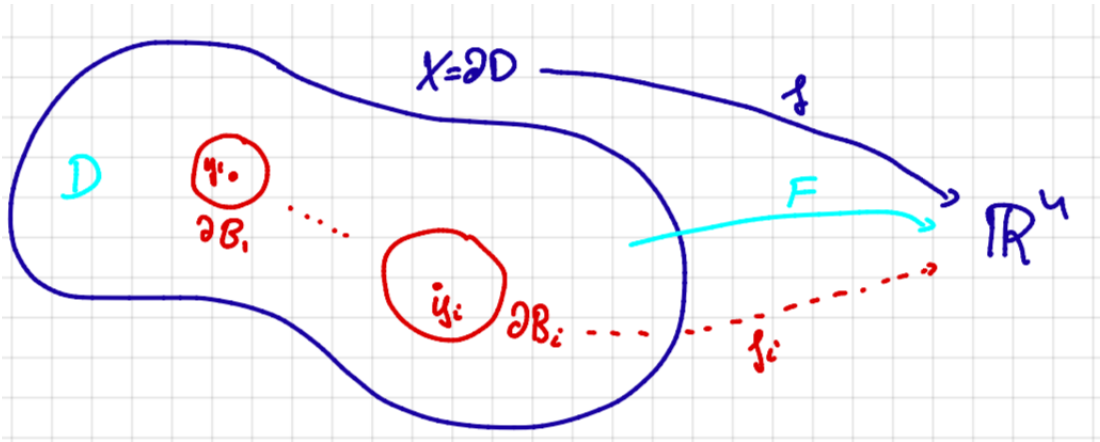


Figure 12.3: We split the contributions of the f_i and f by considering them as restrictions of F.

Now we observe that the subset

$$\tilde{D} := D \setminus (\cup_i \mathrm{Int}(\mathbb{B}_i))$$

is a closed submanifold of D with boundary

$$\partial\tilde{D} = \partial D \,\dot\cup\, \partial\mathbb{B}_1 \dot\cup \,\cdots\, \dot\cup\, \partial\mathbb{B}_m$$

the disjoint union of the boundaries of D and the $\mathbb{B}_i$'s. By the choice of the $\mathbb{B}_i$'s, we have $F^{-1}(z) \cap \tilde{D} = \emptyset$. Hence

$$F^{-1}(z) \cap \tilde{D} = (F_{|\tilde{D}})^{-1}(z) = \emptyset.$$

Hence the **winding number** of $\partial F_{|\tilde{D}}$ at z is **zero**. Since degrees and hence winding numbers are **additive** with respect to **connected components** this yields

$$0 = W_2(\partial F_{|\tilde{D}}, z) = W_2(f, z) + W_2(f_1, z) + \cdots + W_2(f_m, z) \quad \mod 2.$$

Since we are working **modulo** 2, this implies

$$W_2(f, z) = W_2(f_1, z) + \cdots + W_2(f_m, z) \quad \mod 2.$$

Now it **remains to show** $W_2(f_i, z) = 1$ for each $i = 1, \ldots, m$. For then

$$\#F^{-1}(z) = m = \sum_i W_2(f_i, z) = W(f, z) \quad \mod 2.$$

Since z is a **regular value** and $\dim D = n$, dF_{y_i} is an **isomorphism**. Thus, by the **Inverse Function Theorem 3.4**, we can choose the radius of $\mathbb{B}_i$ small enough such that $F_{|\mathbb{B}_i}$ is a **diffeomorphism onto its image** (which contains z). By continuity, this implies also that $f_i = \partial F_{|\mathbb{B}_i}$ is one-to-one onto the boundary of $F(\mathbb{B}_i)$.

By possibly rescaling and translating, we are **reduced to showing**:

Let $\mathbb{B}$ be the closed unit ball in $\mathbb{R}^n$ and $F : \mathbb{B} \to \mathbb{B}$ be a **diffeomorphism**. Let $f = \partial F : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. Then

$$\#F^{-1}(0) = W_2(f, 0) = 1 \quad \mod 2.$$

And this follows from $W_2(f, 0) = \deg_2(f) = \#f^{-1}(v) = 1$ for any $v \in \mathbb{S}^{n-1}$. □

12.3.3 Key lemma: Lifting self-maps of the circle

Now we are almost ready to attack the proof of the Borsuk–Ulam Theorem. The proof will proceed by induction on the dimension. To treat the case $k = 1$, we need one more ingredient which we will discuss next. It will turn out, however, that the effort we put in proving the following lemma will pay off later on.

Lemma 12.18 (Lifting self-maps of the circle) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a smooth map, and let $p : \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C}$ be the map defined by $t \mapsto \exp(2\pi i t) = e^{2\pi i t}$. Then we have:

- There exists a smooth map $g : \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "p"] \\ \mathbb{S}^1 \arrow[r, "f"'] & \mathbb{S}^1. \end{tikzcd}$$

- The map g satisfies

$$g(t+1) = g(t) + q \text{ for all } t \in \mathbb{R} \text{ for some fixed } q \in \mathbb{Z}.$$

- $\deg_2(f) = q$ modulo 2, i.e., $\deg_2(f) = 1$ if and only if q is odd.

Proof of Lemma 12.18: The **idea** is that, given any smooth map $f : \mathbb{S}^1 \to \mathbb{S}^1$, we can **lift** f **locally** and then **patch the pieces together** to get a smooth map, see Figure 12.4,

$$g : \mathbb{R} \to \mathbb{R} \text{ such that } p(g(t)) = f(p(t))$$

where p is the **covering map**

$$p : \mathbb{R} \to \mathbb{S}^1, \ t \mapsto \exp(2\pi i t) = e^{2\pi i t}.$$

If such a map g exists, we must have

$$p(g(t+1)) = f(p(t+1)) = f(p(t)) = p(g(t)) \text{ for all } t.$$

Since $p(t_1) = p(t_2)$ if and only if $t_1 - t_2 \in \mathbb{Z}$, we get

$$g(t+1) - g(t) \in \mathbb{Z}.$$

Since the function $t \mapsto g(t+1) - g(t)$ is **continuous** and takes only values in the **discrete** space $\mathbb{Z}$, it is **locally constant**. Since $\mathbb{R}$ is **connected**, the function must be **constant**. In other words, for all $t \in \mathbb{R}$, we have

$$g(t+1) = g(t) + q \text{ for some } \textbf{fixed } q \in \mathbb{Z}.$$

This q is a **fixed integer** depending only on f. We can actually think of q as measuring **how fast** f wraps $\mathbb{S}^1$ around itself.

- Assuming that the lift g exists, we now show $q = \deg_2(f)$ modulo 2:

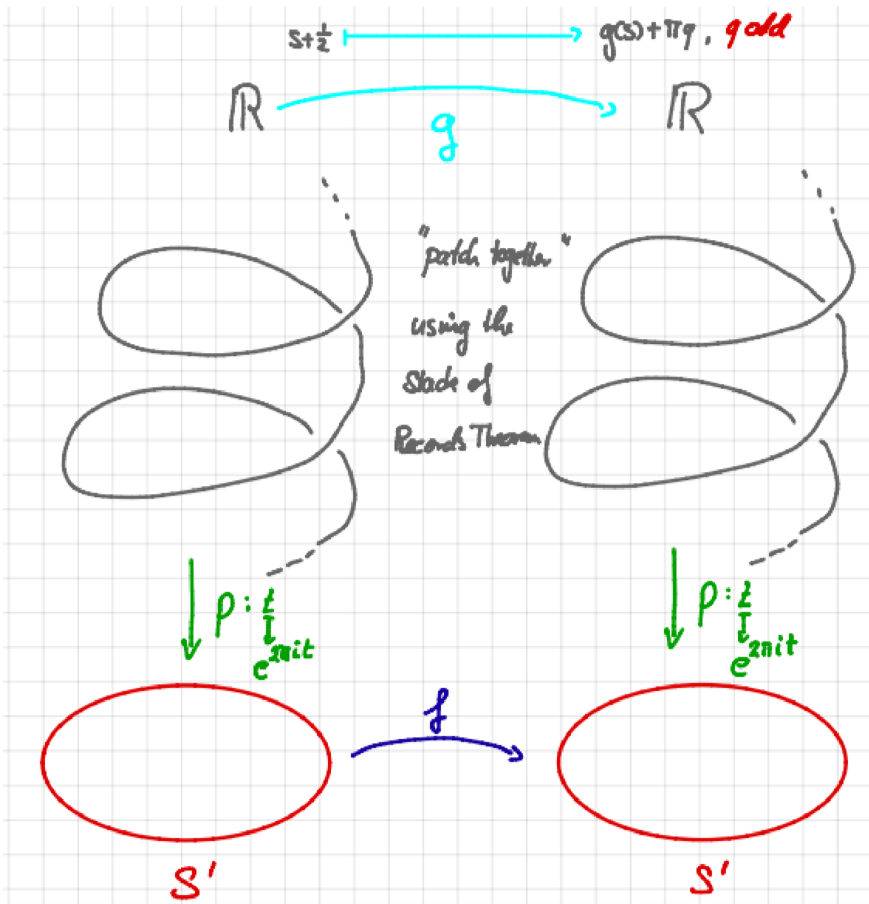


Figure 12.4: We think of $\mathbb{R}$ lying as a spiral above $\mathbb{S}^1$. That makes it easier to imagine how we lift f to a map g, first by local lifts which then are patched together.

First, note that **if** f is **not** surjective, then we can pick a point $y \notin f(\mathbb{S}^1)$. This y is automatically a **regular** value. Since $\#f^{-1}(y) = 0$, we must have $\deg(f) = 0$. In this case, we have $q = 0$, i.e., $g(t+1) = g(t)$. As otherwise $p \circ g$ was surjective and hence f would be surjective. Recall that, since the stereographic projection map $\mathbb{S}^1 \setminus \{y\} \to \mathbb{R}$ is a diffeomorphism and $\mathbb{R}$ is **contractible**, the space $\mathbb{S}^1 \setminus \{y\}$ is contractible. Hence f is a map to a contractible space and therefore **homotopic to a constant map** and has degree 0.

Now we assume that f is **surjective**. Let y be a **regular value** of f. We have $\deg_2(f) = \#f^{-1}(y)$. Hence we must **count the preimages** of y. Let $x \in f^{-1}(y)$ and $s \in \mathbb{R}$ be such that $p(s) = x$. We then have

$$y = f(x) = f(p(s)) = p(g(s)) = p(g(s) + k) \text{ for every **integer** } k.$$

And since $p(u) = p(u')$ if and only if $u - u' \in \mathbb{Z}$, the points of the form $g(s) + k$ are the **only points with** $p(u) = y$. Hence we **need to count** how often we have $g(t) = g(s) + k$ for $t \in [s, s+1]$. See Figure 12.5.

We assume that $q \geq 0$. If $q < 0$, then we use a similar argument with $-q$. By the **Intermediate Value Theorem** in Calculus, we know that the smooth function $g_{|[s,s+1]} \colon [s, s+1] \to \mathbb{R}$ takes the **value** $g(s) + k$ for each integer $k \in \{0, 1, \ldots, q-1\}$ an **odd** number of times and the values $g(s) + k$ for **any other integer** k an **even** number of times.[1] Hence, **modulo** 2, there are q **many points** t in the interval[2] $[s, s+1)$ such that

$$p(g(t)) = p(g(s)) = f(y).$$

Thus, **modulo** 2, there are q **many points** in $f^{-1}(y)$, and we have shown $\deg_2(f) = q$. □

- Now we explain why such **a lift** g **exists**:

We first consider the composite $h = f \circ p$:

$$\begin{tikzcd}
[0,1] \arrow[d, "p"'] \arrow[rd, dotted, "h"] & \mathbb{R} \arrow[d, "p"] \\
\mathbb{S}^1 \arrow[r, "f"'] & \mathbb{S}^1
\end{tikzcd}$$

where we restrict p to $[0,1] \subset \mathbb{R}$. We **make a choice** of a preimage point $s_0 \in \mathbb{R}$ with $p(s_0) = h(0)$. The restriction p_{s_0} of the covering map

$$p_{s_0} \colon (s_0 - 1/2, s_0 + 1/2) \to \mathbb{S}^1 \setminus \{p(s_0 + 1/2)\}$$

is a **diffeomorphism**. Hence we can use the inverse $p_{s_0}^{-1}$ to construct a **local lift** $p_{s_0}^{-1} \circ h$ in a **neighborhood** U_0 **of** 0 in $\mathbb{R}$ small enough such that $h(U_0) \subseteq \mathbb{S}^1 \setminus \{p(s_0 + 1/2)\} =: \mathbb{S}^1_-$:

$$\begin{tikzcd}
U_0 \arrow[r, dotted, "p_{s_0}^{-1} \circ h"] \arrow[d, "p"'] \arrow[rd, dotted, "h"] & \mathbb{R} \arrow[d, "p"'] \\
\mathbb{S}^1 \arrow[r, "f"'] & \mathbb{S}^1_- . \arrow[u, dotted, bend right, "p_{s_0}^{-1}"']
\end{tikzcd}$$

[1]Here we use that y being **regular** implies $\det(df_x) \neq 0$ and hence $dg(t_k) \neq 0$ for all t_k with $g(t_k) = g(s) + k$.

[2]Remember that $p(g(s+1)) = p(g(s))$. Hence we only need to take one of s and $s+1$ into account.

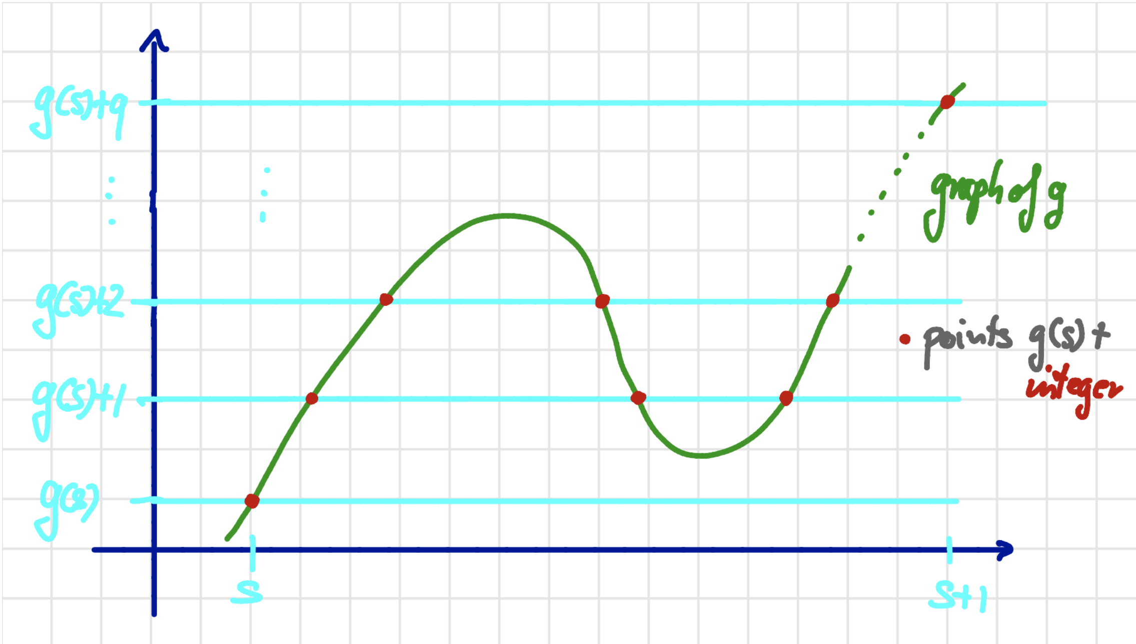


Figure 12.5: An application of an important theorem of Calculus. We count how many times the graph of g hits integer values, marked by the horizontal lines. We know that if the graph drops below a given value it has to pass it again to reach the value $g(s) + q$ eventually. Counting modulo 2 we get q crossings of the horizontal lines.

Now we **continue** this procedure to construct **enough local lifts** to cover all of $[0, 1]$. That this works can be shown as follows:[3] We have just constructed a lift of h on the interval $[0, s]$ for some s with $0 < s \leq 1$. Thus the set

$$D = \{t \in [0, 1] |\ h\colon\ [0, t] \to \mathbb{S}^1 \text{ can be lifted to a map } \tilde{g}\colon\ [0, t] \to \mathbb{R} \text{ beginning at } s_0\}$$

is **not empty**. Since D is bounded by 1, it has a **least upper bound** $d \in [0, 1]$.

- **First claim:** $d \in D$.

To prove the claim, let U be an **open** subset of $\mathbb{S}^1$ containing $h(d)$ such that p maps **each component** of $p^{-1}(U)$ **diffeomorphically** onto U. Since h is **continuous**, there is an **open** subset $W \subset [0, 1]$ containing d with $h(W) \subset U$. By definition of D and d, **there is** a point $s' \in W$ such that $0 < s' < d$ and $s' \in D$. Since we can lift h to $\tilde{g}\colon\ [0, s'] \to \mathbb{R}$, we have a **unique point** $\tilde{g}(s') \in \mathbb{R}$. We also have $\tilde{g}(s') \in p^{-1}(h(s'))$.

Let $V \subset \mathbb{R}$ denote the component of $p^{-1}(U)$ with $\tilde{g}(s') \in V$. Since $h(s')$ and $h(d)$ are **both in** U, we can use $(p_{|V})^{-1}$ to define a **local lift** $(p_{|V})^{-1} \circ h\colon\ [0, s'] \to \mathbb{R}$ of h on $[s', d]$. Now the previous lift of h on $[0, s']$ and the new one on $[s', d]$ **agree at s' by construction**. Hence we

[3] You find a proof of this fact in almost every textbook on Algebraic Topology in the section on covering spaces and lifting of paths. Here we follow the argument in [19, page 88].

can **glue them together** to get a lift $\tilde{g} \colon [0, d] \to \mathbb{R}$ of h on the **entire interval** $[0, d]$.[4] Thus $d \in D$ as claimed.

- **Second claim:** $d = 1$.

Suppose $d < 1$. For the open set $W \subset [0, 1]$ as above, we have $d \in W$. Since W is **open**, it still contains the open interval $(d - 2\varepsilon, d + 2\varepsilon)$ for some **small** $\epsilon > 0$. Then we could use the above argument to **extend the lift** of h to $[0, d + \varepsilon]$. Thus we **would have** $d + \varepsilon \in D$ which **contradicts** that d is the least upper bound of D. Hence we must have $d = 1$ as claimed. □

Hence we have shown that **there is a smooth map** $\tilde{g} \colon [0, 1] \to \mathbb{R}$ such that

$$\begin{tikzcd} {[0,1]} \arrow[r, "\tilde{g}"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "p"] \\ \mathbb{S}^1 \arrow[r, "f"'] & \mathbb{S}^1 \end{tikzcd}$$

commutes. We then have

$$p(\tilde{g}(1)) = f(p(1)) = f(p(0)) = p(\tilde{g}(0))$$

and hence

$$g(1) = g(0) + q \text{ for some fixed } q \in \mathbb{Z}.$$

Finally, we **define** $g \colon \mathbb{R} \to \mathbb{R}$ by setting

$$\begin{cases} g(t) := \tilde{g}(t) \text{ for all } t \in [0, 1], \text{ and} \\ g(t + 1) = g(t) + q \text{ for all } t \in \mathbb{R}. \end{cases}$$

This finishes the proof of the lemma. □

Remark 12.19 **($\mathbb{R}$ is a covering space of $\mathbb{S}^1$)** There is a **deeper reason** why this works. In fact, $\mathbb{R}$ is a **(universal) covering space** of $\mathbb{S}^1$, and continuous paths can always be lifted to a covering space. You will learn more about this phenomenon later.

12.3.4 Proof of the Borsuk–Ulam Theorem

Now we are ready to prove Theorem 12.15. The proof will proceed **by induction**.

- **The case $k = 1$:**

[4]Actually, this gives us only a continuous lift a priori. But we can compose with appropriate smooth bump functions to smoothen out any possible singular points. Since the notation would get extremely annoying, we skip this step.

Since we showed that Theorem 12.15 and Theorem 12.16 are equivalent, it suffices to show that a map $f: \mathbb{S}^1 \to \mathbb{S}^1$ with $f(-x) = -f(x)$ has $\deg_2(f) = 1$. By Lemma 12.18, we can find a smooth map $g: \mathbb{R} \to \mathbb{R}$ with $f \circ p = p \circ g$ and $g(t+1) = g(t) + q$ for some $q \in \mathbb{Z}$.

If **f is odd**, then

$$f(p(t+1/2)) = f(-p(t)) = -f(p(t))$$

where we use $p(t+1/2) = \exp(2\pi i(t+1/2)) = -\exp(2\pi i t) = -p(t)$. Hence

$$p(g(t+1/2)) = f(p(t+1/2)) = -f(p(t)) = -p(g(t)).$$

We also have

$$\begin{aligned} & p(s_1) = -p(s_2) \\ \iff & \exp(2\pi i s_1) = -\exp(2\pi i s_2) \\ \iff & \exp(2\pi i s_1) = \exp(2\pi i s_2 + \pi i) \\ \iff & s_1 = s_2 + m/2 \text{ for some } \textbf{odd } m \in \mathbb{Z}. \end{aligned}$$

This implies

$$g(t+1/2) = g(t) + m/2 \text{ for some } \textbf{odd } m \in \mathbb{Z}.$$

Applied to $t = 1/2$, this yields

$$g(1) = g(1/2 + 1/2) = g(1/2) + m/2 = g(0) + m/2 + m/2 = g(0) + m.$$

Thus $q = m$ is **odd**. Hence, by the previous lemma, we have $\deg_2(f) = q = 1 \mod 2$. This finishes the case $k = 1$.

- **Induction step:**

Assume the theorem is true for $k-1$ and $k \geq 2$. Let $f: \mathbb{S}^k \to \mathbb{R}^{k+1} \setminus \{0\}$ satisfy the symmetry condition (12.1). We consider $\mathbb{S}^{k-1}$ to be the **equator** of $\mathbb{S}^k$, embedded by

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0).$$

- **The idea** is to compute $W_2(f, 0)$ by **counting how often f intersects a line L** in $\mathbb{R}^{k+1}$. By choosing L **disjoint** from the **image of the equator**, we can use the **induction hypothesis** to show that the **equator winds around L an odd number of times**. Finally, it is easy to calculate the intersection of f with L once we know the behavior of f on the equator.

Let $g: \mathbb{S}^{k-1} \to \mathbb{R}^{k+1} \setminus \{0\}$ be the restriction of f to the equator. By **Sard's Theorem 7.1**, we can choose a value $y \in \mathbb{S}^k$ which is **regular for both** smooth maps

$$\frac{g}{|g|}: \mathbb{S}^{k-1} \to \mathbb{S}^k, \text{ and } \frac{f}{|f|}: \mathbb{S}^k \to \mathbb{S}^k.$$

The **symmetry condition** implies that **y is regular for both** these maps if and only if **$-y$ is regular for both** maps, since the derivatives at preimages of y and $-y$ just differ by multiplying with (-1).

Since $\dim \mathbb{S}^{k-1} < \dim \mathbb{S}^k$, the only way y can be a **regular value of** $\frac{g}{|g|}$ is when y is **not** in the image. Hence **neither** y nor $-y$ are in the image of $\frac{g}{|g|}$.

Thus, for the **line** $L := \mathbb{R} \cdot y = \mathrm{span}(y)$, we have

$$y \text{ is a regular value of } g \iff \mathrm{Im}\,(g) \cap L = \emptyset.$$

That y is **regular for** $\frac{f}{|f|}$ means by definition

$$\mathrm{Im}\left(d\left(\frac{f}{|f|}\right)_x\right) = T_y(\mathbb{S}^k).$$

The **tangent space** to $\mathbb{S}^k$ at y is the **orthogonal complement** of the line pointing in direction of y. The map $x \mapsto \frac{f(x)}{|f(x)|}$ is the composite of f and the map $x \mapsto x/|x|$ which is **smooth in dimensions** $k \geq 2$.

The derivative of the latter map satisfies

$$\mathrm{Im}\,(d(x/|x|)_x) = (\mathrm{span}(x))^{\perp} \subset \mathbb{R}^{k+1}, \text{ i.e., } \mathrm{Ker}\,(d(x/|x|)_x) = \mathrm{span}(x).$$

For $f/|f|$, this means

$$\mathrm{Ker}\left(d\left(\frac{f}{|f|}\right)_x\right) = \mathrm{span}(f(x)) \cap \mathrm{Im}\,(d f_x).$$

Thus

$$\begin{aligned}
\mathrm{Im}\left(d\left(\frac{f}{|f|}\right)_x\right) = T_y(\mathbb{S}^k) &\iff \mathrm{Ker}\left(d\left(\frac{f}{|f|}\right)_x\right) = \{0\} \\
&\iff \mathrm{span}(f(x)) \cap \mathrm{Im}\,(d f_x) = \{0\} \\
&\iff \mathrm{span}(f(x)) \not\subset \mathrm{Im}\,(d f_x) \\
&\iff L + \mathrm{Im}\,(d f_x) = \mathbb{R}^{k+1} \\
&\iff f \pitchfork L.
\end{aligned}$$

Summarizing the argument, we have obtained

$$y \text{ is a regular value of } \frac{f}{|f|} \iff f \pitchfork L = \mathrm{span}(y). \tag{12.2}$$

Now we are going to **exploit these two observations** for calculating $W_2(f, 0)$. By definition, we have

$$W_2(f,0) = \deg_2\left(\frac{f-0}{|f-0|}\right) = \deg_2\left(\frac{f}{|f|}\right) = \#\left(\frac{f}{|f|}\right)^{-1}(y) \mod 2.$$

By **symmetry**, we have

$$\#\left(\frac{f}{|f|}\right)^{-1}(y) = \#\left(\frac{f}{|f|}\right)^{-1}(-y).$$

From (12.2) we know

$$\begin{aligned} f^{-1}(L) &= \{x \in \mathbb{S}^k : f(x) \in L\} \\ &= \{x \in \mathbb{S}^k : \frac{f(x)}{|f(x)|} = \pm y\} \\ &= \left(\frac{f}{|f|}\right)^{-1}(y) \cup \left(\frac{f}{|f|}\right)^{-1}(-y). \end{aligned}$$

Thus

$$\#\left(\frac{f}{|f|}\right)^{-1}(y) = \frac{1}{2}\#f^{-1}(L).$$

Hence we need to **calculate the number** $\frac{1}{2}\#f^{-1}(L)$, at least modulo 2.

By **symmetry**, we can do this on the **upper hemisphere** $\mathbb{S}^k_+$ of $\mathbb{S}^k$, i.e., the points on $\mathbb{S}^k$ with $x_{k+1} \geq 0$. Let f_+ be the restriction of f to $\mathbb{S}^k_+$. By the choice of y, **L does not meet the equator**, and hence **no point on the equator** is in $f^{-1}(L)$. This implies

$$\frac{1}{2}\#f^{-1}(L) = \#f_+^{-1}(L).$$

The upper hemisphere is a manifold with **boundary**

$$\partial\mathbb{S}^k_+ = \{x = (x_1, \ldots, x_{k+1}) : \sum_i x_i^2 = 1 \text{ and } x_{k+1} = 0\} = \mathbb{S}^{k-1}$$

being the equator.

Now we would **like to apply** Theorem 12.17 to the f_+ and $g = \partial f_+$ and use the **induction hypothesis**. **However**, the target of f_+ has **dimension** $k+1$, whereas for both the theorem and the induction hypothesis we **need** as target a Euclidean space of **dimension** k. So we need to **fix this**.

The **key** is that the **orthogonal complement of** L in $\mathbb{R}^{k+1}$, denoted by V, is a vector space of **dimension** k. By **choosing a basis** of V, we can **identify** it with $\mathbb{R}^k$.

To complete the argument, let $\pi : \mathbb{R}^{k+1} \to V$ be the **orthogonal projection** onto V. Since g is **symmetric** and π is **linear**,

$$\pi \circ g : \mathbb{S}^{k-1} \to V \text{ is } \textbf{symmetric} : \pi(g(-x)) = \pi(-g(x)) = -\pi(g(x)).$$

Moreover, we have

$$\pi(g(x)) = 0 \iff g(x) \in L, \text{ hence } \boldsymbol{\pi(g(x)) \neq 0} \text{ for all } x \in \mathbb{S}^{k-1}$$

by the definition of π and the **choice of** L. Thus, after choosing a basis for V, we can consider $\pi \circ g$ as a map

$$\pi \circ g : \mathbb{S}^{k-1} \to \mathbb{R}^k \setminus \{0\}.$$

Now we **apply the induction hypothesis** to $\pi \circ g$ and get $W_2(\pi \circ g, 0) = 1$.

To finish, **recall** $f_+ \pitchfork L$ and hence for

$$\pi \circ f_+ : \mathbb{S}^k \to V, \ (\pi \circ f_+) \pitchfork \{0\}.$$

In other words, 0 **is a regular value** of $\pi \circ f_+$. Hence we **can apply** Theorem 12.17 to $\pi \circ f_+$ and its boundary map $\partial(\pi \circ f_+) = \pi \circ g$ to get

$$W_2(\pi \circ g, 0) = \#(\pi \circ f_+)^{-1}(0).$$

But, by the choice of L, we have

$$\pi(f_+(x)) = 0 \iff f_+(x) \in L, \text{ and hence } (\pi \circ f_+)^{-1}(0) = f_+^{-1}(L).$$

This shows

$$W_2(f, 0) = \# f_+^{-1}(L) = W_2(\pi \circ g, 0) = 1. \quad \square$$

Remark 12.20 Going back to the definition of $W_2(f, z)$ and Figure 12.2: we learn from the proof, in particular, that lines **tangential** to $f(X)$ are **not allowed** for calculating $W_2(f, z)$.

Theorem 12.21 (Corollary of the Borsuk–Ulam Theorem) If $f : \mathbb{S}^k \to \mathbb{R}^{k+1} \setminus \{0\}$ is **symmetric about the origin**, i.e., $f(-x) = -f(x)$, then f **intersects every line** through 0 at least **once**.

Proof: Let L be a line in $\mathbb{R}^{k+1}$ through the **origin**. **If** f **never** hits L, then $\# f^{-1}(L) = 0$ and $f \pitchfork L$. By repeating the above proof using this f and L for calculating $W_2(f, 0)$, we would get the **contradiction** to Theorem 12.15

$$W_2(f, 0) = \# f^{-1}(L) = 0. \quad \square$$

In the exercises we study further consequences of the Borsuk–Ulam Theorem. So have a look!

12.4 Linking numbers and the Hopf invariant modulo 2

In this section we will have a first glimpse at the *Hopf invariant* in a modulo 2 version. We will discuss the actual $\mathbb{Z}$-valued Hopf invariant in the exercises later.

There are many different ways to define this tremendously influential invariant. Here we follow Milnor's outline [13, Problems 13-15] of Hopf's original approach using linking numbers.

12.4.1 Mod 2 linking number

We begin with the following definition:

Definition 12.22 (Linking number) For $k \geq 1$, let $X, Y \subset \mathbb{R}^{k+1}$ be two disjoint smooth manifolds. We assume that X and Y are **compact** and without boundary, of dimensions $\dim X = m$ and $\dim Y = n$ such that $m + n = k$. The mod 2 **linking number** $L_2(X, Y) := \deg_2(\lambda)$ of X and Y, see Figure 12.6, is defined to be the mod 2 degree of the map λ, i.e., $L_2(X, Y) := \deg_2(\lambda)$, where λ is given by

$$\lambda : X \times Y \to \mathbb{S}^k, \ (x, y) \mapsto \frac{x - y}{|x - y|}.$$

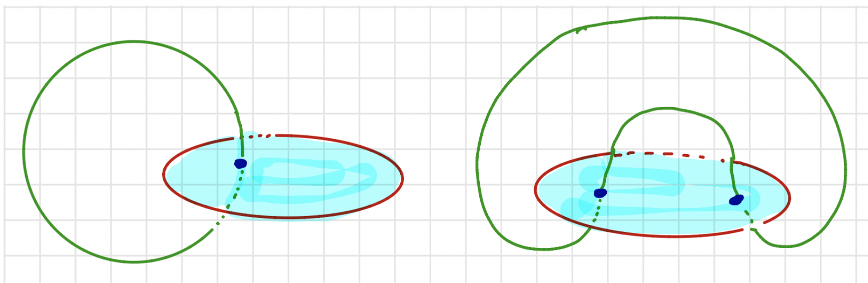


Figure 12.6: The red circle is the boundary of a compact manifold D. In both cases the green curve intersects the manifold D. The linking number detects the intersection. However, multiple intersections may occur. The circles on the left-hand side are linked and cannot be moved apart. The curves on the right-hand side are not linked and can be moved.

Example 12.23 (Linking number of two circles in $\mathbb{R}^3$) Suppose $X, Y \subset \mathbb{R}^3$ are two circles embedded in $\mathbb{R}^3$. Let us try to understand why λ detects how the two circles are linked:

- **First**, let us assume that X and Y are **not** linked. In this case, we can move them completely apart in $\mathbb{R}^3$ using an isotopy. Then $\lambda : X \times Y \to \mathbb{S}^2$ is **not surjective**, since the vectors $x - y$ cannot cover all directions of lines through the origin in $\mathbb{R}^3$. Hence every point p in $\mathbb{S}^2$ which is **not** in the image of λ is a **regular** value. Since $\lambda^{-1}(p) = \emptyset$ and hence $\#\lambda^{-1}(p) = 0$, we have $\deg_2(\lambda) = 0$. For example, if X consists of points with strictly positive first coordinate and Y consists of points with strictly negative first coordinate, then $x - y$ will always have **strictly positive** first coordinate. Hence the **hemisphere** of $\mathbb{S}^2$ of points with negative first coordinate will **not** be in the image of λ. See also Lemma 12.24.

- **Second**, assume that X and Y are **linked once** as on the left-hand side in Figure 12.6. In this case, we can express every direction of lines through the origin in $\mathbb{R}^3$. A little bit of work shows that, if we take orientations of both circles into account and use signs, then every direction is expresses once. This implies that $\#\lambda^{-1}(p) = 1$ for every point $p \in \mathbb{S}^2$.

- Note that, since X and Y are disjoint, the map λ is well-defined and smooth.

- The order of X and Y does not matter modulo 2, i.e., $L_2(X, Y) = L_2(Y, X)$. For the **oriented** linking number that we will study later this will be **different**. Then we have to consider a sign and have $L(Y, X) = (-1)^{(m+1)(n+1)} L(X, Y)$. This will imply that the Hopf invariant vanishes when n is **odd**. See Section 16.4.

Lemma 12.24 (Linking and boundary) Assume that X is the **boundary** of a smooth $(m+1)$-manifold W which is **disjoint** from Y. Then we have $L_2(X, Y) = 0$.

Proof: Since Y does not have a boundary, the product $W \times Y$ is a smooth manifold with boundary $\partial(W \times Y) = \partial W \times Y = X \times Y$. Since W and Y are disjoint, λ extends to a smooth map on $W \times Y$. Then the **Boundary Theorem 12.10** for degrees implies that $\deg_2(\lambda) = 0$. □

Now we extend the definition of the linking number to submanifolds X, Y in $\mathbb{S}^{k+1}$ as follows:

Definition 12.25 (Linking number on spheres) Let X and Y be **compact** submanifolds of $\mathbb{S}^{k+1}$ without boundary with $\dim X + \dim Y = k$. Since $\mathbb{S}^{k+1}$ is connected and since X and Y are closed subsets, there must be a point p which is **not** contained in either X or Y. We identity $\mathbb{S}^{k+1} \setminus \{p\}$ with $\mathbb{R}^{k+1}$ via the diffeomorphism defined by stereographic projection ϕ_p from p. Then we consider the images $\phi_p(X)$ and $\phi_p(Y)$ of X and Y, respectively, under this stereographic projection as submanifolds of $\mathbb{R}^{k+1}$. Now we define the **linking number** $L_2(X, Y)$ of X and Y to be $L_2(\phi_p(X), \phi_p(Y))$ of $\phi_p(X)$ and $\phi_p(Y)$ as in Definition 12.22.

12.4.2 Mod 2 Hopf invariant

The **main situation** we are interested in is the following:

Definition 12.26 (Hopf invariant modulo 2) For $n \geq 1$, consider a smooth map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$. Let $w \neq z \in \mathbb{S}^n$ be two **regular values** for f. Then $f^{-1}(w)$ and $f^{-1}(z)$ are compact submanifolds of $\mathbb{S}^n$ without boundary and their linking number $L_2(f^{-1}(w), f^{-1}(z))$ is defined. We define the **mod 2 Hopf invariant** of f, denoted $H_2(f)$, to be the mod 2 linking number of $f{-}1(w)$ and $f^{-1}(z)$, i.e.,

$$H_2(f) := L_2\left(f^{-1}(w), f^{-1}(z)\right).$$

Our next goal is to show that this number does not depend on the choice of w and z and only depends on the homotopy class of f.

Lemma 12.27 (Linking number is homotopy invariant) Let $F : \mathbb{S}^{2n-1} \times [0,1] \to \mathbb{S}^n$ be a smooth homotopy between $f_0, f_1 : \mathbb{S}^{2n-1} \to \mathbb{S}^n$. Let $w, z \in \mathbb{S}^n$ be regular values for F, and both f_0 and f_1. Then we have the equalities

$$L_2\left(f_0^{-1}(w), f_0^{-1}(z)\right) = L_2\left(f_1^{-1}(w), f_0^{-1}(z)\right) = L_2\left(f_1^{-1}(w), f_1^{-1}(z)\right). \qquad (12.3)$$

Proof: Since w is a regular value for F, the Boundary Theorem 10.16 implies that the subset $F^{-1}(w) \subset \mathbb{S}^{2n-1} \times [0,1] \subset \mathbb{R}^{2n+1}$ is a compact submanifold with boundary $\partial F^{-1}(w)$ given by

$$\begin{aligned}\partial F^{-1}(w) &= F^{-1}(w) \cap (\mathbb{S}^{2n-1} \times \{1\} \sqcup \mathbb{S}^{2n-1} \times \{0\}) \\ &= f_1^{-1}(w) \times \{1\} \sqcup f_0^{-1}(w) \times \{0\}.\end{aligned}$$

Since $f_0^{-1}(z)$ is a compact manifold without boundary, the product $F^{-1}(w) \times f_0^{-1}(z)$ is a compact manifold of with boundary given by

$$\partial(F^{-1}(w) \times f_0^{-1}(z)) = (f_1^{-1}(w) \times \{1\} \times f_0^{-1}(z)) \sqcup (f_0^{-1}(w) \times \{0\} \times f_0^{-1}(z)).$$

Thus the maps $\lambda_0 : f_0^{-1}(w) \times f_0^{-1}(z) \to \mathbb{S}^{2n-2}$ and $\lambda_1 : f_1^{-1}(w) \times f_0^{-1}(z) \to \mathbb{S}^{2n-2}$ are the restrictions of the map

$$\lambda_F : F^{-1}(w) \times f_0^{-1}(z) \to \mathbb{S}^{2n-2}, \; (x, y) \mapsto \frac{x - y}{|x - y|}$$

to the **two boundary components**. By the **Boundary Theorem 12.10** for $\deg_2$, this implies

$$\deg_2(\lambda_0 \sqcup \lambda_1) = 0$$

where $\lambda_0 \sqcup \lambda_1$ denotes the induced map $(f_0^{-1}(w) \times f_0^{-1}(z)) \sqcup (f_1^{-1}(w) \times f_0^{-1}(z)) \to \mathbb{S}^{2n-2}$. Since $\deg_2$ is by construction **additive** on **connected components**, this implies that

$$L_2\left(f_0^{-1}(w), f_0^{-1}(z)\right) + L_2\left(f_1^{-1}(w), f_0^{-1}(z)\right) \equiv 0 \mod 2.$$

Thus we have

$$L_2\left(f_0^{-1}(w), f_0^{-1}(z)\right) \equiv L_2\left(f_1^{-1}(w), f_0^{-1}(z)\right) \mod 2.$$

This proves the first equality in (12.3). Applying the same argument for the second factor proves the second equality. □

Lemma 12.28 The linking number $L_2\left(f^{-1}(w), f^{-1}(z)\right)$ is locally constant as a function of w and of z.

Proof: Because of the symmetry we only prove the assertion for w. The Local Submersion Theorem 4.2 implies that the set of regular values is an **open** subset of $\mathbb{S}^n$. Hence we can choose an $\varepsilon > 0$ so that the **open** ε-neighborhood of w contains only **regular values**. Let v be a point in $\mathbb{S}^n$ with $|v - w| < \varepsilon$. We can choose a family of smooth rotations $r_t : \mathbb{S}^n \to \mathbb{S}^n$ so that $r_1(w) = v$, and so that

- r_t is the identity for $0 \le t < \delta$, for some $0 < \delta < 1$,
- r_t equals r_1 for $1 - \delta < t \le 1$, and
- each $r_t^{-1}(v)$ lies on the great circle from w to v, which implies that it is a regular value for f.

Now we can define a homotopy

$$F : \mathbb{S}^{2n-1} \times [0,1] \to \mathbb{S}^n, \; F(x,t) = r_t(f(x)).$$

By our choice of r_t and v, note that, for each fixed t, v is a **regular value** for the composition $r_t \circ f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$. This implies that v also is a **regular value** for the map F. Now we can apply Lemma 12.27 to deduce the assertion. □

Lemma 12.29 (Hopf invariant is well-defined) For a smooth map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, the number $L_2\left(f^{-1}(w), f^{-1}(z)\right)$ does not depend on the choice of regular values w and z and only depends on the homotopy class of f.

Proof: By Lemma 12.28, the assignment $(w, z) \mapsto L_2\left(f^{-1}(w), f^{-1}(z)\right)$ is a locally constant function. Since Y is **connected**, the assignment is **constant**. Now if $g : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ is a smooth map homotopic to f, then there is a homotopy F between f and g. By **Sard's Theorem 7.1**, the set of regular values for f, g and F are dense in Y. Hence we can find elements w and z in Y which are regular values for f, g and F simultaneously. Now we can apply Lemma 12.27 to deduce that $H_2(f) = H_2(g)$. □

Remark 12.30 (H_2 is a map on homotopy groups) Denoting by $\pi_{2n-1}(\mathbb{S}^n)$ the $(2n-1)$-homotopy group of $\mathbb{S}^n$, we can view the mod 2 Hopf invariant as a map

$$H_2 : \pi_{2n-1}(\mathbb{S}^n) \to \mathbb{Z}/2.$$

12.4.3 The mod 2 Hopf invariant of the Hopf fibration

Now we are going to compute the Hopf invariant for the **Hopf fibration** $\pi : \mathbb{S}^3 \to \mathbb{S}^2$, an extremely important example of a smooth map. We recall the definition of π: We consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, i.e., $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$, and $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e., $\mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$. Then the **Hopf fibration** π is the map $\mathbb{S}^3 \to \mathbb{S}^2$ given by

$$\pi(z_0, z_1) = \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right).$$

We checked in Exercise 2.8 that this actually defines a map $\mathbb{S}^3 \to \mathbb{S}^2$.

We will now prove the following famous result due to Hopf:

Theorem 12.31 (Hopf invariant of the Hopf fibration is non-trivial) We have $H_2(\pi) = 1$ in $\mathbb{Z}/2$.

This result has many important consequences. Here are just a few:

- We can conclude that π is **not** homotopic to a constant map. This is a famous result of Heinz Hopf providing examples of non-contractible maps between spheres where the dimension of the domain is bigger than the codomain.
- We have shown in Exercise 2.8 that the Hopf map realizes $\mathbb{S}^3$ as a **disjoint** union of fibers which each look like $\mathbb{S}^1$. Since the Hopf invariant is the linking number of any two distinct circles on S^3, this shows that **all** these **disjoint circles** are **linked** in each other and cannot be pulled apart.
- Hence π exhibits a very special behavior of maps between spheres that only exists in a handful of dimensions. The latter is a famous result of Frank Adams, known as the Hopf

invariant one problem, and is actually concerned with the integral version of the Hopf invariant that we will study later. Adams' theorem had enormous consequences on the development of mathematics.

Proof of Theorem 12.31: We have shown previously that π is a **submersion**. Thus every point in $\mathbb{S}^2$ is a regular value. Here we consider $a = (0,0,1)$ and $b = (0,1,0)$ on $\mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$. The fiber of π over a is

$$\pi^{-1}(a) = \left\{(z_0, 0) \in \mathbb{S}^3 \subset \mathbb{C}^2 \,:\, |z_0|^2 = 1\right\}.$$

To determine the fiber over b, we write $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$. Then we get

$$\begin{aligned}\pi(z_0, z_1) = (0,1,0) &\Rightarrow 2z_0\bar{z}_1 = i \text{ and } |z_0|^2 = |z_1|^2 = \frac{1}{2}\\ &\Rightarrow y_0 = x_1,\ y_1 = -x_0 \text{ and } x_0^2 + x_1^2 = \frac{1}{2}.\end{aligned}$$

Thus the fiber over b has the form

$$\begin{aligned}\pi^{-1}(b) &= \left\{(z_0, z_1) \in \mathbb{S}^3 \,:\, \bar{z}_1 = \frac{i}{2z_0}\right\}\\ &= \left\{(x_0, y_0, x_1, y_1) \in \mathbb{S}^3 \,:\, y_0 = x_1, y_1 = -x_0\right\}.\end{aligned}$$

By definition of $H_2(\pi)$, we need to choose two distinct **regular values** w and z of π and calculate the mod 2 linking number of $\pi^{-1}(w)$ and $\pi^{-1}(z)$. Since we showed that each value is regular, we can for example choose $w = a = (0,0,1)$ and $z = b = (0,1,0)$ on $\mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathbb{C}\times\mathbb{R}$.

To calculate the linking number of $\pi^{-1}(a)$ and $\pi^{-1}(b)$ we need to choose a point on $\mathbb{S}^3$ disjoint from these two subsets and stereographically project $\mathbb{S}^3$ from this point onto $\mathbb{R}^3$. By our choice of a and b, we get that the north pole $N = (0,0,0,1)$ is neither on $\pi^{-1}(a)$ nor on $\pi^{-1}(b)$. Recall that the formula for the stereographic projection $\phi_N^{-1} : \mathbb{S}^3 \setminus \{N\} \to \mathbb{R}^3$ is, with the notation we use here, given by

$$(x_0, y_0, x_1, y_1) \mapsto \frac{1}{1 - y_1}(x_0, y_0, x_1).$$

Hence we get

$$\mathbb{S}_a := \phi_N^{-1}(\pi^{-1}(a)) = \{\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 \,:\, x_0^2 + x_1^2 = 1 \text{ and } x_2 = 0\}$$

and

$$\mathbb{S}_b := \phi_N^{-1}(\pi^{-1}(b)) = \left\{\mathbf{y} = (y_0, y_1, y_2) \in \mathbb{R}^3 \,:\, y_1 = y_2 \text{ and } y_0^2 + y_1^2 = \frac{(1 - y_0)^2}{2}\right\}.$$

Now we can calculate $H_2(\pi)$ as $\deg_2(\lambda)$ with

$$\lambda : \mathbb{S}_a \times \mathbb{S}_b \to \mathbb{S}^2, (\mathbf{x}, \mathbf{y}) \mapsto \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}.$$

To compute the degree of λ we pick a convenient point of $\mathbb{S}^2$ and determine the fiber over this point. Then we check that we actually picked a regular value.

So let us look at $p = (1, 0, 0)$. The equation $\lambda(\mathbf{x}, \mathbf{y}) = p$ then implies

$$x_1 = y_1 = 0 \text{ and } x_0 - y_0 = |x_0 - y_0|.$$

The latter condition implies that $v_0 - w_0$ is positive. This does not look very helpful at first glance, but we also know

$$1 = x_0^2 + x_1^2 = x_0^2, \text{i.e., } x_0 = \pm 1,$$

and

$$y_0^2 = \frac{(1 - y_0)^2}{2} \iff y_0 = \pm\sqrt{2} - 1.$$

Hence we can check for the four possible permutations of the signs whether they yield $x_0 - y_0 \geq 0$ and get three points: one with $x_0 = 1$, $y_0 = \sqrt{2} - 1$, one with $x_0 = 1$, $y_0 = -\sqrt{2} - 1$, and one with $x_0 = -1$, $y_0 = -\sqrt{2} - 1$. Hence we get three points $(\mathbf{x}, \mathbf{y})$ in $\mathbb{S}_a \times \mathbb{S}_b$ with $\lambda(\mathbf{x}, \mathbf{y}) = p$. Hence, once we have shown that p is a regular value, we will have proved $H_2(\pi) = \deg_2(\lambda) \equiv 1 \mod 2$.

It remains to check the derivatives of λ at these points and to show that p is a **regular value**. Hence we have to show that the determinants at each point are nonzero.

Since $\mathbb{S}_a$ is the unit circle in the xy-plane, the tangent space of $\mathbb{S}_a$ at a point $\mathbf{x}$ is given by

$$T_{\mathbf{x}}\mathbb{S}_a = \{\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{R}^3 \, : \, u_2 = 0 \text{ and } u_0 x_0 + u_1 x_1 = 0\}.$$

Similarly, $\mathbb{S}_b$ lies in the plane P in $\mathbb{R}^3$ of points $\mathbf{y} = (y_0, y_1, y_2)$ with $y_1 = y_2$. Then $\mathbb{S}_b$ is the fiber of the map

$$g_b \colon P \to \mathbb{R}, \; \mathbf{y} = (y_0, y_1, y_1) \mapsto y_0^2 + y_1^2 = \frac{(1 - y_0)^2}{2}.$$

After a simple computation we get

$$\mathbb{S}_b = g_b^{-1}(1) = \{\mathbf{y} = (y_0, y_1, y_1) \in P \, : \, 2y_1^2 + y_0^2 + 2y_0 = 1\}.$$

The derivative of g_b as a map from $P \to \mathbb{R}$ is given by the (2×1)-matrix (we could also consider it as a map $\mathbb{R}^3 \to \mathbb{R}^2$)

$$d(g_b)_{\mathbf{y}} = (2y_0 + 2, 2y_1).$$

Hence we get

$$T_{\mathbf{w}}\mathbb{S}_b = \{\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{R}^3 \, : \, u_2 = u_1 \text{and } u_0(y_0 + 1) + u_1 y_1 = 0\}.$$

Now we calculate the derivative of λ. First we do this as a map

$$\tilde{\lambda} \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \; (\mathbf{x}, \mathbf{y}) \mapsto \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}.$$

This will help us, since $g_3 \circ \tilde{\lambda}$ is constant on $\mathbb{S}_a \times \mathbb{S}_b$. Hence $d\tilde{\lambda}_{(\mathbf{x},\mathbf{y})}$ sends the subspace $T_{\mathbf{x}}\mathbb{S}_a \times T_{\mathbf{y}}\mathbb{S}_b \in \mathbb{R}^3 \times \mathbb{R}^3$ to $T_p S^2 \subset \mathbb{R}^3$.

We determine $d\tilde{\lambda}_{(\mathbf{x},\mathbf{y})}$ by computing its partial derivatives $\frac{\partial \tilde{\lambda}_i}{\partial x_j}(\mathbf{x}, \mathbf{y})$ and $\frac{\partial \tilde{\lambda}_i}{\partial y_j}(\mathbf{x}, \mathbf{y})$ with respect to the variables x_0, x_1, x_2 and y_0, y_1, y_2:

For $i \neq j$, we have

$$\frac{\partial \tilde{\lambda}_i}{\partial x_j}(\mathbf{x},\mathbf{y}) = \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x}-\mathbf{y}|^3} = \frac{\partial \tilde{\lambda}_i}{\partial y_j}(\mathbf{x},\mathbf{y}).$$

For $i = j$, we get

$$\frac{\partial \tilde{\lambda}_i}{\partial x_i}(\mathbf{x},\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|^3} \cdot \begin{cases} (x_1 - y_1)^2 + (x_2 - y_2)^2 & \text{if } i = 0 \\ (x_0 - y_0)^2 + (x_2 - y_2)^2 & \text{if } i = 1 \\ (x_0 - y_0)^2 + (x_1 - y_1)^2 & \text{if } i = 2 \end{cases}$$

and

$$\frac{\partial \tilde{\lambda}_i}{\partial y_i}(\mathbf{x},\mathbf{y}) = -\frac{\partial \tilde{\lambda}_i}{\partial x_i}(\mathbf{x},\mathbf{y}).$$

Now we evaluate these formulae at the points $(\mathbf{x},\mathbf{y})$ with $\lambda(\mathbf{x},\mathbf{y}) = p$. For each such point we found $x_1 = x_2 = y_1 = y_2 = 0$. Hence we get

$$\frac{\partial \tilde{\lambda}_i}{\partial x_j}(\mathbf{x},\mathbf{y}) = 0 = \frac{\partial \tilde{\lambda}_i}{\partial y_j}(\mathbf{x},\mathbf{y})$$

for $i \neq j$,

$$\frac{\partial \tilde{\lambda}_i}{\partial x_i}(\mathbf{x},\mathbf{y}) = \frac{1}{|x_0 - y_0|} \cdot \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1,2 \end{cases}$$

and

$$\frac{\partial \tilde{\lambda}_i}{\partial y_i}(\mathbf{x},\mathbf{y}) = \frac{1}{|x_0 - y_0|} \cdot \begin{cases} 0 & \text{if } i = 0 \\ -1 & \text{if } i = 1,2. \end{cases}$$

Now we are equipped to study the linear map $d\lambda_{(\mathbf{x},\mathbf{y})} \colon T_{\mathbf{x}}\mathbb{S}_a \times T_{\mathbf{y}}\mathbb{S}_b \to T_p\mathbb{S}^2$:

A basis of $T_p\mathbb{S}^2$ is given by the vectors $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

At each of the points $(\mathbf{x},\mathbf{y})$ we found with $\lambda(\mathbf{x},\mathbf{y}) = \mathbf{p}$, the vector $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis of $T_{\mathbf{x}}\mathbb{S}_a$ and the vector $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ provides a basis of $T_{\mathbf{y}}\mathbb{S}_b$. The map $d\lambda_{(\mathbf{x},\mathbf{y})}$ sends $\mathbf{a}$ to $\frac{1}{|x_0 - y_0|} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{b}$ to $\frac{1}{|x_0 - y_0|} \cdot \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$. Hence we have

$$d\lambda_{(\mathbf{x},\mathbf{y})}(\mathbf{a}) = \frac{1}{|x_0 - y_0|} \cdot \mathbf{e}_2 \text{ and } d\lambda_{(\mathbf{x},\mathbf{y})}(\mathbf{b}) = -\frac{1}{|x_0 - y_0|} \cdot \mathbf{e}_2 - \frac{1}{|x_0 - y_0|} \cdot \mathbf{e}_3.$$

These two vectors form a basis of $T_p\mathbb{S}^2$ and we see that $(\mathbf{x},\mathbf{y})$ is a regular point. Since this is true for all points in the fiber of $p \in \mathbb{S}^2$, we conclude that p actually is a regular value. □

12.5 Exercises and more examples

12.5.1 Degree modulo 2

Exercise 12.1 Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps between manifolds with X and Y compact, Y and Z connected. Assume that all three manifolds are boundaryless and $\dim X = \dim Y = \dim Z$. Show that

$$\deg_2(g \circ f) = \deg_2(g) \cdot \deg_2(f).$$

Exercise 12.2 Let $f : X \to Y$ be a smooth map between smooth manifolds with X compact, Y connected and $\dim X = \dim Y$.

(a) Show that, if $\deg_2(f) \neq 0$, then f is surjective.

(b) Show that if Y is not compact, then $\deg_2(f) = 0$.

(c) Let $X = Y = \mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle and assume that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a smooth map without fixed points. Show that f is surjective.

Hint: Show that f is homotopic to the antipodal map $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto -x$. What is $\deg_2(\alpha)$? Use the previous points.

Exercise 12.3 Show that there exists a complex number z such that

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0.$$

Exercise 12.4 Let m be an odd number and let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a complex polynomial. Show that there exists a $w \in \mathbb{C}$ such that $p(w) = 0$.

12.5.2 Borsuk–Ulam Theorem

Exercise 12.5 Let $f_1, \ldots, f_k$ be k smooth real-valued odd functions, i.e., $f_i : \mathbb{S}^k \to \mathbb{R}$ with $f(-x) = -f(x)$. Show that $f_1, \ldots, f_k$ must have a common zero, i.e., there is an $x \in \mathbb{S}^k$ such that $f_1(x) = \cdots = f_k(x)$.

Hint: Use the Borsuk–Ulam Theorem 12.15 and its consequence Theorem 12.21.

Exercise 12.6 Let $g_1, \ldots, g_k$ on $\mathbb{S}^k$ be smooth real-valued functions. Show that there exists a point $p \in \mathbb{S}^k$ such that

$$g_1(p) = g_1(-p), \dots, g_k(p) = g_k(-p).$$

Hint: Use the previous exercise.

Exercise 12.7 Let $p_1, \dots, p_n$ be real polynomials in $n+1$ variables. Assume each p_i is homogeneous of odd order, i.e., there is an odd number m_i such that $p_i(\lambda x) = \lambda^{m_i} p_i(x)$ for all $\lambda \in \mathbb{R}$. We consider each p_i as a smooth function $\mathbb{R}^{n+1} \to \mathbb{R}$ by sending x to $p_i(x)$.

Show that there is a line through the origin in $\mathbb{R}^{n+1}$ on which all the p_i's simultaneously vanish.

Hint: Use the Borsuk–Ulam Theorem 12.15 and the previous exercises.

Exercise 12.8 Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle and $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the two-dimensional sphere.

Show that there is no continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$ with $f(-p) = -f(p)$ for all $p \in \mathbb{S}^2$.

Hint: Assume such a map f existed. Then we could define the continuous map

$$g : \mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \to \mathbb{S}^1, \; g(x, y) := f\left(x, y, \sqrt{1 - x^2 - y^2}\right).$$

Show that g satisfies $g(-q) = -g(q)$ for all $q \in \mathbb{S}^1 = \partial\mathbb{B}^2$. What is the degree modulo 2 of g? Conclude that f cannot exist.

Exercise 12.9 Use the previous exercise to prove the following special case of our previous observations: For every smooth map $\mathbb{S}^2 \to \mathbb{R}^2$, there is a point p on $\mathbb{S}^2$ such that $f(p) = f(-p)$.

Exercise 12.10 Assume the following continuous version of the assertion of Exercise 12.9: For every continuous map $\mathbb{S}^2 \to \mathbb{R}^2$, there is a point p on $\mathbb{S}^2$ such that $f(p) = f(-p)$.

Deduce from this fact the following version of **Invariance of Dimension:**

An open subset in $\mathbb{R}^2$ cannot be homeomorphic to an open subset in $\mathbb{R}^n$ for $n \geq 3$.

13. Tubular Neighborhoods and Transversality

13.1 Normal bundles and tubular neighborhoods

In this section we study two key tools in differential topology. We will see several important application in the next sections. We are going to use a generalization of the Inverse Function Theorem that we will prove first.

13.1.1 A generalization of the Inverse Function Theorem

We begin with the compact case.

Theorem 13.1 (Generalization of the IFT - compact case) Let $f : X \to Y$ be a smooth map that is **one-to-one** on a **compact** submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

is an **isomorphism**. Then f maps an **open** neighborhood of Z in X **diffeomorphically** onto an **open** neighborhood of $f(Z)$ in Y.

Proof: We know that f maps Z **diffeomorphically onto its image** $f(Z)$, since $f : Z \to f(Z)$ is a bijective local diffeomorphism and therefore a diffeomorphism. We would like to show that we can extend this to an **open** neighborhood around Z.

Since df_x is an **isomorphism**, for each $x \in Z$, there exists an open neighborhood U_x in X around x on which $f_{|U_x}$ is a diffeomorphism. The collection $\{U_x\}$ is an open cover of Z. Since Z is **compact**, we can choose a **finite** subcover $\{U_1, \dots, U_n\}$. We set $U := \cup_i U_i$. Restricted to the open subset U, $f_{|U}$ is a **local diffeomorphism**. It remains to shrink U if necessary to ensure that $f_{|U}$ is also **injective**, i.e., we need to show that there is some **open** subset V in X which **contains Z** such that $f_{|V}$ is injective. Then $f_{|U\cap V}$ is an **injective local diffeomorphism**. We have shown previously that this implies that $f_{|U\cap V}$ is a diffeomorphism onto an **open** subset of Y. Since $Z \subset U$ and $Z \subset V$, we also have $Z \subset U \cap V$ and the assertion is proven.

We are going to show that V exists by **assuming the contrary**, i.e., we assume that there exists at least one point $z \in Z$ such that in any small open neighborhood W of z there are points $a \neq b$ with $f(a) = f(b)$. By choosing open neighborhoods W_i smaller and smaller around z and by choosing subsequences $a_i \neq b_i$ with $f(a_i) = f(b_i)$, we can assume that both the a_i and b_i converge to z. Since $f(a_i) = f(b_i)$ for all i and f is **continuous**, we have $f(a_i) \to f(z)$ and $f(b_i) \to f(z)$. But since df_z is an **isomorphism**, the previous Inverse Function Theorem 3.4 implies that there is a small **open** neighborhood W_z in X around z such that $f_{|W_z}$

is a **diffeomorphism**. Since $a_i \to z$ and $b_i \to z$, for N large enough, we have $a_i, b_i \in W_z$ and hence $f(a_i) = f(b_i) \in f(W_z)$ for all $i \geq N$. But since $f_{|W_z}$ is injective, this implies $a_i = b_i$ for all $i \geq N$. This contradicts the choice of the a_i and b_i. □

The existence of **partitions of unity** allows us to move from the compact to the general case. First we recall the following notion from general topology:

Definition 13.2 (Locally finite subcovers) An **open cover** $\{V_\alpha\}$ of a manifold X is called **locally finite** if each point of X possesses a neighborhood that intersects only **finitely** many of the sets V_α.

Partitions of unity can be used to show the following lemma:

Lemma 13.3 (Local finiteness lemma) Every open cover $\{U_\alpha\}$ on a manifold **admits** a locally finite refinement $\{V_\alpha\}$.

Since the above lemma is rather a statement in general topology, we omit its proof here. So we **move on** and use it to generalize the Inverse Function Theorem:

Theorem 13.4 (Generalization of the IFT - general case) Let $f : X \to Y$ be a smooth map that is **one-to-one** on a submanifold Z of X. Suppose that for all $x \in Z$, $df_x : T_x(X) \to T_{f(x)}(Y)$ is an **isomorphism**. **Assume** that f maps Z **diffeomorphically** onto $f(Z)$. Then f maps an **open** neighborhood of Z in X **diffeomorphically** onto an **open** neighborhood of $f(Z)$ in Y.

Proof: For each $x \in Z$, there exists an open neighborhood V_x in X around x on which $f_{|V_x}$ is a **diffeomorphism**, since df_x is an **isomorphism** for all $x \in Z$. Let $U_x = f(V_x)$ be the **open** image in Y. The collection of all U_x is an open cover of $f(Z)$, since each $f(x) \in f(Z)$ lies in some $U_x = f(V_x)$. By the previous lemma, we can choose a **locally finite** subcover $\{U_i\}$ of $f(Z)$ in Y. For each U_i, there is a **local inverse** $g_i : U_i \to V_i \subset X$ of $f_{|V_i}$. We define

$$W := \{y \in U_i : g_i(y) = g_j(y) \text{ whenever } y \in U_i \cap U_j\}.$$

On the subset W, we can define an inverse

$$g : W \to X, \; g(y) = g_i(y) \text{ for any } i.$$

This is well-defined by the construction of W, as $g(y) = g_i(y) = g_j(y)$ whenever $y \in U_i \cap U_j$. Since the g_i's are local inverses of f, we have $f(Z) \subset W$.

It remains to show that W contains an **open** subset which still contains $f(Z)$. Let $x \in Z$. Since $f(x) \in f(Z)$, we can find a k such that $f(x) \in U_k$. If $U_k \subset W$, we are done, since then every point in $f(Z)$ has an open neighborhood which is contained in W. So assume U_k is not contained in W. After shrinking U_j if necessary, we can assume by the **local finiteness** of the cover $\{U_i\}$, that there are only **finitely** many of the U_i's which **intersect** U_k, say $U_1, \dots, U_n$. Then, for $i = 1, \dots, n$, we set C_{ik} to be the **closure** of the set $\{y \in U_i \cap U_k : g_i(y) \neq g_k(y)\}$, i.e.,

$$C_{ik} = \overline{\{y \in U_i \cap U_k : g_i(y) \neq g_k(y)\}}.$$

Since the union of a **finite** collection of closed subsets is closed, $C_k := C_{1k} \cup \cdots \cup C_{nk}$ is **closed** in Y. Hence

$$U := U_k \setminus C_k$$

is **open** in Y. By definition of W and the C_k, we know $U \subset W$. It remains to make sure that $f(x)$ is still in U, i.e., that it does not belong to one of the closures C_{ik}.

Note that $f(x)$ satisfies $g_i(f(x)) = x = g_k(f(x))$ for all $i = 1, \dots, n$. Since df_x is an **isomorphism**, the **usual** Inverse Function Theorem implies that there is a small **open** neighborhood $V_\varepsilon \subset U$ around x such that $f_{|V_\varepsilon}$ is a **diffeomorphism**. Hence, for each $i = 1, \dots, n$, we have

$$g_i(f(x')) = x' = g_k(f(x')) \text{ for all } x' \in V_\varepsilon \cap g_i(U_i) \cap g_k(U_k).$$

Hence the **finite** intersection $f(V_\varepsilon) \cap U_k \cap U_1 \cap \cdots \cap U_n$ is an **open** subset which is **not** contained in any of the sets $\{y \in U_i \cap U_k : g_i(y) \neq g_k(y)\}$. Thus $f(x)$ is **not** contained in C_k. Hence we have shown that $U \subset W$ is an **open** subset **containing** $f(x)$. □

13.1.2 Normal bundle

Let $X \subset \mathbb{R}^m$ be a smooth manifold without boundary. We would like to understand the **geometry** of X with respect to its **environment**. In order to prove the ε-Neighborhood Theorem we introduce an important geometric tool similar to the tangent bundle.

Definition 13.5 (The Normal Bundle) For each $x \in X$, we define $N_x(X)$, the **normal space** of X at x, to be the **orthogonal complement** of $T_x(X)$ in $\mathbb{R}^m$. The **normal bundle** $N(Y)$ is then defined to be the space

$$N(X) = \{(x, v) \in X \times \mathbb{R}^m : v \in N_x(X)\}.$$

There is a **natural projection** map $\sigma : N(X) \to Y$ defined by $\sigma(x, v) = x$.

- **Warning:** Note that, unlike $T(X)$, $N(X)$ is **not intrinsic** to the manifold Y but **depends** on the specific relationship between Y and the surrounding $\mathbb{R}^m$. □

The normal bundle $N(X)$ is actually a smooth **manifold** itself. In order to show this, we must recall a **fact from linear algebra**: Suppose that $A : \mathbb{R}^m \to \mathbb{R}^k$ is a linear map. Its **transpose** is a linear map $A^t : \mathbb{R}^k \to \mathbb{R}^m$ characterised by the dot product equation

$$Av \cdot w = v \cdot A^t w \text{ for all } v \in \mathbb{R}^m, w \in \mathbb{R}^k.$$

Lemma 13.6 (Help from Linear Algebra) If A is **surjective**, then the **transpose** A^t maps $\mathbb{R}^k$ isomorphically onto the **orthogonal complement** of the **kernel** of A.

Proof: First we note that A^t is injective. For if $A^t w = 0$, then $Av \cdot w = v \cdot A^t w = 0$, so that $w \perp A(\mathbb{R}^m)$. Since A is surjective, w must be zero. Now, if $v \in \operatorname{Ker}(A)$, i.e., $Av = 0$, then $0 = Av \cdot w = v \cdot A^t w$. Thus $A^t(\mathbb{R}^k) \perp \operatorname{Ker}(A)$. Hence A^t maps $\mathbb{R}^k$ **one-to-one** into the **orthogonal complement** of $\operatorname{Ker}(A)$. As $\operatorname{Ker}(A)$ has dimension $m-k$, its complement has dimension k, so A^t is surjective, too. □

Now we can prove:

Theorem 13.7 (Normal bundles are manifolds) Let $X \subset \mathbb{R}^m$ be a smooth n-dimensional manifold without boundary. Then $N(X)$ is a smooth **manifold** of dimension m and the projection $\sigma : N(X) \to X$ is a **submersion**.

Proof: We need to **find local parametrizations** for $N(X)$. Let x be any point in X. We learned in the discussion of the Local Immersion Theorem that we can find an open subset $V \subset \mathbb{R}^m$ containing x and local coordinate functions $(u_1, \dots, u_m) : V \to \mathbb{R}^m$ such that

$$X \cap V = \{v \in V : u_{n+1}(v) = \cdots = u_m(v) = 0\}.$$

We **define** φ to be the smooth map

$$\varphi : V \to \mathbb{R}^{m-n},\ v \mapsto (u_{n+1}(v), \dots, u_m(v)).$$

We set $U = \varphi^{-1}(0) = X \cap V$. Note that, since V is open in $\mathbb{R}^m$, we deduce that U is **open** in X. Let $N(U)$ be the normal bundle of U considered as a smooth manifold in $\mathbb{R}^m$. We observe that $N(U)$ equals $N(X) \cap (U \times \mathbb{R}^m)$, thus it is **open in $N(X)$**.

For each $x \in U$, $d\varphi_x : \mathbb{R}^m \to \mathbb{R}^{m-n}$ is **surjective** and has **kernel** $T_x(U) = T_x(X)$ by the **Preimage Theorem 6.3**. Therefore the **transpose** of $d\varphi_x$ maps $\mathbb{R}^{m-n}$ **isomorphically** onto the **orthogonal complement** of $\operatorname{Ker}(d\varphi_x) = T_x(X)$. But this is $N_x(X)$ by definition. Hence we get an **isomorphism**

$$(d\varphi_x)^T : \mathbb{R}^k \xrightarrow{\cong} (T_x(X))^\perp = N_x(X).$$

This shows that the map

$$\psi : U \times \mathbb{R}^{m-n} \to N(U),\ (x, v) \mapsto (x, d\varphi_x^T(v))$$

is **bijective**. It is also an **embedding** of $U \times \mathbb{R}^{m-n}$ into $U \times \mathbb{R}^m$, since it is the **identity** on the first factor and an **injective linear** map on the second factor. Hence ψ is a **diffeomorphism**. Thus we have shown that $N(U)$ is a smooth **manifold** with local parametrization ψ. The **dimension** of $N(U)$ is

$$\dim N(U) = \dim U + m - n = n + m - n = m.$$

Since **every point** of $N(X)$ has such a neighborhood $N(U)$, $N(X)$ is a smooth manifold.
Note that $\sigma \circ \psi : U \times \mathbb{R}^k \to U$ is just the **projection** onto the first factor, which is a **submersion**. Thus $d(\sigma \circ \psi)_{(u,w)}$, is **surjective** at every point (u, w). Hence $d\sigma_u$ is surjective at every u, and σ is a **submersion**. □

13.1.3 The Tubular Neighborhood Theorem

Now we are going to study an important **application** of normal bundles. Let $X \subset \mathbb{R}^n$ be a smooth manifold without boundary. The next theorem will provide us with the desired information about the geometry of how Y sits inside its ambient space. More precisely, our goal is to show that every smooth manifold has a special type of neighborhood. We begin with a lemma that will give us the **existence** of interesting neighborhoods of X in $\mathbb{R}^n$:

Lemma 13.8 **(ε-Neighborhood Lemma)** Let $X \subset \mathbb{R}^n$ be a smooth manifold without boundary. There is a **smooth** function $\varepsilon : X \to \mathbb{R}^{>0}$ such that every **open** subset U of $\mathbb{R}^n$ with $X \subset U$ contains the **open** subspace in $\mathbb{R}^n$

$$X^\varepsilon = \{y \in \mathbb{R}^n : |y - x| < \varepsilon(x) \text{ for some } x \in X\}.$$

If X is **compact**, ε can be chosen to be **constant**.

Proof: For each point $x \in X$, we can find a small radius ε_x such that the open ball $B_{2\varepsilon_x}(x) \subset U$ is contained in U. We set

$$U_x := X \cap B_{\varepsilon_x}(x).$$

- **Claim:** $U_x^{\varepsilon_x} = \{y \in \mathbb{R}^n : |y - x'| < \varepsilon_x \text{ for some } x' \in U_x\} \subset U$.

For, $y \in U_x^{\varepsilon_x}$ means there is an $x' \in U_x$ with $|y-x'| < \varepsilon_x$. But $x' \in U_x$ means $|x'-x| < \varepsilon_x$. Thus the **triangle inequality** implies

$$|y - \alpha| \le |y - x'| + |x' - x| < 2\varepsilon_x.$$

Thus $v \in B_{2\varepsilon_x}(x) \subset U$ by the choice of ε_x. The collection of all U_x forms an open cover $\{U_x\}$ of $X \subset \mathbb{R}^n$. By the existence of partitions of unity for subsets in $\mathbb{R}^n$, we can **choose a partition of unity** $\{\rho_i\}$ subordinate to the cover $\{U_x\}$. Now we define the function

$$\varepsilon : X \to \mathbb{R}^{>0}, \; x \mapsto \sum_i \rho_i(x)\varepsilon_x$$

Note that ε is a **smooth** function, since all the ρ_i's are smooth.

- **Claim:** $X^\varepsilon \subset U$.

Let $y \in X^\varepsilon$. Then there is a $x \in X$ such that $|y - x| < \varepsilon(x)$. For this x, only finitely many of the numbers $\rho_i(x)$ are nonzero, say $\rho_{i_1}(x), \dots, \rho_{i_n}(x)$. This implies $y \in U_{i_1} \cap \dots \cap U_{i_n}$. Let ε_{i_m} be the **maximum** of the **finitely** many numbers $\varepsilon_{i_1}, \dots, \varepsilon_{i_n}$. Then, since $\sum_i \rho_i(x) = 1$, we have $\varepsilon(x) \le \varepsilon_{i_m}$. Hence

$$|y - x| < \varepsilon(x) \le \varepsilon_{i_m} \text{ implies } v \in U_{i_m}^{\varepsilon_{i_m}} \subset U.$$

Thus $X^\varepsilon \subset U$.

If X is **compact**, we can reduce $\{U_x\}$ to a **finite** cover $U_{x_1}, \dots, U_{x_n}$ and let ε be the **maximum** of the ε_{x_j}. □

Now we would like to realize such neighborhoods X^ε of X using the normal bundle $N(X)$ of X in $\mathbb{R}^n$. We define the map

$$h: N(X) \to \mathbb{R}^n, \ (x, v) \mapsto x + v.$$

Since $N(X)$ is a smooth manifold in $\mathbb{R}^n \times \mathbb{R}^n$ and h is just the restriction of the **addition** map on $\mathbb{R}^n$, h is **smooth**. Geometrically, h maps each normal space $N_x(X)$ to the **affine** subspace containing x which is **orthogonal** to T_xX. The map h provides us with the connection of the neighborhood X^ε with the normal bundle in the following way:

Definition 13.9 (Tubular neighborhoods) Let $X \subset \mathbb{R}^n$ be a smooth manifold without boundary. A **tubular nighborhood** of X is an **open** subset X^ε of $\mathbb{R}^n$ containing X such that h maps an **open** subspace $N^\varepsilon(X) \subset N(X)$ **diffeomorphically** onto X^ε where the space $N^\varepsilon(X)$ is given by

$$N^\varepsilon(X) = \{(x, v) \in \mathbb{R}^n : |v| < \varepsilon(x)\}$$

for a **smooth** function $\varepsilon: X \to \mathbb{R}^{>0}$.

A key feature of smooth manifolds embedded in some Euclidean space is that they always have a tubular neighborhood. We will first prove this important fact and then study some of its consequences.

Theorem 13.10 (Tubular Neighborhood Theorem) Let $X \subset \mathbb{R}^n$ be a smooth manifold without boundary. Then X has a **tubular neighborhood** X^ε such that there is a **submersion** $\pi: X^\varepsilon \to X$ for which the restriction of π to X is the **identity**.

Proof: Let again $X_0 = N^0(X) \subset N(X)$ denote the zero-section of $N(X)$, i.e., the subspace $X_0 = \{(x, 0) : x \in X\}$. The map h maps X_0 diffeomorphically onto X. Our goal is to show that this extends to an **open** neighborhood around X_0. To do this we show that $dh_{(x,0)}$ is an **isomorphism** for every point $(x, 0)$ in X_0 as follows:

- First, we observe that, since $h_{|X_0}: X_0 \to X$ is a diffeomorphism, $dh_{(x,0)}$ maps $T_{(x,0)}(X_0) \subset T_{(x,0)}N(X)$ **isomorphically** onto $T_x(X) \subset \mathbb{R}^n$.
- Second, since the restriction of h to the normal space $N_x(X) \subset N(X)$ is given by $v \mapsto x + v$, we see that $dh_{(x,0)}$ maps $T_{(x,0)}(\{x\} \times N_x(X)) \subset T_{(x,0)}(N(X))$ **isomorphically** onto $N_x(X) \subset \mathbb{R}^n$.
- Since we have $T_x\mathbb{R}^n = T_xX \oplus N_xX$, this shows that $dh_{(x,0)}$ is **surjective** for every x.
- Since $\dim T_{(x,0)}(N(X)) = n$, this implies that $dh_{(x,0)}$ is an **isomorphism** for every x.

Hence the assumptions of the **generalized Inverse Function Theorem 13.4** are satisfied and we can conclude that h maps an **open** neighborhood of X_0 in $N(X)$ **diffeomorphically** onto an **open** neighborhood of X in $\mathbb{R}^n$. By the **ε-Neighborhood Lemma 13.8**, every open neighborhood of X contains an X^ε for a smooth function $\varepsilon: X \to \mathbb{R}^{>0}$. It is clear from the definition of h that X^ε is the image the open neighborhood $N^\varepsilon(X)$ of X_0 in $N(X)$ for the same function ε.

Finally, we have shown that there is a smooth map $h^{-1} : X^\varepsilon \to N^\varepsilon(X) \subset N(X)$. We define the map π by

$$\pi = \sigma_{|N^\varepsilon(X)} \circ h^{-1} : X^\varepsilon \to X.$$

Since h^{-1} is a diffeomorphism and σ is a submersion, we conclude that π is **submersion**. It follows directly from the definition of h and σ that restriction of π to $X \subset X^\varepsilon$ is the identity. □

- We emphasize that a key point for the existence of π is that we do not only have X^ε but also that we know that it is the diffeomorphic image of $N^\varepsilon(X)$ under h. Hence we really need to use the normal bundle.
- For some applications, the **important** point of the theorem is not so much the existence of the X^ε, but rather that they come equipped with the **submersion** π.
- It is **crucial** that we can find an **open** neighborhood of X_0 in $N(X)$ which is **diffeomorphic** to an **open** neighborhood of Z in Y. For it is clear that X is diffeomorphic to X_0 which is **closed** in $N(X)$. The difference may become apparent when we look at the **dimensions**: $\dim X^\varepsilon = n$, whereas $\dim X < n$ in general.
- Recall from Lemma 13.8 that, if X is **compact**, then $\varepsilon > 0$ can be chosen constant, and X^ε is the **open** set of points in $\mathbb{R}^n$ with **distance less than** ε from X. If ε is **sufficiently small**, then each point $w \in X^\varepsilon$ possesses a **unique closest point** in X. Writing $\pi(w)$ for this unique point, defines the map $\pi : X^\varepsilon \to X$ in this case. After studying the proof of the theorem, it is a good exercise to check that this actually yields the desired map π.
- Tubular neighborhoods are very **useful**, for example for the **Pontryagin–Thom** construction which is key for proving **more advanced results** in differential topology. We will see some applications in the following sections.

13.1.4 Tubular neighborhood - the relative case

We can actually consider normal bundles more generally whenever we have a submanifold $Z \subset X$ in order to understand the geometry of Z in X:

Definition 13.11 (Normal Bundle to a submanifold) Let $X \subset \mathbb{R}^m$ be a boundaryless manifold, and let Z be a submanifold of X. We define the **normal bundle to Z in X** to be the set

$$N(Z, X) := \left\{(z, v) : z \in Z, v \in T_z(X) \text{ and } v \perp T_z(Z)\right\}.$$

- We think of $N(Z, X)$ as the **relative** normal bundle, since we take the normal space within the tangent space of X, not in the ambient Euclidean space.

Also relative normal bundles are manifolds on their own:

Theorem 13.12 **(Normal bundles are manifolds revisited)** The normal bundle $N(Z, X)$ is a smooth manifold of dimension equal to $\dim X$. The canonical map

$$\sigma : N(Z, X) \to Z, \ \sigma(z, v) = z,$$

is a **submersion**.

Proof: Let z be a point in Z. We can find an open subset $V \subset \mathbb{R}^m$ containing z and local coordinate functions $(u_1, \dots, u_m) : V \to \mathbb{R}^m$ such that

$$Z \cap V = \left\{ v \in Z : u_1(v) = \cdots = u_n(v) = 0 \right\}$$
$$\text{and } X \cap V = \left\{ v \in X : u_{k+1}(v) = \cdots = u_n(v) = 0 \right\}$$

where n is the codimension of Z in $\mathbb{R}^m$ and k is the codimension of Z in X. Let φ be the smooth map given by

$$\varphi = (u_1, \dots, u_n) : V \to \mathbb{R}^n.$$

We set $U := Z \cap V$. We observed above that the map

$$\psi : U \times \mathbb{R}^n \to N_U(Z, \mathbb{R}^m) := (U \times \mathbb{R}^m) \cap N(Z, \mathbb{R}^m),$$
$$(u, v) \mapsto \left(u, d\varphi_u^t(v)\right)$$

is a **local parametrization** of $N(Z, \mathbb{R}^m) = N(Z)$.

By restricting ψ to elements in $U \times \mathbb{R}^k \subset U \times \mathbb{R}^n$, we get a smooth map ϕ defined as the composite

$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[r, "\psi"] \arrow[dr, "\phi"'] & N_U(Z, \mathbb{R}^m) \arrow[d, "\mathrm{id} \times p"] \\ & N_U(Z, X) \end{tikzcd}$$

where $N_U(Z, X) := (U \times \mathbb{R}^m) \cap N(Z, X)$ and p is the map induced by the **orthogonal projection** $p_z : \mathbb{R}^m \to T_z(X)$ at each z. Note that, for a vector $w \in \mathbb{R}^m$ which satisfies $w \perp T_z(Z)$, we have $p(w) \in T_z(X)$ and $p(w) \perp T_z(Z)$. Let $\tilde{\varphi} = (u_{k+1}, \dots, u_n) : V \to \mathbb{R}^{n-k}$. We observe that, by our choice of φ and $\tilde{\varphi}$, we know

$$T_z(Z) = (\text{Ker } d\varphi_z) \subset \text{Ker } (d\tilde{\varphi}_z) = T_z(X)$$

and the orthogonal projection p_z varies smoothly with z. At each $z \in U$, the **dimension of the kernel** of the composite

$$\mathbb{R}^n \xrightarrow{d\varphi_z^T} N_z(Z, \mathbb{R}^m) \xrightarrow{p_z} N_z(Z, X)$$

is

$$\dim \text{Ker } (p_z) = \dim N_z(Z, \mathbb{R}^m) - \dim N_z(Z, X),$$

since $d\varphi_z^T$ is an isomorphism. We can calculate this dimension by

$$\dim N_z(Z, \mathbb{R}^m) - \dim N_z(Z, X) = m - \dim Z - (\dim X - \dim Z) = n - k.$$

Thus, ϕ is a **diffeomorphism** being the **identity** on the factor and a **linear isomorphism** on the second factor. Hence $\phi\colon U \times \mathbb{R}^k \to N_U(Z, Y)$ is a **local parametrization** of $N(Z, X)$. Since $N_U(Z, X)$ is open in $N(Z, X)$ and every point in $N(Z, X)$ lies in such an $N_U(Z, X)$, we conclude that $N(Z, X)$ is a smooth manifold. Its dimension is

$$\dim N(Z, X) = \dim U + \dim \mathbb{R}^k = \dim Z + \dim X - \dim Z = \dim X.$$

We note again that $\sigma \circ \phi\colon U \times \mathbb{R}^k \to U$ is just the **projection** onto the first factor, which is a **submersion**. Thus $d(\sigma \circ \phi)_{(u,v)}$, is surjective at every point (u, v). Hence $d\sigma_u$ is surjective at every u, and σ is a **submersion**. □

Let us look at an example of a normal bundle of an embedded submanifold:

Example 13.13 (Normal bundle to sphere) Consider $\mathbb{S}^{k-1}$ as a **submanifold** of $\mathbb{S}^k$ via the embedding

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0).$$

The **tangent space** $T_p(\mathbb{S}^{k-1})$ is **embedded** in $T_p(\mathbb{S}^k)$ as the **subspace** consisting of vectors with last coordinate being 0. Hence the **orthogonal complement** of $T_p(\mathbb{S}^{k-1})$ in $T_p(\mathbb{S}^k)$ is **spanned by** the vector with coordinates $v_k := (0, \dots, 0, 1)$ (in $T_p(\mathbb{S}^k)$). Hence we can define a map

$$\mathbb{S}^{k-1} \times \mathbb{R} \to N(\mathbb{S}^{k-1}, \mathbb{S}^k),\ (p, \lambda) \mapsto (p, \lambda v_k).$$

This map is a **diffeomorphism with inverse** $(p, \lambda v_k) \mapsto (p, \lambda)$.

- Note that an n-dimensional vector bundle which is **diffeomorphic** to the **product** of the base space with $\mathbb{R}^n$ is called **trivial**. Hence we just showed that $N(\mathbb{S}^{k-1}, \mathbb{S}^k)$ is a **trivial** one-dimensional bundle.

- We get a similar result when we consider $\mathbb{S}^{k-1} \subset \mathbb{R}^k$ for $k \geq 2$. Then, at any $p \in \mathbb{S}^{k-1}$, the unit vector $p/|p|$ spans the normal complement to $T_p(\mathbb{S}^{k-1})$ in $\mathbb{R}^k$. Hence there is a diffeomorphism

$$\mathbb{S}^{k-1} \times \mathbb{R} \to N(\mathbb{S}^{k-1}, \mathbb{R}^k),\ (p, \lambda) \mapsto (p, \lambda p/|p|).$$

  Hence $N(\mathbb{S}^{k-1}, \mathbb{R}^k)$ is a **trivial one-dimensional bundle** over $\mathbb{S}^{k-1}$.

- However, there are a lot of **nontrivial** vector bundles as well. **Important examples** are the tangent bundle $T(\mathbb{S}^2)$ over $\mathbb{S}^2$ and the **universal bundle** over the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^{n+k})$.

- Note that for any $z \in Z$, the preimage $\sigma^{-1}(z) =: N_z(Z, X)$ is the space of **normal vectors** to Z at z in $T_z(X)$ that we have met before.

- **Warning:** As for $N(X)$, $N(Z, X)$ is **not intrinsic** to the manifold Z but **depends** on the specific relationship between Z and the surrounding space X. □

Now we extend the tubular neighborhood theorem to submanifolds:

Theorem 13.14 **(Tubular Neighborhoods - relative version)** Let $X \subset \mathbb{R}^n$ be a smooth manifold **without boundary**, and let Z be a submanifold of X. Then there is a **diffeomorphism** of an **open** neighborhood Z^ε of Z in X to an **open** neighborhood $N^\varepsilon(Z, X)$ of $Z_0 := Z \times \{0\}$ in $N(Z, X)$. There is a **submersion** $Z^\varepsilon \to Z$ which restricts to the identity on Z.

Proof: By the **Tubular Neighborhood Theorem 13.10** applied to $X \subset \mathbb{R}^n$, we have the open neighborhood $X^{\varepsilon_X} \subset \mathbb{R}^n$ for some positive smooth function ε_X and a submersion

$$\pi_X : X^{\varepsilon_X} \to X.$$

We define again a smooth map

$$h: N(Z, X) \to \mathbb{R}^n, \ (z, v) \mapsto z + v.$$

The **inverse image**

$$W := h^{-1}(X^{\varepsilon_X}) \subset N(Z, X)$$

is an **open** neighborhood of Z_0 in $N(Z, X)$. Since $h(z, 0) = z$ for all $z \in Z$, the composition

$$f : W \xrightarrow{h} X^{\varepsilon_X} \xrightarrow{\pi} X$$

is the **identity** when we restrict it to Z_0. By the same argument as above, we can show that $dh_{(z,0)}$ is an **isomorphism** at **every** point of Z_0 in $N(Z, X)$. Since $d\pi_z$ is the identity for all $z \in Z \subset X^{\varepsilon_X}$, $df_{(z,0)}$ is an isomorphism for every $(z, 0) \in Z_0 \subset N(Z, X)$. Hence the assumptions of the **generalized Inverse Function Theorem 13.4**, are satisfied, and we can conclude that there is an **open neighborhood** V of Z_0 in $N(Z, X)$ which is mapped diffeomorphically onto an **open neighborhood** U of Z in X by $f = \pi \circ h$. Then we can find a positive smooth function $\varepsilon : Z \to \mathbb{R}^{>0}$ such that $Z \subset Z^\varepsilon \subset U$ and Z^ε is diffeomorphic to an **open** neighborhood $N^\varepsilon(Z, X)$ of Z_0 in $N(Z, X)$. The composition

$$\pi : Z^\varepsilon \xrightarrow{\cong} N^\varepsilon(Z, X) \xrightarrow{\sigma_{N^\varepsilon(Z,X)}} Z$$

is the desired submersion. □

Finally, we make the following observation that we will use later:

Theorem 13.15 **(Trivial relative normal bundle)** Let $X \subset \mathbb{R}^n$ be a smooth manifold **without boundary**, and let Z be a submanifold of X. Then $N(Z, X)$ is a **trivial** bundle over Z **if and only if** there is a submersion $g : U \to \mathbb{R}^k$ defined on an **open** subset $U \subset X$ such that $Z = g^{-1}(0)$.

Proof: First, we assume that $N(Z, X)$ is trivial, i.e., there is a diffeomorphism $\varphi : Z \times \mathbb{R}^k \to N(Z, X)$. By Theorem 13.14 there are open neighborhoods $Z \subset Z^\varepsilon \subset X$ and $N^\varepsilon(Z, X) \subset N(Z, X)$ with a diffeomorphism $Z^\varepsilon \to N^\varepsilon(Z, X)$. Composition and restriction yield the desired submersion $g : Z^\varepsilon \to N^\varepsilon(Z, X) \to \mathbb{R}^k$ such that $Z = g^{-1}(0)$.

Second, we assume that there is a submersion $g : U \to \mathbb{R}^k$ with $Z = g^{-1}(0)$. Hence, for each $z \in Z \subset U \subset X$, $dg_z : T_zU = T_zX \to \mathbb{R}^k$ is a surjective linear map. This implies by

Lemma 13.6 that the transpose $(dg_z)^t$ maps $\mathbb{R}^k$ isomorphically onto the **orthogonal complement** of $T_z Z$ in $T_z X$ for every $z \in Z$. Thus, by definition of $N(Z, X)$, the map

$$\varphi : Z \times \mathbb{R}^k \to N(Z, X), \ (z, v) \mapsto (z, (dg_z)^t(v))$$

is a diffeomorphism. □

13.2 Whitney Approximation Theorem

In this section we show that every continuous map between smooth manifolds can be approximated by a smooth map. In particular, we will show that every continuous map is homotopic to a smooth one. The **key tool** that makes this work are **tubular neighborhoods**.

13.2.1 Whitney Approximation Theorem for Functions

We begin with the approximation of maps to Euclidean space. We will show that such maps can be approximated in the following sense:

Definition 13.16 **(ε-close functions)** Let X be a smooth manifold and $f, g : X \to \mathbb{R}^n$ two maps. Let $\varepsilon : X \to \mathbb{R}^{>0}$ be a **positive continuous** function. Then we say that f and g are said to be **ε-close** if $|f(x) - g(x)| < \varepsilon(x)$ for all $x \in X$.

Theorem 13.17 **(Approximating maps to Euclidean space)** Let X be a smooth manifold with or without boundary, and let $f : X \to \mathbb{R}^n$ be a **continuous** map. Let $\varepsilon : X \to \mathbb{R}^{>0}$ be a **positive continuous** function. Then there exists a **smooth** map $g : X \to \mathbb{R}^n$ which is **ε-close** to f. Moreover, if f is already smooth on a **closed** subset $A \subset X$, then g can be chosen equal to f on A.

Proof: Let $A \subset X$ be a closed subset and assume that $f_{|A}$ is smooth. By definition, this means that there is an **open** subset $U \subset X$ with $A \subset U$ and a smooth map $F : U \to \mathbb{R}^n$ such that $f_{|A} = F_{|A}$. We define the set U as

$$U_0 := \{x \in X : |F(x) - f(x)| < \varepsilon(x)\}.$$

This is an **open** subset in X, since the function $x \mapsto \varepsilon(x) - |F(x) - f(x)|$ is continuous and U is the inverse image of the open subset $\mathbb{R}^{>0}$ under this function. Note that $X \setminus A \subset U_0$, since $F(x) - f(x) = 0$ for all $x \in A$. If $A = \emptyset$, we set $U_0 := \emptyset$.

Now let $x \in X \setminus A$ and U_x be an **open** neighborhood of x in contained in $X \setminus A$ which is **small enough** such that

$$|F(y) - f(x)| < \frac{1}{2}\varepsilon(x) < \varepsilon(y) \text{ for all } y \in U_x.$$

We can find such a neighborhood, since x is fixed and ε and F are **continuous** maps. The collection $\left\{U_x : x \in X \setminus A\right\}$ of all such neighborhoods for all x is an **open** cover of $X \setminus A$.

Now we can choose a **countable** subcover $\left\{U_{x_i}\right\}_{i=1}^{\infty}$ and we set $U_i := U_{x_i}$ to simplify the notation.

Let $\{\rho_0, \rho_i\}$ be a smooth partition of unity subordinate to the open cover $\{U_0, U_i\}$ of X. We define the map $g: X \to \mathbb{R}^n$ by

$$g(x) := \rho_0(x)F(x) + \sum_{i=1}^{\infty} \rho_i(x) f(x_i).$$

Since F, ρ and the ρ_i are smooth and the $f(x_i)$ are fixed values, g is **smooth**. Moreover, we have $g_{|A} = F_{|A} = f_{|A}$. It remains to show that g and f are ε-close: Since $\sum_{i\geq 0} \rho_i(x) = 1$ and $|F(x) - f(x_i)| < \frac{1}{2}\varepsilon(x)$ for all $x \in X$, we see

$$\begin{aligned} |g(x) - f(x)| &= \left| \rho_0(x)F(x) + \sum_{i=1}^{\infty} \rho_i(x) f(x_i) - \left(\rho_0(x) + \sum_{i=1}^{\infty} \rho_i(x)\right) f(x) \right| \\ &\leq \rho_0(x)|F(x) - f(x)| + \sum_{i=1}^{\infty} \rho_i(x)|f(x_i) - f(x)| \\ &< \rho_0(x)\varepsilon(x) + \sum_{i=1}^{\infty} \rho_i(x)\varepsilon(x) = \varepsilon(x). \quad \square \end{aligned}$$

13.2.2 Whitney Approximation Theorem between manifolds

Now we can extend this result to smooth manifolds embedded in $\mathbb{R}^n$. Recall that two maps $f, g: X \to Y$ are said to be homotopic **relative** to a subset $A \subset X$ if there is a homotopy $H: X \times [0,1] \to Y$ such that $f(x) = H(x,t) = g(x)$ for all $x \in A$ and all $t \in [0,1]$.

Theorem 13.18 (Whitney Approximation Theorem) Let X be a smooth manifold with or without boundary and let $Y \subset \mathbb{R}^n$ be a smooth manifold without boundary. Let $f: X \to Y$ be a **continuous** map. Then f is **homotopic** to a **smooth** map $X \to Y$. Moreover, if f is already smooth on a **closed** subset $A \subset X$, then the homotopy can be chosen relative to A.

Proof: Let $Y^\varepsilon \subset \mathbb{R}^n$ be a **tubular neighborhood** of Y in $\mathbb{R}^n$ and let $\varepsilon: Y \to \mathbb{R}^{>0}$ be the associated positive smooth function. Let $\pi: Y^\varepsilon \to Y$ be the submersion of the Tubular Neighborhood Theorem 13.10 for which the restriction of π to X is the **identity**. We define $\tilde{\varepsilon}$ to be the **positive** continuous function given as the composition

$$\tilde{\varepsilon} = \varepsilon \circ f: X \to \mathbb{R}^{>0}.$$

By the previous theorem, there is a **smooth** map $g: X \to \mathbb{R}^n$ with $g_{|A} = f_{|A}$ and which is $\tilde{\varepsilon}$-close to f. Now we define the map $H: X \times [0,1] \to Y$ as the composition of π with a **straight-line homotopy** between f and g:

$$H(x,t) = \pi\left((1-t)f(x) + tg(x)\right).$$

- **Claim:** H is well-defined.

We know $|g(x) - f(x)| < \tilde{\varepsilon}(x) = \varepsilon(f(x))$ for all $x \in X$. Hence $g(x)$ is contained in the open ball around $f(x)$ with radius $\varepsilon(f(x))$. Thus, $g(x)$ and the whole line segment between $f(x)$ and $g(x)$ are contained in Y^ε for all x by definition of Y^ε. This proves the claim.

Hence we can conclude that H is a homotopy between $H(x, 0) = \pi(f(x)) = f(x)$ and $H(x, 1) = \pi(g(x))$. Moreover, we have $H(x, t) = f(x)$ for all $x \in A$. The map $\pi \circ g$ is **smooth** and the desired map homotopic to f. □

Remark 13.19 (No boundary is important) The assumption on Y being **without** boundary is actually **crucial**. For a simple example, let $X = \mathbb{R}$ and $Y = [0, \infty)$. Consider the **continuous** map $f : X \to Y$ given by $f(x) = |x|$ and let $A = [0, \infty)$. Then there does **not** exist a **smooth** map $g : X \to Y$ with $g_{|A} = f_{|A}$. For f is smooth on the open interval $(0, \infty)$ with $\frac{d}{dx} f(x) = 1$ for all $x \in (0, \infty)$. However, g would have to satisfy $g(x) \in Y$, i.e., $g(x) \geq 0$ for all $x \in X$. From what we learned in Calculus we can deduce that such a map g cannot be smooth. As a consequence we get that there is **no smooth** homotopy $h : X \times [0, 1] \to Y$ with $h(x, t) = f(x)$ for all $x \in A$ and all $t \in [0, 1]$.

Theorem 13.20 (Homotopic maps are smoothly homotopic) Let X be a smooth manifold with or without boundary and let Y be a smooth manifold without boundary. Let $f, g : X \to Y$ be **smooth** maps. Then, if f and g are homotopic, they are **smoothly** homotopic. Moreover, if f and g are homotopic are **relative** to some closed subset $A \subset X$, then they are **smoothly** homotopic relative to A.

Proof: Let $h : X \times [0, 1] \to Y$ be a **continuous** homotopy between f and g relative to A. We would like to apply Whitney's Theorem 13.18 to h. **However**, $X \times [0, 1]$ may not be a manifold with boundary in our sense. Hence, we first extend h to a continuous map $\tilde{h} : X \times \mathbb{R} \to Y$ by setting

$$\tilde{h}(x, t) = \begin{cases} h(x, t), & t \in [0, 1] \\ h(x, 0), & t \leq 0 \\ h(x, 1), & t \geq 1. \end{cases}$$

The restriction of $\tilde{h}$ to $X \times \{0\} \cup X \times \{1\}$ is **smooth**, since it is equal to $f \circ \pi_1$ on $X \times \{1\}$ and $g \circ \pi_1$ where $\pi_1 : X \times \mathbb{R} \to X$ denotes the projection. If h is a homotopy **relative** to A, then $\tilde{h}$ is also smooth on $A \times [0, 1]$.

Now, since $X \times \mathbb{R}$ is a smooth manifold, with or without boundary, Whitney's Theorem 13.18 implies that there is a **smooth** map $H : X \times \mathbb{R} \to Y$ whose restriction to $X \times \{0\} \cup X \times \{1\} \cup A \times [0, 1]$ agrees with $\tilde{h}$. By restricting H to $X \times [0, 1]$ we get the desired **smooth** homotopy between f and g relative to A. □

13.2.3 Compact manifolds are not contractible

The ideas we used to prove Brouwer's fixed point Theorem 11.7 together with the approximation theorems of this section lead to the following stronger form of Theorem 12.8:

Theorem 13.21 (Compact manifolds are not contractible) Let X be a connected **compact** smooth manifold of dimension at least one without boundary. Then the identity map from X to X is **not** homotopic to a **constant** map.

- The theorem says that no compact manifold of dimension $n \geq 1$ has the **homotopy type** of a point.
- In **Algebraic Topology**, we learn about this fact when we show that the nth singular homology group of a connected **compact manifold** is nontrivial, more specifically we have $H_n(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Since **homology** is **invariant under homotopy** and $H_n(\{x_0\}; \mathbb{Z}/2) = 0$ for $n \geq 1$, there cannot be a homotopy equivalence between X and a one-point space.
- Here, however, we give a more **geometric explanation**. For the theorem follows again from the **classification of one-manifolds** in Theorem 11.1 together with Whitney's Approximation Theorem 13.18.

Proof: Assume there was a smooth homotopy $H: X \times [0,1] \to X$ between the identity on X and a constant map with image $x_0 \in X$, i.e., $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in X$. By **Sard's Theorem 10.18**, the map H has at least one **regular value** $y \in X$. We note that, since $\dim X \geq 1$, we must have $y \neq x_0$. This follows from the fact that $dH_{(x,1)}$ fails to be surjective, since $H(-,1): X \to X$ is **constant**.

The fiber $H^{-1}(y)$ is a smooth manifold by the Preimage Theorem 10.16. Since X is **compact**, so is $X \times [0,1]$. Since $H^{-1}(y)$ is closed in $X \times [0,1]$, $H^{-1}(y)$ is compact as well. Since $H^{-1}(y)$ has codimension $\dim X$ in $X \times [0,1]$, we have $\dim H^{-1}(y) = 1$.

Moreover, $H^{-1}(y)$ may have a **boundary**, since X has no boundary and hence $X \times [0,1]$ is a manifold with boundary. The Preimage Theorem 10.16 for manifolds **with boundary** implies that this boundary has the form

$$\partial H^{-1}(y) = \partial(X \times [0,1]) \cap H^{-1}(y) = (X \times \{0\} \cup X \times \{1\}) \cap H^{-1}(y).$$

Since H satisfies $H(x,0) = x$ for all $x \in X$, we have

$$(X \times \{0\}) \cap H^{-1}(y) = (y, 0).$$

For $t = 1$, however, $H(-,1)$ is **constant** with $H(x,1) = x_0$ for all $x \in X$. Since we know $x_0 \neq y$, we have

$$(X \times \{1\}) \cap H^{-1}(y) = \emptyset.$$

Thus we have $\partial H^{-1}(y) = (y,0)$. Hence the boundary of $H^{-1}(y)$ consists of a **single** point. This **contradicts** Lemma 11.2 which says that a **compact** one-manifold **with boundary** always has an **even** number of boundary points.

By **Whitney's Approximation Theorem 13.18**, we know that if there was a **smooth** homotopy between the identity and a constant map, then there was a **continuous** homotopy between the identity and a constant map. Since such a smooth homotopy cannot exist, we get the theorem. □

13.3 Ehresmann Fibration Theorem

In this section we study another consequence of the existence of tubular neighborhoods.

We begin with an important type of maps:

Definition 13.22 (Locally trivial fibrations) Let $f: X \to Y$ be a smooth map between manifolds. Then f is called a **locally trivial fibration** if for each $y \in Y$ there is an open neighborhood $U \subset Y$ of y and a diffeomorphism $\varphi: f^{-1}(U) \to U \times f^{-1}(y)$ such that the diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "f_{|f^{-1}(U)}"'] & & U \times f^{-1}(y) \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

commutes where p_U denotes the projection onto the first factor.

Here are some **examples** and **remarks**:

- Every **projection** $p_X: X \times Z \to X$ is a locally trivial fibration. In fact, it is globally trivial.
- Every **vector bundle** $\pi: E \to Y$ is a locally trivial fibration.
- Assume we have a vector bundle $\pi: E \to Y$ such that each fiber $\pi^{-1}(y)$ has a metric $|\cdot|_y$. Then we can form the **sphere bundle** $f: S(E) \to Y$ as follows: we set $S(E) = \{v \in E : |v|_y = 1 \text{ if } \pi(v) = y\} \subset E$, and f is just the restriction to $S(E)$. Then f is a locally trivial fibration.
- If Y is **connected**, then the **fibers** of a locally trivial fibration $f: X \to Y$ are all **diffeomorphic**.
- The **Hopf fibration** $f: \mathbb{S}^3 \to \mathbb{S}^2$ is a locally trivial fibration. This will become clear after we have proven the main theorem of this section.

The following famous theorem gives us a sufficient criterion for a map to be locally trivial based on the notions we have studied before:

Theorem 13.23 (Ehresmann Fibration Theorem) Let $f: X \to Y$ be a **proper submersion**. Then f is a locally trivial fibration. In particular, if X is **compact**, every submersion is a locally trivial fibration.

- The second assertion follows from the first, since every continuous map with a compact domain is proper.

- Ehresmann's theorem is highly influential in many areas of geometry and topology. For example, in complex and algebraic geometry it implies that the higher direct images of a constant sheaf along f form a local system on Y (see [20]).
- Let $y_0 \in Y$ be a given basepoint in Y. The theorem also tells us that we can think of a proper submersion $f: X \to Y$ as a **family** of diffeomorphic copies of the fiber $f^{-1}(y_0)$ varying over the points in Y.

Proof of Theorem 13.23: The assertion is clear if X is the empty set. So assume X and $f^{-1}(y)$ are nonempty for $y \in Y$. Since f is a submersion, y is a regular value for f. Hence $Z := f^{-1}(y)$ is a submanifold of X. Then the relative version of the **Tubular Neighborhood Theorem 13.14** says that there is a neighborhood Z^ε of Z which is **open** in X and a **submersion** $\pi: Z^\varepsilon \to Z$ for which the restriction to Z is the **identity**. Then we can form the map

$$q := (\pi, f_{|Z^\varepsilon}): Z^\varepsilon \to Z \times Y.$$

Since f is **proper**, $Z = f^{-1}(y)$ is **compact**. Moreover, the Preimage Theorem 4.7 tells us that Z is of dimension $\dim X - \dim Y$ and $T_z(Z) = \mathrm{Ker}\,(df_z)$. This implies that the derivative dq_z of q at any point $z \in Z \subset Z^\varepsilon$ is an **isomorphism**, as df_z is surjective onto T_yY and $d\pi_z$ is surjective onto T_zZ. Hence dq_z is a **surjective linear** map and since the dimension of T_zZ^ε is equal to $\dim X = \dim Z + \dim Y$, dq_z must be an isomorphism. Thus we can apply the **generalized Inverse Function Theorem 13.4** to $Z \subset Z^\varepsilon$ and q. Hence we get that there is a neighborhood $Z \subset W \subset Z^\varepsilon$ of Z which is **open** in Z^ε such that $q_{|W}$ is a **diffeomorphism** onto an open neighborhood of $q(Z)$ in $Z \times Y$.

Finally, since f is **proper**, it is a **closed** map. Hence, Lemma 13.24 below implies that there is an **open** neighborhood $U \subset Y$ around y such that $Z \subset f^{-1}(U) \subset W$. Then we have $q(f^{-1}(U)) = U \times Z$ and have shown that $\varphi := q_{f^{-1}(U)}: f^{-1}(U) \to U \times f^{-1}(y)$ is a **diffeomorphism** such that $f_{|f^{-1}(U)} = p_U \circ \varphi$. □

Lemma 13.24 (Closed maps) Let $f: X \to Y$ be a continuous map between topological spaces. Then f is **closed** if and only if for all $y \in Y$ and open subset $W \subseteq X$ satisfying $f^{-1}(y) \subseteq W$, there is an **open** neighborhood U of y satisfying $f^{-1}(U) \subseteq W$.

Proof: We only show the implication we need. The other direction is left as an exercise. Since W is open, its complement $X \setminus W$ is closed in X. Since f is **closed**, the image $f(X \setminus W)$ is closed in Y. Hence the complement $U := Y \setminus f(X \setminus W)$ is **open** in Y. As $f^{-1}(y) \subset W$, we know $y \notin f(X \setminus W)$ and hence $y \in U$ and $f^{-1}(y) \subseteq f^{-1}(U) \subseteq W$. □

13.4 Thom's Transversality Theorem

We are going to review what we have learned about transversality and show that transversality is actually a generic property. This is a quite long and technical chapter. You may want to skip some details for the first reading and get back to them later. The good news is that the results we prove here will lay the ground for the intersection theory we will develop in the next chapter. So hang in there, it is worth it!

- **Main application:**

Our main application will be to reach our goal to define an interesting **intersection theory** for smooth manifolds which helps us deciding difficult questions. For example, we would like to use it to show that $\mathbb{S}^2$ and $\mathbb{R}\mathrm{P}^2$ are not homeomorphic. So assume we have a **compact** smooth k-dimensional manifold (without boundary), Y an n-dimensional manifold, and Z a **closed** m-dimensional submanifold of Y. Let $f: X \to Y$ be a smooth map which is **transverse** to Z. Then the **Preimage Theorem 6.2** tells us that $f^{-1}(Z)$ is a $k+m-n$-dimensional submanifold of X. Since it is a closed subset in the compact space X, $f^{-1}(Z)$ is also **compact**.

In the **special case** $m = n - k$, $f^{-1}(Z)$ is a compact manifold of **dimension** 0. Thus it is a **finite** set of points. This is the starting point for intersection theory. For we can ask: **How many** points are in the preimage $f^{-1}(Z)$? Actually, we will define the **mod 2-intersection number** $I_2(f, Z)$ as the number $\# f^{-1}(Z)$ modulo 2. The reason why we have to compute this number modulo 2 is due to our wish to make $I_2(f, Z)$ **homotopy invariant**. Now we would like to do this with a **general** smooth map. **However**, the assumption that f meets Z **transverse** is **crucial** for the above to work. The **goal** of this section is to show that every smooth map $f: X \to Y$ **can be replaced up to homotopy** with one a map that is **transverse** to Z, and the resulting intersection number will **not depend** on the chosen homotopic map. This will make it possible to define $I_2(f, Z)$ in the next chapter for an arbitrary smooth map. □

13.4.1 Thom's Transversality Theorem

We start with the following extension of Sard's Theorem 10.18:

> **Theorem 13.25 (Thom's Transversality Theorem)** Let $F: X \times S \to Y$ be a smooth map of manifolds, where **only** X has a **boundary**. Let Z be a submanifold of Y without boundary. If both F and ∂F are **transverse to Z**, then **for almost every** $s \in S$, **both** f_s and ∂f_s are **transverse to Z** where f_s denotes the map $x \mapsto f_s(x) = F(x, s)$.

- Recall that the **difference** between requiring that F is transverse to Z versus f_s is transverse to Z is that, for F, the image of $T_{(x,s)}(X \times S)$ under $dF_{(x,s)}$ has to be big enough, whereas for f_s we look at the **potentially smaller** image of $T_{(x,s)}(X \times S)$ under $d(f_s)_x$. Similarly for ∂F and ∂f_s.

Proof: Since both F and ∂F are transverse to Z, the **Preimage Theorem 10.17** implies that $W := F^{-1}(Z)$ is a submanifold of $X \times S$ with boundary

$$\partial W = W \cap \partial(X \times S) = W \cap (\partial X \times S).$$

Let $\pi: X \times S \to S$ be the natural projection map. We will show that whenever $s \in S$ is a **regular value** for the **restriction** $\pi: W \to S$, then $f_s \pitchfork Z$, and whenever s is a **regular value** for $\partial\pi: \partial W \to S$, then $\partial f_s \pitchfork Z$. By **Sard's Theorem 10.18** for manifolds with boundary **almost every** $s \in S$ is a **regular value for both** maps, so the theorem follows.

- **Claim:** $f_s \pitchfork Z$.

Suppose $f_s(x) = z \in Z$. Because $F(x,s) = z$ and $F \pitchfork Z$ by the **assumption**, we know that

$$dF_{(x,s)}(T_{(x,s)}(X \times S)) + T_z(Z) = T_z(Y).$$

Hence, given any vector $a \in T_z(Y)$, there exists a vector $b \in T_{(x,s)}(X \times S)$ such that

$$dF_{(x,s)}(b) - a \in T_z(Z).$$

We need to find a vector $v \in T_x(X)$ such that

$$d(f_s)_x(v) - a \in T_z(Z),$$

as that would show that $d(f_s)_x(T_x(X)) + T_z(Z) = T_z(Y)$. Since

$$T_{(x,s)}(X \times S) = T_x(X) \times T_s(S),$$

we can **write** b **as a pair** (w, e) for vectors $w \in T_x(X)$ and $e \in T_s(S)$. **If** e is **zero**, we are **done** as we will now explain: Since f_s is the restriction of F to $X \times \{s\}$, it follows that $d(f_s)_x$ is the **restriction** of $dF_{(x,s)}$ to $T_x(X) \times \{0\} \subset T_x(X) \times T_s(S)$, i.e., the diagram

$$\begin{tikzcd}
T_x(X) \times \{0\} \arrow[r, hook] \arrow[dr, "d(f_s)_x"'] & T_x(X) \times T_s(S) \arrow[d, "dF_{(x,s)}"] \\
 & T_z(Y)
\end{tikzcd}$$

commutes and hence

$$dF_{(x,s)}(b) = dF_{(x,s)}(w, 0) = d(f_s)_x(w).$$

However, e need not be zero. But we may **use the projection** π to **modify** w and e as follows: It is an exercise to check that

$$d\pi_{(x,s)} : T_x(X) \times T_s(S) \to T_s(S)$$

is just **projection** onto the second factor. In fact, this holds for every projection map from a product of manifolds. Now we use the **assumption** that s is a **regular value of** π. For this implies that the restriction of $d\pi_{(x,s)}$ to $T_{(x,s)}(W)$

$$\begin{tikzcd}
T_{(x,s)}(W) \arrow[r, hook] \arrow[dr, "d\pi_{(x,s)})_{|T_{(x,s)}(W)}"'] & T_x(X) \times T_s(S) \arrow[d, "d\pi_{(x,s)}"] \\
 & T_s(S)
\end{tikzcd}$$

is **surjective**. In particular, the **fiber** over $e \in T_s(S)$ is **nonempty**, and **there is** some vector of the form (u, e) **in** $T_{(x,s)}(W)$.

The restriction $F_{|W} : W = F^{-1}(Z) \to Z$ is a map to Z, so the vector $dF_{(x,s)}(u, e)$ is an **element in** $T_z(Z)$. Consequently, the **vector** $v := w - u \in T_x(X)$ is the vector in $T_x(X)$ we were **looking for**:

$$\begin{aligned}
d(f_s)_x(v) - a &= dF_{(x,s)}(w - u, 0) - a \\
&= dF_{(x,s)}((w, e) - (u, e)) - a \\
&= (dF_{(x,s)}(w, e) - a) - dF_{(x,s)}(u, e),
\end{aligned}$$

and we remember that **both** $dF_{(x,s)}(w, e) - a$ and $dF_{(x,s)}(u, e)$ **belong to** $T_z(Z)$. Hence $d(f_s)_x(v) - a$ is in $T_z(Z)$ as we **wished to prove**.

- **Claim:** $\partial f_s \pitchfork Z$.

This is a special instance of what we just proved, for the case of the boundaryless manifold ∂X and the map $\partial F : (\partial X) \times S \to Y$. □

13.4.2 Transversality is generic

This shows that transversality for smooth maps $X \to \mathbb{R}^N$ is **generic** in the following sense:

- Let $f : X \to \mathbb{R}^N$ be a smooth map. Let S be an **open ball** in $\mathbb{R}^M$, and define

$$F : X \times S \to \mathbb{R}^N, \; F(x, s) = f(x) + s.$$

The derivative of F at (x, s) is

$$dF_{(x,s)} = (df_x, \mathrm{Id}_{\mathbb{R}^N}) : T_x(X) \times \mathbb{R}^N \to \mathbb{R}^N.$$

Thus $dF_{(x,s)}$ is **surjective** at any (x, s). Hence F is a **submersion**. This implies that F is **transverse to every submanifold** $Z \subset \mathbb{R}^N$. Now we can apply the **Transversality Theorem 13.25** we have just proven:

Let $Z \subset \mathbb{R}^N$ be a manifold. Since F and ∂F are transverse to Z, for **almost every** $s \in S$, the map $f_s(x) = f(x) + s$ is **transverse to** Z. Moreover, the map

$$X \times [0, 1] \to \mathbb{R}^N, \; (x, t) \mapsto f_{ts}(x) = f(x) + ts$$

provides a smooth **homotopy** between f and f_s.

This shows us that transversality is **generic** for maps $X \to \mathbb{R}^N$. We would like to **generalize** this result to an **arbitrary** boundaryless smooth manifold $Y \subset \mathbb{R}^N$ and smooth map $f : X \to Y$. Given a submanifold $Z \subset Y$, we have **just learned** how to vary a smooth map $f : X \to Y \subset \mathbb{R}^N$ as a **family of maps** $X \to \mathbb{R}^N$ such that $f_s \pitchfork Z$ for arbitrarily small s, where we consider Z as a submanifold in $\mathbb{R}^N$. It **remains** to **project** these maps **down** to Y such that a **small perturbation** f_s of f **remains transversal** to the given submanifold $Z \subset Y$. We can do this using **normal bundles** and **tubular neighboorhoods** we constructed in the previous section.

As a first consequence we get:

Theorem 13.26 (Creating families of submersions) Let $f : X \to Y$ be a smooth map where Y is a manifold without boundary. Let S be an **open ball** in $\mathbb{R}^N$. Then **there is** a smooth map $F : X \times S \to Y$ such that $F(x, 0) = f(x)$, and for any **fixed** $x \in X$, the map

$$F(x, -) : S \to Y, \; s \mapsto F(x, s) \text{ is a } \textbf{submersion}.$$

In particular, **both** F and ∂F are **submersions**.

Proof: Let $Y \subset \mathbb{R}^N$ and S be the **open unit ball** in $\mathbb{R}^N$. We define

$$F: X \times S \to Y, \ F(x,s) = \pi(f(x) + \varepsilon(f(x))s). \tag{13.1}$$

Recall the map

$$\pi: Y^\varepsilon \to Y$$

from the **Tubular Neighborhood Theorem 13.10**. Since π restricts to the identity on Y, we have

$$F(x,0) = \pi(f(x) + 0) = f(x).$$

For **fixed** x, the map

$$\varphi: S \to Y^\varepsilon, \ s \mapsto f(x) + \varepsilon(f(x))s$$

is the **translation** of a **linear** map. Thus $d\varphi_s$ is just given by **multiplying a vector** in $T_s(S) = \mathbb{R}^M$ by the real number $\varepsilon(f(x)) > 0$ to get a vector in $T_{\varphi(s)}(Y^\varepsilon) \subset \mathbb{R}^N$. This means that the derivative $d\varphi_s$ is just given by multiplying the canonical submersion $\mathbb{R}^M \to \mathbb{R}^N$ with the number $\varepsilon(f(x))$, and therefore $d\varphi_s$ is **surjective**. Thus φ is a **submersion**.

As the **composition** of two submersions is a submersion, we get that $\pi \circ \varphi$

$$\begin{tikzcd} & Y^\varepsilon \arrow[d, "\pi"] \\ S \arrow[ur, "\varphi"] \arrow[r, "F(x,-)"'] & Y \end{tikzcd}$$

that is the map

$$F(x,-) = \pi \circ \varphi: S \xrightarrow{\varphi} Y^\varepsilon \xrightarrow{\pi} Y, \ s \mapsto F(x,s) \text{ is a } \textbf{submersion}.$$

Hence the restriction $F_{\{x\} \times S}: \{x\} \times S \to Y$ of F to the submanifold $\{x\} \times S$ is a **submersion for every** $x \in X$. Since every point of $X \times S$ lies in one of these submanifolds, F must be a **submersion** as well. The same argument applied to ∂F and ∂X, shows that ∂F is a submersion. □

Now we can prove that **transversality is generic**:

Theorem 13.27 (Transversality Homotopy Theorem) Let $f: X \to Y$ be a smooth map. Let Y be a smooth manifold without boundary and let Z be a boundaryless submanifold Z of Y. Then there **exists** a smooth map $g: X \to Y$ such that g is **homotopic** to f and **transverse** to Z, i.e., such that

$$g \simeq f \text{ and } g \pitchfork Z \text{ and } \partial g \pitchfork Z.$$

Proof: For the family of mappings F of Theorem 13.26 the **Transversality Theorem 13.25** implies that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ for **almost all** $s \in S$. But each f_s is homotopic to f, the **homotopy** being

$$X \times I \to Y, \ (x,t) \mapsto F(x,ts). \quad \square$$

13.4.3 The Extension Theorem

In fact, Tubular Neighborhood Theorem 13.10 allows us to prove a **stronger form** of the Transversality Homotopy Theorem 13.27. In order to be able to formulate it, we need some terminology.

Definition 13.28 (Transversality on subsets) Let $f: X \to Y$ be a smooth map, $Z \subset Y$ a submanifold, and $C \subset X$ be a **subset**. We will say f is **transverse to Z on C**, if the **transversality** condition

$$\mathrm{Im}\,(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y) \tag{13.2}$$

is satisfied for all $x \in C \cap f^{-1}(Z)$.

- Note that, even if C is a **submanifold**, the condition of Definition 13.28 is **different** than requiring $f_{|C} \pitchfork Z$ since (13.2) involves $\mathrm{Im}\,(df_x) = df_x(T_x(X))$, **not** $\mathrm{Im}\,(d(f_{|C})_x) = df_x(T_x(C))$, which is **smaller in general**.

Now we can formulate an important technical result.

Theorem 13.29 (Thom's Extension Theorem) Let $f: X \to Y$ be a smooth map, Y without boundary, and Z a **closed** submanifold of Y without boundary. Let C be a closed subset of X. Assume that $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then **there exists** a smooth map $g: X \to Y$ **homotopic to f**, such that $g \pitchfork Z$ and $\partial g \pitchfork Z$, and on an **open** neighborhood of C we have $g = f$.

Since the **boundary** ∂X is **closed** in X, we obtain the **important special case**:

Theorem 13.30 (Extension of maps on boundaries) Let $f: X \to Y$ be a smooth map such that $\partial f \pitchfork Z$. Then there **exists** a map $g: X \to Y$ **homotopic** to f such that $\partial g = \partial f$ and $g \pitchfork Z$. In particular, suppose there is a smooth map $h: \partial X \to Y$ transverse to Z. Then, **if** h extends to **any** smooth map $X \to Y$, **it extends** to a smooth map $f: X \to Y$ that is **transverse** to Z.

For the proof of the Extension Theorem we need a lemma:

Lemma 13.31 (Extending functions) If U is an **open** subset which **contains** the closed set **C in X**, then there **exists** a smooth function $\gamma: X \to [0,1]$ that is **equal** to **1 outside** U but that is 0 on a **neighborhood of C**.

Proof: Let C' be any **closed** set contained in U that contains C in its **interior**, and let $\{\rho_i\}$ be a **partition of unity** subordinate to the open cover $\{U, X \setminus C'\}$. Then we just take γ to be the **sum** of those ρ_i that **vanish outside** of $X \setminus C'$. □

Proof of the Extension Theorem 13.29:

First we show that $f \pitchfork Z$ on a neighborhood of C i.e. an open subset containing C. If $x \in C$ but $x \notin f^{-1}(Z)$, then since Z is closed, $X \setminus f^{-1}(Z)$ is a neighborhood of x on which $f \pitchfork Z$ automatically.

If $x \in f^{-1}(Z)$, then there is a neighborhood W of $f(x)$ in Y and a submersion $\varphi\colon W \to \mathbb{R}^k$ such that $f \pitchfork Z$ at a point $x' \in f^{-1}(Z \cap W)$ precisely when $\varphi \circ f$ is regular at x'. But if $\varphi \circ f$ is regular at x, so it is regular in a neighborhood of x. Thus $f \pitchfork Z$ on a neighborhood of every point $x \in C$, and so

$$f \pitchfork Z \text{ on a } \textbf{neighborhood } U \text{ of } C \text{ in } X.$$

Second, let γ be the function in Lemma 13.31 for the closed subset C and the open neighborhood U of C in X. We set $\tau := \gamma^2$. We have

$$d\tau_x = 2\gamma(x) d\gamma_x, \text{ and hence } \gamma(x) = 0 = \tau(x) \Rightarrow d\tau_x = 0.$$

Now we modify the map $F\colon X \times S \to Y$ which we defined in (13.1) in proving the Homotopy Theorem 13.27, where S is the unit ball in $\mathbb{R}^M$. We set

$$G\colon X \times S \to Y,\ G(x,s) := F(x, \tau(x)s).$$

- **Claim:** $G \pitchfork Z$.

To prove the claim we first **assume** $\tau(x) \neq 0$. Now suppose that $(x,s) \in G^{-1}(Z)$. Then the map

$$S \to Y,\ r \mapsto G(x,r),$$

is a submersion, since it is the composition of the

$$\text{diffeomorphism } r \mapsto \tau(x)r \text{ with the submersion } r \mapsto F(x,r).$$

Hence G is regular at (x,s), so certainly $G \pitchfork Z$ at (x,s). To show the claim when $\tau(x) = 0$, we need to check that the image of the derivative $dG_{(x,s)}$ is big enough. To do this, we introduce the map

$$m\colon X \times S \to X \times S,\ (x,s) \mapsto (x, \tau(x)s).$$

We would like to calculate the derivative of m. To do so we apply the product rule to the second coordinate and remember that $\tau\colon X \to [0,1]$, i.e. $\tau(x)$ and $d\tau_x(v)$ are both in $\mathbb{R}$ for any $v \in T_x(X)$. Then we get

$$dm_{(x,s)}(v,w) = (v, \tau(x)\cdot w + d\tau_x(v)\cdot s)$$

where w and s are vectors in $\mathbb{R}^N$.

We observe that $G = F \circ m$. Hence in order to calculate the derivative of G, we can apply the chain rule. Since we are interested in the case where $\tau(x) = 0$ and $d\tau_x = 0$ we get

$$dG_{(x,s)}(v,w) = dF_{(x,s)}(v,0).$$

Moreover, since $F(x,0) = f(x)$ for all x by construction of F, we know $F_{|X\times\{0\}} = f$. This implies

$$dF_{(x,s)}(v,0) = dF_{(x,0)}(v,0) = df_x(v).$$

Hence we get

$$dG_{(x,s)}(v,w) = df_x(v)$$

and therefore

$$\mathrm{Im}\,(dG_{(x,s)}) = \mathrm{Im}\,(df_x(v)) \subset T_{f(x)}(Y). \tag{13.3}$$

Now $\tau(x) = 0$, implies $x \in U$ by definition of γ and τ. But by the choice of U above, this implies $f \pitchfork Z$ at x. Hence (13.3) implies $G \pitchfork Z$ at (x,s). This proves the claim.

- The same argument shows $\partial G \pitchfork Z$.

Now we can apply the **Transversality Theorem 13.25** to $G\colon X \times S \to Y$ and conclude that we can pick and fix an s for which the map

$$g(x) := G(x,s) \text{ satisfies } g \pitchfork Z \text{ and } \partial g \pitchfork Z.$$

The map G is then a **homotopy**

$$f = F_{|X\times\{0\}} = G_{|X\times\{0\}} \sim G_{|X\times\{s\}} = g.$$

Finally, if x belongs to the neighborhood of C on which $\tau = 0$, then we even have

$$g(x) = G(x,s) = F(x,0) = f(x). \quad \square$$

Let us summarize what we have achieved:

- **(Summary)** We have proven three key results about transversality:

  (a) The Transversality Theorem 13.25 says that when a **homotopy** F is **transversal** to Z, then, in this homotopy family, **almost every** $f_s = F(-,s)$ is **transversal** to Z.

  (b) The Transversality Homotopy Theorem 13.27 says that given a map f and a submanifold Z, then **there exists** a map g **transversal** to Z and g is **homotopic** to f.

  (c) The Extension Theorem 13.29 says that, given a map f which is transversal to Z on **a subset** C, then we can always **replace** f with a homotopic map g which is **transversal** to Z **everywhere** (not only on C) and $f = g$ on an **open** set containing C.

  (a) is a generalization of Sard's Theorem 10.18. For **(b)** and **(c)**, the key for the proof is the **Tubular Neighborhood Theorem 13.10**.

13.5 Exercises and more examples

Exercise 13.1 Prove the *General Position Lemma*: Let X and Y be submanifolds of $\mathbb{R}^N$. For almost every $a \in \mathbb{R}^N$ the translate $X + a$ intersects Y transversally.

Exercise 13.2 Let X be a compact submanifold of $\mathbb{R}^n$, and $w \in \mathbb{R}^n$. Let $\sigma : N(X) \to X$ denote the normal bundle.

(a) Show that there exists a (not necessarily unique) point $x \in X$ closest to w, and prove that $w - x \in N_x(X)$.

Hint: If $c(t)$ is a curve on X with $c(0) = x$, then the smooth function $|w - c(t)|^2$ has a minimum at 0. Now use that we have shown in Exercise 2.13 that there is a unique correspondence between tangent vectors at x and velocity vectors at 0 of curves $c : (-a, a) \to X$ with $c(0) = x$.

(b) Let $h : N(X) \to \mathbb{R}^n$, $h(x, v) = x + v$, be the map used in the proof of the Tubular Neighborhood Theorem 13.10. We know that h maps a neighborhood of X in $N(X)$ diffeomorphically onto $X^\varepsilon \subset \mathbb{R}^n$, where $\varepsilon > 0$ is constant. We define the smooth map $\pi = \sigma \circ h^{-1}$. Use the previous point to show that, if $w \in X^\varepsilon$, then $\pi(w)$ is the unique point of X closest to w.

Exercise 13.3 Let X be a submanifold of $\mathbb{R}^N$. Show that 'almost every' vector space V of any fixed dimension k in $\mathbb{R}^N$ intersects X transversally, i.e.,

$$V + T_x(X) = \mathbb{R}^N \text{ for every } x \in X.$$

Hint: Use the fact that the set $S \subset \left(\mathbb{R}^N\right)^k$ consisting of all linearly independent k-tuples of vectors in $\mathbb{R}^N$ is open in R^{Nk}. Show that the map $\mathbb{R}^k \times S \to \mathbb{R}^N$ defined by

$$((t_1, \dots, t_k), v_1, \dots, v_k) \mapsto t_1 v_1 + \dots + t_k v_k$$

is a submersion, and apply previous results.

Exercise 13.4 The following is a harder problem, but it is an interesting application of the Transversality Theorem and tubular neighborhoods. So try it!

(a) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with $n > 1$, and let $K \subset \mathbb{R}^n$ be compact and $\varepsilon > 0$. Show that there exists a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that dg_x is never 0, and $|f(x) - g(x)| < \varepsilon$ for all $x \in K$.

Hint: Let $M(n)$ be the space of $n \times n$-matrices. First show that the map $F : \mathbb{R}^n \times M(n) \to M(n)$, defined by $F(x, A) = df_x + A$, is a submersion. Pick A so that $F_A \pitchfork \{0\}$ for $F_A : x \mapsto (x, A)$ as in the main text. Now use this knowledge to construct g. At some point along this way you will have used $n > 1$. Make sure you see where and how it has been used.

(b) Show that this result is false for $n = 1$, i.e., find f, ε, $K \subset \mathbb{R}$ such that we cannot find such a g.

Hint: You could contemplate on the Mean Value Theory.

14. Intersection Theory modulo 2

14.1 Intersection Numbers modulo 2

A classical geometric approach to classifying maps is to study their fibres. This is directly related to other fundamental problems in mathematics. For example, if $f : X \to Y$ is a map defined by an equation and given a value $y \in Y$, the set $\{x \in X : f(x) = y\}$ is the set of solutions of the equation $f(x) = y$. In **geometric terms**, we could rephrase the question which x solve equation $f(x) = y$ by asking how f meets or **intersects** the subspace $\{y\}$ in Y.

Building on the methods we have developed so far, we are going to **exploit this geometric idea** to derive interesting and **powerful invariants**. We will start with intersection numbers modulo 2, i.e., intersection numbers with values in $\mathbb{Z}/2$. In order to define a $\mathbb{Z}$-valued invariant we will have to introduce orientations later.

14.1.1 Intersecting manifolds

Let us start with a natural situation.

Definition 14.1 (Intersecting manifolds) Two **submanifolds** X and Z inside Y have **complementary dimension** if

$$\dim \mathbf{X} + \dim \mathbf{Z} = \dim \mathbf{Y}.$$

We assume all manifolds are **without boundary** for the moment. If $X \pitchfork Z$, the Preimage Theorem 4.7 tells us that their intersection $X \cap Z$ is manifold with codim $(X \cap Z)$ in X being equal to codim Z in Y. Since codim $Z = \dim X$, $X \cap Z$ **is a zero-dimensional manifold**. If we further assume that both X and Z are **closed** and that **at least one** of them, say X, is **compact**, then $X \cap Z$ must be a **finite set of points**. We are going to think of this **finite number of points** in $X \cap Z$ as the **intersection number** of X and Z, denoted by $\#(X \cap Z)$.

We would like to generalize the notion of intersection numbers. A first obstacle is that if X and Z do **not intersect transversally**, then it makes in general no sense to count the points in $X \cap Z$. Hence, once again, transversality is key.

Luckily, we have learned how to **move or deform** manifolds to make intersections transversal: we can alter them in **homotopic families**. And since **embeddings form a stable class** of maps, i.e., for any homotopy i_t of an embedding i_0, there is an $\varepsilon > 0$ such that i_t is still an embedding for all $t < \varepsilon$, any small homotopy of i gives us another embedding $X \hookrightarrow Y$ and thus produces an image manifold that is a **diffeomorphic copy** of X adjacent to the original.

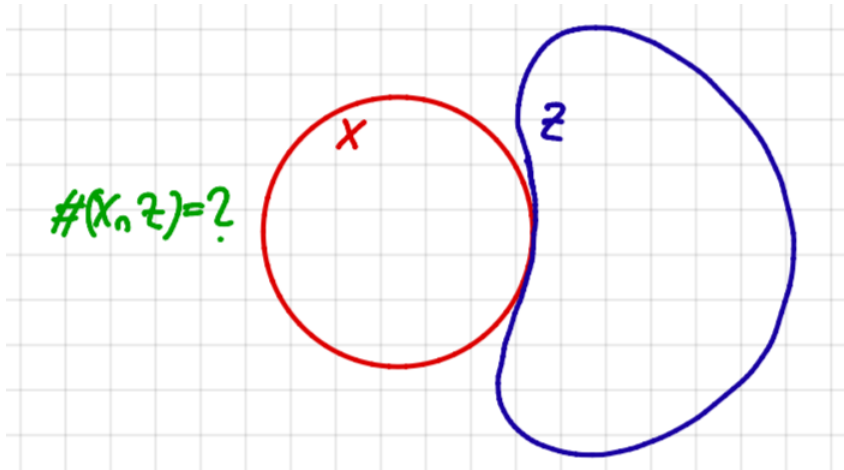


Figure 14.1: A tangential or non-transverse intersection is not stable. Any slight perturbation will change the number of intersection points. We need to avoid such situations.

But we still have to be **careful**. For the intersection number may depend on **how** we move or deform the manifold.

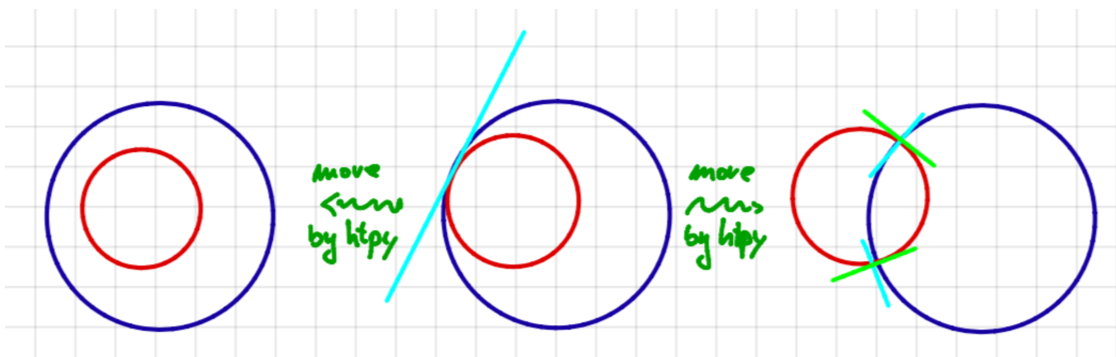


Figure 14.2: We can move the two circles in the plane and either move them apart or make them intersect transversely. In both cases we get a stable number of intersection points. However, 0 is not equal 2, unless we work modulo 2. That is what we are going to do.

For example, take **two circles** in $\mathbb{R}^2$. Assume that they intersect **non-transversally**, i.e., they touch each other in a point such that both tangent spaces agree and together just span a line. Then we can move the circles by a simple translations $x \mapsto x + ta$ in direction a such that they intersect either in **two** points or in **no** points. In both cases, the intersection is **transversal**, but the intersection numbers do **not agree**. We observe, **however**, that the **parity** of the intersection numbers is **preserved**, i.e., **up to a multiple of** 2 the intersection numbers after moving into a transversal intersection **agree**.

14.1.2 Intersection Number modulo 2

This observation is the starting point for the following generalization:

Definition 14.2 (Mod 2 Intersection numbers) Let X be a **compact** manifold, and let $f: X \to Y$ be a smooth map **transverse** to the **closed** manifold Z in Y. Assume $\dim X + \dim Z = \dim Y$. Then $f^{-1}(Z)$ is a closed submanifold of X of codimension equal to $\dim X$. Hence $f^{-1}(Z)$ is of **dimension zero**, and therefore a **finite** set. We define the **mod 2 intersection number** of the map f with Z, denoted $I_2(f, Z)$, to be **the number of points** in $f^{-1}(Z)$ modulo 2:

$$I_2(f, Z) := \# f^{-1}(Z) \mod 2.$$

For an **arbitrary** smooth map $g: X \to Y$, we **can choose** a map $f: X \to Y$ that is **homotopic to** g and **transverse** to Z by the **Transversality Homotopy Theorem 13.27**. Then we **define** $I_2(g, Z) := I_2(f, Z)$.

- **Independence of the chosen homotopy:**

We have made choices in the definition and we **need to check** that the intersection number does **not depend** on these choices. We will check this in Lemma 14.3 and Lemma 14.4. The **key** technical result that allows us to show independence is the **Extension Theorem 13.29** which says the following: Let $f: X \to Y$ be a smooth map, Y boundaryless, and Z a closed submanifold of Y without boundary. Let C be a **closed** subset of X. Assume that $f \pitchfork Z$ **on** C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then **there exists** a smooth map $g: X \to Y$ **homotopic to** f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$, and on a **neighborhood of** C we have $g = f$.

We will now prepare our argument with the following observation: Let X, Y and $Z \subset Y$ are boundaryless manifolds. The product $X \times [0, 1]$ is then a manifold with boundary. We let C be the **boundary** of $X \times [0, 1]$, i.e., C is the **closed** subset

$$C := \partial(X \times [0, 1]) = X \times \{0\} \cup X \times \{1\}.$$

Now we **apply** the **Extension Theorem 13.29** to the case of a smooth homotopy

$$F: X \times [0, 1] \to Y.$$

Then ∂F, i.e., F restricted to the boundary of $X \times [0, 1]$, is given by the two maps

$$f_0 = F(-, 0): X \to Y \text{ and } f_1 = F(-, 1): X \to Y.$$

The **two conditions** $F \pitchfork Z$ on C and $\partial F \pitchfork C$ on $C \cap \partial X$ are thus **equivalent**, and mean $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$. Hence, assuming $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$, the **Extension Theorem 13.29** says that **there is** a smooth map

$$G: X \times [0, 1] \to Y \text{ with } \mathbf{G \pitchfork Z} \text{ and } \partial G \pitchfork Z,$$

and $G = F$ on a **neighborhood of** $C = \partial(X \times [0, 1])$. The latter means that

$$G \text{ is } \textbf{still a homotopy} \text{ between } f_0 = G(-, 0) \text{ and } f_1 = G(-, 1).$$

Now we are ready to prove the following crucial observation:

Lemma 14.3 (Mod 2 Intersection Numbers are well-defined) If $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic and both **transverse** to Z, then $I_2(f_0, Z) = I_2(f_1, Z)$.

Proof: Let $F : X \times I \to Y$ be a homotopy of f_0 and f_1. By the above discussion, we may assume that $F \pitchfork Z$. By the **Preimage Theorem 10.16** with boundary, this implies $F^{-1}(Z)$ is a submanifold of $X \times [0,1]$ such that

$$\operatorname{codim} F^{-1}(Z) \text{ in } X \times [0,1] = \operatorname{codim} Z \text{ in } Y.$$

Hence

$$\begin{aligned} \dim F^{-1}(Z) &= \dim(X \times [0,1]) + \dim Z - \dim Y \\ &= \dim X + 1 + \dim Z - \dim Y \\ &= 1 \end{aligned}$$

since we assume that $\dim X + \dim Z = \dim Y$. Moreover, the boundary of $F^{-1}(Z)$ is

$$\partial F^{-1}(Z) = F^{-1}(Z) \cap \partial(X \times [0,1]) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}.$$

Since X is **compact**, $F^{-1}(Z)$ is **compact**. Hence Lemma 11.2 implies that $\partial F^{-1}(Z)$ must have an **even** number of points. Thus, computing **mod** 2, we get

$$I_2(f_0, Z) = \# f_0^{-1}(Z) = \# f_1^{-1}(Z) = I_2(f_1, Z). \quad \square$$

We can generalise this a bit further.

Lemma 14.4 (All homotopic maps have equal intersection numbers) If $g_0 : X \to Y$ and $g_1 : X \to Y$ are arbitrary homotopic maps, then $I_2(g_0, Z) = I_2(g_1, Z)$.

Proof: As before, we can choose maps $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$ such that $g_0 \sim f_0$, $I_2(g_0, Z) = I_2(f_0, Z)$, and $g_1 \sim f_1$, $I_2(g_1, Z) = I_2(f_1, Z)$. Since homotopy is a **transitive** relation[1], we have

$$f_0 \sim g_0 \sim g_1 \sim f_1, \text{ and hence } f_0 \sim f_1.$$

By the **previous Lemma 14.3**, this implies

$$I_2(g_0, Z) = I_2(f_0, Z) = I_2(f_1, Z) = I_2(g_1, Z). \quad \square$$

Remark 14.5 (Intersection number and Brouwer degree mode 2) Assume X is compact, Y is connected and $\dim X = \dim Y$. Let $f : X \to Y$ be a smooth map. Then we recover the mod 2 Brouwer degree as a mod 2 intersection number as follows. We let $Z = \{y\}$ for any $y \in Y$ and get from the definition of both sides that

$$I_2(f; y) = \deg_2(f).$$

[1]Recall that we proved that it is an equivalence relation

The difference in the two approaches to the degree is that in Section 12.1.3 we used the **Isotopy Lemma 12.3** to show that we can use any regular value for y to calculate the degree. For $I_2(f; y)$, we use the machinery of Thom Transversality of Section 13.4 to replace f with a homotopic map if necessary. While the approach in Section 12.1.3 is more direct and simpler, Thom Transversality provides us with a more general theory.

14.2 Intersection of manifolds and self-intersections

Now that we have a **solid** notion of intersection numbers modulo 2 for maps and submanifolds, let us return to the **situation we started with**:

Definition 14.6 (Mod 2 intersection numbers of submanifolds) Let X be a **compact** submanifold of Y and Z a **closed** submanifold of Y. Assume the **dimensions are complementary**, i.e., $\dim X + \dim Z = \dim Y$. Then we can define the **mod 2 intersection number of X with Z**, denoted by $I_2(X, Z)$, by

$$I_2(X, Z) := I_2(i, Z)$$

where $i : X \hookrightarrow Y$ is the **inclusion**. Note that if $X \pitchfork Z$, then

$$I_2(X, Z) = \#(X \cap Z).$$

In general, we have to **move or deform** X into a **transverse position**. That is, we choose a homotopy $i_t : X \times [0, 1] \to Y$ such that i_1 is still an **embedding** and such that i_1 is transverse to Z. This means that $i_1(X)$ is still a submanifold in Y, and we have $i_1(X) \pitchfork Z$. Then we **define** $I_2(X, Z)$ to be $I_2(i_1, Z) = \#(i_1(X) \cap Z)$.

Here some **important situations**:

Example 14.7 (Obstruction to pull apart circles) If $I_2(X, Z) \neq 0$, then **no matter** how X is moved or deformed, it **cannot** be pulled entirely away from Z.

- For **example**, on the **torus** $Y = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C} \times \mathbb{C}$, the **two circles** $\mathbb{S}^1 \times \{1\}$ and $\{1\} \times \mathbb{S}^1$ have complimentary dimensions and **nonzero** mod 2 intersection number.

Example 14.8 (Self-intersection number) If $\dim Y = 2 \dim X$, then we may consider $I_2(X, X)$ as the **mod 2 self-intersection number** of X.

- An important and interesting **example** is the central circle X on the open **Möbius band** Y. See Figure 14.3. In **Exercise 14.6** we show that $I_2(X, X) = 1$. Without actually calculating the intersection number, we can already deduce from the **Intermediate Value Theorem** in Calculus that we cannot continuously deform X such that it does not intersect itself anymore.

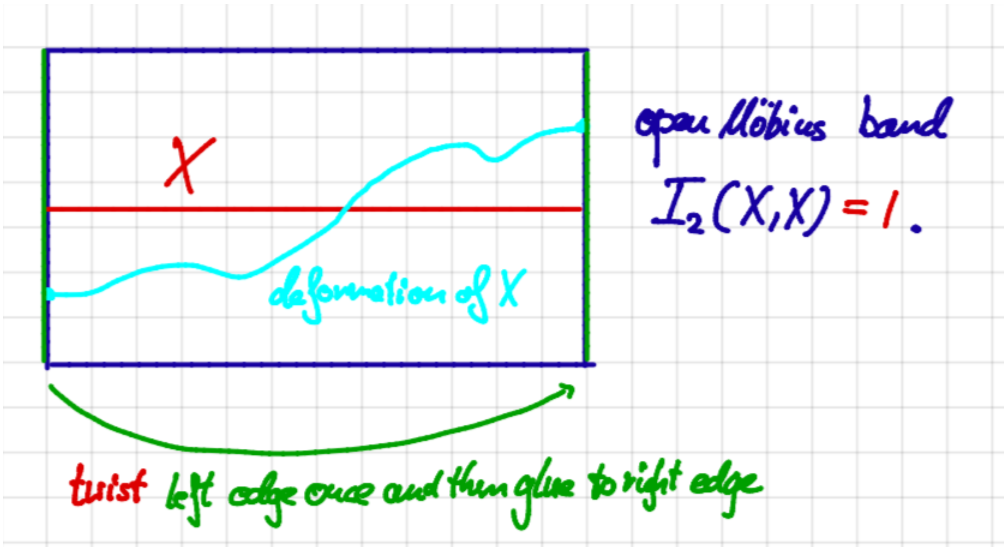


Figure 14.3: We can construct the Möbius band by taking a rectangle and glue to opposite sides after a twist. The central line then becomes a central circle on the band. If we move it a bit from its initial position, the new curve has to intersect the original central line. This is a consequence of the Intermediate Value Theorem, since the two ends of the curve must be on opposite sides of the central line because of the twist.

14.3 Obstruction to extensions to boundaries

If X happens to be the **boundary** of some W in Y, then $I_2(X, Z) = 0$. For if $Z \pitchfork X$, then, roughly speaking, Z **leaves** W as often as it **enters**. Hence $\#(X \cap Z)$ should be **even**. This heuristic can be made rigorous as follows:

Theorem 14.9 (Boundary Theorem for intersection numbers) Suppose that X is the **boundary** of some **compact** manifold W and $g: X \to Y$ is a smooth map. **If** g can be **extended** to all of W, **then** $I_2(g, Z) = 0$ for every **closed** submanifold Z in Y which satisfies $\dim X + \dim Z = \dim Y$.

Proof: Let $G: W \to Y$ be an extension of g, i.e., G is smooth and $\partial G = g$. From the **Transversality Homotopy Theorem 13.27**, we obtain a map $F: W \to Y$ **homotopic** to G with $F \pitchfork Z$ and $\partial F \pitchfork Z$. We write $f := \partial F$. Then $f \sim g$ and hence

$$I_2(g, Z) = I_2(f, Z) = \#f^{-1}(Z) \quad \text{mod } 2.$$

Now $F^{-1}(Z)$ is a **compact submanifold** whose codimension in W is the same as the codimension of Z in Y. Here we use again that X is the boundary of W, for this implies $\dim W = \dim \partial W + 1 = \dim X + 1$, and hence $\dim F^{-1}(Z) = \dim X + 1 - \dim Y + \dim Z = 1$. This shows that $F^{-1}(Z)$ is a compact one-dimensional manifold with boundary, so $\#\partial(F^{-1}(Z))$ is **even** and hence

$$\#\partial(F^{-1}(Z)) = \#(\partial F)^{-1}(Z) = \#f^{-1}(Z) \text{ is } \textbf{even}. \qquad \square$$

This has an interesting consequence:

Theorem 14.10 (Obstruction for extending maps) Let W be a **compact** manifold and $f : \partial W \to Y$ be a smooth map. **If** there is at least one closed submanifold $Z \subset Y$ such that $I_2(f, Z) \neq 0$, then g **cannot be extended** to a smooth map $W \to Y$ on all of W.

We will see some applications of the previous two theorems in the exercises.

14.4 Intersecting circles on $\mathbb{S}^n$ versus $\mathbb{R}\mathrm{P}^n$

In this section we discuss an example for how intersection theory can be used to show the existence of non-trivial elements in the fundamental group. Recall from Section 9.2 real projective n-space $\mathbb{R}\mathrm{P}^n$ which consists of the set of equivalence classes $[x_0 : x_1 : \dots : x_n]$ of $(n+1)$-tuples of real numbers with the equivalence relation

$$(x_0, x_1, \dots, x_n) \sim (\lambda x_0, \lambda x_1, \dots, \lambda x_n) \text{ for } \lambda \in \mathbb{R} \setminus \{0\}.$$

Recall that we showed that $\mathbb{R}\mathrm{P}^n$ is an n-dimensional smooth manifold. Alternatively, we can describe $\mathbb{R}\mathrm{P}^n$ as a the quotient of $\mathbb{S}^n$ under the equivalence relation $x \sim -x$, i.e., we identify antipodal points on $\mathbb{S}^n$ to obtain $\mathbb{R}\mathrm{P}^n$:

$$\mathbb{R}\mathrm{P}^n = \mathbb{S}^n/(x \sim -x),$$

and we have a smooth map

$$\mathbb{S}^n \to \mathbb{R}\mathrm{P}^n,\ x \mapsto [x]$$

which sends a point x in $\mathbb{S}^n$ to its equivalence class $[x]$ in $\mathbb{R}\mathrm{P}^n$. One can show that $\mathbb{R}\mathrm{P}^1$ and $\mathbb{S}^1$ are diffeomorphic. However, this is the **exception** special to dimension one as we will now show. First we observe that the intersection theory for circles on $\mathbb{S}^n$ is rather simple:

Example 14.11 (Intersecting circles on $\mathbb{S}^2$) Let us look at the concrete case for $\mathbb{S}^2$. We define a smooth map

$$j : \mathbb{S}^1 \to \mathbb{S}^2,\ (\cos t, \sin t) \mapsto (\cos t, \sin t, 0)$$

which maps the circle on the intersection of the sphere with the xy-plane in $\mathbb{R}^3$. This map is **homotopic** to the **constant** map $\mathbb{S}^1 \to \mathbb{S}^2, t \mapsto (0, 0, 1)$. We can show this directly using the homotopy $(\cos t, \sin t, s) \mapsto (s\cos t, s\sin t, 1-s)$. Hence we can **smoothly move** $j(\mathbb{S}^1)$ **away** from itself and get that, on $\mathbb{S}^2$, the self-intersection number $I_2\left(j(\mathbb{S}^1), j(\mathbb{S}^1)\right)$ is **zero**.

Note that if we define, for example, a map $j_2 : \mathbb{S}^1 \to \mathbb{S}^2$, $t \mapsto (0, \sin t, \cos t)$, then $j(\mathbb{S}^1) \cap j_2(\mathbb{S}^1)$ consists of **two** points: the points $(0, 1, 0)$ and $(0, -1, 0)$. Hence **modulo** 2, we still get $I_2(j(\mathbb{S}^1), j_2(\mathbb{S}^1)) = 0$.

Recall from Section 8.2 that a space X is called **simply-connected** if every continuous map $\mathbb{S}^1 \to X$ is homotopic to a constant map. See also Exercise 8.4 and Exercise 11.2. We proved in Theorem 8.16 that $\mathbb{S}^n$ is simply-connected for $n \geq 2$. We will now use this fact to show that $\mathbb{S}^n$ and $\mathbb{RP}^n$ are not homeomorphic for $n \geq 2$. In fact, we will prove a stronger statement via the following result:

Theorem 14.12 ($\mathbb{RP}^n$ is not simply-connected) For every $n \geq 1$, there is a loop $\mathbb{S}^1 \to \mathbb{RP}^n$ which is **not** homotopic to a constant map.

Proof: The case $n = 1$ follows from $\mathbb{RP}^1 \cong \mathbb{S}^1$ and the fact that $\mathbb{S}^1$ is **not** simply-connected. This follows from $\deg_2(z \mapsto z) = 1$. So now we assume $n \geq 2$. Our strategy is to find a smooth map $f : \mathbb{S}^1 \to \mathbb{RP}^n$ and a suitable closed submanifold $Z \subset \mathbb{RP}^n$ such that $I_2(f, Z)$ is defined and **non-zero**. Then we know that f is **not** homotopic to a constant map. For, if it was, then f would be homotopic to a constant map whose image was **disjoint** to Z. Then we would have $f \pitchfork Z$, and hence $I_2(f, Z) = 0$. This is not possible if our strategy works out. Then we use **Whitney's Approximation Theorem 13.20** which implies that if there was a **continuous** homotopy between f and a constant map, then there **also** was a **smooth** homotopy. Since the latter does **not exist**, the former cannot exist either.

So let us find suitable f and Z. Recall that, given a smooth manifold X, a smooth map $f : \mathbb{S}^1 \to X$ is equivalent to a **smooth** map on the **open** interval $g : (-\varepsilon, 2\pi + \varepsilon) \to X$ such that $g(0) = g(2\pi)$ for some $\varepsilon > 0$. Hence we may define a map f by

$$f : \mathbb{S}^1 \to \mathbb{RP}^n,\ t \mapsto [\cos(t/2) : \sin(t/2) : 0 : \ldots : 0].$$

Now we embed $\mathbb{S}^{n-1}$ as a submanifold of $\mathbb{S}^n$ by requiring the coordinate x_0 to be zero and let $Z \subset \mathbb{RP}^n$ be its image under the quotient map $\mathbb{S}^n \to \mathbb{RP}^n$:

$$Z = \left\{[x] = [x_0 : x_1 : \ldots : x_n] \in \mathbb{RP}^n | x_0 = 0\right\}.$$

We can check that Z is a **submanifold** of dimension $n - 1$ of $\mathbb{RP}^n$ by restricting the standard charts (V_i, ϕ_i) for $\mathbb{RP}^n$: Recall that we defined in Section 9.2, for $0 \leq i \leq n$, the subsets

$$V_i = \left\{[x] \in \mathbb{RP}^n : x_i \neq 0\right\}$$

and homeomorphisms

$$\phi_i^{-1} : V_i \to \mathbb{R}^n,\ [x_0 : \ldots : x_i : \ldots : x_n] \mapsto \frac{1}{x_i}(x_0, \ldots, \hat{x}_i, \ldots, x_n).$$

Then the subsets $V_i^Z : Z \cap V_i$ and maps $(\phi_i^{-1})_{|V_i^Z} : V_i^Z \to \mathbb{R}^{n-1}$ for $1 \leq i \leq n$ provide charts for Z. Since $\mathbb{S}^{n-1}$ is a closed subset in $\mathbb{S}^n$, Z is **closed** in $\mathbb{RP}^n$ by definition of the quotient topology on $\mathbb{RP}^n$. Hence the assumptions for applying intersection theory are satisfied.

The **only point** of Z which is hit by the image of f is the point $[0 : 1 : 0 : \ldots : 0] \in Z$. Since we describe f as a map $[0, 2\pi] \to \mathbb{RP}^n$ which **extends** to a smooth map over the boundary points of $[0, 2\pi]$, we get $f^{-1}(Z) = \{\pi\} \subset [0, 2\pi]$. Hence the **preimage** $f^{-1}(Z)$ consists of **exactly one point**. Now we need to check that f and Z are **transverse** in $\mathbb{RP}^n$. We can do this by studying the impact of the **derivative** df_π of f at π and the tangent space of Z **locally**, i.e., in a chart around the point $[0 : 1 : 0 : \ldots : 0]$. Under the map $\phi_1 : \mathbb{R}^n \to V_1$, the point $[0 : 1 : 0 : \ldots : 0] \in V_1$ corresponds to the origin. For $\mathbb{S}^1$, we choose the local parametrization $\psi : (\pi - \varepsilon, \pi + \varepsilon) \to \mathbb{S}^1$, $t \mapsto (\cos(t/2), \sin(t/2))$. Then we consider the diagram

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "f"] & \mathbb{R}\mathrm{P}^n \arrow[d, "\phi_1^{-1}"] \\
(\pi-\varepsilon, \pi+\varepsilon) \arrow[u, "\psi"] \arrow[r, "\tilde{f}"'] & \mathbb{R}^n
\end{tikzcd}$$

where $\tilde{f}$ is defined as the composition of the other maps such that the diagram commutes. We can compute $\tilde{f}$ as

$$\tilde{f}(t) = \frac{1}{\sin(t/2)} \cdot (\cos(t/2), 0, \dots, 0).$$

The derivative of $\tilde{f}$ at $t = \pi$ is then the linear map

$$d\tilde{f}_\pi : \mathbb{R} \to \mathbb{R}^n,\ x \mapsto (-1/2) \cdot (x, 0, \dots, 0)$$

given by multiplying x with the number $\tilde{f}'(t) = -\frac{1}{2}\frac{1}{\sin^2(t/2)}$ for $t = \pi$ and embedding for the first coordinate. The key consequence of this computation is that the **image** of $d\tilde{f}_\pi$ in $\mathbb{R}^n$ is the **line** $\mathbb{R} \times \{0\} \subset \mathbb{R}^n$. On the other hand, the **tangent space** at the origin of the image of Z under ϕ_1^{-1} is the subspace $\{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$. Thus this tangent space and the image of $d\tilde{f}_\pi$ together span all of $\mathbb{R}^n$. By definition of the smooth structure on the (abstract) manifold $\mathbb{R}\mathrm{P}^n$, the map $\phi_1 : \mathbb{R}^n \to V_1$ is a **diffeomorphism**. Thus the induced map on tangent spaces is an **isomorphism**. Since tangent spaces are determined locally, this shows that f and Z are transverse in $\mathbb{R}\mathrm{P}^n$. Thus we have proved

$$I_2(f, Z) = \# f^{-1}(Z) = 1$$

as we set out to show. □

Theorem 14.12 implies that the fundamental group of $\mathbb{R}\mathrm{P}^n$ is non-trivial. Using techniques from algebraic topology we can say even more:

Remark 14.13 (Fundamental group of $\mathbb{R}\mathrm{P}^n$) In Exercise 14.7 we will show that $2[f] = 0$ in $\pi_1(\mathbb{R}\mathrm{P}^2, x_0)$, where x_0 is the point $[1 : 0 : \dots : 0]$. Hence $[f]$ generates a subgroup $\mathbb{Z}/2$ inside $\pi_1(\mathbb{R}\mathrm{P}^2)$. One can then check that $\pi_1(\mathbb{R}\mathrm{P}^2, x_0) \cong \mathbb{Z}/2$ and extend this computation, for example using induction and the Seifert–van Kampen Theorem, to show

$$\pi_1(\mathbb{R}\mathrm{P}^n, x_0) \cong \mathbb{Z}/2 \text{ for } n \geq 2.$$

Example 14.14 (Intersecting circles on $\mathbb{R}\mathrm{P}^2$) Let us look at the concrete case $n = 2$. The map f of the above proof is given by

$$f : \mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2,\ t \mapsto [\cos(t/2) : \sin(t/2) : 0].$$

In Exercise 14.8, we study the map f in more detail and check that f is proper and injective. The submanifold Z

$$Z = \left\{ [x] = [x_0 : x_1 : x_2] \in \mathbb{R}\mathrm{P}^2 | x_0 = 0 \right\}$$

corresponds to the image of $\mathbb{S}^1$ under the map

$$\mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2,\ t \mapsto [0 : \sin(t/2) : \cos(t/2)].$$

The intersection of the image of f and Z therefore is the **intersection** of the images of the **two circles** in $\mathbb{R}\mathrm{P}^2$. This intersection consists of the **single** point

$$[0 : 1 : 0] \in f(\mathbb{S}^1) \cap Z.$$

Hence, in $\mathbb{R}\mathrm{P}^2$, there is **exactly one** intersection point. If we had considered the map and submanifold corresponding to f and Z for $\mathbb{S}^2$ instead of $\mathbb{R}\mathrm{P}^2$, we would have gotten two intersection points: $(0,1,0)$ and $(0,-1,0)$. Hence on $\mathbb{S}^2$, the corresponding **mod** 2 intersection number is **zero**. This fits well with the fact that $\mathbb{S}^2$ is simply-connected. On $\mathbb{R}\mathrm{P}^2$, however, we have $I_2(f,Z) = 1$ which implies that it is **impossible** to move $f(\mathbb{S}^1)$ and Z within $\mathbb{R}\mathrm{P}^2$ such that **they do not meet**.

By Lemma 8.14, Theorem 8.16 and Theorem 14.12 imply that there cannot be a homotopy equivalence between $\mathbb{S}^n$ and $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$. Since there cannot be a homeomorphism between spaces which are not homotopy equivalent, we have proven the following theorem:

Theorem 14.15 **($\mathbb{S}^n$ and $\mathbb{R}\mathrm{P}^n$ are not equivalent for $n \geq 2$)** For $n \geq 2$, there is **no homotopy equivlance** between $\mathbb{S}^n$ and $\mathbb{R}\mathrm{P}^n$. In particular, there is **no homeomorphism** between $\mathbb{S}^n$ and $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$.

14.5 $\mathbb{R}^n$ as a commutative division algebra

We will now discuss an important and at first glance maybe surprising **application** of Theorem 14.15. An **algebra** structure on $\mathbb{R}^n$ is a $\mathbb{R}$-bilinear multiplication map

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, (a,b) \mapsto ab.$$

Note that it is not required that this multiplication has an identity element. Such a map is called a **division algebra** structure if there are no zero-divisors, i.e., $ab = 0$ implies $a = 0$ or $b = 0$. It is called **commutative** if $ab = ba$ for all $a, b \in \mathbb{R}^n$. **Examples** of finite-dimensional commutative division algebras over $\mathbb{R}$ are given by $\mathbb{R}$ itself and $\mathbb{C} = \mathbb{R}^2$. However, there is not much more:

Theorem 14.16 **(Commutative algebra structures on $\mathbb{R}^n$)** There is **no commutative** division algebra structure on $\mathbb{R}^n$ for $n \geq 3$.

Proof:[2] **Suppose** there is a commutative division algebra structure on $\mathbb{R}^n$. Then we can define a map by

$$f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1},\ f(x) = x^2/|x^2|.$$

This is **well-defined**, since $x \neq 0$ implies $x^2 \neq 0$ in a **division algebra**. The map f is **smooth**, since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is **bilinear** and hence smooth. Since f satisfies

[2] The following proof follows Hatcher's book [6] on Algebraic Topology.

$f(-x) = f(x)$ for all x, f induces a **smooth map** on the quotient

$$\bar{f} : \mathbb{RP}^{n-1} \to \mathbb{S}^{n-1}.$$

- **Claim:** $\bar{f}$ is **injective**.

To show the claim we assume $f(x) = f(y)$. This implies

$$x^2 = \alpha^2 y^2 \text{ for } \alpha = \sqrt{\frac{|x^2|}{|y^2|}} > 0.$$

Thus we have

$$x^2 - \alpha^2 y^2 = 0.$$

Using **commutativity** and the fact that α is a **real** number and multiplication is $\mathbb{R}$-**bilinear** we see that this equation **factors** into

$$(x + \alpha y)(x - \alpha y) = 0.$$

Since we assume that the multiplication has **no zero-divisors**, this implies

$$x = \pm \alpha y.$$

Since $|x| = |y| = 1$ and α is a **real** number, this implies $|\alpha| = 1$ and hence

$$x = \pm y.$$

Thus x and y determine the same point in $\mathbb{RP}^{n-1}$. This proves the **claim** that $\bar{f}$ is injective.

Now we use the nice topological properties of $\mathbb{RP}^{n-1}$ and $\mathbb{S}^{n-1}$. First we observe that $\bar{f}$ is a map between **compact Hausdorff** spaces. Since $\bar{f}$ is **injective**, it is a **homeomorphism** onto its **image**. Moreover, since $\mathbb{RP}^{n-1}$ is **compact**, its image in $\mathbb{S}^{n-1}$ is also compact and hence **closed**. Since $\mathbb{RP}^{n-1}$ and $\mathbb{S}^{n-1}$ are smooth manifolds of the **same dimension**, Invariance of Domain[3], which we stated in Theorem 11.12, implies that the image is also **open** in $\mathbb{S}^{n-1}$.[4] Since $\mathbb{S}^{n-1}$ is a **connected**, $\bar{f}$ must be a **homeomorphism**. By Theorem 14.15, we know that for $n \geq 3$ there **cannot** be a homeomorphism between $\mathbb{RP}^{n-1}$ and $\mathbb{S}^{n-1}$. Hence our initial **assumption** must have been **wrong**. □

There is a fascinating much deeper story attached to this problem:

Remark 14.17 (Adams' Theorem: Hopf invariant one) There is a vast generalisation of the above result: Let $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a division algebra structure on $\mathbb{R}^n$, without assuming it is commutative. For $n = 4$, there are the Hamiltonians, or Quaternions, $\mathbb{H} \cong \mathbb{R}^4$ with a multiplication which is associative and almost as good as the one in $\mathbb{C}$ and $\mathbb{R}$, but it is **not** commutative. For $n = 8$, there are the Octonions $\mathbb{O} \cong \mathbb{R}^8$. The multiplication is neither commutative nor associative. And **that's it!** This is a deep result. And surprisingly it is equivalent to a topological problem on the behavior of tangent spaces on spheres. It was solved first by Adams. The prove goes way beyond

[3]Let $U \subset \mathbb{R}^n$ be an open subset and $h : U \to \mathbb{R}^n$ an injective continuous map. Then $h(U)$ is **open** in $\mathbb{R}^n$.

[4]Alternatively, we could show that f is a submersion and use that submersions are open maps by Exercise 4.1.

the methods of this class, unfortunately. However, it turned out to be not that hard using complex K-theory.[a] A key tool for the argument is the **Hopf invariant**. We have seen a mod 2 version of the Hopf invariant in Section 12.4 and we will define the $\mathbb{Z}$-valued version later via a series of exercises in Section 16.4 using intersection theory and linking numbers.

[a] One may read about this proof in [16, Lecture 26].

14.6 Exercises and more examples

Exercise 14.1 Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps between manifolds with X compact, and let $W \subset Z$ be a submanifold. Assume that g is transversal to W, so that $g^{-1}(W)$ is a submanifold in Y. Show that

$$I_2(f, g^{-1}(W)) = I_2(g \circ f, W).$$

In particular, verify that if one of these two intersection numbers is defined, then the other one is defined as well.

Exercise 14.2 Let X be a compact manifold without boundary.

(a) Assume $\dim X \geq 1$: Show that if $f : X \to Y$ is homotopic to a constant map, then $I_2(f, Z) = 0$ for all complementary dimensional closed submanifolds Z in Y.

Hint: Show that if $\dim Z < \dim Y$, then f is homotopic to a constant map $X \to \{y\}$ for some $y \notin Z$. You may assume that Y is connected.

(b) For $\dim X = 0$, show that the corresponding assertion of the previous point is wrong.

Hint: If X consists of just one point, for which Z will $I_2(f, Z) \neq 0$?

(c) Show that $\mathbb{S}^1$ is not simply-connected. Recall that we call a manifold X *simply-connected* if it is connected and if every map of the circle $\mathbb{S}^1$ into X is homotopic to a constant map.

Hint: Consider the identity map.

Exercise 14.3 **(a)** Show that intersection theory is trivial in contractible boundaryless manifolds: if Y is boundaryless and contractible, i.e., its identity map is homotopic to a constant map, and $\dim Y > 0$, then $I_2(f, Z) = 0$ for every smooth map $f : X \to Y$ such that X compact and Z closed, $\dim X \geq 1$ and $\dim X + \dim Z = \dim Y$. In particular, intersection theory is trivial in Euclidean space.

(b) Prove that no compact boundaryless manifold - other than the one-point space - is contractible.

Hint: Consider the identity map.

Exercise 14.4 **(a)** Let $f : X \to \mathbb{S}^k$ be a smooth map with X compact and $0 < \dim X < k$. Show that, for all closed submanifolds $Z \subset \mathbb{S}^k$ of dimension complementary to X, $I_2(f, Z) = 0$.

Hint: Use Sard's Theorem 7.1 to show that there exists a $p \notin f(X) \cap Z$. Now use a stereographic projection and the previous exercises.

(b) Show that $\mathbb{S}^2$ and the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ are not diffeomorphic.

Exercise 14.5 Two compact manifolds X and Z of the same dimension in Y are called **cobordant** in Y if there exists a compact manifold with boundary $W \subset Y \times [0, 1]$ such that

$$\partial W = X \times \{0\} \cup Z \times \{1\}.$$

The manifold W is called a **cobordism** between X and Z. See Figure 14.4.

(a) Show that if we can deform X into Z, i.e., if there is a smooth homotopy from the embedding $i_0 : X \hookrightarrow Y$ of X in Y to an embedding $i_1 : X \hookrightarrow Y$ with $i_1(X) = Z$ such that each i_t is an embedding, then X and Z are cobordant.

Note that the standard image of a cobordism, a pair of pants as in Figure 14.4, illustrates that the converse is false: X and Z are cobordant, but we cannot deform X into Z, since X has one connected component whereas Z has two.

(b) Show that if X and Z are cobordant in Y, then for every compact submanifold C in Y with dimension complementary to X and Z, i.e., $\dim X + \dim C = \dim Z + \dim C = \dim Y$ (where $\dim X = \dim Z$ because they are cobordant), we have

$$I_2(C, X) = I_2(C, Z).$$

Hint: Let f be the restriction to W of the projection map $Y \times [0, 1] \to Y$, and use the Boundary Theorem 14.9.

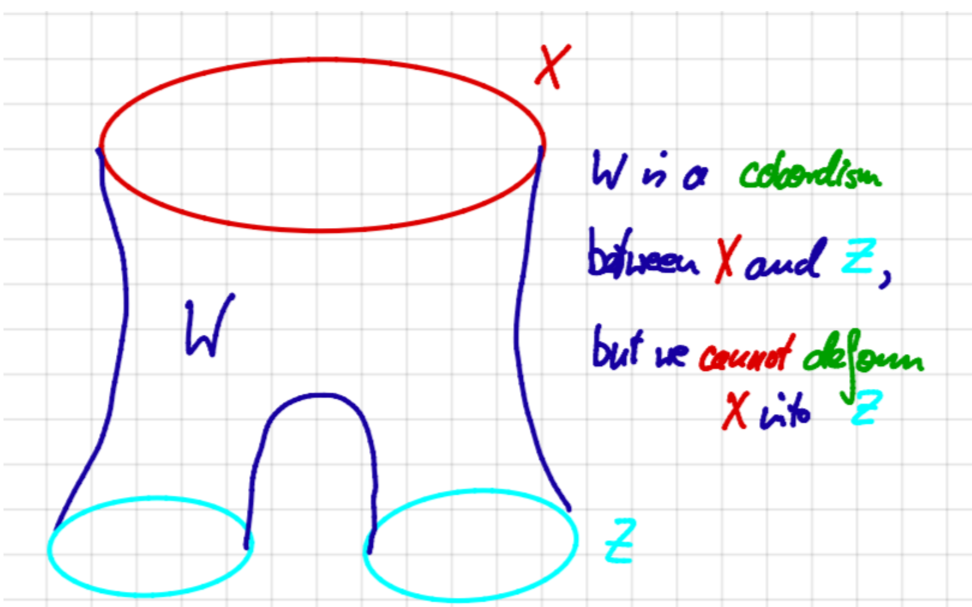


Figure 14.4: A pair of pants is a cobordism between the upper boundary X and the lower boundary Z. While X is connected, Z has two connected components.

Exercise 14.6 Let $Z \subset X$ be a compact submanifold of X with $\dim Z = \frac{1}{2}\dim X$. Show that if there is a submersion $g : U \to \mathbb{R}^k$ defined on an open subset $U \subset X$ such that $Z = g^{-1}(0)$, then $I_2(Z, Z) = 0$.

Hint: Use Theorem 13.14 and Theorem 13.15.

The following exercises require some computations for abstract manifolds. They shed light on very interesting phenomenons and provide the proof for important results we mentioned in the main text.

Exercise 14.7 Let X be the open Möbius band given as the quotient of $[-1,1]\times(-1,1)$ modulo the equivalence relation $(-1,x) \sim (1,-x)$. Let Z denote the central circle given by $Z = \{(x,0) : x \in [-1,1]\} \subset X$.

(a) Show that $I_2(Z, Z) = 1$. See Figure 14.3.

(b) Conclude using Exercise 14.6 that there cannot be a submersion $g : X \to \mathbb{R}$ such that $Z = g^{-1}(0)$.

Exercise 14.8 Let $f : \mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2$ be the map defined by sending $t \in [0, 2\pi]$ to $[\cos(t/2) : \sin(t/2) : 0]$.

(a) Show that f is a smooth embedding, i.e., explain why it is smooth, injective, and proper.

(b) Write Z for the image of f in $\mathbb{R}\mathrm{P}^2$ and consider it as a submanifold of $\mathbb{R}\mathrm{P}^2$. Prove that $I_2(Z, Z) = 1$.

(c) Conclude that f is not homotopic to a constant map.

Aside: Recall that, since every continuous map on $\mathbb{S}^2$ is homotopic to a constant map, this implies that there cannot be a homeomorphism between $\mathbb{S}^2$ and $\mathbb{R}\mathrm{P}^2$.

Aside: Note that the map $[0, 2\pi] \to \mathbb{S}^2$ defined by sending t to the point $(\cos(t/2), \sin(t/2), 0)$ is not a loop on $\mathbb{S}^2$. Hence the image has a boundary and our intersection theory fails to work. And if we remove the boundary points and define the map on $(0, 2\pi)$, then we loose compactness and even closedness, so intersection theory will fail as well. However, if we look at the loop $[0, 2\pi] \to \mathbb{S}^2$ defined by sending t to $(\cos(t), \sin(t), 0)$ on $\mathbb{S}^2$, then we see that the vertical loop intersects the original loop twice. And we can convince us that any other loop would intersect the original loop in an even number of points. Hence the mod 2-self intersection number of the loop is indeed zero.

(d) We write $2f$ for the map $2f : \mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2$ defined by sending $t \in [0, 2\pi]$ to $[\cos(t) : \sin(t) : 0]$. Show that $2f$ is homotopic to a constant map by constructing a concrete homotopy. Do you see why the homotopy works for $2f$ but not for f?

Aside: The observations for f and $2f$ are an indicator for the fact that the fundamental group of $\mathbb{R}\mathrm{P}^2$ is cyclic of order two, i.e., $\pi_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/2$, where f

represents the non-trivial element and $[2f] = 0$. Apart from the formula for $2f$ this is another explanation of our choice for the name $2f$.

15. Orientation

Our next goal is to **improve** our definition of degree and **intersection number** and remedy the defect that they only have even and odd values.

15.1 Towards integer-valued invariants

One of the reasons for this limitation was that a **homotopy** can move a **non-transversal** intersection into, for example, **either** an **empty** intersection **or** an intersection in **two points**. The idea to handle this phenomenon is to take into account in which **direction** the intersection happens and to **count** intersection points **with signs**:

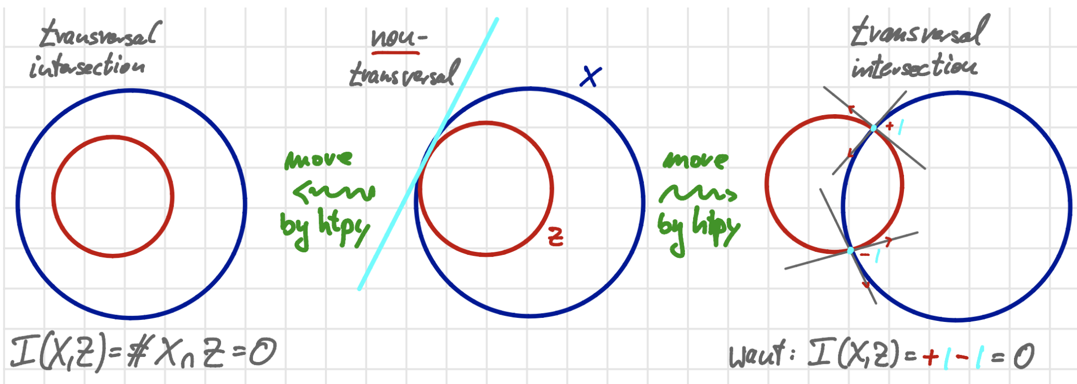


Figure 15.1: We can again move the two circles in the plane and either move them apart or make them intersect transversely. In both cases we get a stable number of intersection points. However, 0 is still not equal 2. But this time we keep track of how the points are oriented by looking at how the relation of the directions of tangent vectors change.

The technical solution to implement this idea is to **introduce orientations**. We will see that, unfortunately, **not all** manifolds are **orientable**. For those manifolds that orientable, however, we will introduce and study integer-valued invariants in the next chapters. In order to get a first idea why the claimed solution might be reasonable we look back at an important example first.

- **A look back: Mod 2 Brouwer Degree of maps $\mathbb{S}^1 \to \mathbb{S}^1$**

In Section 12.3.3 we studied self-maps of the circle. Given a smooth map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, we showed that it has a lift $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(t+1) = g(t) + q$ for all t for some fixed $q \in \mathbb{Z}$. The integer q is a number that is attached to f. We then showed that $\deg_2(f) = q$ modulo 2. **How nice would it be** if we had a notion of a degree for f with integer values so that we **could write** an equation like

$$\deg(f) = q \text{ in } \mathbb{Z}.$$

If we knew in addition that the number $\deg(f)$ is **homotopy invariant**, i.e., it only depends on the class of f under the equivalence relation on the set of maps given by homotopy, then we could use the degree to distinguish between possibly infinitely many different homotopy classes of maps $\mathbb{S}^1 \to \mathbb{S}^1$, or even maps $\mathbb{S}^n \to \mathbb{S}^n$.

So let us think about how we came up with the formula $\deg_2(f) = q$ modulo 2 was to look at the graph of g in Figure 12.5. Then we used the Intermediate Value Theorem and observed that the graph crosses the horizontal lines $g(s) + k$ for an integer $k \in \{0, 1, \dots, q-1\}$ an **odd** number of times and lines $g(s)+k$ for **any other integer** k an **even** number of times. We would like to improve this count by taking all the values with $g(t) = g(s) + k$ into account. But there might be many more such values than just q. How can we fix this?

The **idea** is to remember **how** the graph of g crosses the lines $g(s)+k$: when t **increases** then $g(t)$ **either increases** while crossing $g(s) + k$ **or decreases**. In other words, the **derivative** of g is **either positive or negative** at t_k with $g(t_k) = g(s) + k$. Note that we cannot have $g'(t_k) = 0$, since y is regular and hence $\det(df_x) \neq 0$ and therefore also $g'(t) \neq 0$ for all t with $p(g(t)) = y$.

Now, if $g'(t_k) > 0$, we may **count** t_k with **value** $+1$, if $g'(t_k) < 0$, we may count t_k with **value** -1. See Figure 15.2. Counting all $t \in [s, s+1)$ with $g(t) = g(s)+k$ for $k \in \mathbb{Z}$, this gives us the desired value q. Hence if we can improve our definition of the degree such that we can use this way of counting points on the graph of g, we would get $\deg(f) = q$.

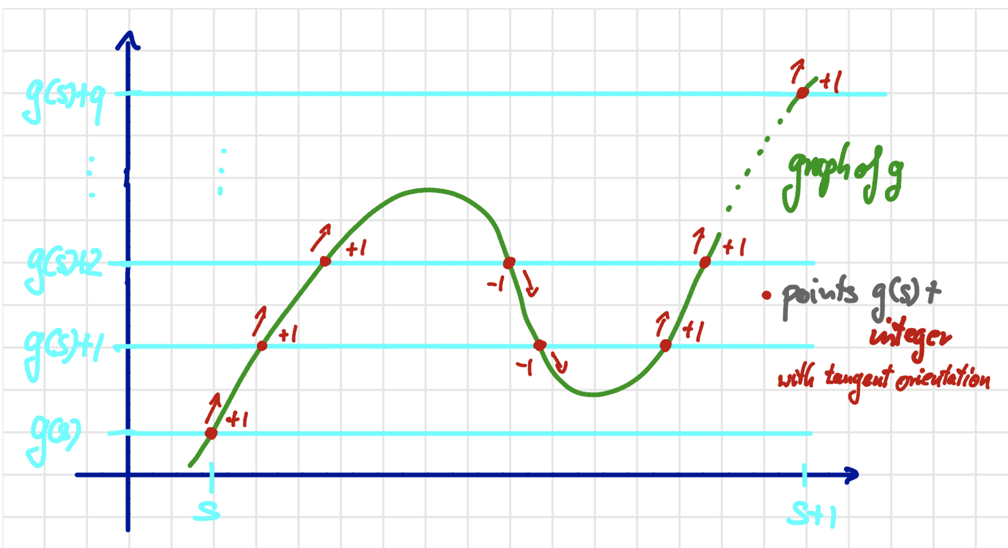


Figure 15.2: The idea is to keep track of how the graph passes the horizontal lines: if g increases while passing the horizontal line we count the intersection as $+1$ and if g decreases we count it as -1.

To generalise this procedure, we need to describe $g'(t_k) > 0$ or $g'(t_k) < 0$ in more general terms. In **higher dimensions**, we cannot just attach a sign to $g'(t)$ or $f'(t)$. However, an **alternative way** to describe the **sign of the derivative** of $g'(t)$ is to say that g preserves the orientation of the coordinate axes. More generally, thinking of the axes as the tangent spaces of copies of $\mathbb{R}$, we could say that the derivative of g does or does not preserve the **orientation** of the tangent spaces. A line has **exactly two** orientations, depending on if we walk in one direction or the other. It turns out that this perspective can be **generalised to higher dimensions**. Since the derivatives are linear transformations, we can look at their **determinants**. Then we can ask whether the determinant has a positive or a negative sign. Geometrically, this sign will detect whether or not the linear transformation given by the derivative preserves or reverses the orientation of the respective tangent spaces. In order to make this idea work we first have to define what an orientation of a vector space is, and then we need to transfer this to smooth manifolds. This is what we are going to do next.

15.2 Orientation

We begin with a look back at Linear Algebra.

15.2.1 Orientation on vector spaces

An **orientation** for a finite dimensional real vector space V is an **equivalence class of ordered bases** where the relation is defined as follows: the ordered basis $(v_1, \dots, v_n)$ has the **same orientation** as the basis $(v'_1, \dots, v'_n)$ if the matrix A with

$$v'_i = Av_i \text{ for all } i \text{ has determinant } \det(A) > 0.$$

It has the **opposite orientation** if $\det(A) < 0$. The fact that this an **equivalence relation** follows from the multiplicativity of the determinant function. Thus **each finite dimensional** vector space **has precisely two orientations**, corresponding to the **two equivalence classes of ordered bases**.

So an orientation of V is a choice of an equivalence class of ordered bases. To make it easier to talk about the choice of orientation, we attach to the chosen orientation a **positive sign** and a **negative sign** to the other orientation. We say then that an ordered basis is **positively oriented** (respectively negatively oriented) if its equivalence class belongs to the orientation $+1$ (respectively -1). We often confuse an orientations with their corresponding signs $+1$ or -1.

Remark 15.1 (Warning) The **ordering** of the basis elements is **essential**. Interchanging the positions of two basis vectors **changes the sign** of the orientation since the the determinant of the corresponding permutation matrix is negative.

In the case of the **zero dimensional** vector space it is convenient to define an **orientation** as the **symbol** $+1$ or -1.

Example 15.2 (Standard orientation on $\mathbb{R}^n$) The vector space $\mathbb{R}^n$ has a **standard orientation** given by the ordered basis $(e_1, \dots, e_n)$. We always **assign $+1$** to the **standard orientation** of $\mathbb{R}^n$.

If $\varphi : V \to W$ is an **isomorphism** of vector spaces, then φ **either preserves or reverses** the orientation. For, given two ordered bases β and β' of V belonging to the the same equivalence class, the ordered bases $\varphi(\beta)$ and $\varphi(\beta')$ either still belong to the same equivalence class of ordered bases of W or not. Whether φ preserves or reverses the orientation is **determined by**

its determinant. If $\det(\varphi)$ is positive, then φ preserves orientations, and if $\det(\varphi)$ is negative, then φ reserves orientations.

15.2.2 Orientation on manifolds

Now we translate orientations from vector spaces to orientations for manifolds. The idea is to orient each tangent space. However, this needs to be done in a compatible way.

Definition 15.3 (Orientation on manifolds) Let X be a smooth manifold. An **orientation** of X is a **smooth choice of orientations** for all the **tangent spaces** T_xX at each point $x \in X$. That means: around each point $x \in X$ there exists a local parametrization $\phi : U \to X$ such that the isomorphism $d\phi_u : \mathbb{R}^k \to T_{\phi(u)}X$ **preserves orientations at each point** u of $U \subseteq \mathbb{H}^k$ where the orientation on $\mathbb{R}^k$ is always assumed to be the standard one. We use the **symbol** $\mathfrak{o}_X$ to denote the data of an orientation on X. We will sometimes write $(X, \mathfrak{o}_X)$ to express that X is oriented by the orientation $\mathfrak{o}_X$.

We now introduce an important distinction to our terminology and distinguish between the possibility to make a choice or that such a choice actually has been made:

Definition 15.4 (Orientable and oriented manifolds) A smooth manifold X is called **orientable** if a **smooth choice of orientations** of its tangent spaces **exists**. A smooth manifold is called **oriented** if it is **orientable** and a **choice of orientation** has been made.

Hence an oriented manifold is a **pair** consisting of a manifold together with a chosen orientation.

Example 15.5 For **zero-dimensional** manifolds, orientations are very simple: to each point $x \in X$ we assign an **orientation number** $+1$ or -1.

Definition 15.6 (Maps and orientations) A smooth map $f : X \to Y$ between oriented smooth manifolds is called **orientation preserving** if its derivative $df_x : T_x(X) \to T_{f(x)}(Y)$ preserves orientations at every point $x \in X$.

Lemma 15.7 (Diffeomorphisms and orientation) Let X and Y be oriented smooth manifolds and let $f : X \to Y$ be a diffeomorphism. Then f is either preserves or reverses orientation.

Proof: We show in Exercise 15.4 that if df_x preserves orientation at one point x, then f preserves orientation at every point. □

Example 15.8 The n-dimensional sphere $\mathbb{S}^n$ is orientable and has a standard orientation as we will explain in Example 15.19.

Example 15.9 (Reflections and orientation) Let $r_i : \mathbb{R}^n \to \mathbb{R}^n$ be the **reflection** of the ith coordinate, i.e.,

$$r_i(x_1, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n).$$

Then r_i is an isomorphism which reverses the orientation. The **composition** $r_i \circ r_j$ of two reflections is an isomorphism which preserves the orientation. This shows that $-\mathrm{Id}_{\mathbb{R}^n}$, the negative of the identity of $\mathbb{R}^n$, preserves the orientation if n is **even** and $-\mathrm{Id}_{\mathbb{R}^n}$ reverses the orientation if n is **odd**. Now we can consider the reflection $r_i : \mathbb{S}^n \to \mathbb{S}^n$ on the n-sphere given by

$$r_i(x_1, \dots, x_{n+1}) = (x_1, \dots, -x_i, \dots, x_{n+1}).$$

Since the derivative of r_i is a reflection on a vector space, we see that r_i is a diffeomorphism which reverses the orientation.

Since the antipodal map on $\mathbb{S}^n$ is induced by a composition of $n+1$ reflections, the above observation yields the following result:

Lemma 15.10 (Orientation and the antipodal map) The **antipodal map** $a : \mathbb{S}^n \to \mathbb{S}^n$, $a(x) = -x$, is a diffeomorphism which **preserves** the orientation if n is **odd** and **reverses** the orientation if n is **even**.

We just learned that a manifold **may** or may **not** be orientable. To assign $+1$ or -1 to the orientation of $T_x(X)$ for every point is a locally constant function. If X is orientable this assignment is continuous. If X is in addition connected, then this assignment must be constant. Hence on every connected component of an orientable manifold, the orientation is constant $+1$ or -1.

Here is a rigorous proof of this fact:

Lemma 15.11 (Orientable manifolds have exactly two orientations) A **connected**, **orientable** manifold with boundary admits **exactly two orientations**.

Proof: Assume we are given two orientations on X. In fact, we know that there are **at least two**, since given one, we can reverse signs everywhere and get another orientation. We need to show that there are **not more** than two. We do this by showing that the **set of points at which two orientations agree** and the **set where they disagree** are **both open**. Consequently, two orientations of a **connected** manifold are **either identical or opposite**.

Since X is **orientable**, we can choose local parametrizations $\phi : U \to X$ and $\phi' : U' \to X$ around $x \in X$ with $\phi(0) = x = \phi'(0)$ such that $d\phi_u$ preserves the first orientation and $d\phi'_{u'}$ preserves the second, for all $u \in U$ and $u' \in U'$. After possibly shrinking we can assume $\phi(U) = \phi'(U')$ (replace U and U' with $\phi^{-1}(\phi(U) \cap \phi'(U'))$ and $\phi'^{-1}(\phi(U) \cap \phi'(U'))$, respectively). If the two orientations of $T_x(X)$ **agree**, then the map

$$d(\phi^{-1} \circ \phi')_0 : \mathbb{R}^k \to \mathbb{R}^k$$

is an **orientation preserving isomorphism**. Thus the **determinant** of $d(\phi^{-1} \circ \phi')_0$ is **positive**.

Hence the function

$$\varphi: U' \to \mathbb{R},\ u' \mapsto \det(d(\phi^{-1} \circ \phi')_{u'})$$

satisfies $\varphi(0) > 0$.

Since the derivative **depends continuously** on u' and the determinant function is continuous, φ is **continuous**. Hence, since $\varphi(0) > 0$, there is an open neighborhood V' around 0 in U' on which $\varphi_{|V'} > 0$. But this implies that the orientations of $T_x(X)$ induced by ϕ and ϕ', respectively, agree for all x in the open subset $\phi'(V')$. Since **every point** on X has such an **open neighborhood**, the set of points where the **orientations agree** is **open**.

If the orientations on $T_x(X)$ induced by ϕ and ϕ', respectively, do **not** agree, the same argument shows that the set of points where the orientations do not agree is open. □

Definition 15.12 (Reversed orientation) Hence if $(X, \mathfrak{o}_X)$ is an oriented manifold X, then we can talk about the manifold X with the **opposite orientation**. We write $-\mathfrak{o}_X$ for the opposite orientation. We will often just **write $-X$** for the oriented manifold $(X, -\mathfrak{o}_X)$.

The product of oriented manifolds inherits an orientation as follows:

Definition 15.13 (Product orientation) Assume X and Y are **oriented** and one of them is without boundary. Then $X \times Y$ is a manifold with boundary and **inherits an orientation** in the following way: At a point $(x, y) \in X \times Y$, let $\alpha = (v_1, \dots, v_k)$ and $\beta = (w_1, \dots, w_m)$ be ordered bases of $T_x(X)$ and $T_y(Y)$, respectively. We denote the ordered basis $((v_1, 0), \dots, (v_k, 0), (0, w_1), \dots, (0, w_m))$ of $T_x(X) \times T_y(Y) = T_{(x,y)}(X \times Y)$ by $(\alpha \times 0, 0 \times \beta)$. Now it comes handy that we related orientations of ordered bases to signs. For we can define the orientation of $T_x(X) \times T_y(Y)$ simply by determining a sign by setting

$$\operatorname{sign}(\alpha \times 0, 0 \times \beta) := \operatorname{sign}(\alpha) \cdot \operatorname{sign}(\beta).$$

Remark 15.14 (Classification) **Not all** manifolds are orientable. One of the most famous examples of a non-orientable manifold is the **Möbius band**. We will give a rigorous proof of the non-orientability in Theorem 15.17. Other examples of non-orientable manifolds include the **Klein bottle** and the **real projective plane** $\mathbb{R}\mathrm{P}^2$. We will discuss the latter case in Theorem 15.16. As a consequence, we see that the question whether a smooth manifold is orientable or not helps **classifying manifolds** up to diffeomorphism: There is the class of orientable manifolds, and the class of non-orientable manifolds.

Now we study how orientations behave with respect to smooth maps:

Theorem 15.15 (Pullback of an orientation) Let X and Y be smooth manifolds with or without boundary. Assume that $Y = (Y, \mathfrak{o}_Y)$ is **oriented** and that there is a **local diffeomorphism** $f: X \to Y$. Then there is a **unique** orientation on X such that f **preserves orientations**. We call the induced orientation on X the **pullback orientation**

induced by f and denote it by $f^*\mathfrak{o}_Y$. Moreover, if $g: Y \to Z$ is another smooth map and $(Z, \mathfrak{o}_Z)$ is an oriented manifold, then the pullback orientation on X induced by $g \circ f$ is the same as the orientation on X obtained by pulling back the orientation on Z via g to Y and then further to X, i.e.,

$$(g \circ f)^*\mathfrak{o}_Z = f^*(g^*\mathfrak{o}_Z) \text{ as orientations on } X.$$

In other words, the pullback of orientations behaves **functorially**.

Proof: Since f is a local diffeomorphism, at every point $x \in X$, there are open subsets $U \subset X$ and $V \subset Y$ with $x \in U$ and $f(x) \in V$ such that $f_{|U}: U \to V$ is a diffeomorphism. By choosing U and V small enough, we can assume that there are local parametrizations $\phi: W \to U \subset X$ and $\psi: W \to V \subset Y$ such that $f \circ \phi = \psi$ with $W \subset \mathbb{R}^n$ open. Since Y is oriented, we can choose ψ such that $d\psi_w$ determines the orientation on T_yY for all $y = \psi(w)$ in V. Since df_x is an **isomorphism** for all x by assumption, we can **define an orientation** on T_xX by requiring $\det(df_x) > 0$. This defines an orientation on T_xX for all $x \in U$. It remains to check that this orientation on the tangent spaces for points in $U \subset X$ does not depend on the choice of ϕ. So let $\phi': W' \to U' \subset X$ be another local parametrization of X with $U \cap U' \neq \emptyset$. Then we can find a local parametrization $\psi': W' \to V' \subset Y$ of Y such that $f_{|U'}: U' \to V'$ is a diffeomorphism. Since Y is oriented, the orientation of T_yY determined by $d\psi_w$ and by $d\psi_{w'}$, with $\psi(w) = y = \psi(w')$ are the same for all y in $V \cap V'$. In other words, we have

$$\det(d(\psi^{-1} \circ \psi')_{w'} > 0 \text{ for all } w' \in W \cap W' \subset \mathbb{R}^n.$$

Since $\det(df_x) > 0$, this implies

$$\det(d(\phi^{-1} \circ \phi')_{w'} > 0 \text{ for all } w' \in W \cap W' \subset \mathbb{R}^n$$

as well. Hence the orientation of T_xX via $d\phi_w$ and $d\phi'_{w'}$ is the same for all $w, w' \in W \cap W$ with $\phi(w) = x = \phi'(w')$. This shows that we have defined an orientation on X. The facts that f preserves orientations, uniqueness and functoriality follow from the construction of the induced orientation. □

Here is an interesting **application** of Theorem 15.15:

Theorem 15.16 (Orientability of real projective space) Real projective space $\mathbb{R}\mathrm{P}^n$ is **orientable** if and only if n is **odd**.

Proof: Let $q: \mathbb{S}^n \to \mathbb{R}\mathrm{P}^n$, $x \mapsto [x]$, be the canonical projection, and let $a: \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map. We have $q(x) = q(-x)$ for all $x \in \mathbb{S}^n$, i.e., we have $q = q \circ a$. Both q and a are local diffeomorphisms. Hence, if $\mathbb{R}\mathrm{P}^n$ is oriented by an orientation $\mathfrak{o}_{\mathbb{R}\mathrm{P}^n}$, then the orientation on $\mathbb{S}^n$ induced by q and by $q \circ a$ must be the same. Using the functoriality of the pullback orientation, we need to have i.e., we must have

$$q^*\mathfrak{o}_{\mathbb{R}\mathrm{P}^n} = (q \circ a)^*\mathfrak{o}_{\mathbb{R}\mathrm{P}^n} = a^*(q^*\mathfrak{o}_{\mathbb{R}\mathrm{P}^n}).$$

This shows that $\mathbb{R}\mathrm{P}^n$ can be oriented only if a **preserves orientation**. By Lemma 15.10 this is the case if only if n is odd. This proves that $\mathbb{R}\mathrm{P}^n$ is **not** orientable if n is **even**.

Now we assume that n is **odd**. We will now **define an orientation** of $\mathbb{R}\mathrm{P}^n$. At every point $[x] \in \mathbb{R}\mathrm{P}^n$, the fiber under q consists of two antipodal points $q^{-1}([x]) = \{x, -x\} \subset \mathbb{S}^n$. Given a point $x \in \mathbb{S}^n$, we can use stereographic projection from a point $\neq \pm x$ on $\mathbb{S}^n$ as **one** local parametrization ϕ for both x and $-x$. Let β_x be a basis of $T_x\mathbb{S}^n$. We now **define** an orientation on $T_{[x]}\mathbb{R}\mathrm{P}^n$ by requiring that the image of β_x in $T_{[x]}\mathbb{R}\mathrm{P}^n$ under dq_x has the same sign as β_x in $T_x\mathbb{S}^n$, i.e., we set

$$\operatorname{sign}(dq_x(\beta_x)) := \operatorname{sign}(\beta_x).$$

We have to check that this definition is independent of the choice of representing $[x] \in \mathbb{R}\mathrm{P}^n$ by x and $-x$ in $\mathbb{S}^n$. By Lemma 15.10, the antipodal map $a\colon \mathbb{S}^n \to \mathbb{S}^n$, $x \mapsto -x$, preserves orientations. Considering the commutative diagram (15.1), this implies that the orientations of $T_x\mathbb{S}^n$ and $T_{-x}\mathbb{S}^n$ are such that the respective images of the standard bases of $\mathbb{R}^n$ under $d\phi_{\phi^{-1}(x)}\colon \mathbb{R}^n \to T_x\mathbb{S}^n$ and $d\phi_{\phi^{-1}(-x)}\colon \mathbb{R}^n \to T_{-x}\mathbb{S}^n$ are either **both** positively or **both** negatively oriented.

$$\begin{tikzcd} & T_x\mathbb{S}^n \arrow[dd, "da_x"] \arrow[dr, "dq_x"] & \\ \mathbb{R}^n \arrow[ur, "d\phi_{\phi^{-1}(x)}"] \arrow[dr, "d\phi_{\phi^{-1}(-x)}"'] & & T_{[x]}\mathbb{R}\mathrm{P}^n \\ & T_{-x}\mathbb{S}^n \arrow[ur, "dq_{-x}"'] & \end{tikzcd} \tag{15.1}$$

This shows that we have

$$\operatorname{sign}(a_x(\beta_x)) = \operatorname{sign}(\beta_x)$$

where $a_x(\beta_x) =: \beta_{-x}$ is a basis of $T_{-x}\mathbb{S}^n$. Thus, the orientations on $T_{[x]}\mathbb{R}\mathrm{P}^n$ determined by $T_x\mathbb{S}^n$ and $T_{-x}\mathbb{S}^n$ via dq_x and dq_{-x}, respectively, are the same. Since $\mathbb{S}^n$ is oriented as explained in Example 15.19, this defines an orientation on $\mathbb{R}\mathrm{P}^n$ for n **odd**. □

We now look at an important example of a non-orientable manifold.

Theorem 15.17 The Möbius band is **not** orientable.

Proof: Let X denote the Möbius band. Suppose that X is orientable and assume that we have chosen an orientation. We will now show that this leads to a contradiction to previously obtained results. Let $Z \subset X$ be the central circle. Then Z is orientable as a circle, and it is a submanifold of codimension one. By Exercise 15.6, this implies that the normal bundle $N(Z, X)$ is trivial. By Theorem 13.15, this implies that there is a submersion $g\colon U \to \mathbb{R}^1$ defined on an **open** subset $U \subset X$ such that $Z = g^{-1}(0)$. By Exercise 14.7, such a g does **not exist**. This shows that X cannot be equipped with an orientation. □

It is now a **long** and **technical** endeavour to check how orientations behave under the main constructions and relate to the concepts we have developed so far. We will go through them **one by one**.

15.3 Induced orientation on the boundary

Let X be an oriented smooth manifold with boundary. Then the boundary submanifold ∂X inherits an orientation as follows:

At every point $x \in \partial X$, $T_x(\partial X)$ is a subspace of **codimension one** in $T_x(X)$. Its **orthogonal complement** in $T_x(X)$, is a line which contains **exactly two unit vectors**: one is pointing **inward** into $T_x(X)$, the other one is pointing **outward** away from $T_x(X)$.

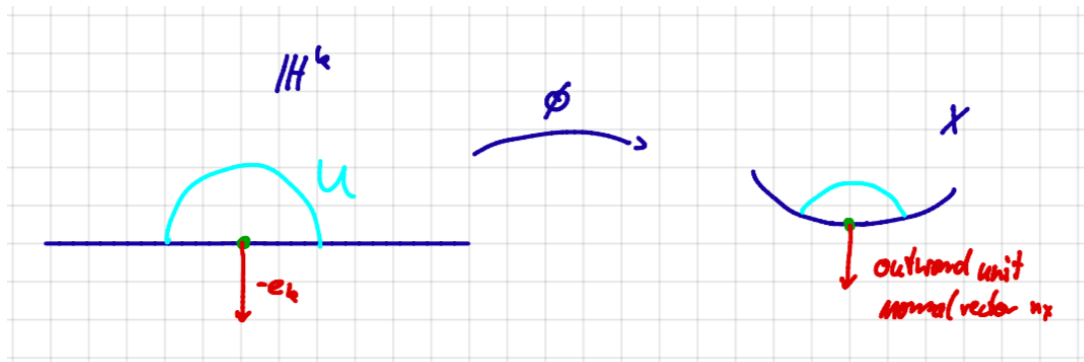


Figure 15.3: We define an outward normal vector n_x as the image $d\phi_0(-e_k)$ of the outward pointing standard basis vector $-e_k$.

This can be **made precise** by choosing a local parametrization $\phi\colon U \to X$ around x with $U \subset \mathbb{H}^k$ open and $\phi(0) = x$. The derivative $d\phi_0\colon \mathbb{R}^k \to T_x(X)$ is by definition of $T_x(X)$ an isomorphism. In $\mathbb{R}^k$, there are two unit vectors: $e_k = (0, \dots, 0, 1)$ **pointing into** $\mathbb{H}^k$, and $-e_k = (0, \dots, 0, -1)$ **pointing out** of $\mathbb{H}^k$. See Figure 15.3. Using the **Gram–Schmidt process** we can **orthonormalize the image** of e_k under $d\phi_0$ with respect to $T_x(\partial X)$ and get the inward pointing unit normal vector. The orthonormalization with respect to $T_x(\partial X)$ of $d\phi_0(-e_k)$ is the outward pointing unit normal vector.[1]

We denote the **outward pointing unit normal vector** by n_x. We checked in Exercise 10.4 that the construction of n_x does **not depend on the choice** of ϕ and that the assignment $x \mapsto n_x$ is a **smooth** map on ∂X. See Figure 15.4.

Now we are ready to **orient** $T_x(\partial X)$:

Definition 15.18 (Boundary orientation) We define the sign of an ordered basis $(v_1, \dots, v_{k-1})$ in $T_x(\partial X)$ to be the sign of the ordered basis $(n_x, v_1, \dots, v_{k-1})$ in $T_x(X)$, i.e., we set

$$\operatorname{sign}(v_1, \dots, v_{k-1}) \text{ in } T_x(\partial X) := \operatorname{sign}(n_x, v_1, \dots, v_{k-1}) \text{ in } T_x(X).$$

Since both the assignment $x \mapsto n_x$ and the choice of sign for ordered bases on $T_x(X)$ vary smoothly, this defines an orientation on ∂X which is called the **boundary orientation**.

Example 15.19 (Orientation of the sphere) The n-dimensional sphere $\mathbb{S}^n$ is an orientable smooth manifold. The standard orientation of $\mathbb{S}^n$ is given as the **boundary orientation** induced from the orientation of the unit disk $\mathbb{D}^{n+1}$ defined by the standard orientation of $T_x(\mathbb{D}^{n+1}) = \mathbb{R}^{n+1}$. See Figure 15.4.

[1] Note that the **inner product** on $T_x(X)$ is induced by the standard inner product on $\mathbb{R}^N$, where $X \subset \mathbb{R}^N$ and hence $T_x(X) \subset \mathbb{R}^N$.

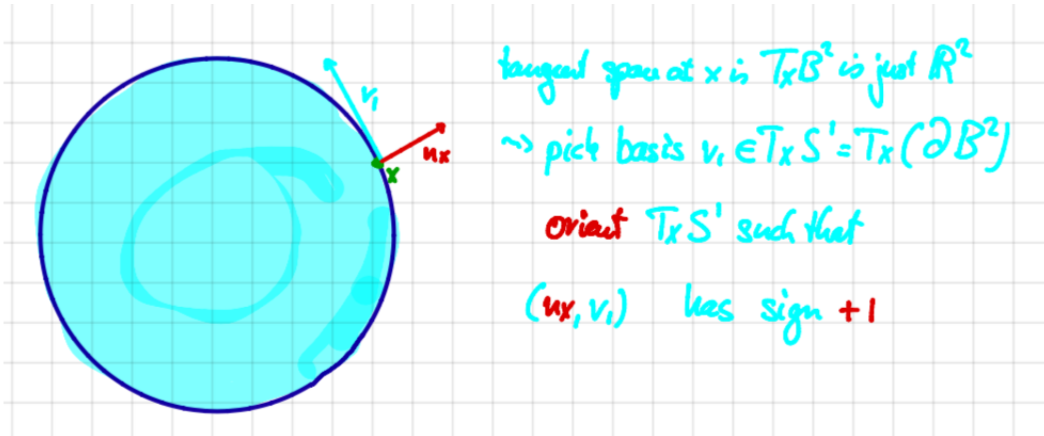


Figure 15.4: The outward pointing unit vector n_x is the vector orthogonal to the tangent line at x to $\mathbb{S}^1$ and pointing away from the origin.

Lemma 15.20 (Orientation of one-manifolds) Let us apply what we just learned to the case of a **one-manifold with boundary**. The boundary ∂X is **zero** dimensional. The orientation of the zero-dimensional vector space $T_x(\partial X)$ **equals the sign** of the basis of $T_x(X)$ consisting of the **outward-pointing unit vector n_x**.

Example 15.21 (Unit interval) Let us look at the **compact interval** $X = [0, 1]$ with its standard orientation inherited from being a subset in $\mathbb{R}$. Note that local parametrizations of $[0, 1]$ are given by

$$\phi\colon [0, 1) \to [0, 1], x \mapsto x$$

around $0 \in [0, 1]$ and

$$\psi\colon [0, 1) \to [0, 1], x \mapsto 1 - x$$

around $1 \in [0, 1]$. Hence, at $x = 1$, the **outward-pointing normal vector** is $1 \in \mathbb{R} = T_x(X)$. The **basis** consisting of this vector is **positively oriented**. At $x = 0$ the **outward-pointing normal vector** is the **negatively** oriented $-1 \in \mathbb{R} = T_0(X)$. Thus the **orientation of $T_1(\partial X)$ is $+1$**, and the **orientation of $T_0(\partial X)$ is -1**. Reversing the orientation on $[0, 1]$ simply reverses the orientations at each boundary point. Thus the sum of both orientation numbers at the boundary points of $[0, 1]$ is always zero.

Since, by Theorem 11.1, every **compact one-manifold with boundary** is diffeomorphic is the disjoint union of copies of $[0, 1]$, we conclude:

Lemma 15.22 (Boundary orientation of one-manifolds) The **sum of the orientation numbers** at the **boundary points** of any **compact oriented one-dimensional** manifold with boundary is **zero**. In particular, the **boundary** points of a **smooth path** γ on an oriented manifold X, i.e., a smooth map $\gamma\colon [0, 1] \to X$, must have **opposite orientation signs**.

> **Remark 15.23** This is an **important** observation which makes it possible to define **homotopy invariant** degree of a smooth map with values in $\mathbb{Z}$.

15.4 Oriented Homotopy

As an application of product and boundary orientations, we would like to orient the product $[0,1] \times X$ for a **boundaryless smooth oriented** manifold X which is the **domain of all homotopies** on X. This will be crucial for the homotopy invariance of the integer valued Brouwer degree in the next section.

We just learned that products and boundaries inherit orientations. For each $t \in [0,1]$, the **slice** $X_t := \{t\} \times X$ is diffeomorphic to X, and the **orientation on X_t** should be such that the **diffeomorphism**

$$X \to X_t,\ x \mapsto (t,x) \text{ **preserves orientations**.}$$

For future applications, we are particularly interested in the **orientation of the boundary**

$$\partial([0,1] \times X) = \{0\} \times X \cup \{1\} \times X.$$

So let us try to **understand the induced orientation** on the **boundary**:

We start with X_1: We see from the local parametrization ψ above that along X_1 the outward-pointing normal vector is

$$n_{(1,x)} = (1,0) = (1,0,\ldots,0) \in T_1([0,1]) \times T_x(X),$$

If $\beta = (v_1,\ldots,v_k)$ is an ordered basis of $T_x(X)$, then $0 \times \beta = ((0,v_1),\ldots,(0,v_k))$ is an ordered basis of $T_x(X_1)$. By definition of the **boundary orientation**, $(n_{(1,0)}, (0\times\beta))$ is **positively** oriented **if and only if** β is **positively** oriented. In terms of signs:

$$\operatorname{sign}(n_{(1,0)}, (0 \times \beta)) = \operatorname{sign}(\beta).$$

If we calculate the orientation induced from the product structure, then we get

$$\operatorname{sign}((1,0),(0 \times \beta)) = (+1) \cdot \operatorname{sign}(\beta) = \operatorname{sign}(\beta).$$

We learn from these two equations, that the **boundary orientation of X_1** is just the **orientation of X** as a copy in the product $[0,1] \times X$.

This sounds obvious, **but pay attention**:

We see from the local parametrization ϕ that along X_0 the outward-pointing normal vector is

$$n_{(0,x)} = (-1,0) = (-1,0,\ldots,0) \in T_0([0,1]) \times T_x(X).$$

Hence the **orientation** on $T_0([0,1])$ is **opposite** to the **standard orientation** of $\mathbb{R}$. Hence the formula for product orientations yields

$$\operatorname{sign}((-1,0), 0 \times \beta)) = \operatorname{sign}(-1) \cdot \operatorname{sign}(\beta) = -\operatorname{sign}(\beta).$$

Thus the **boundary orientation on** X_0 is the **reverse** of its **orientation as a copy of** X in the product $[0,1] \times X$.

Thus the orientation on the boundary is

$$\partial([0,1] \times X) = X_1 \cup (-X_0).$$

We will also express this fact by using the **notation**

$$\partial([0,1] \times X) = X_1 - X_0.$$

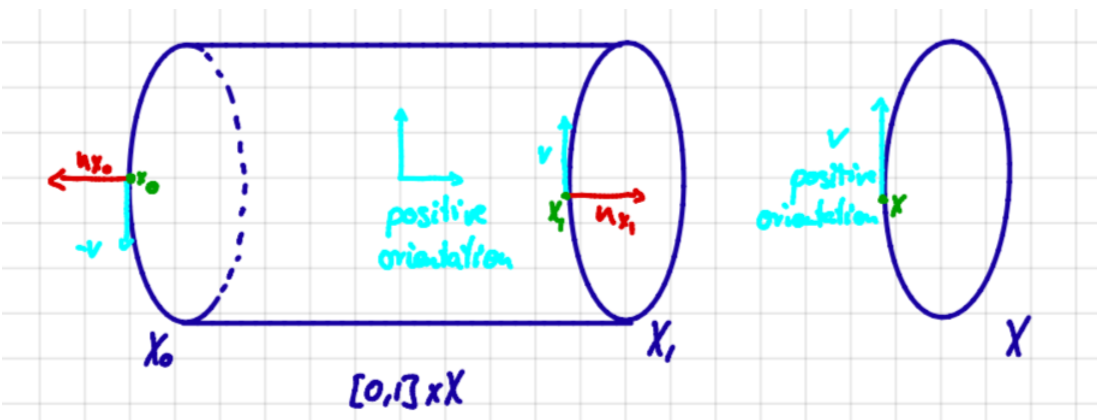


Figure 15.5: The vectors n_{x_0} and n_{x_1} point in opposite directions. This provides the two boundary circles with opposite orientations. This fact will turn out to be crucial for the definition of intersection numbers later.

15.5 Orientation of transverse preimage

Our next goal is to **orient preimages**. In order to do so, we will have to look at direct sums of vector spaces, and we need to orient those guys.

Definition 15.24 (Orientation on a direct sum of vector spaces) Suppose that $V = V_1 \oplus V_2$ is a direct sum of vector spaces. Then orientations on any two of these vector spaces automatically induces a **direct sum orientation** on the third, as follows: Choose ordered bases β_1 of V_1 and β_2 of V_2. Let $\beta = (\beta_1, \beta_2)$ be the combined ordered basis of V (in this order!). For orientations or signs to be compatible with the structure as a direct sum, we require the formula

$$\operatorname{sign}(\beta) := \operatorname{sign}(\beta_1) \cdot \operatorname{sign}(\beta_2).$$

It follows immediately from the way matrices on direct sums are put together that the above formula determines an orientation on the third space if two orientations are given.[2] Note again that the **order** of the summands V_1 and V_2 is **crucial**.

Now let $f : X \to Y$ be a smooth map and $Z \subset Y$ be a submanifold with $f \pitchfork Z$ and $\partial f \pitchfork Z$. We assume that X, Y, and Z are all **oriented** and Y and Z are without boundary. We would like to define a **preimage orientation** on the manifold with boundary $S = f^{-1}(Z)$:

If $f(x) = z \in Z$ and z is a regular value, then, by Lemma 4.9, we have

$$T_x(S) = (df_x)^{-1}(T_z(Z)) \subset T_x(X).$$

Let $N_x(S; X)$ be the **orthogonal complement** to $T_x(S)$ in $T_x(X)$. By definition, we have a direct sum decomposition

$$N_x(S; X) \oplus T_x(S) = T_x(X).$$

Hence, by Definition 15.24 on the orientation of a direct sum, we only need to **choose an orientation on $N_x(S; X)$** to obtain a **direct sum orientation** on $T_x(S)$. Since $f \pitchfork Z$, we have

$$\begin{aligned} T_z(Y) &= df_x(T_x(X)) + T_z(Z) \\ &= df_x(N_x(S; X) \oplus T_x(S)) + T_z(Z) \\ &= df_x(N_x(S; X)) \oplus T_z(Z) \end{aligned}$$

where we use $df_x(T_x(S)) = T_z(Z)$ for the last step. Thus the orientations on Z and Y induce a **direct sum orientation** on $df_x(N_x(S; X))$. It remains to show that this also induces an **orientation on $N_x(S; X)$**: We have

$$\{0\} \subset T_z(Z) \Rightarrow \operatorname{Ker}(df_x) \subset (df_x)^{-1}(T_z(Z)) = T_x(S),$$

and hence the **restriction** of df_x to $N_x(S; X)$ is in fact an **isomorphism onto its image**. Therefore, the induced orientation on $df_x(N_x(S; X))$ defines an orientation on $N_x(S; X)$ via the isomorphism df_x. Since the orientations on X, Y and Z vary smoothly and df_x also depends smoothly on x, the **induced orientation** on $T_x(S)$ varies **smoothly with x**.

We put this into a formula in the following definition:

Definition 15.25 (Preimage orientation) Let $f : X \to Y$ be a smooth map and $Z \subset Y$ be a submanifold with $f \pitchfork Z$ and $\partial f \pitchfork Z$. Let X, Y, and Z be **oriented** and Y and Z be without boundary. We choose an ordered basis β_V for the vector space V which runs through the spaces $T_x f^{-1}(Z)$, $T_x X$, $T_{f(x)} Z$, $T_{f(x)} Y$, and $N_x(f^{-1}(Z); X)$. We define the sign of $\beta_{N_x(f^{-1}(Z);X)}$, and thereby the orientation of $N_x(f^{-1}(Z); X)$, such that

$$\operatorname{sign}\left(df_x\left(\beta_{N_x(f^{-1}(Z);X)}\right)\right) \cdot \operatorname{sign}\left(\beta_{T_{f(x)}Z}\right) = \operatorname{sign}\left(\beta_{T_{f(x)}Y}\right).$$

Then we define the sign of $\beta_{T_x f^{-1}(Z)}$ and hence the orientation of $f^{-1}(Z)$ such that

$$\operatorname{sign}\left(\beta_{N_x(f^{-1}(Z);X)}\right) \cdot \operatorname{sign}\left(\beta_{T_x f^{-1}(Z)}\right) = \operatorname{sign}\left(\beta_{T_x X}\right).$$

[2] Note that this not only means that orientations on V_1 and V_2 determine an orientation on V, but also orientations on V and, say, V_2 determine an orientation on V_1.

Remark 15.26 (Orthogonality is not important) Note that we did not really use that $N_x(S; X)$ is orthogonal to $T_x(S)$. All we needed was a direct sum decomposition $H \oplus T_x(S) = T_x(X)$ with a space H with an orientation induced by the orientation of X. We will exploit this fact in the proof below.

15.6 Example: Fibers of the Hopf fibration

Recall the **Hopf fibration** π that we have seen previously: We consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, i.e., $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$, and $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e., $\mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$. Then the Hopf fibration π is the map $\mathbb{S}^3 \to \mathbb{S}^2$ given by

$$\pi(z_0, z_1) = \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right).$$

Consider the point $a = (0, 0, 1)$ on $\mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$. In a previous exercise we have determined the fiber $\pi^{-1}(a)$. Now we would like to compute its **orientation** as a **preimage** under π.

Recall that the fiber over a is

$$\pi^{-1}(a) = \{(z_0, 0) \in \mathbb{S}^3 \subset \mathbb{C}^2 : |z_0|^2 = 1\}.$$

Let $q = (x_0, y_0, 0, 0) \in \pi^{-1}(a)$ be a point in the fiber over a. The tangent space $T_q\mathbb{S}^3$ is the vector space

$$T_q\mathbb{S}^3 = \{\mathbf{u} \in \mathbb{R}^4 : \mathbf{u} \perp q\} = \operatorname{span}\left\{ q^\perp = \begin{pmatrix} -y_0 \\ x_0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_3^4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_4^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The orientation of $T_q\mathbb{S}^3$ as a **boundary** of the unit ball is such that the **outward** pointing vector q together with the basis vectors of $T_q\mathbb{S}^3$ form a **positively oriented** basis of $\mathbb{R}^4$. The **determinant** of the matrix

$$\begin{pmatrix} x_0 & -y_0 & 0 & 0 \\ y_0 & x_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which expresses the basis $(q, q^\perp, \mathbf{e}_3^4, \mathbf{e}_4^4)$ in the standard basis of $\mathbb{R}^4$ equals $x_0^2 + y_0^2 = 1 > 0$. In particular, it is **positive** and the basis $(q^\perp, \mathbf{e}_3^4, \mathbf{e}_4^4)$ is a **positively oriented** basis of $T_q\mathbb{S}^3$.

The tangent space $T_q\pi^{-1}(a)$ equals the kernel of $d\tilde{\pi}_q$, where we consider the map $\tilde{\pi} : \mathbb{R}^4 \cong \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ using the same formula as for π, i.e., $\pi = \tilde{\pi}_{|\mathbb{S}^3}$. In Exercise 4.10, we computed this map as represented by the matrix

$$d\tilde{\pi}_q = 2 \cdot \begin{pmatrix} 0 & 0 & x_0 & y_0 \\ 0 & 0 & y_0 & -x_0 \\ x_0 & y_0 & 0 & 0 \end{pmatrix}.$$

The **kernel** of this map is the **span** of the vector $q^\perp$ that we have just seen above. The **normal space** $N_q(\pi^{-1}(a); \mathbb{S}^3) \subset T_q\mathbb{S}^3$ of vectors which are orthogonal to $T_q\pi^{-1}(a)$ is the span of $(\mathbf{e}_3^4, \mathbf{e}_4^4)$. The map $d\tilde{\pi}_q$ sends $\mathbf{e}_3$ and $\mathbf{e}_4$ to, respectively,

$$d\tilde{\pi}_q(\mathbf{e}_3^4) = 2\begin{pmatrix} x_0 \\ y_0 \\ 0 \\ 0 \end{pmatrix}, \quad d\tilde{\pi}_q(\mathbf{e}_4^4) = 2\begin{pmatrix} y_0 \\ -x_0 \\ 0 \\ 0 \end{pmatrix}.$$

These two vectors form a **basis** $(d\tilde{\pi}_q(\mathbf{e}_3^3), d\tilde{\pi}_q(\mathbf{e}_4^4))$ of $T_a\mathbb{S}^2$.

We need to **check the orientation** of this basis: The tangent space $T_a\mathbb{S}^2$ has a basis $(\mathbf{e}_1^3, \mathbf{e}_2^3)$ as a subspace in $\mathbb{R}^3$. This basis is **positively oriented** since, together with the **outward** pointing vector $a = \mathbf{e}_3^3$, the basis $(\mathbf{e}_3^3, \mathbf{e}_1^3, \mathbf{e}_2^3)$ is a **positively oriented** basis of $\mathbb{R}^3$.[3] The matrix A which expresses $(d\tilde{\pi}_q(\mathbf{e}_3^3), d\tilde{\pi}_q(\mathbf{e}_4^4))$ in terms of the basis $(\mathbf{e}_1^3, \mathbf{e}_2^3)$ is given by

$$A = 2 \cdot \begin{pmatrix} x_0 & y_0 \\ y_0 & -x_0 \end{pmatrix}.$$

We see that $\det A = 4(-x_0^2 - y_0^2) = -4 < 0$ is **negative**. Hence the basis $(d\tilde{\pi}_q(\mathbf{e}_3^3), d\tilde{\pi}_q(\mathbf{e}_4^4))$ is a **negatively oriented** basis of $T_a\mathbb{S}^2$. This defines an orientation on the normal space $N_q(\pi^{-1}(a); \mathbb{S}^3)$ **by declaring** the orientation of the basis $(\mathbf{e}_3^4, \mathbf{e}_4^4)$ to be **negative**.

Finally, the **orientation of $T_q\pi^{-1}(a)$** is such that the direct sum

$$N_q(\pi^{-1}(a); \mathbb{S}^3) \oplus T_q\pi^{-1}(a) = T_q\mathbb{S}^3$$

induces the **given orientation** on $T_q\mathbb{S}^3$.

We make this explicit by looking at the basis $(\mathbf{e}_3^4, \mathbf{e}_4^4, q^\perp)$ of $N_q(\pi^{-1}(a); \mathbb{S}^3) \oplus T_q\pi^{-1}(a)$. As a basis of $T_q\mathbb{S}^3$, $(\mathbf{e}_3^4, \mathbf{e}_4^4, q^\perp)$ is **positively oriented** since it arises by **two permutations** from the positively oriented basis $(q^\perp, \mathbf{e}_3^4, \mathbf{e}_4^4)$. Since the sign of $(\mathbf{e}_3^4, \mathbf{e}_4^4)$ is **negative** as a basis of $N_q(\pi^{-1}(a); \mathbb{S}^3)$, **we need** that $q^\perp$ also has **negative sign**. Hence the vector $q^\perp$ provides a **negatively oriented** basis of $T_q\pi^{-1}(a)$. **Comparing** this orientation with the **standard orientation** of $\mathbb{S}^1 \subset \mathbb{C} \subset \mathbb{C}^2$, we see that $\pi^{-1}(a)$ has the **opposite orientation**.

We will compute the orientation of the fiber of π at another point in Exercise 15.8. You should try to solve this exercise even though it is a bit involved.

15.7 Orientation on boundary of preimage

Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ and $\partial f \pitchfork Z$, where X, Y, and Z are all oriented, Y and Z are boundaryless, and X has a boundary.

Then the manifold $\partial f^{-1}(Z)$ acquires **two orientations**:

- an orientation as the **boundary** of the manifold $f^{-1}(Z)$, and

[3] We make two permutations which lead to multiplying with $(-1)^2 = +1$.

- an orientation as the **preimage** of Z under the map $\partial f : \partial X \to Y$.

It turns out that these two orientations may not agree. However, there is a formula that relates them:

Theorem 15.27 **(Orientation on boundary of preimage)** Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ and $\partial f \pitchfork Z$, where X, Y, and Z are all oriented, Y and Z are without boundary, while X has a boundary. Then the orientation of the boundary of $f^{-1}(Z)$ satisfies the formula:

$$\partial(f^{-1}(Z)) = (-1)^{\operatorname{codim} Z}(\partial f)^{-1}(Z).$$

This means the orientations of $\partial f^{-1}(Z)$, induced by being a boundary or by being a preimage, are the **same** if codim Z is **even**, and **opposite** if codim Z is **odd**.

Proof: Denote $f^{-1}(Z)$ again by S. Let H be a subspace of $T_x(\partial X)$ complementary to $T_x(\partial S)$, i.e.,

$$H \oplus T_x(\partial S) = T_x(\partial X).$$

Note that H is also **complementary** to $T_x(S)$ in $T_x(X)$, i.e.,

$$H \oplus T_x(S) = T_x(X).$$

For we have

$$H \cap T_x(S) = \{0\} \text{ and } T_x(S) \cap T_x(\partial X) = T_x(\partial S),$$

and

$$\dim H = \dim T_x(\partial X) - \dim T_x(\partial S) = \dim T_x(X) - \dim T_x(S).$$

Hence we may use H to define the **direct sum orientation** of both S and ∂S at x.

Since $H \subset T_x(\partial X) \subset T_x(X)$, the maps df_x and $d(\partial f)_x$ agree on H, i.e.,

$$df_x(H) = d(\partial f)_x(H).$$

As in the case of $N_x(S; X)$, since

$$\{0\} \subset T_z(Z) \Rightarrow \operatorname{Ker}(df_x) \subset f^{-1}(T_z(Z)) = T_x(S),$$

the intersection $\operatorname{Ker}(df_x) \cap H$ is $\{0\}$. Hence the **restrictions** of df_x and $d(\partial f)_x$ to H are **isomorphisms** onto their common image.

Thus $f \pitchfork Z$ and $\partial f \pitchfork Z$ imply that we have two direct sum decompositions $df_x(H) \oplus T_z(Z) = T_z(Y) = d(\partial f)_x(H) \oplus T_z(Z)$, and the two orients of H via these direct sums agree.

To conclude, we obtained that H has a well-defined orientation. Hence we can use this unique orientation on H to orient

$$S \text{ via } H \oplus T_x(S) = T_x(X) \text{ and } \partial S \text{ via } H \oplus T_x(\partial S) = T_x(\partial X).$$

It remains to check how this orientation of $T_x(\partial S)$ relates to the orientation of the boundary induced from the orientation of $T_x(S)$. Let n_x be the **outward unit vector** to ∂S in $T_x(S)$, and let $\mathbb{R} \cdot n_x$ represent the **one-dimensional subspace** spanned by n_x. We **orient** this space by assigning the **sign $+1$** to the basis (n_x). Even though n_x need not be perpendicular to all of $T_x(\partial X)$, it suffices to know that n_x lies in the half-space **pointing away** from $T_x(X)$ to know that the orientations of $\mathbb{R} \cdot n_x$, $T_x(\partial X)$ and $T_x(X)$ are **related by the direct sum**

$$T_x(X) = \mathbb{R} \cdot n_x \oplus T_x(\partial X).$$

Now we use that H is **complementary to both** $T_x(S)$ in $T_x(X)$ and $T_x(\partial S)$ in $T_x(\partial X)$ and plugg this into the above direct sum to get

$$H \oplus T_x(S) = \mathbb{R} \cdot n_x \oplus H \oplus T_x(\partial S).$$

This equation is already **almost what we need**, since we would like to compare the orientations $T_x(S)$ and $\mathbb{R} \cdot n_x \oplus T_x(\partial S)$. For doing so, we need to move $\mathbb{R} \cdot n_x$ passed H. If $\dim H = m$, H has m basis vectors $(w_1, \ldots, w_m)$. Remembering the rule for orienting direct sums, this means we have to **apply exactly m transpositions** to the ordered set

$$(n_x, w_1, \ldots, w_m) \text{ to get to } (w_1, \ldots, w_m, n_x).$$

This results in m shifts of signs. Hence we get

$$H \oplus T_x(S) = (-1)^{\operatorname{codim} Z} H \oplus \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Since H appears on both sides as the **first summand**, we can disregard it for the computation and get that if ∂S is oriented as a preimage under ∂f, then its orientation relates to the one of $T_x(S)$ by

$$T_x(S) = (-1)^{\operatorname{codim} Z} \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Now, if ∂S is **oriented as a boundary**, then we have

$$T_x(S) = \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Thus

$$\text{sign}\,(\partial S) \text{ as a } \textbf{boundary} = (-1)^{\operatorname{codim} Z} \cdot \text{sign}\,(\partial S) \text{ as a } \textbf{preimage}. \quad \square$$

15.8 Example: Simply-connected manifolds are orientable

The following theorem shows that a lot of manifolds are orientable. Recall that a manifold X is called **simply-connected** if it is connected and every smooth map $\mathbb{S}^1 \to X$ is homotopic to a constant map.

Theorem 15.28 (Simply-connected implies orientable) Every simply-connected manifold is **orientable**.

Proof: We start by **picking any point** $x \in X$, and **choose an orientation** for the tangent space $T_x(X)$. Since $T_x(X)$ is a vector space, this is always possible. Now let $y \in X$ be any other point in X. Since X is simply-connected, X is connected. By a previous exercise, since X is a **smooth manifold**, X is therefore even **path-connected**. Hence there is a smooth map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. For every point in $z \in \gamma([0,1])$ we choose a local parametrization $\phi_z\colon V_z \to U_z$ around z. By **shrinking V_z if necessary**, we can assume that each V_z is an **open ball** in $\mathbb{R}^k$.

The sets $U_z \cap \gamma([0,1])$ is open in $\gamma([0,1])$, and the collection of $\{U_z \cap \gamma([0,1])\}$ for all $z \in \gamma([0,1])$ is an open covering of $\gamma([0,1])$. Since $[0,1]$ is compact and γ continuous, the image $\gamma([0,1])$ is compact. Hence **finitely many** of the U_z suffice to cover $\gamma([0,1])$. We label these open sets $U_1, \dots, U_m$ and **order them** such that $U_i \cap U_{i+1} \neq \emptyset$ and $x \in U_1$, $y \in U_m$.

For U_1, we **choose the orientation** which is **compatible** with the chosen orientation of $T_x(X)$. This means: Let $\phi_1\colon U_1 \to X$ be the associated local parametrization with $\phi_1(0) = x$.

- **If** $d(\phi_1)_0\colon \mathbb{R}^k \to T_x(X)$ is **orientation preserving**, we orient the vector space $T_a(U_1)$ such that $d(\phi_1)_{\phi_1^{-1}(z)}\colon \mathbb{R}^k \to T_a(X)$ is orientation preserving for all $a \in U_1$.
- **If** $d(\phi_1)_0\colon \mathbb{R}^k \to T_x(X)$ **reverses orientation**, we first replace ϕ_1 with $\tilde{\phi}_1\colon V_1 \to X$, $v \mapsto \phi_1(-v)$. This new map $\tilde{\phi}_1$ is also a local parametrization of X with domain V_1, since V_1 is an open ball in $\mathbb{R}^k$ and ϕ_1 is therefore symmetric with respect to the origin.

Hence after possibly replacing ϕ_1 with $\tilde{\phi}_1$, we **can assume** that $d(\phi_1)_0$ is **orientation preserving**, and we orient all $T_a(U_1)$ as above. For U_2, we choose the orientation which is compatible with the orientation of the $T_a(X)$ for all points $a \in U_1 \cap U_2$: **If** $d(\phi_2)_{\phi_2^{-1}(a)}$ is **orientation preserving** on $T_a(X)$ for $a \in U_1 \cap U_2$, we orient $T_a(X)$ such that $d(\phi_2)_{\phi_2^{-1}(a)}\colon \mathbb{R}^k \to T_a(X)$ is orientation preserving for all $a \in U_2$. **If** it is **not** orientation preserving, then we replace $\phi_2(v)$ with $\phi_2(-v)$. **Continuing** this way, we obtain an orientation for U_m and therefore **an orientation for $T_y(X)$** after **finitely many steps**. See Figure 15.6.

It remains to show that the induced orientation on $T_y(X)$ **does not depend on the choice** of γ and the U_i's. So let $\omega\colon [0,1] \to X$ be another smooth path with $\omega(0) = x$ and $\omega(1) = y$. As for γ, we choose open sets $W_1, \dots, W_l$ covering all points in $\omega([0,1])$ with $x \in W_1$ and $y \in W_l$ and $W_i \cap W_{i+1} \neq \emptyset$. Then we orient $T_y(X)$ following the same procedure using the W_i's. Arriving at y, we **do not know a priori** whether the orientation of $T_y(X)$ induced by γ and the orientation of $T_y(X)$ induced by ω agree or not. **Now we use** that X is **simply-connected**.

For, walking **first along γ** and then **back on ω** defines, after readjusting the speed and smoothing things out, a **loop** $\alpha\colon [0,1] \to X$ with $\alpha(0) = x = \alpha(1)$, i.e., a smooth map $\alpha\colon \mathbb{S}^1 \to X$. Walking along α, we obtain an **isomorphism**

$$J(\alpha)\colon T_x(X) = T_{\alpha(0)}(X) \xrightarrow{\cong} T_{\alpha(1)}(X) = T_x(X)$$

by composing

$$T_x(X) \xrightarrow{d(\phi_1)_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\phi_2)_\bullet} T_z(X) \xrightarrow{d(\phi_2)_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\phi_2)_\bullet} \cdots \xrightarrow{d(\psi_{m-1})_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\psi_m)_\bullet} T_x(X)$$

where the subscript $\bullet$ stands for the varying points at which we take derivatives.

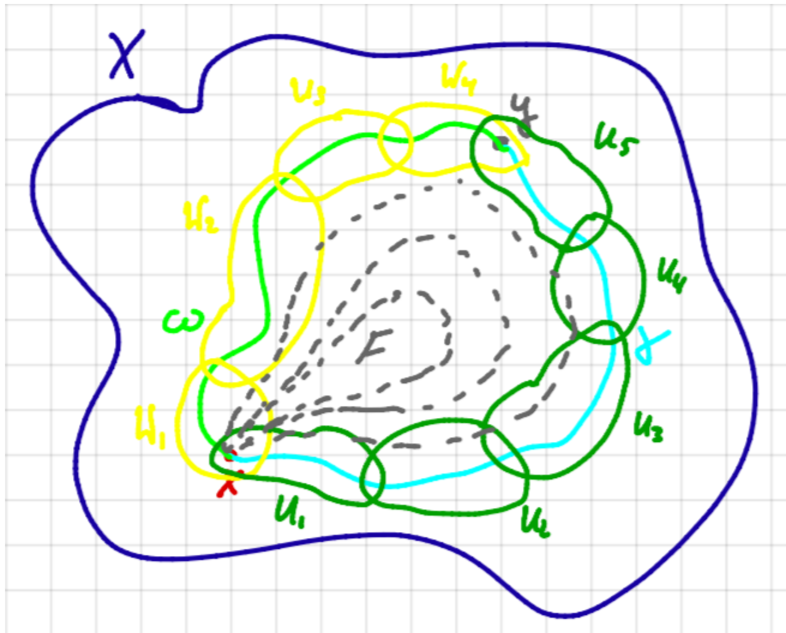


Figure 15.6: We choose a path from x to y and open sets which overlap the path. Then we orient the tangent spaces such that everything is compatible on overlaps. Simply-connectedness will make sure that the choice of path and open sets did not matter, since we can shrink any path to a point.

The **isomorphism** $J(\alpha)$ is **either** **orientation preserving** **or** **reversing**. If it **preserves** the orientation, then its determinant is **positive**. If it **reverses** the orientation, then its determinant is **negative**. And $J(\alpha)$ is **orientation preserving** if and only if the **two orientations** on $T_y(X)$ induced by γ and ω, respectively, **agree**.

Since X is **simply-connected**, α is homotopic to the constant map $c_x : \mathbb{S}^1 \to \{x\}$. Let $F : \mathbb{S}^1 \times [0,1] \to X$ be a homotopy from α to c_x. Since $\mathbb{S}^1 \times [0,1]$ is **compact**, its image in X is compact and we can add **finitely many** open subsets to the collection $U_1, \ldots, U_m, W_1, \ldots, W_l$ to cover $F(\mathbb{S}^1 \times [0,1])$ with the codomains of local parametrizations.

For each $t \in [0,1]$, $F(-,t)$ defines a **smooth loop** from x to x. Using the above procedure for orienting tangent spaces along a path, we obtain an isomorphism

$$J(F(-,t)) \colon T_x(X) = T_{F(0,t)}(X) \xrightarrow{\cong} T_{F(1,t)}(X) = T_x(X) \text{ for each } t \in [0,1].$$

Taking the **determinant** of $J(F(-,t))$ defines a **map**

$$[0,1] \to \mathbb{R},\ t \mapsto \det(J(F(-,t)))$$

which is **continuous**, since **each point** of X is contained an **open neighborhood** on which the **orientation is determined by the derivatives** of local parametrizations, and these **derivatives vary smoothly** with the base-points.

Since each $J(F(-,t))$ is an isomorphism, its determinant is either strictly positive > 0 or strictly negative < 0. Since $[0,1]$ is **connected** and $t \mapsto \det(J(F(-,t)))$ is **continuous**, we have

$$\textbf{either } \det(J(F(-,t))) > 0 \textbf{ or } \det(J(F(-,t))) < 0 \textbf{ for all } t \in [0,1].$$

But we know that, for $t = 1$, $F(-, 1) = c_x$ is the **constant loop at** x. Thus

$$\det(J(F(-, 1))) = \det(\mathrm{Id}_{T_x(X)}) > 0.$$

Hence we must have $\det(J(F(-, t))) > 0$ **for all** $t \in [0, 1]$. In other words, $J(F(-, t))$ must be orientation preserving for all t, and in particular, $J(\alpha)$ is **orientation preserving**. This shows that the orientation of $T_y(X)$ does not depend on the choice of γ. □

15.9 Summary

Let us summarise the key points we should remember from this chapter:

- An **orientation** of a vector space is a **choice of a sign**, $+1$ or -1, for an **equivalence of orderings of a bases**. We can think of it as choosing a positive and negative direction.
- An **orientation on a manifold** is a **smooth choice of orientations** of the **tangent spaces** for each point. Such a smooth choice **may or may not exist**. Hence manifolds can be orientable or not.
- Orientations help us **classifying manifolds**: there is a box with orientable and a box with non-orientable manifolds.
- For any compact oriented **one-dimensional** manifold with **boundary**, the **sum** of the orientation numbers at the boundary points is **zero**.
- The boundary of a **cylinder** has opposite orientations:

  $$\partial([0, 1] \times X) = X_1 - X_0.$$

  This is the key point for defining **homotopy invariant intersection numbers** soon.
- There is a formula for the boundary of preimages:

  $$\begin{aligned} &\operatorname{sign}(\partial f^{-1}(Z)) \text{ as a boundary} \\ &= (-1)^{\operatorname{codim} Z} \cdot \operatorname{sign}(\partial f^{-1}(Z)) \text{ as a preimage.} \end{aligned}$$

- An important class of orientable manifolds consists of simply-connected manifolds.

15.10 Exercises and more examples

Exercise 15.1 Let $\beta = (v_1, \dots, v_k)$ be an ordered basis of a vector space V.

(a) Show that replacing one v_i by a multiple cv_i yields an equivalently oriented ordered basis if $c > 0$, and an oppositely oriented one if $c < 0$.

(b) Show that transposing two elements, i.e., interchanging the places of v_i and v_j for $i \neq j$, yields an oppositely oriented ordered basis.

(c) Show that subtracting from one v_i a linear combination of the others yields an equivalently oriented ordered basis.

(d) Suppose that V is the direct sum of V_1 and V_2. Show that the direct sum orientation of V from $V_1 \oplus V_2$ equals $(-1)^{(\dim V_1)(\dim V_2)}$ times the orientation from $V_2 \oplus V_1$.

Exercise 15.2 The upper half space $\mathbb{H}^k$ is oriented by the standard orientation of $\mathbb{R}^k$. Thus $\partial \mathbb{H}^k$ acquires a boundary orientation. But $\partial \mathbb{H}^k$ may also be identified with $\mathbb{R}^{k-1}$. Show that the boundary orientation agrees with the standard orientation of $\mathbb{R}^{k-1}$ if and only if k is even.

Exercise 15.3 In this exercise we study the orientation on spheres:

(a) Write down the orientation of $\mathbb{S}^2$ as the boundary of the closed unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$, by specifying a positively oriented ordered basis for the tangent space at each $(a, b, c) \in \mathbb{S}^2$.

(b) Show that the boundary orientation of $\mathbb{S}^k$ equals the orientation of $\mathbb{S}^k = g^{-1}(1)$ as the preimage under the map

$$g\colon \mathbb{R}^{k+1} \to \mathbb{R},\ x \mapsto |x|^2.$$

Exercise 15.4 Suppose that $f\colon X \to Y$ is a diffeomorphism of connected oriented manifolds with boundary. Show that if $df_x\colon T_x(X) \to T_{f(x)}(Y)$ preserves orientation at one point x, then f preserves orientation at every point.

Exercise 15.5 Let X and Z be transversal submanifolds in Y and assume X, Z and Y are oriented. Let $i\colon X \hookrightarrow Y$ be the inclusion of X into Y, $j\colon Z \hookrightarrow Y$ be the inclusion of Z into Y. We orient the intersection $X \cap Z$ as the preimage $i^{-1}(Z)$, and the intersection $Z \cap X$ as the preimage $j^{-1}(X)$. Show that the orientations of $X \cap Z$ and $Z \cap X$ are related by

$$X \cap Z = (-1)^{(\operatorname{codim} X)(\operatorname{codim} Z)} Z \cap X.$$

Hint: Show that the orientation of $S = X \cap Z$ at any y is induced by the direct sum

$$(N_y(S, X) \oplus N_y(S, Z)) \oplus T_y(S) = T_y(Y).$$

What happens when you consider $Z \cap X$ instead?

Exercise 15.6 Let X be an oriented manifold and let $Z \subset X$ be a submanifold of codimension one. Show that Z is orientable if and only if the relative normal bundle $N(Z, X)$ is trivial.

Exercise 15.7 **(a)** Let V be a vector space. Show that both orientations on V define the same product orientation on $V \times V$.

(b) Let X be an orientable manifold. Show that the product orientation on $X \times X$ is the same for all choices of orientation on X.

(c) Suppose that X is not orientable. Show that $X \times Y$ is never orientable, no matter what manifold Y may be. In particular, $X \times X$ is not orientable.

Hint: First show that $X \times \mathbb{R}^m$ is not orientable, and then use that every Y has an open subset diffeomorphic to $\mathbb{R}^m$.

(d) Prove that there exists a natural orientation on some neighborhood of the diagonal Δ in $X \times X$, whether or not X can be oriented.

But note that Δ itself is orientable if and only if $X \times X$ is orientable. Why?

Hint: Cover a neighborhood of Δ by local parametrizations $\phi \times \phi \colon U \times U \to X \times X$, where $\phi \colon U \to X$ is a local parametrization of X, then apply the previous observations.

Exercise 15.8 Recall the *Hopf fibration* π that we have seen previously: We consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, i.e., $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$, and $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e., $\mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$. Then the *Hopf fibration* π is the map $\mathbb{S}^3 \to \mathbb{S}^2$ given by

$$\pi(z_0, z_1) = \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right).$$

Consider the point $b = (0, 1, 0)$ on $\mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$. Recall that that computed the fiber $\pi^{-1}(b)$ in a previous exercise. Determine the orientation of the fiber $\pi^{-1}(b)$ as a preimage under π.

16. The integer-valued Brouwer Degree

We have seen in Section 12.1 that the mod 2 degree is a powerful invariant of smooth maps between manifolds. Now we remedy the defect that it was merely an element in $\mathbb{Z}/2$ and define an integer-valued degree. We will then study several applications.

16.1 A well-defined invariant

Let X and Y be **oriented** n-dimensional smooth manifolds without boundary. Let $f : X \to Y$ be a smooth map. We assume that X is **compact** and that Y is **connected**.

Then the degree of f is defined as follows: Let $x \in X$ be a **regular point** of f. With our assumptions this means that $df_x : T_x X \to T_{f(x)} Y$ is a linear **isomorphism** between **oriented** vector spaces. The **sign** of df_x **is defined** to be $+1$ if df_x preserves orientations and it is defined to be -1 if df_x reverses orientations. See Figure 16.1. Now, for a **regular value** $y \in Y$ of f, we **define** the integer

$$\deg(f; y) := \sum_{x \in f^{-1}(y)} \operatorname{sign} df_x.$$

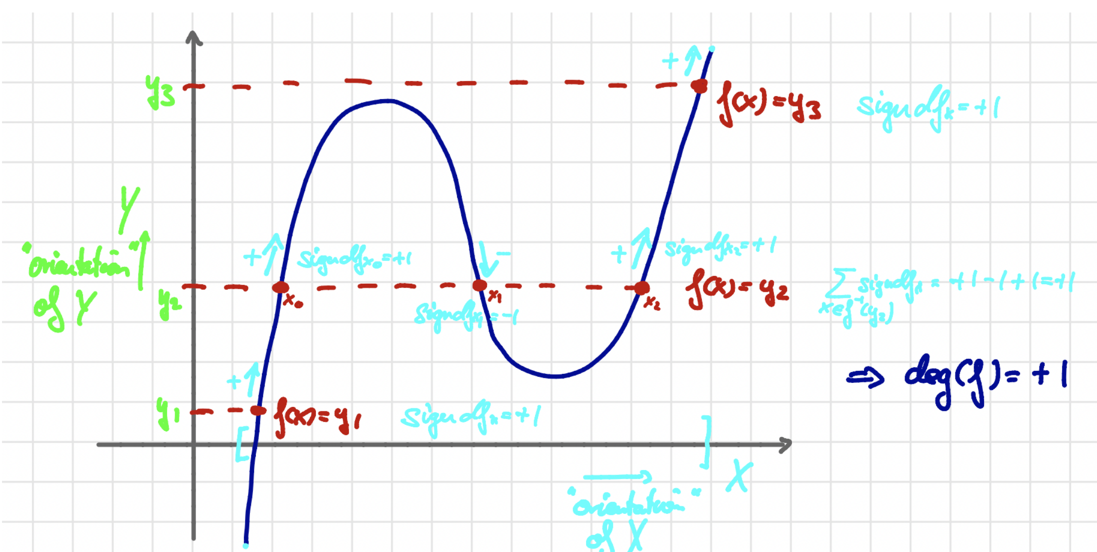


Figure 16.1: We calculate the degree by counting the number of points in the fiber of a regular value. The number is well-defined if we take signs determined by orientation into account. In dimension one, this boils down to looking at the sign of the derivative or by checking whether the graph hits the horizontal line corresponding to a value from below or above.

In the following we keep the above assumptions.

Lemma 16.1 **(deg is locally constant)** The function $Y \to \mathbb{Z}$, $y \mapsto \deg(f; y)$ is locally constant on the set of regular values of f.

Proof: Let y be a regular values of f. By the **Stack of Records Theorem 4.18**, we can find a neighborhood U of y such that the preimage $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$, where each V_i is an open set in X mapped by f diffeomorphically onto U. Hence, for all points $z \in U$, we have $\#f^{-1}(\{z\}) = n$. It remains to take **orientations** into account. Since $f_{|V_i} : V_i \to U$ is a diffeomorphism, we know that

$$df_{x_i} : T_{x_i}(X) \to T_y(Y)$$

is an isomorphism. Now both $T_{x_i}(X)$ and $T_y(Y)$ are oriented, and hence df_{x_i} is either orientation preserving or reversing. But by our definition of orientations on manifolds, we have

- **either** $\det(df_{x_i}) > 0$ and hence, for all $z \in U$, $\det(df_{w_i}) > 0$, where w_i is the unique point in V_i with $f(w_i) = z$; in other words, df_{w_i} **preserves orientations for all points** $w_i \in V_i$;
- **or** $\det(df_{x_i}) < 0$ and hence, for all $z \in U$, $\det(df_{w_i}) < 0$, where w_i is the unique point in V_i with $f(w_i) = z$; in other words, df_{w_i} **reverses orientations for all points** $w_i \in V_i$.

Thus the orientation number is the same for all points in V_i. Hence the **sum of orientation numbers** of the points in $f^{-1}(z)$ **is the same** for all points $z \in U$. Consequently, the function

$$Y \to \mathbb{Z}, \ y \mapsto \deg(f; y)$$

is **locally constant** on the subspace of regular values. □

Now we show a very useful theorem about the degree which generalizes Theorem 12.10.

Theorem 16.2 **(Boundary Theorem for deg)** Assume that $X = \partial W$ is the boundary of a **compact** oriented manifold W and that X is **oriented as the boundary** of W. If $f : X \to Y$ can be **extended** to a smooth map $F : W \to Y$, then $\deg(f; y) = 0$ for every regular value y of f.

Proof: First we suppose that y is a **regular value** for **both** F and f. By Theorem 10.16, $F^{-1}(y)$ is a **compact** submanifold of W of dimension $\dim W - \dim Y = 1$ **with boundary** given by

$$\partial F^{-1}(y) = \partial W \cap F^{-1}(y) = X \cap (F_{|X})^{-1}(y) = f^{-1}(y).$$

We also see that **only** the **boundary points** of $F^{-1}(y)$ lie on X. By Theorem 11.1, the one-dimensional manifold $F^{-1}(y)$ is the disjoint union of **finitely** many connected components which are diffeomorphic to either $\mathbb{S}^1$ or $[0, 1]$. Since $\mathbb{S}^1$ does not have boundary points, only the boundary points of the components diffeomorphic to $[0, 1]$ lie on $X = \partial W$. Let A be one such component diffeomorphic to $[0,1]$ and let a and b denote the boundary points of A, i.e., $\partial A = \{a\} \cup \{b\} \subset X$. We will show that

$$\operatorname{sign} df_a + \operatorname{sign} df_b = 0. \tag{16.1}$$

Since $f^{-1}(y) = (F_{|\partial W})^{-1}(y)$ consists of finitely many boundary points, this implies that the sum of the signs of all df_x for $x \in f^{-1}(y)$ is zero as claimed.

To show Equation 16.1 we look at how the tangent spaces at the boundary points of A are oriented. So let $p \in A$ and let $(v_1, v_2, \dots, v_n, v)$ be a **positively oriented** basis for T_pW such that v is tangent to the one-dimensional submanifold A, i.e., v forms a basis of the one-dimensional vector subspace $T_pA \subset T_pW$, while $(v_1, \dots, v_n)$ is an ordered basis of the **normal space** $N_p(A;W) \subset T_pW$. Since we have a **direct sum** decomposition $N_p(A;W) \oplus T_pA = T_pW$, we get the equation

$$\operatorname{sign}(v_1, \dots, v_n) \cdot \operatorname{sign}(v) = \operatorname{sign}(v_1, v_2, \dots, v_n, v) = +1$$

Thus we have

$$\operatorname{sign}(v) = +1 \text{ in } T_pA \iff \operatorname{sign}(v_1, \dots, v_n) = +1 \text{ in } N_p(A;W).$$

Since A is a component of $F^{-1}(y)$, it inherits an orientation as a **preimage** as explained in Section 15.5. This means that $\operatorname{sign}(v_1, \dots, v_n)$ in $N_p(A;W)$ is **determined** by the effect of dF_p on $(v_1, \dots, v_n)$. As in Definition 15.25 we have

$$\operatorname{sign}(v_1, \dots, v_n) = \operatorname{sign}(dF_p(v_1, \dots, v_n))$$

where the right-hand side is determined by the given orientation of T_yY. Thus, we conclude that v is a **positively oriented** basis of T_pA **if and only if** dF_p sends the basis $(v_1, \dots, v_n)$ to a **positively oriented** basis in T_yY.

The vector $v(p)$ depends smoothly of p. Now let $\varphi\colon [0,1] \to A$ be a **diffeomorphism** with $\varphi(0) = a$ and $\varphi(1) = b$. Since 0 and 1 have opposite orientations in $[0,1]$, the points a and b have **opposite** orientations, i.e., we have

$$\operatorname{sign} v(a) + \operatorname{sign} v(b) = 0.$$

Now, at the boundary points $p \in X$, dF_x restricts to df_x. We just learned that the sign of $v(x)$ is **positive** if and **only if** $\operatorname{sign} df_x = +1$, i.e., $\operatorname{sign} v(x) = \operatorname{sign} df_x$ at boundary points. Hence we conclude

$$\operatorname{sign} df_a + \operatorname{sign} df_b = 0.$$

Summing over all arcs in $F^{-1}(y)$, we have proved the assertion when y is a **regular value** for F.

Now suppose that y_0 is a **regular value** for f, but **not** for F. The function $y \mapsto \deg(f;y)$ is constant within some open neighborhood U of y_0 by Lemma 16.1. By Sard's Theorem 7.1 the open subset U contains a regular value for F. Hence we can choose a regular value y for F within U and get

$$\deg(f;y_0) = \deg(f;y).$$

The previous case then implies the assertion of the theorem. □

Lemma 16.3 (Homotopy Lemma for deg) Let $f, g : X \to Y$ be two smooth maps. Assume f and g are smoothly **homotopic**. If $y \in Y$ is a **regular value** for **both** f and g, then

$$f^{-1}(y) = g^{-1}(y).$$

Proof: Let $F : [0,1] \times X \to Y$ be a smooth homotopy between $F_0 = f$ and $F_1 = g$. Recall that the boundary of $[0,1] \times X$ is given by $\{0\} \times X \cup \{1\} \times X$ where the orientation of $\{1\} \times X$ is the one of X and the orientation of $\{0\} \times X$ is the opposite one. Since the degree is additive on connected components, we deduce that the degree at y of the restriction $\partial F := F_{|\partial([0,1]\times X} : \{0\} \times X \cup \{1\} \times X \to Y$ of F to the boundary of $[0,1] \times X$ is equal to the difference

$$\deg(g; y) - \deg(f; y).$$

Since ∂F can be extended to the compact oriented manifold $[0,1] \times X$, this difference must be zero by Theorem 16.2. □

Now we can prove the following key result:

Theorem 16.4 (The Brouwer Degree is well-defined) Let X and Y be **oriented** n-dimensional smooth manifolds without boundary, with X **compact** and Y **connected**. Let $f : X \to Y$ be a smooth map. The number $\deg(f; y)$ does **not depend** on the choice of the regular value y, and we denote it by $\deg f$. Moreover, $\deg f$ only depends on the homotopy class of f, i.e., if f_0 and f_1 are smoothly homotopic then $\deg f_0 = \deg f_1$.

Proof: Let y and z be two regular values of f. By the **Isotopy Lemma 12.3** we can choose a diffeomorphism $h : X \to Y$ which isotopic to the identity and with $h(y) = z$. Every diffeomorphism either preserves or reverses the orientation at every point. Hence we know sign $dh_x = \pm 1$ for all x. Since h is **isotopic** to the identity, we must have sign $dh_x = +1$ for all x. Thus h preserves orientation, and we get

$$\deg(f; y) = \deg(h \circ f; h(y)).$$

Since h is homotopic to the identity, f is homotopic to $h \circ f$. Hence

$$\deg(h \circ f; z) = \deg(f; z)$$

by Lemma 16.3. Thus we get $\deg(f; y) = \deg(f; z)$.

Now let $f_1 : X \to Y$ be a smooth map which is smoothly homotopic to $f_0 = f$. By Sard's Theorem 7.1 we can choose a $y \in Y$ which is regular value for both f_0 and f_1. By the first assertion, we can use y to calculate $\deg(f_0) = \deg(f_0; y)$ and $\deg(f_1) = \deg(f_1; y)$. Then Lemma 16.3 implies that $\deg(f_0) = \deg(f_1)$. □

In the exercises we are going to show that the degree is multiplicative in the following sense:

Lemma 16.5 (The degree is multiplicative) Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps between manifolds with X and Y compact, Y and Z connected. Assume that all three manifolds are oriented, without boundary and that $\dim X = \dim Y = \dim Z$. Then we have

$$\deg(g \circ f) = \deg(g) \cdot \deg(f).$$

Proof: This is Exercise 16.9. □

As in Remark 12.7, we observe that there is no contradiction with our observations in Section 4.4:

Remark 16.6 (Still no contradiction to previous observations) We emphasise again that the fact that $\deg(f)$ is well-defined is no contradiction to our observation in Remark 4.23. We proved that the sum of preimage points **counted with orientation signs** is constant. The number $\# f^{-1}(y)$ itself, however, given by counting the finite number of points in the fiber without signs may vary.

16.2 Examples

In all examples we keep the assumption that X and Y are smooth manifolds without boundary, X is compact, Y is connected and $\dim X = \dim Y$.

Example 16.7 (Constant maps have degree zero) Let $f : X \to Y$ be a constant map with value y_0. Then $\deg(f) = 0$. For, any value $y \neq y_0$ is a regular value and $f^{-1}(y) = \emptyset$.

Example 16.8 (Degree of a diffeomorphism) Let $f : X \to Y$ be a diffeomorphism. Then we have $\# f^{-1}(y) = 1$ for every $y \in Y$, and $\deg(f) = +1$ if f **preserves** the orientation and $\deg(f) = -1$ if f **reverses** the orientation.

In particular, we get:

Lemma 16.9 (Obstruction to a homotopy to the identity) An **orientation reversing diffeomorphism** of a compact boundaryless manifold is **not** smoothly homotopic to the identity.

Example 16.10 (Reflection on $\mathbb{S}^n$) An example of an **orientation reversing** diffeomorphism is provided by the **reflection** $r_i : \mathbb{S}^n \to \mathbb{S}^n$ which we have seen in the exercises before:

$$r_i(x_1, \dots, x_{n+1}) = (x_1, \dots, -x_i, \dots, x_{n+1}).$$

By Example 15.9 and Example 16.8 we have $\deg(r_i) = -1$.

Example 16.11 (Degree of self-maps of $\mathbb{S}^1$) Recall that the restriction of complex multiplication $z \to z^m$ defines a smooth map $f_m : \mathbb{S}^1 \to \mathbb{S}^1$ for every $m \in \mathbb{Z}$. For $m \neq 0$, let us calculate the derivative $d(f_m)_z : T_z(\mathbb{S}^1) \to T_{f_m(z)}(\mathbb{S}^1)$. We use the parametrization $\phi : t \mapsto (\cos t, \sin t)$. We have the commutative diagram

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "f_m"] & \mathbb{S}^1 \\
\mathbb{R} \arrow[u, "\phi"] \arrow[r, "t \mapsto mt"'] & \mathbb{R}. \arrow[u, "\phi"']
\end{tikzcd}$$

Taking derivatives yields, where we note that $t \mapsto mt$ is a linear map and therefore equal to its derivative:

$$\begin{tikzcd}
T_z(\mathbb{S}^1) \arrow[r, dotted, "d(f_m)_z"] & T_{z^m}(\mathbb{S}^1) \\
\mathbb{R} \arrow[u, "d\phi_t"] \arrow[r, "t \mapsto mt"'] & \mathbb{R}. \arrow[u, "d\phi_{mt}"']
\end{tikzcd}$$

In order to determine $d(f_m)_z$, recall that $d\phi_t$ has the form

$$d\phi_t : \mathbb{R} \to \mathbb{R}^2, \ s \mapsto (-\sin t, \cos t) \cdot s$$

and hence at $z = \phi(t)$:

$$T_z(\mathbb{S}^1) = (-\sin t, \cos t) \cdot \mathbb{R}.$$

Putting these information together we obtain

$$\begin{aligned}
&d(f_m)_z : T_z(\mathbb{S}^1) \to T_{z^m}(\mathbb{S}^1), \\
&(-\sin t, \cos t) \cdot s \mapsto m \cdot (-\sin(mt), \cos(mt)) \cdot s
\end{aligned}$$

which means that $d(f_m)_z$ is the **linear** map given by **multiplication** by m. Hence, when $m > 0$, f_m wraps the circle uniformly around itself m times preserving orientation. The map is **everywhere regular** and **orientation preserving**, so its **degree is the number of preimages** of any point. And that number is m. Similarly, when $m < 0$ the map is everywhere regular but orientation reversing. As each point has $|m|$ preimages, the degree is $-|m| = m$. Finally, when $m = 0$ the map is constant, so its degree is **zero**.

Remark 16.12 (One homotopy class $\mathbb{S}^1 \to \mathbb{S}^1$ for each integer) One immediate consequence of this calculation (which could not have been proven with the mod 2 degree) is the interesting fact that the circle admits an **infinite** number of **homotopically distinct maps** since $\deg(z^m) = m$ implies that f_n and f_m are **not** homotopic if $n \neq m$. We provide a more complete picture of the self-maps of $\mathbb{S}^1$ in Theorem 16.19.

Example 16.13 (Self-maps of $\mathbb{S}^2$ and complex polynomials) Let $p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$ be a monic polynomial with complex coefficients $a_1, \ldots, a_m$. We may consider p as a smooth map $\mathbb{C} \to \mathbb{C}$, and as in Equation 4.3, it induces a smooth map $f : \mathbb{S}^2 \to \mathbb{S}^2$ which sends ∞ to ∞, i.e., f sends the north pole to the north pole. We claim that the map f has **degree** m.

To justify our claim we first observe that the homotopy

$$H(z,t) = tz^m + (1-t)(a_1 z^{m-1} + \cdots + a_m)$$

induces a **smooth homotopy** $F : \mathbb{S}^2 \times [0,1] \to \mathbb{S}^2$ between the maps induced by p and the monomial z^m. The invariance of the degree under homotopy then shows that we can assume that $p(z) = z^m$ is a monomial. Then the construction of f and the computation of Example 16.11 yield the claim. We recommend to look at Exercise 16.6, Exercise 16.7 and Exercise 16.8 which study this situation further. Hopf's Degree Theorem 17.1 yields a more general statement.

Theorem 16.14 (Self-maps of $\mathbb{S}^k$) Let $k \geq 1$ and $m \in \mathbb{Z}$ be an integer. Then there is a smooth map $f_m : \mathbb{S}^k \to \mathbb{S}^k$ with $\deg(f_m) = m$.

Proof: For $m = 0$ it suffices to take any constant map. For $k = 1$ and $m \in \mathbb{Z}$, we proved the assertion in Example 16.11. We can generalize this case as follows. First we assume that m is **positive**, i.e., we assume $m \geq 1$. We can describe the coordinates of any point in $\mathbb{S}^k$ as the tuple $\left(x, \sqrt{1-|x|^2}\cos t, \sqrt{1-|x|^2}\sin t\right)$ where $x = (x_1, \ldots, x_{k-1})$ denotes the first $k-1$ coordinates and $|x|^2 = x_1^2 + \cdots x_{k-1}^2$. We define the smooth map $f_m : \mathbb{S}^k \to \mathbb{S}^k$ by

$$\left(x, \sqrt{1-|x|^2}\cos t, \sqrt{1-|x|^2}\sin t\right) \mapsto \left(x, \sqrt{1-|x|^2}\cos(mt), \sqrt{1-|x|^2}\sin(mt)\right).$$

The map f_m is **surjective** and by Sard's Theorem 7.1 we can find a **regular value** $p \in \mathbb{S}^k$. The preimage of p consists of exactly m points. It remains to check the sign of the orientation at each preimage point $q \in f^{-1}(p)$. With respect to the bases of $T_q\mathbb{S}^k$ and $T_{f_m(q)}\mathbb{S}^k$ that are induced by the standard basis in $\mathbb{R}^k$, the derivative $d(f_m)_q$ at q is given by the $(k \times k)$-matrix which consists of the $(k-1 \times k-1)$-identity matrix and the integer m in the bottom right-hand corner. This matrix has determinant $m > 0$. Thus, the orientation is preserved at each point in $q \in f^{-1}(p)$, and we get $\deg(f_m) = m$.

Second we assume that $m < 0$ is **negative**. Then we may consider the composition $f_{|m|} \circ r_1$ where r_1 denotes the **reflection** in the first coordinate. Since the degree is multiplicative by Lemma 16.5, this map has degree m by the first case and Example 16.10. □

Remark 16.15 (Homotopy groups of spheres - surjectivity) Let $[\mathbb{S}^k, \mathbb{S}^k]$ denote the set of smooth maps $\mathbb{S}^k \to \mathbb{S}^k$ modulo homotopy. By Theorem 16.4 we can think of the degree as a map

$$\deg : [\mathbb{S}^k, \mathbb{S}^k] \longrightarrow \mathbb{Z}.$$

By Theorem 16.14, this map is **surjective**. We will later prove Hopf's Theorem 17.1 which tells us that this map is **injective** as well.

In the exercises we are going ro prove the following two applications of the degree:

Theorem 16.16 (Fixed points of self-maps of the sphere) Let $f : \mathbb{S}^k \to \mathbb{S}^k$ be a smooth map. Assume that $\deg(f) \neq (-1)^{k+1}$. Then f must have a **fixed point**.

Proof: We prove the assertion in Exercise 16.10. □

Theorem 16.17 (Self-map of the sphere of odd degree) Let $f : \mathbb{S}^k \to \mathbb{S}^k$ be a smooth map. Assume $\deg(f)$ is **odd**. Then f must send **some** pair of **antipodal points to antipodal points**, i.e., there is **at least one pair** of antipodal points $x_0, -x_0$ such that $f(-x_0) = -f(x_0)$.

Proof: We prove the assertion in Exercise 16.12. □

- Note that Theorem 16.16 does **not** follow from the Borsuk–Ulam Theorem 12.15, but rather is a **partial converse**.

Example 16.18 (Self-maps of the torus) Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the two-dimensional torus. We have seen that we also can view it as a quotient of a square where we identify opposite sides. Yet another presentation of $\mathbb{T}^2$ is as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ where two points $p, q \in \mathbb{R}^2$ are identified if $p - q \in \mathbb{Z}^2$. We equip this space with the quotient topology induced by the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. Now let n be a fixed positive integer. We consider the two **linear maps** $\mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrices

$$A = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$

Since A and B send $\mathbb{Z}^2$ to $\mathbb{Z}^2$, they induce well-defined maps

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2.$$

We denote the induced maps on $\mathbb{T}^2$ by the same symbols A and B respectively. The effect of A on $\mathbb{T}^2$ is to **revolve n times** around the circle $\mathbb{S}^1 \times \{y\}$ in $\mathbb{T}^2$, for some $y \in \mathbb{S}^1$. while the effect of B on $\mathbb{T}^2$ is to revolve n times around the circle $\{x\} \times \mathbb{S}^1$ in $\mathbb{T}^2$, for some $x \in \mathbb{S}^1$, Hence, for any point $q \in \mathbb{T}^2$, the **preimages** $A^{-1}(q)$ and $B^{-1}(q)$ consist of **exactly n points**.

The derivative dA_p of A at any point $p \in \mathbb{T}^2$ equals the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ represented by A, i.e., $dA_p = A$, and similarly $dB_p = B$. In particular, we get

$$\det(dA_p) = \det(A) = n = \det(B) = \det(dB_p).$$

Since $n > 0$, the determinant of dA_p and dB_p is **positive** at every point p which is sent to q. Thus we can conclude

$$\deg(A) = n = \deg(B).$$

16.3 Hopf Degree Theorem in dimension one

We return to self-maps of $\mathbb{S}^1$. We learned that there is a homotopy class of maps $\mathbb{S}^1 \to \mathbb{S}^1$ for every integer m. Actually, the following theorem, the one-dimensional case of a famous theorem of Hopf, shows that the degree is a **bijective map**

$$\deg : [\mathbb{S}^1, \mathbb{S}^1] \to \mathbb{Z},\ f \mapsto \deg(f),$$

where $[\mathbb{S}^1, \mathbb{S}^1] = \mathrm{Hom}(\mathbb{S}^1, \mathbb{S}^1)/\sim$ denotes the set of equivalence classes of smooth maps from $\mathbb{S}^1$ to $\mathbb{S}^1$ **modulo the homotopy relation**.

Theorem 16.19 **(Hopf Degree Theorem in dimension one)** Two smooth maps $f_0, f_1 : \mathbb{S}^1 \to \mathbb{S}^1$ are **homotopic** **if and only if** they have the **same degree**.

- We generalize the assertion to all n in Theorem 17.1.
- Note that the statement also holds for **continuous** maps as follows: Since we know that every continuous map between smooth manifolds is homotopic to a smooth map by Whitney's Approximation Theorem 13.18, all we need to extend the assertion is a definition of the degree for merely continuous maps. This can be done using the techniques of **algebraic topology**, e.g., singular homology. See for example [17].

Proof of Theorem 16.19: We already know that if f_0 and f_1 are homotopic, then we have $\deg(f_0) = \deg(f_1)$. So assume $\deg(f_0) = \deg(f_1)$, and we **need to show** $f_0 \sim f_1$. Recall from Figure 12.4 the map p defined by

$$p : \mathbb{R} \to \mathbb{S}^1,\ t \mapsto e^{2\pi i t},$$

and showed that every smooth map $f : \mathbb{S}^1 \to \mathbb{S}^1$ can be lifted[1] to a map $g : \mathbb{R} \to \mathbb{R}$ with

$$g(t+1) = g(t) + q \text{ for some } q \in \mathbb{Z} \text{ such that } f(p(t)) = p(g(t)).$$

- **Claim:** $q = \deg(f)$.

If we can show the claim, then we **get a homotopy** $f_0 \sim f_1$ as follows: Assume we have two maps f_0 and f_1 with $\deg(f_0) = q = \deg(f_1)$. Let g_0 and g_1 be smooth maps $\mathbb{R} \to \mathbb{R}$ which both satisfy

$$g_0(t+1) = g_0(t) + q \text{ and } g_1(t+1) = g_1(t) + q$$

such that $f_0(p(t)) = p(g_0(t))$, $f_1(p(t)) = p(g_1(t))$. Then the smooth map

$$g_s(t) := s g_1 + (1-s) g_0 \text{ also satisfies } g_s(t+1) = g_s(t) + q.$$

Note $g_s(t)$ defines a **homotopy** G from g_0 to g_1 by $G(t,s) = g_s(t)$. Then G defines a **homotopy**

$$G : \mathbb{R} \times [0,1] \to \mathbb{R} \text{ with } G(t+1, s) = G(t,s) + q \text{ for all } t, s$$

[1] The idea was to lift piecewise locally and to patch the pieces together.

which induces a well-defined homotopy

$$F: \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1, (z,s) \mapsto p(G(t,s)) \text{ for any } t \in p^{-1}(z).$$

Hence the above $g_s(t)$ induces a **homotopy** from $f_0 = p \circ g_0$ to $p \circ g_1 = f_1$.

It remains to **prove the claim**:

- First, note that **if** f is **not** surjective, then we can pick a point $y \notin f(\mathbb{S}^1)$. Thus y is automatically a **regular** value. Since $\# f^{-1}(y) = 0$, we must have $\deg(f) = 0$. In this case, we need to have $q = 0$, i.e., $g(t+1) = g(t)$. For otherwise $p \circ g$ was surjective and hence f would be surjective.

  Note that, since the stereographic projection map $\mathbb{S}^1 \setminus \{y\} \to \mathbb{R}$ is a diffeomorphism and $\mathbb{R}$ is **contractible**, this shows that $\mathbb{S}^1 \setminus \{y\}$ is contractible. Hence f is a map to a contractible space and therefore **homotopic to a constant map** and has degree 0.

- Now we assume that f is **surjective**. Let $y \in \mathbb{S}^1$ be a **regular value** of f, and let $z \in f^{-1}(y)$. Since p is surjective, there is a $t \in \mathbb{R}$ with $p(t) = z$. Since y is a regular value, f is a local diffeomorphism around z. Its derivative is related to the one of g by the **chain rule**

  $$df_z \circ dp_t = dp_{g(t)} \circ dg_t.$$

  The derivative of $p: \mathbb{R} \to \mathbb{S}^1$ at any t is

  $$dp_t: \mathbb{R} \to T_{p(t)}(\mathbb{S}^1), \ w \mapsto 2\pi \cdot (-\sin(2\pi t), \cos(2\pi t)) \cdot w.$$

  Hence the determinant of dp_t at any t is positive (in fact equal $+2\pi$). Thus the sign of the determinant of df_z equals the sign of $dg_t \in \mathbb{R}$:

  $$\operatorname{sign} df_z = \operatorname{sign} dg_t.$$

  As above, let $y \in \mathbb{S}^1$ be a regular value of f and $z \in f^{-1}(y)$. Let us fix a $t_0 \in \mathbb{R}$ with $p(t_0) = z$. When we walk from t_0 to $t_0 + 1$ we need to **count** how many preimages of y we collect along the way, **with their orientation (!)**.

- We start with the case $q = 0$, i.e., $g(t+1) = g(t)$. It will actually teach us the key ideas we need to remember from this proof.

  We need to count how often $g(s) = g(t_0)$ with $dg_s = g'(s) > 0$ and how often $g(s) = g(t_0)$ with $dg_s = g'(s) < 0$. Note that since y is **regular**, dg_s is always $\neq 0$ for such an s.

  Since g is a smooth function $\mathbb{R} \to \mathbb{R}$, this is now just an exercise in Calculus. Using the **periodicity** of g, i.e., that $g'(t_0)$ must have the same sign as $g'(t_0 + 1)$, we see that there are **exactly as many** points s with $g(s) = g(t_0)$ and $dg_s = g'(s) > 0$ as there are points with $g(s) = g(t_0)$ and $dg_s = g'(s) < 0$. Thus $\deg(f) = 0$. See Figure 16.2.

- Now assume $q > 0$, and $g(t+1) = g(t) + q$.

  Again, we walk from t_0 to $t_0 + 1$ and sum up the orientation numbers of all the preimages of y that we collect along the way. This corresponds to counting how often we have $g(s) = g(t_0) + i$ for some $i = 0, 1, \dots, q-1$ and $s \in [t_0, t_0 + 1]$.

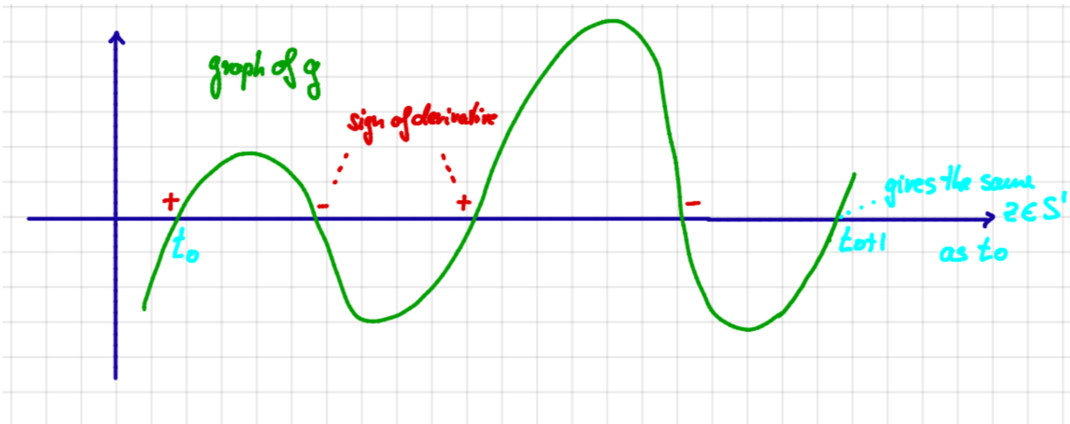


Figure 16.2: We count the intersection points with the sign of the derivative. If $q = 0$, there are as many crossing points from below with positive derivative as there are crossing points from above with negative derivative.

Let us look at one interval $[g(t_0) + i, g(t_0) + i + 1]$ at a time. We would like to know how many $s \in [t_0, t_0 + 1]$ are sent to either $g(t_0) + i$ or $g(t_0) + i + 1$ together with the sign of the derivative.

Therefore we look at the preimage $g^{-1}([g(t_0)+i, g(t_0)+i+1])$. This set is a disjoint union of closed intervals. For each of these intervals the start and endpoints are sent to either $g(t_0) + i$ or $g(t_0) + i + 1$. Let us think of the graph of g passing $g(t_0) + i$ with a positive sign of the derivative as **going in with** $+1$ and passing $g(t_0)+i+1$ with a positive sign of the derivative as **going out** $+1$, and the other two alternatives as the ones with -1. Then we see that the graph has to go in with $+1$ for a first time, and has to go out with $+1$ for a last time (since the graph starts at $g(t_0) \leq g(t_0) + i$ and ends at $g(t_0) + q \geq g(t_0) + i + 1$). In between those two points, the graph is going out with -1 as often as it goes in $+1$ and goes in with -1 as often as it goes out with $+1$. See Figure 16.3.

Thus in total the orientation numbers for $g^{-1}([g(t_0)+i, g(t_0)+i+1])$ add up to $+2$. Repeating this for all $i = 0, 1, \dots, q - 1$ gives a sum of orientation numbers equal to q, since we have to account for that we counted the inner points twice. Since the sum of orientation numbers of f equals the one of g, this shows $\deg(f) = q$.

- Finally, if $q < 0$, the same argument works with signs and directions reversed. □

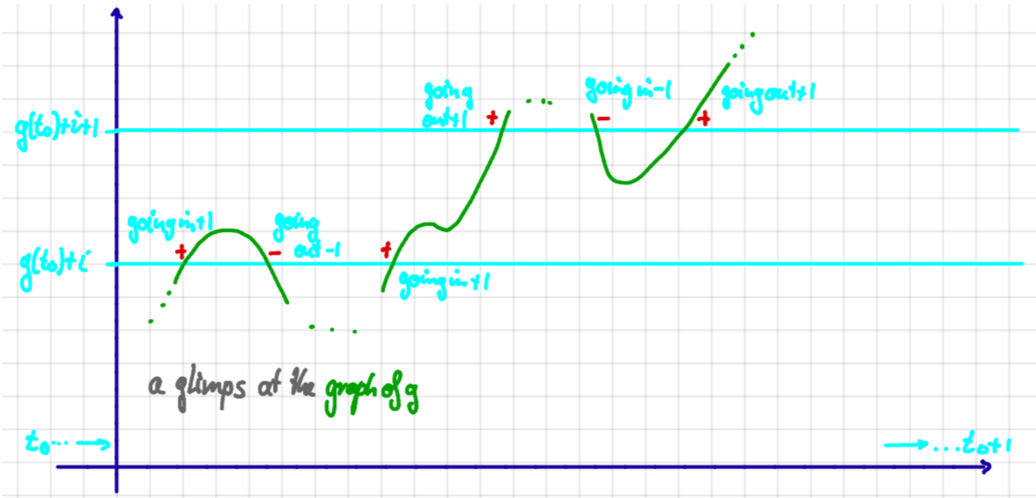


Figure 16.3: We count the intersection points with the sign of the derivative. If $q > 0$, there are exactly q more crossing points from below with positive derivative than there are crossing points from above with negative derivative.

16.4 Linking number and the Hopf invariant via exercises

In the following exercises, we will study some basic properties of the *Hopf invariant*. Recall that we have already defined the linking number and the Hopf invariant in a mod 2-version in Section 12.4. Now we improve the constructions to get $\mathbb{Z}$-valued invariants. In particular, we show that the Hopf invariant of the Hopf fibration equals one. This is a much stronger result than what we proved previously in Theorem 12.31.

As in Definition 12.22 we define the linking number of submanifolds in Euclidean space as follows: For $k \geq 1$, let $X, Y \subset \mathbb{R}^{k+1}$ be two disjoint smooth manifolds. The *linking map*

$$\lambda : X \times Y \to \mathbb{S}^k$$

is defined by

$$\lambda(x, y) = \frac{x - y}{|x - y|}.$$

Note that, since X and Y are disjoint, this map is well-defined and smooth. Now we assume that X and Y are compact, **oriented**, and without boundary, of dimensions $\dim X = m$ and $\dim Y = n$ such that $m + n = k$. Then the *linking number* $L(X, Y)$ of X and Y is defined to be the degree of λ, i.e.,

$$L(X, Y) := \deg(\lambda).$$

Exercise 16.1 **(a)** Show that

$$L(Y, X) = (-1)^{(m+1)(n+1)} L(X, Y).$$

Hint: Check what happens with the orientation numbers at points when we switch the order of X and Y, and think of our computation of the degree of the antipodal map on $\mathbb{S}^k$.

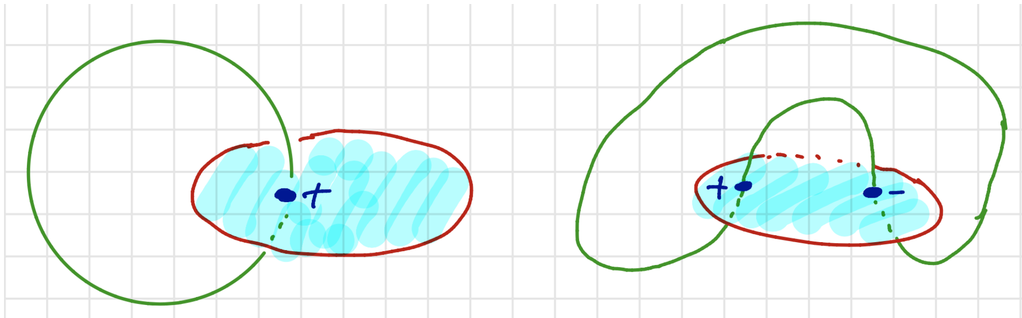


Figure 16.4: The red circle is the boundary of a compact manifold D. In both cases the green curve intersects the manifold D. The linking number detects the intersection. The circles on the left-hand side are linked and cannot be moved apart. However, multiple intersections may occur. The curves on the right-hand side are not linked and can be moved. The linking number detects this by counting the linking points with signs. See also Example 12.23.

(b) Assume that X is the boundary of an oriented manifold W which is disjoint from Y. Show that this implies $L(X, Y) = 0$.

Hint: Use the Boundary Theorem 16.2 for degrees.

As in Definition 12.25, we extend the definition of the linking number to submanifolds X, Y in $\mathbb{S}^{k+1}$: Assume X and Y are compact, **oriented** and boundaryless, and $\dim X + \dim Y = k$. Since the sphere is connected and X and Y are closed subsets, there must be a point p which is not contained in either X or Y. We identity $\mathbb{S}^{k+1} \setminus \{p\}$ with $\mathbb{R}^{k+1}$ via the diffeomorphism defined by stereographic projection from p. Then we consider X and Y as submanifolds of $\mathbb{R}^{k+1}$ and define the linking number $L(X, Y)$ as above.

For $n \geq 1$, consider a smooth map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$. Let $w \neq z \in \mathbb{S}^n$ be two regular values for f. Then $f^{-1}(w)$ and $f^{-1}(z)$ are compact, oriented, boundaryless submanifolds of $\mathbb{S}^n$ and their linking number $L\left(f^{-1}(w), f^{-1}(z)\right)$ is defined.

The number

$$H(f) := L\left(f^{-1}(w), f^{-1}(z)\right)$$

is called the *Hopf invariant* of f. This is a **famous invariant** that played a crucial role in the development of mathematics. As in Section 12.4.2 one can show that $H(f)$ does not depend on the choice of w and z and only depends on the homotopy class of f. We skip this verification here. Denoting by $\pi_{2n-1}(\mathbb{S}^n)$ the $(2n-1)$-homotopy group of $\mathbb{S}^n$, we can view the Hopf invariant as a map

$$H : \pi_{2n-1}(\mathbb{S}^n) \to \mathbb{Z}.$$

Exercise 16.2 We are going to study some of the basic properties of the Hopf invariant in this exercise:

(a) Show that if n is odd, then $H(f) = 0$.

(b) Let $g : \mathbb{S}^n \to \mathbb{S}^n$ be a smooth map. Consider the composition

$$\mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n \xrightarrow{g} \mathbb{S}^n.$$

Show that

$$H(g \circ f) = H(f) \cdot \deg(g)^2.$$

Now we are going to compute the Hopf invariant for the *Hopf fibration* $\pi : \mathbb{S}^3 \to \mathbb{S}^2$. We have met π several times before and have solved several of the following problems in the main text and previous exercises. We collect and reprove all the results for computing $H(\pi)$. We recommend that you solve each of the tasks again to get more practice. The calculation of $H(\pi) = 1$ is a key new step which goes beyond the computation modulo 2.

Exercise 16.3 We consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, i.e., $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$, and $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$, i.e., $\mathbb{S}^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$. Then the *Hopf fibration* π is the map $\mathbb{S}^3 \to \mathbb{S}^2$ given by

$$\pi(z_0, z_1) = \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right).$$

(a) Check that this actually defines a map $\mathbb{S}^3 \to \mathbb{S}^2$.

(b) Show that $\pi(z_0, z_1) = \pi(w_0, w_1)$ if and only if there is a complex number α with $|\alpha|^2 = \alpha\bar{\alpha} = 1$ such that $(w_0, w_1) = (\alpha z_0, \alpha z_1)$.

(c) Show that, for every point $p \in \mathbb{S}^2$, the fiber $\pi^{-1}(p)$ is diffeomorphic to $\mathbb{S}^1$.

(d) Show that each point in $\mathbb{S}^2$ is a regular value for π.

(e) Consider $a = (0, 0, 1)$ and $b = (0, 1, 0)$ on $\mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$. Determine the fibers $\pi^{-1}(a)$ and $\pi^{-1}(b)$ together with their orientations as preimages under π.

Hint: We have worked out the orientation of $\pi^{-1}(a)$ in Section 15.6. Follow the same outline to figure it out for $\pi^{-1}(b)$. Or see Exercise 15.8.

(f) Show that $H(\pi) = 1$.

Hint: You may want to choose a and b as regular values for π and compute $H(\pi)$ as the degree of $\lambda : S_a \times S_b \to \mathbb{S}^2$ with $S_a = \phi_N^{-1}(\pi^{-1}(a))$ and $S_b = \phi_N^{-1}(\pi^{-1}(b))$ where ϕ_N^{-1} denotes the stereographic projection from $N = (0, 0, 0, 1) \in \mathbb{S}^3$. To calculate $\deg(\lambda)$, pick a regular value for λ, say $p = (1, 0, 0)$. Then, for a point $(\mathbf{v}, \mathbf{w})$ with $\lambda(\mathbf{v}, \mathbf{w}) = p$, compute $d\lambda_{(\mathbf{v},\mathbf{w})} : T_{\mathbf{v}}S_a \times T_{\mathbf{w}}S_b \to T_p\mathbb{S}^2$. The final step is to check if $d\lambda_{(\mathbf{v},\mathbf{w})}$ preserves or reverses orientations. To do this, use the previous point to determine the orientations of S_a and S_b.

16.5 Exercises and more examples

16.5.1 The Degree

Exercise 16.4 Let $a\colon \mathbb{S}^k \to \mathbb{S}^k$, $a(x) = -x$, be the antipodal map on $\mathbb{S}^k$.

(a) Show that a has degree $(-1)^{k+1}$.

Hint: Consider the reflections

$$r_i(x_1, \dots, x_{k+1}) = (x_1, \dots, -x_i, \dots, x_{k+1}).$$

Recall that reflections are diffeomorphisms which reverse orientations.

(b) Show that a is homotopic to the identity if and only if k is odd.

Hint: Recall and use that we showed in Exercise 8.3 that a is homotopic to the identity if k is odd.

Aside: Note that we would not have been able to make the *only if*-conclusion with the mod 2-degree, since in $\mathbb{Z}/2$ we cannot distinguish between 1 and -1.

Exercise 16.5 Let $X \subset \mathbb{R}^N$ be a smooth manifold. Recall that a **vector field** on X is a smooth section of $\pi\colon T(X) \to X$, i.e., a smooth map $\sigma\colon X \to T(X)$ such that $\pi \circ \sigma = \mathrm{Id}_X$. An equivalent way to describe such a section is to give a map $s\colon X \to \mathbb{R}^N$ such that $s(x) \in T_x(X) \subset \mathbb{R}^N$ for all x (with corresponding $\sigma(x) = (x, s(x))$). A point $x \in X$ is a zero of the vector field σ if $\sigma(x) = (x, 0)$ or equivalently $s(x) = 0$. Prove the following important theorem:

The n-dimensional sphere $\mathbb{S}^n$ admits a vector field v *without zeros*, i.e., $v(x) \neq 0$ for all $x \in \mathbb{S}^n$, **if and only if** n is *odd*.

Hint: Note that we have proven most of the statement in previous exercises, see Exercise 8.7. You can assume those results. What is new is the *only if*-part for which we need the improved degree. To show this, write the antipodal map as a composition of reflections.

Exercise 16.6 Use the degree to prove the Fundament Theorem of Algebra: Given a monic polynomial with complex coefficients

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m.$$

Show that there exists a $w \in \mathbb{C}$ such that $p(w) = 0$.

Exercise 16.7 Explain why the degree cannot be used to prove that there is a zero in $\mathbb{R}$ for every monic real polynomial.

Exercise 16.8 Let
$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$
be a monic complex polynomial. At every point $z_0 \in \mathbb{C}$, there is an integer $m \geq 0$ such that we can factor
$$p(z) = (z - z_0)^m q(z) \text{ with } q(z_0) \neq 0.$$
Note that z_0 is a zero of p if and only if $m > 0$. In this case we call m the *multiplicity* of the zero z_0. In this exercise we relate multiplicities and degrees as follows:

Let $\mathbb{D}_0 \subset \mathbb{C}$ be a disk with center z_0 and radius r small enough such that there is no other zero of p in $\mathbb{D}_0$ except z_0. Recall that we showed that this is possible when we discussed Milnor's proof of the FTA in Section 4.4.

(a) Check that the formula $g(z) = z_0 + rz$ defines an orientation-preserving diffeomorphism $\mathbb{S}^1 \to \partial\mathbb{D}_0$ where $\mathbb{D}_0$ inherits an orientation from $\mathbb{C} = \mathbb{R}^2$.

(b) Explain why $\frac{p}{|p|} : \partial\mathbb{D}_0 \to \mathbb{S}^1$ and $\frac{p\circ g}{|p\circ g|} : \mathbb{S}^1 \to \mathbb{S}^1$ have the same degree.

(c) Find a smooth homotopy $h_t : \mathbb{S}^1 \to \mathbb{S}^1$ between $h_1 = \frac{p\circ g}{|p\circ g|}$ and $h_0(z) = c \cdot z^m$ with c being the constant $c = \frac{q(z_i)}{|q(z_i)|}$.

(d) Conclude that $\deg\left(\frac{p}{|p|}\right) = m$.

Exercise 16.9 Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps between manifolds with X and Y compact, Y and Z connected. Assume that all three manifolds are oriented and boundaryless and $\dim X = \dim Y = \dim Z$. Show that
$$\deg(g\circ f) = \deg(g) \cdot \deg(f).$$

Exercise 16.10 Let $\mathbb{S}^k \xrightarrow{f} \mathbb{S}^k$ be a smooth map. Show that if $\deg(f) \neq (-1)^{k+1}$, then f must have a *fixed point*.

Hint: Assume that f had no fixed point. Show that then the composition of f with the antipodal map would be homotopic to the identity map.

Exercise 16.11 Let $k \geq 1$ be an *odd* integer. Let $q : \mathbb{S}^k \to \mathbb{R}\mathrm{P}^k$, $x \mapsto [x]$, be the canonical projection. Recall from Theorem 15.16 that $\mathbb{R}\mathrm{P}^k$ is compact and orientable so that $\deg(q)$ is defined. Show that the degree of q is ± 2, and that it is $+2$ if the orientation on $\mathbb{R}\mathrm{P}^k$ is chosen such that q preserves orientation.

Hint: Use that q does not change after composition with the antipodal map.

Exercise 16.12 In this exercise we prove Theorem 16.17: Let $\mathbb{S}^k \xrightarrow{f} \mathbb{S}^k$ be a smooth map. If $\deg(f)$ is *odd*, then f must send *some* pair of antipodal points to antipodal points.

To prove the claim we assume that $f : \mathbb{S}^k \to \mathbb{S}^k$ sends no pair of antipodal points to antipodal points. We will show that this implies that $\deg(f)$ must be even.

(a) Assume that $f : \mathbb{S}^k \to \mathbb{S}^k$ sends no pair of antipodal points to antipodal points. Show that then f and $f \circ a$ are both smoothly homotopic to the map $g : \mathbb{S}^k \to \mathbb{S}^k$ defined by $g(x) := \frac{1}{2}(f(x) + f(-x))$, and deduce that $\deg(f) = \deg(g) = \deg(f \circ a)$.

(b) Now further assume that k is *even*. Show that this implies $\deg(g) = 0$.

(c) Now assume that k is *odd*. Use Exercise 16.9 and Exercise 16.11 to deduce that this implies $\deg(g)$ is *even*.

(d) Deduce the claim and thereby prove Theorem 16.17.

17. Hopf Degree Theorem

We proved previously that there is exactly one homotopy class of maps $\mathbb{S}^1 \to \mathbb{S}^1$ for every integer $n \in \mathbb{Z}$. By our classification of one-manifolds, we can read this also as follows:

For every **compact**, connected, boundaryless one-manifold X, there is **exactly one** homotopy class of maps $X \to \mathbb{S}^1$ for every integer $n \in \mathbb{Z}$.

Now we are going to generalize this result to higher dimensions. Recall that the degree of a map $f : X \to \mathbb{S}^k$ as in the theorem is defined as

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x)$$

where y is a regular value of f and $\operatorname{sign}(df_x)$ is $+1$ if df_x preserves orientations and -1 if df_x reverses orientations. We will refer to this sign rule as **our usual orientation convention**.

Theorem 17.1 (Hopf Degree Theorem) Let X be a **compact**, connected, **oriented** smooth k-manifold without boundary. Then two continuous maps $X \to \mathbb{S}^k$ are **homotopic** if and only if they have the **same degree**. In other words, the degree defines an **injective** map $[X, \mathbb{S}^k] \to \mathbb{Z}$.

Together with Theorem 16.14 and since every continuous map between spheres is homotopic to a smooth map we then get:

Corollary 17.2 (Homotopy groups of spheres - injectivity) Let $\pi_k(\mathbb{S}^k)$ denote the kth homotopy group of $\mathbb{S}^k$. The degree defines a **bijection**

$$\deg : \pi_k(\mathbb{S}^k) \to \mathbb{Z}.$$

Remark 17.3 Note that the situation is similar, though **different** for **non-orientable** manifolds: Two maps of a compact, connected, **non-orientable**, boundaryless k-manifold X to $\mathbb{S}^k$ are **homotopic** if and only if they have the **same degree modulo 2**.

17.1 Strategy for proving Hopf's theorem

Now we start our march towards a proof Hopf's theorem. We will follow the guideline of Guillemin-Pollack [5]. But it is worth noting that there are many different ways to prove this theorem. In particular, there is Pontryagin's proof as presented in Milnor's book [13] which uses an extremely important and interesting concept, called cobordism. We recommend to have a look at that proof as well.

- **(Strategy for the proof)** Assume given two maps f_0 and f_1 from X to $\mathbb{S}^k$.
  - Set $W := X \times [0,1]$, define $f : \partial W \to \mathbb{S}^k$ by $f := f_0$ on $X \times \{0\}$ and $f := f_1$ on $X \times \{1\}$. Then $\deg(f) = \deg(f_1) - \deg(f_0) = 0$, and a homotopy between f_0 and f_1 is a **global extension of f to W**.
  - Show the **Extension Theorem 17.13**: $f : \partial W \to \mathbb{S}^k$ has a **global extension** $W \to \mathbb{S}^k$ **if** and only if $\deg(f) = 0$, for any compact, connected, oriented $k+1$-manifold W. We **knew already**: existence of global extensions $\Rightarrow \deg(f) = 0$.
  - For the proof of the Extension Theorem 17.13, use the **Isotopy Lemma 12.3** to move W inside some ball $\mathbb{B} \subset \mathbb{R}^{k+1}$ with $\mathrm{Int}(W) \subset \mathbb{B}$. This **reduces** to checking an extension statement on **balls and spheres**.
  - Use **winding numbers** to show that a map which is **homotopic** to a **constant map** on the **boundary** of a ball $\mathbb{B}$ **extends** to all of $\mathbb{B}$.
  - Show the **Special Case** $X = \mathbb{S}^k$: For $f : \mathbb{S}^k \to \mathbb{S}^k$,

    $$\deg(f) = 0 \Rightarrow f \sim \textbf{constant} \text{ map}.$$

    This follows by **induction on the dimension** k of $\mathbb{S}^k$. We have shown previously that $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ are homotopic if and only if $\deg(f) = \deg(g)$.

    The induction step is actually a **zigzag** argument using **winding numbers**. The **Isotopy Lemma** is frequently used to move points into appropriate open neighborhoods and balls.

In order to make this strategy work, we need to prove a series of technical results. This will occupy the rest of the chapter. Two main technical ingredients are **isotopies** which allow to move points, and **winding numbers** which help us **calculating degrees**. We will also need the following version of the Isotopy Lemma:

Lemma 17.4 (Linear Isotopy Lemma) Suppose that E is a linear isomorphism of $\mathbb{R}^k$ that **preserves** orientation. Then there exists a **homotopy** E_t consisting of linear isomorphisms such that $E_0 = E$ and E_1 is the **identity**. If E **reverses** orientation, then there exists a homotopy E_t consisting of linear isomorphisms such that $E_0 = E$ and E_1 is the **reflection map**

$$E(x_1, x_2, \ldots, x_k) = (-x_1, x_2, \ldots, x_k).$$

Proof: First we remark that it **suffices** to deal with the case that E **preserves orientations**.

For if E is orientation reversing, then $r_1 \circ E$ preserves orientations. Then if there is a homotopy F between $r_1 \circ E$ and Id, then, after composing all maps with r_1, $r_1 \circ F$ is a homotopy between $E = r_1 \circ r_1 \circ E$ and r_1.

So let E be a **linear** isomorphism of $\mathbb{R}^k$ that **preserves** orientations. The proof is **by induction** on the **dimension** k. We need to check **two initial cases**:

First, let $k = 1$. Then $E : \mathbb{R} \to \mathbb{R}$ is given by **multiplication** by a real number $\lambda > 0$. Then $E_t = t \cdot 1 + (1-t) \cdot \lambda$ is a homotopy between $E = \lambda$ and $\mathrm{Id} = 1$. Note that each E_t is nonzero and therefore a linear isomorphism.

Now let $k = 2$ and assume that E has **only complex eigenvalues**. Then $E_t = tE + (1-t)\mathrm{Id}$ is a linear homotopy between Id and E. Moreover, each E_t is a linear isomorphism. To show this we show that $\det(E_t) \neq 0$ for all $t \in [0,1]$. If $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we get

$$\begin{aligned}\det(E_t) &= (t(a-1)+1)(t(d-1)+1) - t^2bc \\ &= t^2(a-1)(d-1) + t(a+d-2) + 1 - t^2bc \\ &= t^2(ad - bc - a - d + 1) + t(a+d-2) + 1.\end{aligned}$$

The **discriminant** of this quadratic equation in t is

$$\begin{aligned}&(a+d-2)^2 - 4(ad - bc - a - d + 1) \\ =&(a+d)^2 - 4(a+d) + 4 - 4(ad-bc) + 4(a+d) - 4 \\ =&(a+d)^2 - 4(ad-bc).\end{aligned}$$

But this is exactly the **discriminant** of the equation

$$t^2 + t(a+d) - (ad-bc) = 0$$

which is the **characteristic polynomial** in t of E. By assumption, this polynomial has only complex roots, i.e., its discriminant is **negative**. Hence there is **no real** t such that $\det(E_t) = 0$.

Now we show the **induction step**: So **assume** $k \geq 2$ and the assertion to be **true in all dimensions** $< k$. Then E has **either** at least **one real** eigenvalue **or** at least **one complex** eigenvalue. Let $V \subset \mathbb{R}^k$ be the corresponding **eigenspace**, which is either one- or two-dimensional. Then E maps V into itself. Hence $\mathbb{R}^k$ splits into a direct sum $\mathbb{R}^k = V \oplus W$. By choosing a basis of $\mathbb{R}^k$ consisting of a basis of V and one for W, we can represent E as a matrix of the form

$$E = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Here A is either a 1×1- or a 2×2-matrix determined by the type of the eigenvalues. Then we can define a **linear homotopy** E_t by

$$E_t = \begin{pmatrix} A & tB \\ 0 & C \end{pmatrix}.$$

Since E is a linear isomorphism and the **determinant is multiplicative**, we have

$$0 \neq \det(E) = \det(A)\det(C) = \det(E_t).$$

Thus E_t is also a **linear isomorphism for every** t. For $t = 0$, we see that E_0 maps V to V by A and W to W by C. Since $\dim W$ is strictly less than k, we can apply the **induction hypothesis** to C and W and the **initial cases** to A and V, respectively. Hence we have a homotopy C_t consisting of linear isomorphisms between C and the identity and a homotopy A_t between A and the identity. Then

$$\begin{pmatrix} A_t & tB \\ 0 & C_t \end{pmatrix}$$

is a **linear homotopy** between E and the identity map of $\mathbb{R}^k$. □

17.1.1 Winding numbers revisited

As for many results on maps between spheres, the winding number is useful concept. We used it before with values modulo 2. Now we need an **integral version**:

Definition 17.5 (Integer-valued winding numbers) Let X be a **compact**, **oriented** k-dimensional smooth manifold, and let $f : X \to \mathbb{R}^{k+1}$ be a smooth map. The **winding number** of f, denoted $W(f, z)$, around any point $z \in \mathbb{R}^{k+1} \setminus f(X)$ is defined as the degree of the map

$$u : X \to \mathbb{S}^k, \; x \mapsto \frac{f(x) - z}{|f(x) - z|}.$$

As a formula:

$$W(f, z) = \deg(u).$$

The winding number will be the **main tool** in the proof of Hopf's theorem. In order to exploit it effectively, we investigate some of **its properties**:

Lemma 17.6 (Step 1) Let $f : U \to \mathbb{R}^k$ be a smooth map defined on an **open** subset U of $\mathbb{R}^k$, and let x be a regular point, with $f(x) = z$. Let $\mathbb{B}$ be a sufficiently small **closed ball** centred at x, and define $\partial f : \partial\mathbb{B} \to \mathbb{R}^k$ to be the **restriction** of f to the **boundary** of $\mathbb{B}$. Then we have

$$W(\partial f, z) = \begin{cases} +1 \text{ if } f \text{ preserves orientation at } x, \\ -1 \text{ if } f \text{ reverses orientation at } x. \end{cases}$$

Proof: After possibly translating things, we can assume $x = 0 = z$, which keeps the notation simpler. We set $A = df_0$. We are going to show that $W(A, 0)$ can be used to calculate $W(\partial f, 0)$. This will follow if we show that we can choose $\mathbb{B}$ small enough such that there is a homotopy $F_t : \partial\mathbb{B} \times [0, 1] \to \mathbb{S}^{k-1}$ between $Ax/|Ax|$ and $\partial f(x)/|\partial f(x)|$. For then

$$W(\partial f, 0) = \deg\left(\frac{\partial f(x)}{|\partial f(x)|}\right) = \deg\left(\frac{Ax}{|Ax|}\right) = W(A, 0).$$

Now we are going to **construct the homotopy** F_t: By **Taylor theory**, we can write

$$f(x) = Ax + \varepsilon(x), \text{ where } \varepsilon(x)/|x| \to 0 \text{ when } x \to 0. \tag{17.1}$$

We define

$$f_t(x) = Ax + t\varepsilon(x) \text{ for } t \in [0, 1].$$

Then, f_t is a **homotopy** between $f_0(x) = Ax$ and $f_1(x) = f(x)$.

Since $x = 0$ is a **regular point**, we know that A is an **isomorphism**. Hence the image of the unit ball in $\mathbb{R}^k$ under A strictly contains a **closed ball** of some radius $r > 0$. Since every linear isomorphism is a **diffeomorphism**, we also know that A maps **boundaries to boundaries**, i.e., $\mathbb{S}^{k-1}$ to the boundary of the closed ball of radius r. Hence

$$|Ax| > r \text{ for all } x \in \mathbb{S}^{k-1}.$$

As a consequence,

$$|A(x/|x|)| > c \text{ and thus } |Ax| > |rx| \text{ for all } x \in \mathbb{R}^k \setminus \{0\}.$$

Now we use (17.1). Since $\varepsilon(x)/|x| \to 0$ as $x \to 0$, we can choose a ball $\mathbb{B}$ small enough such that

$$\varepsilon(x)/|x| < \frac{r}{2} \text{ for all } x \in \partial\mathbb{B}.$$

Then we have

$$|f_t(x)| = |Ax| - t|\varepsilon(x)| > r|x| - \frac{r}{2}|x| = \frac{r}{2}|x|,$$
$$i.e., |f_t(x)| > 0 \text{ for all } x \in \partial\mathbb{B}.$$

Hence we can define the desired homotopy F_t by

$$F_t : \partial\mathbb{B} \times [0, 1] \to \mathbb{S}^k, \ x \mapsto \frac{f_t(x)}{|f_t(x)|}.$$

Now we compute $W(A, 0)$. Therefor we apply the **Linear Isotopy Lemma 17.4** and get that A is homotopic to the **identity** if it **preserves** orientations, and homotopic to the **reflection** map $(x_1, x_2, \dots, x_k) \mapsto (-x_1, x_2, \dots, x_k)$ if it **reverses** orientations. In the former case, we have $W(A, 0) = +1$, and in the latter case $W(A, 0) = -1$. □

This result determines how local diffeomorphisms can wind. Now we are going to use this information to **count preimages**:

Lemma 17.7 (Step 2) Let $f : \mathbb{B} \to \mathbb{R}^k$ be a smooth map defined on some **closed ball** $\mathbb{B}$ in $\mathbb{R}^k$. Suppose that z is a **regular value** of f that has no preimages on the **boundary sphere** $\partial\mathbb{B}$, and let $\partial f : \partial\mathbb{B} \to \mathbb{R}^k$ be its restriction to the boundary. Then the **number of preimages** of z, counted with our usual orientation convention, equals the **winding number** $W(\partial f, z)$.

Proof: By the **Stack of Records Theorem 4.18**, we know that $f^{-1}(z)$ is a finite set $\{x_1, \dots, x_n\}$, and we can choose **disjoint balls** $\mathbb{B}_i$ around each x_i. Since $f^{-1}(z)$ is disjoint from $\partial\mathbb{B}$ by assumption, we can shrink these balls such that $\mathbb{B}_i \cap \partial\mathbb{B} = \emptyset$ and so that each $\mathbb{B}_i$ is **sufficiently small** so that **Step 1** can be applied. Let $\partial f_i = f_{|\partial\mathbb{B}_i}$. Then **Step 1**, i.e., Lemma 17.6, implies that the **number of preimage points**, counted with our usual orientation convention, equals $\sum_{i=1}^n W(\partial f_i, z)$. Let $\mathbb{B}' := \mathbb{B} \setminus \cup_i \mathbb{B}_i$ and consider the map

$$u\colon \partial\mathbb{B} \to \mathbb{S}^{k-1},\ x \mapsto \frac{f(x) - z}{|f(x) - z|}.$$

Since $f(x) \neq z$ on $\mathbb{B}'$, this map **extends** to all of $\mathbb{B}'$. This implies

$$W(f_{|\partial\mathbb{B}'}, z) = \deg(u) = 0.$$

The **orientations** of the boundaries **are related** by

$$\partial\mathbb{B}' = \partial\mathbb{B} \cup_{i=1}^n (-\partial\mathbb{B}_i).$$

This implies

$$W(f_{|\partial\mathbb{B}'}, z) = W(\partial f, z) - \sum_{i=1}^n W(\partial f_i, z).$$

Hence in total we get $W(\partial f, z) = \sum_{i=1}^n W(\partial f_i, z)$. □

Lemma 17.8 (Step 3) Let $\mathbb{B}$ be a **closed ball** in $\mathbb{R}^k$, and let $f:\ \mathbb{R}^k \setminus \mathrm{Int}(\mathbb{B}) \to Y$ be a smooth map defined outside the open ball $\mathrm{Int}(\mathbb{B})$. Let $\partial f:\ \partial\mathbb{B} \to Y$ be the **restriction to the boundary**. Assume that ∂f is **homotopic to a constant map**. Then f **extends** to a smooth map defined on all of $\mathbb{R}^k$ into Y.

Proof: For simplicity, we assume that $\mathbb{B}$ is centred at 0. Then we can write every **non-zero** point $x \in \mathbb{B}$ **uniquely** as $x = ty$ for some $y \in \partial\mathbb{B}$ and some $t \in [0,1]$. By assumption, there is a **homotopy** $g_t:\ \partial\mathbb{B} \to Y$ with $g_1 = \partial f$ and g_0 being a **constant map**.

Now we define the map $F:\ \mathbb{R}^k \to Y$ by setting

$$F(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^k \setminus \mathrm{Int}(\mathbb{B}) \\ g_t(x) & \text{if } x \in \mathbb{B} \text{ and } x = ty \text{ for some } y \in \partial\mathbb{B} \text{ and } t \in [0,1]. \end{cases}$$

Note that F is **well-defined** on $\mathbb{R}^k \setminus \mathrm{Int}(\mathbb{B})$, since f and g_t agree on $\partial\mathbb{B} = \mathbb{B} \cap (\mathbb{R}^k \setminus \mathrm{Int}(\mathbb{B}))$ where we have $f = \partial f = g_1$. Note also that $F(0)$ is well-defined as the **constant** value of g_0.

Now it remains to use **smooth bump function** to turn F into a **smooth** homotopy.[1] □

17.2 The Special Case

[1] It is already smooth except, possibly, on $\partial\mathbb{B}$.

Lemma 17.9 (The Special Case) Let $f: \mathbb{S}^k \to \mathbb{S}^k$ be a smooth map of **degree zero**. Then f is homotopic to a **constant map**.

Before we prove this case, we look at a **consequence**:

Theorem 17.10 (Winding number zero) Any smooth map $f: \mathbb{S}^k \to \mathbb{R}^{k+1} \setminus \{0\}$ having **winding number zero** with respect to the origin is homotopic to a **constant map**.

Proof of Theorem 17.10: By assumption, the **degree** of the map $f/|f|$ is **zero**. By the **special case**, this implies that $f/|f|$ is homotopic to a constant map. But $f/|f|$ and f are homotopic via the homotopy

$$F: \mathbb{S}^k \times [0,1] \to \mathbb{R}^{k+1} \setminus \{0\},\ (x,t) \mapsto tf(x) + (1-t) \cdot f/|f|$$

Since homotopy is a **transitive relation**, f is also homotopic to a **constant map**. □

Proof of the special case Lemma 17.9:

The proof is **by induction** on the dimension k. We have **established the case** $k = 1$ in Theorem 16.19. So we **assume** the special case being **true for** $k-1$ and want to deduce it for k. We need to prove a lemma first:

Lemma 17.11 Let $f: \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map with 0 as a **regular value**. Suppose that $f^{-1}(0)$ is **finite** and that the sum of preimage points in $f^{-1}(0)$ is **zero** when **counted with** the usual **orientation** convention. Assuming the special case in dimension $k-1$. Then there **exists a map** $g: \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = f$ **outside a compact set**. In particular, the homotopy $tf + (1-t)g$ from g to f is constant outside this compact set.

Proof: Since $f^{-1}(0)$ is a **finite**, we can choose a **ball** $\mathbb{B}$ centred at the origin with $f^{-1}(0) \subset \mathrm{Int}(\mathbb{B})$. By assumption, the sum of preimage points is zero when counted with the usual orientation convention. By **Step 2**, i.e., Lemma 17.7, the map $\partial f: \partial\mathbb{B} \to \mathbb{R}^k \setminus \{0\}$ has winding number zero. Since $\partial\mathbb{B}$ is diffeomorphic to $\mathbb{S}^{k-1}$, we may consider ∂f as a map from $\mathbb{S}^{k-1}$ to $\mathbb{R}^k \setminus \{0\}$.

Since we are assuming the special case being true in dimension $k-1$, we can apply its consequence, i.e., Theorem 17.10, in that dimension. Thus, ∂f is homotopic to a constant map. Hence

$$f_{|\mathbb{R}^k \setminus \mathrm{Int}(\mathbb{B})}: \mathbb{R}^k \setminus \mathrm{Int}(\mathbb{B}) \to \mathbb{R}^k \setminus \{0\}$$

is a map to which we can apply **Step 3**, i.e., Lemma 17.8. This implies that f extends to a smooth map $g: \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ with $f = g$ outside the compact space $\mathbb{B}$. □

Now we get back to the proof of the special case. So we are given a smooth map $f: \mathbb{S}^k \to \mathbb{S}^k$ with $\deg(f) = 0$.

The **idea of the proof** is to show that f is homotopic to a map $h: \mathbb{S}^k \to \mathbb{S}^k \setminus \{b\}$, where b is some point in $\mathbb{S}^k$. But $\mathbb{S}^k \setminus \{b\}$ is diffeomorphic to $\mathbb{R}^k$ via **stereographic projection** from

b. Since $\mathbb{R}^k$ is **contractible**, this implies h is **homotopic to a constant map**. Then f is also homotopic to a constant map.

So we need to show:

- **Claim:** f is **homotopic** to a smooth map $g: \mathbb{S}^k \to \mathbb{S}^k \setminus \{b\}$.

By **Sard's Theorem 7.1**, we can choose distinct regular values a and b of f. By the **Stack of Records Theorem 4.18**, the preimage sets are **finite**, say $f^{-1}(a) = \{a_1, \dots, a_n\}$ and $f^{-1}(b) = \{b_1, \dots, b_m\}$.

Moreover, we can find an **open neighborhood** U of a_1 such that U is diffeomorphic to $\mathbb{R}^k$ via a diffeomorphism $\alpha: \mathbb{R}^k \to U$ and such that $b_i \notin U$ for all $i = 1, \dots, m$.

Since $k > 1$, we can apply the **Isotopy Theorem 12.13**, which is an extension of the **Isotopy Lemma 12.3**, to the points $\{a_2, \dots, a_n\}$ in $Y := \mathbb{S}^k \setminus \{b\}$ to get a diffeomorphism which is **isotopic to the identity**, compactly supported, and **moves the points a_i into U**.

Since homotopy is a **transitive relation**, we can therefore assume that U is an open neighborhood of $f^{-1}(a)$ with $b \notin f(U)$.

Now let $\beta: \mathbb{S}^k \setminus \{b\} \to \mathbb{R}^k$ be a **diffeomorphism** with $\beta(a) = 0$. Then

$$\beta \circ f \circ \alpha: \mathbb{R}^k \xrightarrow{\alpha} U \xrightarrow{f} \mathbb{S}^k \setminus \{b\} \xrightarrow{\beta} \mathbb{R}^k$$

is a smooth map from $\mathbb{R}^k$ to $\mathbb{R}^k$. Since a is a **regular value** of f, 0 is a **regular value** of $\beta \circ f \circ \alpha$. Moreover, since $f^{-1}(a)$ is **finite**, $(\beta \circ f \circ \alpha)^{-1}(0)$ is **finite** as well.

Now we use the **assumption** $\deg(f) = 0$. For this means that the **number of preimages of a under f is zero** when **counted with** our usual **orientation** convention. Hence the **number of preimages** of 0 under $\beta \circ f \circ \alpha$ is **zero** when **counted with** the usual **orientation** convention.

Thus, we can **apply Lemma 17.11** to $\beta \circ f \circ \alpha: \mathbb{R}^k \to \mathbb{R}^k$ and get a map $g: \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = \beta \circ f \circ \alpha$ **outside** a **compact** set B and g is **homotopic** to $\beta \circ f \circ \alpha$ on $\mathbb{R}^k$.

Since α and β are **diffeomorphisms**, this implies that f is homotopic to $\beta^{-1} \circ g \circ \alpha^{-1}$ as a map from U to $\mathbb{S}^k \setminus \{b\}$. As $g = \beta \circ f \circ \alpha$ outside B, we have

$$\beta^{-1} \circ g \circ \alpha^{-1} = f \text{ on } U \setminus \alpha^{-1}(B).$$

Thus, the map

$$h: \mathbb{S}^k \to \mathbb{S}^k \setminus \{b\}$$

defined by setting

$$h = \begin{cases} f & \text{on } \mathbb{S}^k \setminus \alpha^{-1}(B) \\ \beta^{-1} \circ g \circ \alpha^{-1} & \text{on } \alpha^{-1}(B) \end{cases}$$

is smooth, and h is the desired map homotopic to f. This proves the special case. □

17.3 The Extension Theorem for degree zero maps

Next we need to learn how to extend maps defined on the boundary to the whole manifold. This is possible for maps of degree zero. This is an important result. We start with the simpler case of maps to Euclidean space:

Lemma 17.12 (Extending maps to Euclidean spaces) Let W be a **compact** smooth manifold **with boundary**, and let $f: \partial W \to \mathbb{R}^k$ be a smooth map. Then f can be **extended** to a globally defined map $F: W \to \mathbb{R}^k$.

Proof: As always we assume that W is a subset of some $\mathbb{R}^N$. Since W is **compact**, it is a **closed** subset of $\mathbb{R}^N$, and so is ∂W. Since f is a smooth map defined on a closed subset of $\mathbb{R}^N$, it may be **locally extended** to a smooth map on **open sets**. Since ∂W is **compact and boundaryless**, we can apply the **Tubular Neighborhood Theorem 13.10** to **extend** f to a map F defined on a neighborhood U of ∂W in $\mathbb{R}^N$.

Now we choose a **smooth bump function** ρ that is constant 1 on ∂W and 0 **outside** some **compact** subset of U. Then we can **extend** f to all of W by letting it be

$$\rho \cdot F \text{ on } U, \text{ and } 0 \text{ outside of } U.$$

This is a smooth function defined on all of $\mathbb{R}^N$ with values in $\mathbb{R}^k$ and being $f = 1 \cdot F$ on ∂W. □

Now we apply this lemma to maps with values in spheres:

Theorem 17.13 (Extension Theorem for degree zero maps) Let W be a **compact**, connected, **oriented** $k+1$-dimensional smooth manifold **with boundary**, and let $f: \partial W \to \mathbb{S}^k$ be a smooth map. Then f **extends** to a **globally** defined map $F: W \to \mathbb{S}^k$ with $\partial F = f$ if and only if $\deg(f) = 0$.

Proof: We already know that if f can be extended to all of W, then $\deg(f) = 0$. It remains to show the **opposite direction**.

So let f be as in the theorem, and assume $\deg(f) = 0$. By Lemma 17.12 above, we can extend f to a smooth map $F: W \to \mathbb{R}^{k+1}$. By the **Transversality Extension Theorem 13.29**, we can assume that 0 is a regular value of F. Since W is **compact** of dimension $k+1$, we know that $F^{-1}(0)$ is a **finite** set. Hence we can apply the corollary to the **Isotopy Lemma 12.3** to this finite set, and move $F^{-1}(0)$ inside $\mathrm{Int}(\mathbb{B})$ where $\mathbb{B}$ is a closed ball contained $\mathrm{Int}(W)$.

In particular, since $F^{-1}(0) \subset \mathrm{Int}(\mathbb{B})$, the map $\frac{F}{|F|}$ extends to $W' := W \setminus \mathrm{Int}(\mathbb{B})$. Hence

$$W(F/|F|, 0) = \deg(F/|F|) = 0.$$

On the other hand, we know by our **assumption** that

$$W(F_{|\partial W}, 0) = W(f, 0) = \deg(f) = 0,$$

where we use $f = f/|f|$, since f has **values in** $\mathbb{S}^k$.

Now let

$$\partial F = F_{\partial\mathbb{B}} : \partial\mathbb{B} \to \mathbb{R}^{k+1} \setminus \{0\}$$

be the restriction to the boundary. By the definition of W' and of **boundary orientations**, we have

$$\partial W' = (\partial W) \cup (-\partial\mathbb{B}).$$

Hence we get

$$W(F_{|\partial W'}, 0) = W(F_{|\partial W}, 0) - W(F_{|\partial\mathbb{B}}, 0)$$

and therefore $W(F_{|\partial\mathbb{B}}, 0)$ by our previous observations.

Now Theorem 17.10, i.e., the consequence of the special case, implies that ∂F is homotopic to a constant map. By **Step 3**, i.e., Lemma 17.8, this implies that ∂F **extends** to a map $G : W \to \mathbb{R}^{k+1} \setminus \{0\}$. Then the map $G/|G| : W \to \mathbb{S}^k$ is the **global extension** of f. □

17.4 The proof of Hopf's theorem

Let f_0 and f_1 be two maps $X \to \mathbb{S}^k$ and let $W := X \times [0,1]$. We define a map $f : \partial W \to \mathbb{S}^k$ by setting

$$f = \begin{cases} f_0 & \text{on } X \times \{0\} \\ f_1 & \text{on } X \times \{1\}. \end{cases}$$

By the **Extension Theorem 17.13**, f extends to a map on all of W if and only if $\deg(f) = 0$. By definition, such an extension would be a homotopy between f_0 and f_1. Thus we have

$$f_0 \sim f_1 \iff \deg(f) = 0.$$

It remains to relate $\deg(f)$ to $\deg(f_0)$ and $\deg(f_1)$. But, since $\partial W = (X \times \{1\}) \cup (X \times \{0\})$ with the **opposite orientation** on $X \times \{0\}$, it follows that

$$\deg(f) = \deg(f_1) - \deg(f_0).$$

Thus

$$f_0 \sim f_1 \iff \deg(f_1) = \deg(f_0). \quad \square$$

18. Vector Fields and the Poincaré–Hopf Index Theorem

18.1 Vector Fields

Let $X \subset \mathbb{R}^N$ be a smooth manifold. Recall from the exercises that a **vector field** on X is a smooth assignment of a vector tangent to X at each point x, i.e., a smooth map $\mathbf{v}\colon X \to \mathbb{R}^N$ such that $\mathbf{v}(x) \in T_x(X)$ for every x.

Vector fields play a crucial role in many applications, for example in mathematical physics. To provide efficient tools to understand them is an important motivation for differential topology. A priori one might expect that vector fields are free to move in every way they want. It turns out, however, that the topology of X provides pretty **strong restrictions** for what is allowed and what is not. This is the content of the Poincaré–Hopf Index Theorem 18.16 that we are going to prove. Since $\mathbf{v}(x)$ varies **smoothly** with x, the most interesting points $x \in X$ are where $\mathbf{v}(x) = 0$. For, in a small neighborhood of such a point, $\mathbf{v}$ can change directions radically. Hence the main points we need to investigate are the **zeros** of the vector field.

We have already seen some examples of vector fields in the exercises. Here are some more examples:

Example 18.1 (Vector field on an open subset) Let $U \subset \mathbb{R}^k$ be an open subset. Recall that the tangent space T_uU is just $\mathbb{R}^k$ for every $u \in U$. Hence a vector field $\mathbf{v}$ on U is just a smooth map $U \to \mathbb{R}^k$. Let $e_1, \dots, e_k$ be the standard basis of $\mathbb{R}^k$. Then $\mathbf{v}$ is of the form $\mathbf{v} = \sum_{i=1}^k v_i e_i$ for smooth functions $v_i\colon U \to \mathbb{R}$.

Example 18.2 (A vector field on a sphere) On the n-dimensional sphere $\mathbb{S}^n$, let $p_N = (0, \dots, 0, 1) \in \mathbb{S}^n$ denote the north pole. We can construct a vector field $\mathbf{v}$ on $\mathbb{S}^n$ by setting

$$\mathbf{v}(x) = p_N - (p_N \cdot x)x$$

where $\cdot$ denotes the inner product defined by considering p_N and x as vectors in $\mathbb{R}^{n+1}$. To check that this actually defines a vector field, it suffices to check that $\mathbf{v}(x)$ is **perpendicular** to every x:

$$\mathbf{v}(x) \cdot x = p_N \cdot x - (p_N \cdot x)(x \cdot x) = 0$$

where we use $x \cdot x = 1$ for every $x \in \mathbb{S}^n$.

Example 18.3 (Gradient vector field of a function) Let $X \subset \mathbb{R}^N$ be a smooth k-dimensional manifold, and let $f\colon X \to \mathbb{R}$ be a smooth real-valued **function** on X. For each $x \in X$, the derivative $df_x\colon T_xX \to \mathbb{R}$ defines a **linear functional** on T_xX. Recall from linear algebra that for any such functional there is a vector $\mathbf{v}(x) \in T_xX$ such that

$$df_x(w) = \mathbf{v}(x)^t \cdot w$$

where $\mathbf{v}(x)^t$ denotes the **transpose** of $\mathbf{v}(x)$ and $\cdot$ is matrix multiplication. We may think of this assignment $x \mapsto \mathbf{v}(x) \in T_xX$ as a vector field. This is called the **gradient field** of f and we denote it by $\mathbf{grad}(f)$. We note that if $z \in X$ is a **zero** of $\mathbf{grad}(f)$, then $df_z(w) = (\mathbf{grad}(f)(z))^t \cdot w = 0$ for all w. Hence $df_z = 0$ and z is a **critical** point of f. Conversely, if z is a critical point of f, then $df_z = 0$ and $\mathbf{grad}(f)(z)$ must be zero. Thus, z is a **zero** the gradient field of f if and only if z is a **critical point**.
In the **special case** $X = \mathbb{R}^k$, the gradient field of f can be represented by the **Jacobian** $(1 \times k)$-matrix at x, i.e.,

$$\mathbf{grad}(f)(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_k}(x)\right),$$

i.e., for the standard basis vectors $e_1, \dots, e_k$ of $\mathbb{R}^k$ we have

$$\mathbf{grad}(f)(x) = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(x)e_i.$$

18.2 Index of a vector field at a zero

We begin our investigation in **Euclidean space**, i.e., we assume first $X = \mathbb{R}^n$.

Definition 18.4 **(Index at the origin)** Assume that the vector field $\mathbf{v}$ on $\mathbb{R}^n$ has an **isolated zero** at the **origin**, i.e., there is a small radius $\varepsilon > 0$ such that $\mathbf{v}$ has no other zeros in the closed ball $\bar{\mathbb{B}}^n_\varepsilon$ around the origin with radius ε. Then we can define the map

$$\bar{\mathbf{v}}\colon \mathbb{S}^{n-1}_\varepsilon \to \mathbb{S}^{n-1},\ y \mapsto \frac{\mathbf{v}(y)}{|\mathbf{v}(y)|}.$$

We equip both spheres with the **standard orientation**, i.e., they are oriented as the boundary of $\bar{\mathbb{B}}^n_\varepsilon$ and $\bar{\mathbb{B}}^n_1$, respectively. We define the **index** of $\mathbf{v}$ at 0, denoted $\mathrm{ind}_0(\mathbf{v})$, to be the **degree** of $\bar{\mathbf{v}}$:

$$\mathrm{ind}_0(\mathbf{v}) := \deg(\bar{\mathbf{v}}).$$

- We can think of the map $y \mapsto \mathbf{v}(y)/|\mathbf{v}(y)|$ as a measure of the variation of the **direction** of $\mathbf{v}$ around the origin.
- The choice of the **radius does not matter** for the definition of ind_0, since if we choose another radius, say $\varepsilon' < \varepsilon$, then $\bar{\mathbf{v}}$ can be extended to a compact manifold $W = \bar{\mathbb{B}}^n_\varepsilon \setminus \mathbb{B}^n_{\varepsilon'}$ where $\bar{\mathbb{B}}$ denotes the **closed** ball and $\mathbb{B}$ the **open** ball. The boundary of W is $\partial W = \mathbb{S}^{n-1}_\varepsilon - \mathbb{S}^{n-1}_{\varepsilon'}$. Hence the **Boundary Theorem 16.2** implies that the two degrees that arise from using $\mathbb{S}^{n-1}_\varepsilon$ and $\mathbb{S}^{n-1}_{\varepsilon'}$, respectively, agree.

- On $\mathbb{R}^2$ we can easily construct vector fields with a zero with **arbitrary index** as follows: In the plane of complex numbers the polynomial $z \mapsto z^k$ defines a smooth vector field with a zero of index k at the origin, and the function $z \mapsto \bar{z}^k$ defines a vector field with a zero of index $-k$. See Figure 18.1.

- For more examples of vector fields and for some curves that the fields are tangent to see Figure 18.2 and Figure 18.3.

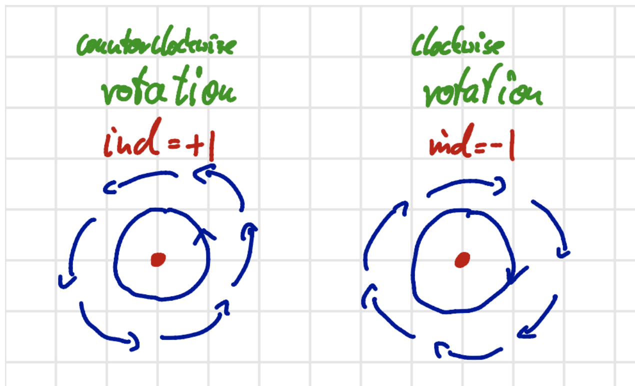


Figure 18.1: Vector fields arising from the maps $z \mapsto z^1$ and $z \mapsto \bar{z}$ with indices 1 and -1, respectively.

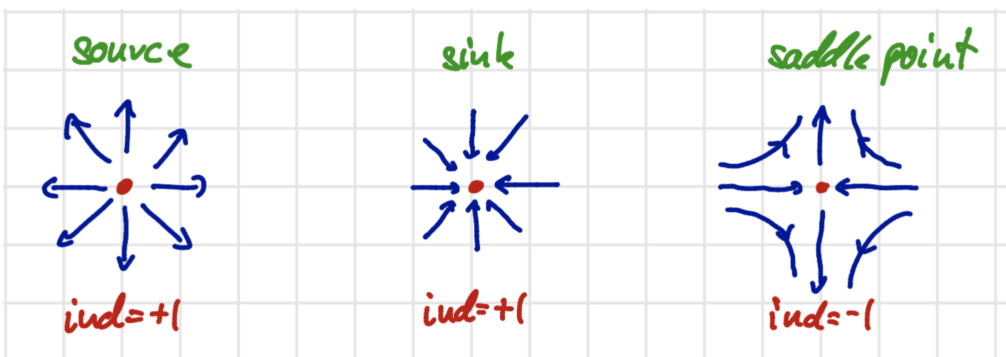


Figure 18.2: Source is the vector field on $\mathbb{R}^2$ given by $(x, y) \mapsto (x, y)$. The induced map on $\mathbb{S}^1$ is the identity. Hence the index is $+1$. The sink is the vector field on $\mathbb{R}^2$ given by $(x, y) \mapsto (-x, -y)$. The induced map on $\mathbb{S}^1$ is the antipodal map which is homotopic to the identity in this case. Hence the index of the sink is also $+1$. The saddle point field is the vector field on $\mathbb{R}^2$ given by $(x, y) \mapsto (y, x)$. The induced map on $\mathbb{S}^1$ reverses the orientation. Hence the index of saddle point at the origin is -1.

Now let X be a smooth manifold and let $\mathbf{v}$ be a vector field on X. We would like to transfer the definition of the index of $\mathbf{v}$ at a point $x \in X$ from Euclidean space to X. The idea is of how to do this is the familiar one: we use a local parametrization to transfer information between X and Euclidean space.

First we introduce a useful construction:

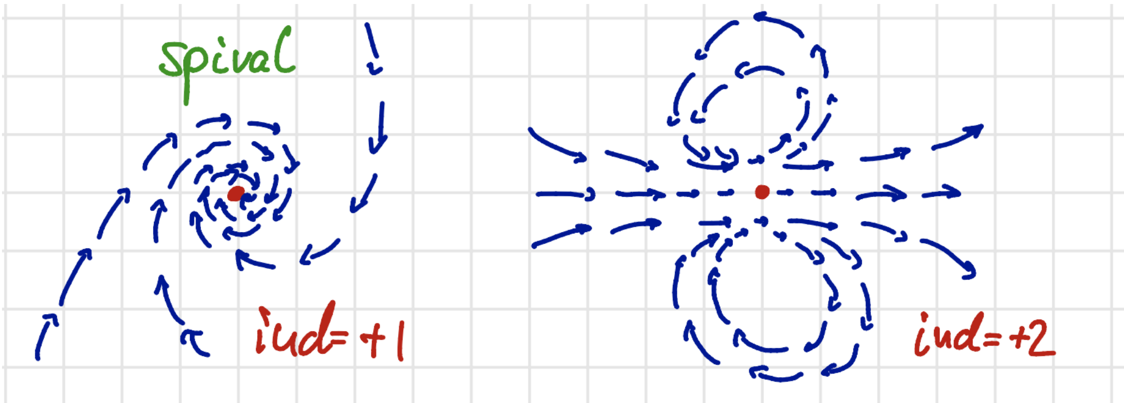


Figure 18.3: Two types of spiral movements. The left-hand spiral is just a scaled circular movement. Since the length of the vectors does not matter the index is +1. The right-hand vector field on $\mathbb{R}^2$ is given by $(x, y) \mapsto (x^2 - y^2, 2xy)$. Considering the first coordinate as the real and the second as the imaginary part of points $z = (x, y)$ in the complex plane, this map is just $z \mapsto z^2$. Hence the index is +2.

Definition 18.5 (Pullback of vector fields) Let X and Y be smooth manifolds and let $\mathbf{v}$ be a vector field on Y. Assume that there is a **diffeomorphism** $f : X \to Y$. Then we define the **pullback** vector field, denoted $f^*\mathbf{v}$, by assigning to x the vector which corresponds to the value of $\mathbf{v}$ at $f(x)$:

$$f^*\mathbf{v}(x) := df_x^{-1}(\mathbf{v}(f(x))).$$

Example 18.6 (Gradient vector field - pullback) Let $X \subset \mathbb{R}^N$ be a smooth k-dimensional manifold, and let $f : X \to \mathbb{R}$ be a real-valued **function** on X. Recall from Example 18.3 the gradient field of f. Let $\phi : U \to X$ be a local parametrization of X. We will now have a closed look at the pullback vector field $\phi^*\mathbf{grad}(f) : U \to \mathbb{R}^k$ on U. By definition we have

$$\phi^*\mathbf{grad}(f)(u) = d\phi_u^{-1}(\mathbf{grad}(f)(\phi(u))).$$

To compute the right-hand side, let $(e_1, \dots, e_k)$ denote the standard basis of $\mathbb{R}^k$. For $u \in U$, the vectors $d\phi_u(e_1), \dots, d\phi_u(e_k)$ form a basis of $T_{\phi(u)}X$. We need to determine how $f_{\phi(u)}$ acts on each $d\phi_u(e_i)$. By definition of the induced derivative we have

$$df_{\phi(u)}(d\phi_u(e_i)) = d(f \circ \phi)_u(d\phi_u^{-1} \circ d\phi_u(e_i)) = d(f \circ \phi)_u(e_i)$$

where $d(f \circ \phi)_u$ is the derivative of the smooth map $U \xrightarrow{\phi} X \xrightarrow{f} \mathbb{R}$. Hence, with respect to the basis $(d\phi_u(e_1), \dots, d\phi_u(e_k))$, we can represent $df_{\phi(u)}$ by the Jacobian $(1 \times k)$-matrix

$$df_{\phi(u)} = \left(\frac{\partial(f \circ \phi)}{\partial u_1}(u), \dots, \frac{\partial(f \circ \phi)}{\partial u_k}(u) \right).$$

The **vector** $\mathbf{grad}(f)(\phi(u)) \in T_{\phi(u)}X$ such that $df_{\phi(u)}(w) = (\mathbf{grad}(f)(\phi(u)))^t \cdot w$ for all

$w \in T_{\phi(u)}X$, is then given by

$$\mathbf{grad}(f)(\phi(u)) = \sum_{i=1}^{k} \frac{\partial(f \circ \phi)}{\partial u_i}(u) d\phi_u(e_i) \in T_{\phi(u)}X.$$

For a **basis** vector $d\phi_u(e_j) \in T_{\phi(u)}X$, we then get

$$\begin{aligned} df_{\phi(u)}(d\phi_u(e_j)) &= (\mathbf{grad}(f)(\phi(u)))^t \cdot d\phi_u(e_j) \\ &= \left(\sum_{i=1}^{k} \frac{\partial(f \circ \phi)}{\partial u_i}(u) d\phi_u(e_i) \right)^t \cdot d\phi_u(e_j) \in \mathbb{R}. \end{aligned} \tag{18.1}$$

In order to obtain $\phi^*\mathbf{grad}(f)$ it remains to apply $d\phi_u^{-1}$ and to express the effect of $\phi^*\mathbf{grad}(f)(u)$ in terms of the standard basis $e_1, \ldots, e_k$ of $\mathbb{R}^k$. For the latter we observe that we have

$$\left(\phi^*\mathbf{grad}(f)(u)\right)^t \cdot e_j = (\mathbf{grad}(f)(\phi(u)))^t \cdot d\phi_u(e_j) \in \mathbb{R}.$$

Moreover, Equation 18.1 tells us which real number that is. Thus, in terms of the standard basis of $\mathbb{R}^k$, we can express $\phi^*\mathbf{grad}(f)(u)$ as follows: We define smooth functions $g_{ij} : U \to \mathbb{R}$ by the formula

$$g_{ij}(u) = (d\phi_u(e_i))^T \cdot d\phi_u(e_j) \in \mathbb{R}.$$

Then we get, for all $u \in U$,

$$(\phi^*\mathbf{grad}(f))(u) = \sum_{j=1}^{k} \left(\sum_{i=1}^{k} \frac{\partial(f \circ \phi)}{\partial u_i}(u) g_{ij}(u) \right) \cdot e_j \in \mathbb{R}^k. \tag{18.2}$$

Finally, recall from Example 18.1 that every vector field $\mathbf{v}$ on $U \subset \mathbb{R}^k$ is of the form $\mathbf{v} = \sum_j v_j e_j$ for smooth functions $v_j : U \to \mathbb{R}$. Then we can write $\phi^*\mathbf{grad}(f)$ as

$$\phi^*\mathbf{grad}(f) = \sum_j v_j e_j$$

for the smooth functions

$$v_j = \sum_{i=1}^{k} \frac{\partial(f \circ \phi)}{\partial u_i} g_{ij}.$$

Now we extend the definition of the index to arbitrary smooth manifolds:

Definition 18.7 **(Index of a zero)** Let X be a smooth manifold and let $\mathbf{v}$ be a vector field on X with an isolated zero at $z \in X$. Let $\phi : U \to X$ be a local parametrization with $\phi(0) = z$. We define the **index** of $\mathbf{v}$ at z to be

$$\mathrm{ind}_z(\mathbf{v}) := \mathrm{ind}_0(\phi^*\mathbf{v}).$$

We need to show that this definition **does not depend** on the choice of the local parametrization ϕ. We do this in two steps via the following two lemmas:

Lemma 18.8 Every **orientation preserving** diffeomorphism $f: \mathbb{R}^n \to \mathbb{R}^n$ is smoothly isotopic to the **identity**.

Proof: We may assume that $f(0) = 0$. Since the evaluation at a point x in $T_x\mathbb{R}^n = \mathbb{R}^n$ of the derivative of f at 0 can be defined by

$$df_0(x) = \lim_{t\to 0} f(tx)/t,$$

we **define** an isotopy $F: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by the formula

$$\begin{cases} F(x,t) := f(tx)/t & \text{for } 0 < t \le 1, \\ F(x,0) := df_0 & \text{for } t = 0. \end{cases}$$

We need to check that F is **smooth** even when $t \to 0$. To do so we write $f(x) = x_1g_1(x) + \cdots + x_ng_n(x)$ for smooth functions $g_i: \mathbb{R}^n \to \mathbb{R}$ defined by

$$g_i(x_1,\dots,x_n) := \int_0^1 \frac{\partial f}{\partial x_i}(tx_1,\dots,tx_n)dt.$$

We then have

$$F(x,t) = x_1g_1(tx) + \cdots + x_ng_n(tx) \text{ for all } t \in [0,1].$$

Thus f is smoothly isotopic to the **linear** map df_0. Since f is an **orientation preserving** diffeomorphism, we know $\det df_0 > 0$. Since $\mathrm{GL}_n(\mathbb{R})^+$ is **path-connected**, this implies that df_0 is **smoothly isotopic** to the identity. □

- Note that the assertion of Lemma 18.8 is quite different from the situation for maps $\mathbb{S}^n \to \mathbb{S}^n$ on the sphere $\mathbb{S}^n$ where there may exist an orientation preserving diffeomorphism which is **not** smoothly homotopic to the identity.

That the index of a zero of a vector field on a smooth manifold is well-defined will now follow from the next lemma:

Lemma 18.9 Let U and U' be open neighborhoods of the origin in $\mathbb{R}^n$. Let $\mathbf{v}$ be a vector field on U with an isolated zero at the origin. Assume there is a **diffeomorphism** $f: U' \to U$ with $f(0) = 0$. Then the **index** of $\mathbf{v}$ at 0 is equal to the index of $f^*\mathbf{v}$ at 0:

$$\mathrm{ind}_0(\mathbf{v}) = \mathrm{ind}_0(f^*\mathbf{v}).$$

Proof: Since we only need to study the zeros in a small open neighborhood of the origin, we may assume that U and U' are open balls around the origin with sufficiently small radii ε and ε', respectively. **First** we **assume** that f **preserves orientation**. Then by Lemma 18.8 we can find a smooth isotopy

$$f_t: U' \times [0,1] \to \mathbb{R}^n, \text{ with } f_0 = \mathrm{id},\ f_1 = f, \text{ and } f_t(0) = 0 \text{ for all } t.$$

For each t, we set $\mathbf{v}_t := f_t^*\mathbf{v}$ on $f_t^{-1}(U) \subset \mathbb{R}^n$. Since f_t is a smooth **isotopy**, each f_t is a diffeomorphism and the vector field $\mathbf{v}_t$ is well-defined with an isolated zero at the origin for every $t \in [0,1]$. For each $t \in [0,1]$, let $\varepsilon_t > 0$ be the maximal radius such that $\mathbf{v}_t$ has no zeros other than the origin in $\mathbb{B}^n_{\varepsilon_t}$. The function $[0,1] \to \mathbb{R}^+$, $t \mapsto \varepsilon_t$ is continuous and hence takes a minimum when t varies in $[0,1]$. Let $\varepsilon > 0$ be that minimum. Then we have a well-defined smooth map

$$\bar{\mathbf{v}}_t : \mathbb{S}^{n-1}_\varepsilon \to \mathbb{S}^{n-1},\ y \mapsto \frac{\mathbf{v}_t(y)}{|\mathbf{v}_t(y)|}.$$

As $\mathbf{v}_0 = \mathbf{v}$ and $\mathbf{v}_1 = f^*\mathbf{v}$, we deduce

$$\mathrm{ind}_0(\mathbf{v}) = \deg(\bar{\mathbf{v}}_0) = \deg(\bar{\mathbf{v}}_1) = \mathrm{ind}_0(f^*\mathbf{v})$$

from the invariance of deg under homotopy. This proves the assertion for an orientation preserving diffeomorphism.

Now we **assume** that f **reverses orientation**. Then we can write f as the composition of an orientation preserving diffeomorphism and a reflection $r : \mathbb{R}^n \to \mathbb{R}^n$ which switches the sign of exactly one coordinate, i.e., r sends $(x_1, \dots, x_i, \dots, x_n)$ to $(x_1, \dots, -x_i, \dots, x_n)$. By the first case, we may therefore assume that $f = r$. We write $\mathbf{v}' = f^*\mathbf{v}$. Then the associated map $\bar{\mathbf{v}}'$ on the ε-sphere $\mathbb{S}^{n-1}_\varepsilon \to \mathbb{S}^{n-1}$ with $\bar{\mathbf{v}}'(y) = \mathbf{v}'(y)/|\mathbf{v}'(y)|$ satisfies

$$\bar{\mathbf{v}}' = r^{-1} \circ \bar{\mathbf{v}} \circ r$$

since r is a linear isomorphism and therefore $dr_0 = r$. Using the definition of the degree this shows $\deg(\bar{\mathbf{v}}) = \deg(\bar{\mathbf{v}}')$ and completes the proof of the lemma. □

Now we can show that the definition of the index is well-defined:

Lemma 18.10 (Index of a zero is well-defined) Let X be a smooth manifold and let $\mathbf{v}$ be a vector field on X with an isolated zero at $z \in X$. Then the index of $\mathbf{v}$ at z does **not** depend on the **choice** of a local parametrization around z.

Proof: Let $\phi : U \to X$ and $\psi : U' \to X$ be local parametrizations with $\phi(0) = z = \psi(0)$. By choosing U and U' small enough, the composition $f := \psi^{-1} \circ \phi$ defines a **diffeomorphism** $f : U' \to U$. We check using the definition of the pullback that we have

$$\phi^*\mathbf{v} = (\psi^{-1} \circ \phi)^*(\psi^*\mathbf{v}).$$

By Lemma 18.9, we then get

$$\mathrm{ind}_0(\phi^*\mathbf{v}) = \mathrm{ind}_0\left((\psi^{-1} \circ \phi)^*(\psi^*\mathbf{v})\right) = \mathrm{ind}_0(\psi^*\mathbf{v}).$$

This proves the assertion. □

Example 18.11 (Index sum on spheres) Recall the the vector field $\mathbf{v}$ on the n-dimensional sphere $\mathbb{S}^n$ defined in Example 18.2 given by

$$\mathbf{v}(x) = p_N - (p_N \cdot x)x$$

where $p_N \in \mathbb{S}^n$ denotes the north pole. and $\cdot$ denotes the inner product defined by considering p_N and x as vectors in $\mathbb{R}^{n+1}$. We compute the effect of $\mathbf{v}$ on $x = (x_1, \dots, x_{n+1}) \in \mathbb{S}^n$

as

$$\mathbf{v}(x) = (-x_1x_{n+1}, -x_2x_{n+1}, \dots, -x_nx_{n+1}, 1 - x_{n+1}^2). \tag{18.3}$$

This shows that $\mathbf{v}$ has **exactly two zeros**: the north pole $p_N = (0, \dots, 0, 1)$ and the south pole $p_S = (0, \dots, 0, -1)$. Now **we compute the index** of $\mathbf{v}$ at both zeros. By definition, we compute the index at a zero by choosing a local parametrization and pulling back the vector field to $\mathbb{R}^n$. For this example, it is convenient to use the **stereographic projections** from the poles of $\mathbb{S}^n$. First, we look at the zero at p_S. Let $\phi_N : \mathbb{R}^n \to \mathbb{S}^n \setminus p_N$ be the stereographic projection form the north pole. We then restrict ϕ_N to the open lower hemisphere on $\mathbb{S}^n$ as an open neighborhood V_S of p_S which does not contain any other zeros of $\mathbf{v}$. This has the advantage that the restriction of ϕ_N to V_S is a diffeomorphism of V_S to the open ball $U_S := \mathbb{B}_1^n \subset \mathbb{R}^n$. Thus the boundary of the closure $\overline{U}_S$ of U_S is just the unit sphere $\mathbb{S}^{n-1} \in \mathbb{R}^n$. Hence ϕ_N send a point $u \in \partial\overline{U}_S$ to $\phi_N(u) = (u_1, \dots, u_n, 0) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Thus, by Equation 18.3, we get

$$\mathbf{v}(\phi_N(u)) = (0, \dots, 0, 1) \text{ for all } u \in \partial\overline{U}_S.$$

This does not mean that $\phi_N^*\mathbf{v}$ is constant, since we still have to apply the derivative $(d(\phi_N)_u)^{-1}$. We computed this derivative in Exercise 2.7. The formula there yields

$$\phi_N^*\mathbf{v}(u) = \begin{pmatrix} 1 & 0 & \dots & 0 & u_1 \\ 0 & 1 & \dots & 0 & u_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & u_n \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ for all } u \in \partial\overline{U}_S$$

Thus, on the **boundary** of the closure of the neighborhood around the zero p_S, $\overline{\phi_N^*\mathbf{v}}$ acts as the **identity**. This shows that the index of $\mathbf{v}$ at p_S is $+1$:

$$\operatorname{ind}_S(\mathbf{v}) = \operatorname{ind}_0(\phi_N^*\mathbf{v}) = \deg(\mathrm{id}_{\mathbb{S}^{n-1}}) = +1.$$

To compute the index at the north pole p_N we use the stereographic projection ϕ_S and use the **upper hemisphere** as a neighborhood V_N of p_N. We then get again $\mathbf{v}(\phi_N(u) = (0, \dots, 0, 1)$ for all $u \in \partial\overline{U}_N$. **However**, the vector field $\phi_S^*\mathbf{v}$ now acts as

$$\phi_S^*\mathbf{v}(u) = \begin{pmatrix} 1 & 0 & \dots & 0 & -u_1 \\ 0 & 1 & \dots & 0 & -u_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & -u_n \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \end{pmatrix} \text{ for all } u \in \partial\overline{U}_N$$

This shows that $\overline{\phi_S^*\mathbf{v}}$ acts as the **antipodal map** $a_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$. Hence the index of $\mathbf{v}$ at N is equals the degree of the antipodal map on $\mathbb{S}^{n-1}$ which is $(-1)^n$ by Exercise 16.4:

$$\operatorname{ind}_N(\mathbf{v}) = \operatorname{ind}_0(\phi_S^*\mathbf{v}) = \deg(a_{\mathbb{S}^{n-1}}) = (-1)^n.$$

We observe that, in this example, the **sum of the indices** of the zeros of the vector field $\mathbf{v}$ is 0 if n is **odd** and is 2 if n is **even**. We will see in the next section that this sum is actually **independent** of the particular **choice** of vector field.

18.3 The Euler characteristic - algebraic topology in a nutshell

In order to state the main result of this chapter we have to take a brief detour to algebraic topology and introduce the Euler characteristic. There are of course many different ways to define the Euler characteristic of a smooth manifold X. However, the definitions that are accessible with the techniques developed in this book, for example as the self-intersection number of the diagonal embedded into $X \times X$, do not convey the property that we are interested here: the Euler characteristic is a purely topological invariant, i.e., it only depends on the topology of X and not the structure as a smooth manifold. Other definition, for example via triangulation, are more intuitive and seem more elementary. To make them precise and independent of choices, however, requires a lot of work. Since we assume the reader to be interested or familiar with algebraic topology anyway we will define the Euler characteristic via singular homology.

We will now briefly introduce singular homology groups. We refer the reader to any general introduction to algebraic topology for any details and further explanation and motivation, for example [6] or [17]. One may view the underlying idea as to map certain **model spaces** with precisely understood properties into the topological spaces we would like to understand. There are different choices for these model spaces, for example one can use spheres and define homotopy groups. For singular homology one uses spaces with a very nice **combinatorial behavior**. A rigorous way to do this is In algebraic topology is the following.

For $n \geq 0$, the **standard n-simplex Δ^n** is the set $\Delta^n \subset \mathbb{R}^{n+1}$ defined by

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \, : \, \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for all } i \right\}.$$

The standard simplices are related by **face maps** for $0 \leq i \leq n$ which can be described as

$$\phi_i^n(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

with the 0 inserted at the ith coordinate where t_0 is the 0th coordinate. Let X be a topological space. A **singular n-simplex** in X is a continuous map $\sigma \colon \Delta^n \to X$. We denote by $\mathrm{Sing}_n(X)$ the set of all n-simplices in X. For example, $\mathrm{Sing}_0(X)$ is just the set of points of X. But, in general, $\mathrm{Sing}_n(X)$ carries more interesting information for $n \geq 1$. For $0 \leq i \leq n$, we can use the face maps ϕ_i^n to define maps

$$d_i^n \colon \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X), \; \sigma \mapsto \sigma \circ \phi_i^n$$

by sending an n-simplex σ to the $n-1$-simplex defined by precomposition with the ith face inclusion. The image $d_i^n(\sigma) = \sigma \circ \phi_i^n$ is called the ith face of σ.

To make this construction accessible to algebraic tools, we define the group $S_n(X)$ of **singular n-chains** in X as the free abelian group generated by n-simplices, i.e.,

$$S_n(X) := \mathbb{Z}\mathrm{Sing}_n(X).$$

Thus an n-chain is a finite $\mathbb{Z}$-linear combination of simplices. We define the **boundary operator** as the homomorphism of abelian groups **determined by**

$$\partial_n \colon S_n(X) \to S_{n-1}(X), \; \partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i d_i^n \sigma.$$

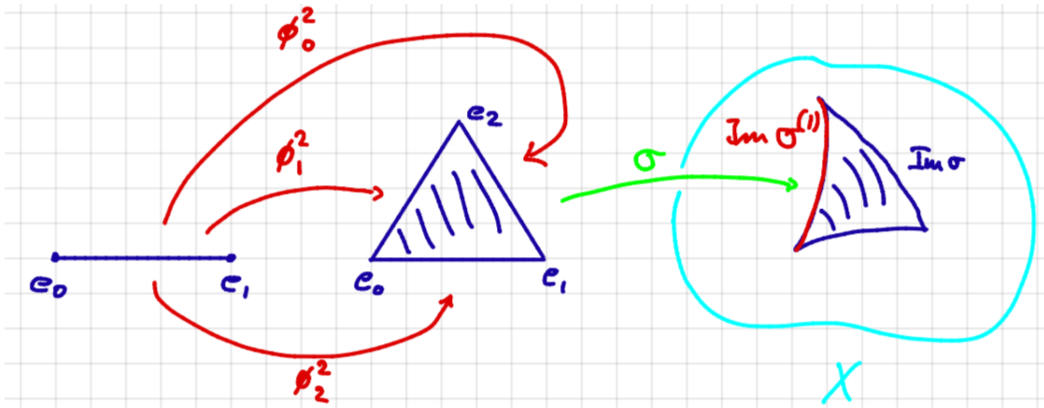


Figure 18.4: The face maps from the one-dimensional simplex into the two-dimensional one. There are three maps corresponding to the three edges of Δ^2. Then σ maps the simplices to X.

A key property of the operators ∂_n is that their composition vanishes, i.e., $\partial_{n+1} \circ \partial_n = 0$. This implies that the sequence of pairs $\{S_n(X), \partial_n\}$ forms a **chain complex**:

$$\cdots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0.$$

This allows to make the following definition:

Definition 18.12 (Singular homology) The nth **singular homology** group of X with coefficients in $\mathbb{Z}$ is defined to be the quotient group

$$H_n(X; \mathbb{Z}) = \frac{\operatorname{Ker}(\partial : S_n(X) \to S_{n-1}(X))}{\operatorname{Im}(\partial : S_{n+1}(X) \to S_n(X))}.$$

- The construction of singular homology is **functorial**, i.e., a continuous map $f : Y \to Y$ induces a homomorphism of abelian groups $f_* : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$.
- For each n, the group $H_n(X; \mathbb{Z})$ only depends on the **homotopy type** of X, i.e., $f : Y \to Y$ is a homotopy equivalence, then $f_* : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$ is an isomorphism.
- This follows from the fact that the assignment $f \mapsto f_*$ is invariant under homotopy, i.e., $f \simeq g$ implies $f_* = g_*$.
- **Intuitively**, the nth singular homology measures the following property of X: A n-cycle α, i.e., an element in the kernel of ∂_n, is a closed n-dimensional loop on X. The cycle α vanishes in homology if it is a boundary, i.e., if there is an element β in $S_{n+1}(X)$ such that $\partial_{n+1}(\beta) = \alpha$. In other words, α vanishes if it can be filled by a chain of one dimension higher. Hence we may think of a nonzero element in $H_n(X; \mathbb{Z})$ as a tool to **detect an n-dimensional hole** in X.
- In particular, the group $H_0(X; \mathbb{Z})$ is determined by the number m of connected components of X, i.e., $H_0(X; \mathbb{Z})$ is a sum of m copies of $\mathbb{Z}$. The $H_1(X; \mathbb{Z})$ is an abelianization of the fundamental group of X, i.e., it is determined by the closed loops on X up to continuous deformations.

- The homology of the n-dimensional sphere are given by $H_0(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z}$, $H_n(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z}$, and $H_i(\mathbb{S}^n;\mathbb{Z} = 0$ for all $i \neq 0, n$.

Remark 18.13 (Relative homology and attaching cells) Given a subspace $Y \subset X$, there are relative homology groups denoted $H_n(X, Y;\mathbb{Z})$ which, roughly speaking, measure only the features of the complement of Y in X. They are defined as the homology of the induced chain complex $\{S_n(X)/S_n(Y), \bar{\partial}_n\}_n$. A **special case** of this situation is of particular interest for us: Suppose we are given a map $f : \mathbb{S}^{n-1} \to Y$. Then we can attach $\bar{\mathbb{B}}^n$ to Y via f by forming the pushout diagram

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \\ \bar{\mathbb{B}}^n \arrow[r] & Y \cup_f \bar{\mathbb{B}}^n. \end{tikzcd}$$

We think of this process as obtaining X by **gluing** an n-dimensional **cell** to Y. In this case, the relative homology groups are given as follows:

$$H_j(X, Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$$

The Euler characteristic may now be viewed as a very rough summary of the information the homology groups contain. First we reduce the information of homology groups to their rank, i.e., we let b_i denote the rank of $H_i(X;\mathbb{Z})$ as an abelian group. The number b_i is called the ith **Betti number** of X. If X is a k-dimensional manifold, then we have $b_i = 0$ for all $i \geq k+1$. Hence we may take the alternating sum of the Betti numbers:

Definition 18.14 (Euler characteristic) Let X be a smooth k-dimensional manifold. The **Euler characteristic** of X, denoted by $\chi(X)$, is defined as the alternating sum of the Betti numbers of X, i.e.,

$$\chi(X) := \sum_{i=1}^{k} (-1)^i b_i = \sum_{i=1}^{k} (-1)^i \cdot \operatorname{rank} H_i(X;\mathbb{Z}).$$

- Since, for all i, the groups $H_i(X;\mathbb{Z})$ only depend on the topology of X, it follows that Euler characteristic of X only **depends on the topology** of X. In fact, $\chi(X)$ only depends on the **homotopy type** of X.

The Euler characteristic can often be computed by other methods. The following situation is of particular interest in Morse theory which will play again a role later in this chapter.

Remark 18.15 (Euler characteristic of a cell complex) Assume that the manifold X is homotopy equivalent to a **cell complex** Y, i.e., a space which is obtained by successively gluing i-dimensional cells together for $i = 1, \ldots, k$. Let c_i denote the number of i-

dimensional cells we attach. In this case, the Euler characteristic of X can be computed as

$$\chi(X) = \sum_{i=1}^{k} (-1)^i c_i. \tag{18.4}$$

Equation 18.4 comes quite close to the familiar **formula of Euler** which says that the alternating sum $v - e + f$ is an invariant of a surface S, where v, e, f denote the number of vertices, edges and faces, respectively, of a polygon which covers S. For example, the Euler characteristic of the 2-sphere is two and the Euler characteristic of the torus is zero.

18.4 The Poincaré–Hopf Index Theorem

We are now ready to state the following **famous result**:

Theorem 18.16 (Poincaré–Hopf Index Theorem) Let X be a **compact** smooth manifold and let $\mathbf{v}$ be a vector field on X with only **finitely** many isolated zeros. Then the **sum of the indices** of $\mathbf{v}$ at its zeros is equal to the **Euler characteristic** of X.

- Theorem 18.16 consists of two remarkable statements: First, the sum of the indices is independent of the vector field, i.e., the sum is entirely determined by the manifold X. Secondly, it is just the topology of X that matters since the latter determines the Euler characteristic of X.
- The Euler characteristic is a **topological** invariant while the nature of the index is **analytic**. Hence we may think of the theorem as a version of an **index theorem**. The generalizations of such results are extremely influential and important in many fields of mathematics, mostly for the reason that they provide a bridge between different branches.
- We have seen in Example 18.11 that the index sum for some vector field on $\mathbb{S}^{2n}$ is 2. Hence the index sum for **any** vector field on $\mathbb{S}^{2n}$ must be 2. In particular, every vector field on $\mathbb{S}^{2n}$ must have **at least one zero**.
- Similarly, every vector field on $\mathbb{S}^{2n+1}$ has index sum 0.
- The theorem can be extended to manifolds with boundary by requiring that the vector field has to point **outwards** at every boundary point. We will omit the proof of this extended result here.
- The **Eisenbud–Levine–Khimshiashvili signature formula** provides an **algebraic method** to compute the index at isolated zeros by assigning a quadratic form to the germ of a smooth $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ and an isolated zero at the origin, see [2] and [3].

As an immediate consequence of Theorem 18.16, we observe:

Corollary 18.17 (Vanishing of Euler characteristic) If a **compact** manifold X admits a vector field **without zeros**, then the **Euler characteristic** of X must be **zero**.

- For example, we know from Exercise 8.7 that **odd**-dimensional spheres admit **non-vanishing** vector fields. This again shows $\chi(\mathbb{S}^{2k+1}) = 0$.

Corollary 18.18 (Non-vanishing of Euler characteristic implies existence of zeros) Let X be a **compact** smooth manifold. If $\chi(X) \neq 0$, then every vector field on X must have at least one zero.

In fact, the dimension of the manifold has a strong influence on the Euler characteristic.

Corollary 18.19 (Odd dimensional manifolds have vanishing Euler characteristic) Let X be a **compact** smooth manifold. If the dimension n of X is **odd**, then the index sum of any vector field on X and the **Euler characteristic** of X are **zero**.

Proof: Let $\mathbf{v}$ be a vector field on X. Then we can multiply each $\mathbf{v}(x)$ by -1 to get a new field we denote by $-\mathbf{v}$. If z is a zero of $\mathbf{v}$, z is a zero of $-\mathbf{v}$ as well. The induced maps $\mathbb{S}^{n-1}_\varepsilon \to \mathbb{S}^{n-1}$ we use to compute the indices at z of $\mathbf{v}$ and $-\mathbf{v}$, respectively, differ by a composition with the **antipodal map** $a_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$. This changes the degree by multiplication by $(-1)^n$ which is -1, since n is **odd**. Thus, by Theorem 18.16, the index sum of $\mathbf{v}$ and $-\mathbf{v}$ satisfy

$$\sum_{z\in\mathbf{v}^{-1}(0)} \mathrm{ind}_z(\mathbf{v}) = \sum_{z\in(-\mathbf{v})^{-1}(0)} \mathrm{ind}_z(-\mathbf{v}) = -\sum_{z\in\mathbf{v}^{-1}(0)} \mathrm{ind}_z(\mathbf{v}).$$

Hence the **index sum** must be zero. By Theorem 18.16 this implies that the Euler characteristic is zero as well. □

Example 18.20 (Nowhere vanishing vector field on the torus) Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ be the two-dimensional torus. We define a vector field $\mathbf{v}$ on $\mathbb{T}$ by

$$\mathbf{v}\colon \mathbb{T} \to \mathbb{R}^4, \; x = (x_1, y_1, x_2, y_2) \mapsto (-y_1, x_1, -y_2, x_2).$$

We can verify that $\mathbf{v}(x) \in T_x\mathbb{T} = (T_{(x_1,y_1)}\mathbb{S}^1) \times (T_{(x_2,y_2)}\mathbb{S}^1) \subset \mathbb{R}^4$. Moreover, we see $\mathbf{v}(x)$ is never zero. Hence there are nowhere vanishing vector fields on the torus. We deduce from this example and Theorem 18.16 that the index sum of any vector field on the torus and that the Euler characteristic of the torus are both zero.

Remark 18.21 (Hopf's Theorem on non-vanishing vector fields) In fact, it is a theorem of Hopf in [8] that the vanishing of the index sum is not just a necessary condition for the existence of a non-vanishing vector field but also sufficient. That is, if the index sum of sum of some vector field on a **connected** compact oriented smooth manifold X is zero, then there exists a vector field on X with no zeros at all. We will discuss a proof of this result in Section 18.7.

18.5 Poincaré–Hopf Theorem - Independence

We split the proof of Theorem 18.16 into two steps: the assertion on the independence of the index sum of the vector field, and the identification with the Euler characteristic. We begin with independence and will prove the following theorem:

Theorem 18.22 (Poincaré–Hopf Index Theorem - Independence) Let X be a **compact** smooth manifold and let $\mathbf{v}$ be a vector field on X with only **finitely** many isolated zeros. If X has a boundary we require that $\mathbf{v}$ points outward at all boundary points of X. Then the **index sum** does not depend on the choice of the vector field.

We prove Theorem 18.22 in several steps. The key idea is show that the index sum equals the degree of a particular map that only depends on X.

Definition 18.23 (Gauss map) Let $X \subset \mathbb{R}^n$ be a **compact** smooth **n-dimensional** manifold **with** boundary. The **Gauss map**

$$g : \partial X \to \mathbb{S}^{n-1}$$

is defined by sending $x \in \partial X$ to the **outward** pointing unit **normal** vector at x.

In the following we are going to use without mentioning the following observation:

Remark 18.24 The assumption that $X \subset \mathbb{R}^n$ is an **n-dimensional** manifold implies that its tangent space, which is an n-dimensional subspace of $\mathbb{R}^n$, is equal to $\mathbb{R}^n$ at every point. Hence a vector field on X is a smooth map $\mathbf{v} : X \to \mathbb{R}^n$.

Now we prove the first case of Theorem 18.22.

Lemma 18.25 Let $X \subset \mathbb{R}^n$ be a **compact** smooth **n-dimensional** manifold with boundary. Let $\mathbf{v} : X \to \mathbb{R}^n$ be a vector field on X with finitely many isolated zeros and such that $\mathbf{v}$ **points outward** at all boundary points of X. Then the **index sum** is equal to the **degree of the Gauss map**. In particular, the index sum does **not** depend on the choice of $\mathbf{v}$.

Proof: Let $z_1, \ldots, z_k$ be the zeros of $\mathbf{v}$ which are **not** on the boundary of X. After removing an ε-ball around each zero for a sufficiently small ε we obtain again a smooth manifold with boundary which we denote by Y. Since there are **no zeros** of $\mathbf{v}$ on the **boundary** of X by assumption, the assignment $y \mapsto \bar{\mathbf{v}}(y) = \mathbf{v}(y)/|\mathbf{v}(y)|$ defines a **smooth** map $Y \to \mathbb{S}^{n-1}$. Since $\bar{\mathbf{v}}$ is defined on all of Y, the **Boundary Theorem 16.2** implies that the **degree** of the **restriction** of $\bar{\mathbf{v}}$ to ∂Y is **zero**.

Since the degree is additive with respect to connected components, the degree of the **restriction** of $\bar{\mathbf{v}}$ to the **boundary** ∂Y equals the **sum of the degrees** of the restrictions of $\bar{\mathbf{v}}$ to the components of ∂Y. One of the components of ∂Y is ∂X. By assumption, $\mathbf{v}(x)$ **points outward** at all points $x \in \partial X$ on the **boundary**. This implies $\bar{\mathbf{v}}_{|\partial X}$ is **homotopic to the Gauss map** g which sends $x \in \partial X$ to the outward unit **normal** vector at x. We just need to smoothly normalize the outward pointing vector $\mathbf{v}(x)$. Since the degree is invariant under homotopy, this implies $\deg(\bar{\mathbf{v}}_{|\partial X}) = \deg(g)$.

The other boundary components of ∂Y are $n-1$-dimensional ε-spheres with the **boundary orientation** induced by the orientation of X. **By construction** of Y, the boundary orientation is such that the normal vector pointing **into** the ε-sphere completes to a positive oriented basis. That is, the orientation of each of the ε-spheres, which are the components of ∂Y, is the **opposite** of the standard orientation. Thus, by definition of the index of a zero of $\mathbf{v}$, the degree of the restriction of $\bar{\mathbf{v}}$ to the other boundary components of ∂Y equals $-\sum_{i=0}^{k} \operatorname{ind}_{z_i}(\mathbf{v})$, the **negative** of the index sum. Hence, in total, we have shown

$$\deg(g) - \sum_{i=0}^{k} \operatorname{ind}_{z_i}(\mathbf{v}) = \deg(\bar{\mathbf{v}}_{|\partial Y}) = 0.$$

This proves the lemma. □

In order to extend Lemma 18.25 to arbitrary manifolds we will now study the derivative of the vector field.

Definition 18.26 (Nondegenerate zeros - Euclidean case) Let $U \subset \mathbb{R}^n$ be an **open** subset. Let $\mathbf{v}: U \to \mathbb{R}^n$ be a vector field on U and let $z \in U$ be a zero of $\mathbf{v}$. Then the vector field $\mathbf{v}$ is said to be **nondegenerate** at z if the linear transformation $d\mathbf{v}_z : \mathbb{R}^n \to \mathbb{R}^n$ is an **isomorphism**.

- Note that it follows from the Inverse Function Theorem 13.4 that, if $\mathbf{v}$ is nondegenerate at z, then z is an **isolated** zero.

Example 18.27 (Gradient vector field - nondegenerate zeros - Euclidean case) Let $U \subset \mathbb{R}^n$ be an **open** subset. Let $f : U \to \mathbb{R}$ be a smooth function and let $\mathbf{v} := \mathbf{grad}(f) : U \to \mathbb{R}^n$ denote its gradient vector field. Recall from Example 18.3 that $\mathbf{v}(u)$ is represented by the $(1 \times k)$-Jacobian matrix with respect to the standard basis of $\mathbb{R}^n$. Hence the derivative $d\mathbf{v}_u$ is represented by the **Hessian** matrix of f. This shows that a **zero** of $\mathbf{grad}(f)$ is **nondegenerate** if and only if z is a **nondegenerate critical point** of f, i.e., the **Hessian** matrix of f at z is **invertible**.

Lemma 18.28 Let $U \subset \mathbb{R}^n$ be an **open** subset. Let $\mathbf{v}: U \to \mathbb{R}^n$ be a vector field on X and let $u \in U$ be a zero of $\mathbf{v}$. Assume that $\mathbf{v}$ is **nondegenerate** at u. Then we have $\operatorname{ind}_z(\mathbf{v}) = \det d\mathbf{v}_z / |\det d\mathbf{v}_z|$. That is, the index of $\mathbf{v}$ at u is either $+1$ or -1 according to whether the determinant of $d\mathbf{v}_z$ is **positive** or **negative**.

Proof: We may assume $z = 0$. We can find a sufficiently small **open** ball $\mathbb{B}$ around the origin inside U. We may then consider $\mathbf{v}_{|\mathbb{B}}$ as a **diffeomorphism** into $\mathbb{R}^n$. Now we can apply Lemma 18.8 and the method used in the proof of Lemma 18.9. If $\mathbf{v}$ preserves orientation, then we can smoothly deform it into the identity without introducing any new zeros. In this case the index equals $+1$. If $\mathbf{v}$ reverses orientation, then we can smoothly deform it into a reflection without introducing any new zeros. In this case the index equals -1. This proves the lemma. □

Lemma 18.29 (Derivative of a vector field) Let $X \subset \mathbb{R}^N$ be a smooth n-dimensional manifold, $\mathbf{w}: X \to \mathbb{R}^N$ a vector field on X, and let z be a zero of $\mathbf{w}$. Then the derivative $d\mathbf{w}_z : T_zX \to \mathbb{R}^N$ has image $T_zX \subset \mathbb{R}^N$, and we may consider $d\mathbf{w}_z$ as a linear transformation from T_zX into itself. If this linear transformation has **determinant** $\neq 0$, then z is an **isolated zero** of $\mathbf{w}$ with index equal $\det d\mathbf{w}_z/|\det d\mathbf{w}_z|$, i.e, with index $+1$ or -1 according to whether the determinant of $d\mathbf{w}_z$ is **positive** or **negative**.

Proof: We choose a local parametrization $\phi: U \to X$ around z. Let e^i denote the ith basis vector of $\mathbb{R}^n$. For a point $u = (u_1, \ldots, u_n) \in U$ and writing $\phi = (\phi^1, \ldots, \phi^N)$, we set

$$t^i(u) := d\phi_u(e^i) = \begin{pmatrix} \frac{\partial \phi^1}{\partial u_i}(u) \\ \vdots \\ \frac{\partial \phi^N}{\partial u_i}(u) \end{pmatrix} =: \frac{\partial \phi}{\partial u_i}(u). \tag{18.5}$$

By definition of the tangent space and the derivative, the $t^1(u), \ldots, t^n(u)$ then form a **basis** of the tangent space $T_{\phi(u)}X$. We need to determine the image of the $t^i(u)$ under the linear transformation $d\mathbf{w}_{\phi(u)}$. We fix an i and then have

$$d\mathbf{w}_{\phi(u)}(t^i(u)) = d(\mathbf{w}\circ\phi)_u(e^i) = \frac{\partial(\mathbf{w}\circ\phi)}{\partial u_i}(u) \tag{18.6}$$

using the shortened notation of (18.5) on the right-hand side. Now we let $\mathbf{v} = \phi^*\mathbf{w}$ denote pullback vector field on U along ϕ. Since U is an open subset of $\mathbb{R}^n$, the tangent space T_uU at any $u \in U$ is just $\mathbb{R}^n$. We may then write $\mathbf{v}$ as $\mathbf{v} = \sum_{i=1}^n v_i e^i$ with suitable smooth functions $v_1, \ldots, v_n : U \to \mathbb{R}$. By definition we have $\mathbf{v}(u) = d\phi_u^{-1}\circ\mathbf{w}\circ\phi(u)$ and $t^i(u) = d\phi_u(e^i)$ so that

$$\mathbf{w}(\phi(u)) = d\phi_u(\mathbf{v}(u)) = \sum_i v_i(u)t^i(u).$$

Taking partial derivatives with respect to u_i and using the **product rule** then yields

$$\frac{\partial(\mathbf{w}\circ\phi)}{\partial u_i}(u) = \sum_j \frac{\partial v_j}{\partial u_i}(u)t^j(u) + \sum_j v_j(u)\frac{\partial t^j}{\partial u_i}(u). \tag{18.7}$$

Now we apply (18.7) at the **zero** $\bar{u} = \phi^{-1}(z)$ of $\mathbf{v}$, i.e., $\mathbf{v}(\bar{u}) = \sum_i v_i(\bar{u})e^i = 0$. Since the $e^1, \ldots, e^n$ form a **basis** of $\mathbb{R}^n$, we must have $v_1(\bar{u}) = \ldots = v_n(\bar{u}) = 0$. Thus, at the zero $\bar{u}$ we just get

$$d\mathbf{w}_z(t^i(\bar{u})) = \frac{\partial(\mathbf{w}\circ\phi)}{\partial u_i}(\bar{u}) = \sum_j \frac{\partial v_j}{\partial u_i}(\bar{u})t^j(\bar{u}). \tag{18.8}$$

This shows that, at the **zero** z, the image of the basis vector t^i under $d\mathbf{w}_z$ is a linear combination of the basis vectors of T_zX. Thus, the image of $d\mathbf{w}_z$ in $\mathbb{R}^N$ is contained in the subspace T_zX. This proves the first assertion of the lemma.

Moreover, (18.8) also shows that the **determinant** of the linear transformation

$$d\mathbf{w}_z : T_zX \to T_zX$$

equals the determinant of the matrix with (i, j)th entry $\frac{\partial v_j}{\partial u_i}(\bar{u})$. Thus we get $\det d\mathbf{w}_z = \det d\mathbf{v}_{\bar{u}}$. Since ϕ is a **diffeomorphism** between open neighborhoods of the zeros $\bar{u}$ of $\mathbf{v}$ and z of $\mathbf{w}$, the **indices** $\operatorname{ind}_z(\mathbf{w})$ and $\operatorname{ind}_{\bar{u}}(\mathbf{v})$ are equal. Moreover, by the Inverse Function Theorem 13.4 the zero z is **isolated** if $\det d\mathbf{w}_z \neq 0$, since then there is a small open neighborhood of z such that z is the only point where $\mathbf{w}(x)$ vanishes. By Lemma 18.28 we know $\operatorname{ind}_{\bar{u}}(\mathbf{v})$ is $+1$ or -1 according to whether $\det d\mathbf{v}_{\bar{u}}$ is positive or negative. Thus, $\operatorname{ind}_z(\mathbf{w})$ is $+1$ or -1 according to whether $\det d\mathbf{w}_z$ is positive or negative. This proves the lemma. □

Lemma 18.29 allows us, in particular, to extend the definition of nondegenerate zeros to arbitrary vector fields.

Definition 18.30 (Nondegenerate zeros - general case) Let $X \subset \mathbb{R}^n$ be a smooth manifold and $\mathbf{v}$ be a vector field on X. Let $z \in X$ be a zero of $\mathbf{v}$. Then the zero z is said to be **nondegenerate** if the linear transformation $d\mathbf{v}_z : T_z X \to T_z X$ is an **isomorphism**.

Nondegenerate zeros of the gradient field of a function can be detected by studying the Hessian matrix:

Lemma 18.31 (Gradient vector field - nondegenerate zeros) Let $X \subset \mathbb{R}^n$ be a smooth k-dimensional manifold. Let $f : X \to \mathbb{R}$ be a real-valued function on X and $\mathbf{v} := \mathbf{grad}(f)$ be its gradient field defined in Example 18.3. A zero z of $\mathbf{grad}(f)$ is **nondegenerate** if and only if z is a **nondegenerate critical point** of f, i.e., the **Hessian** matrix of f at z, computed in any local coordinate system, is **invertible**.

Proof: We already know that $z \in X$ is a zero of $\mathbf{grad}(f)$ if and only if z is a critical point of f. Let $z \in X$ be a zero of $\mathbf{grad}(f)$. Let $\phi : U \to X$ be a local parametrization of X with $\phi(u) = z$. Then it follows from Equation 18.2 that the derivative $d(\phi^*\mathbf{grad}(f))_u$ of $\phi^*\mathbf{grad}(f)$ at u is represented, with respect to the standard basis of $\mathbb{R}^k$, by a matrix which is the product of the **Hessian** matrix of $f \circ \phi$ at u and the real $(k \times k)$-matrix $G(u)$ with (i, j)-entry $g_i j(u)$. By definition, z is a nondegenerate critical point of f if and only if the Hessian matrix of $f \circ \phi$ at u is invertible. Thus, to prove the assertion it suffices to show that $\det G(u) \neq 0$.
Let A_u denote the matrix which represents $d\phi_u$ with respect to the bases $e_1, \ldots, e_k$ of $\mathbb{R}^k$ and $d\phi_u(e_1), \ldots, d\phi_u(e_k)$ of $T_{\phi(u)}X$. Then $d\phi_u(e_i)^t$ is the ith row and $d\phi_u(e_j)$ is the jth column of M_u. Thus, by definition of $G(u)$ and the $g_{ij}(u)$, we have $G(u) = M_u^t \cdot M_u$. Since M_u is invertible, this shows that $G(u)$ is an invertible real symmetric matrix. Hence it is **positive definite** and all its eigenvalues are strictly positive. In particular, we see that $\det G(u) > 0$. □

Remark 18.32 (Existence of vector fields with finitely many zeros on compact manifolds) Since functions with only nondegenerate critical points are generic by Theorem 7.13, it follows from Lemma 18.31 that there are vector fields on every smooth manifold with **only nondegenerate** zeros. In particular, every smooth manifold admits a vector field with **only isolated** zeros. Moreover, if X is **compact**, then the set of isolated zeros is closed and hence finite. Thus, on every compact manifold **there exists** a vector field with **only finite many zeros** which are all isolated.

Now we let $X \subset \mathbb{R}^n$ be a **compact** k-dimensional smooth manifold **without** boundary. For

$\varepsilon > 0$, we let

$$X^\varepsilon = \{y \in \mathbb{R}^n : |y - x| < \varepsilon(x) \text{ for some } x \in X\}$$

be the **open** subspace in $\mathbb{R}^n$ containing X of the **ε-Neighborhood Lemma 13.8**. We then write

$$\bar{X}^\varepsilon = \{y \in \mathbb{R}^n : |y - x| \leq \varepsilon(x) \text{ for some } x \in X\}$$

for the **closure** of X^ε. Note that if we choose $\varepsilon > 0$ small enough, then $\bar{X}^\varepsilon$ is a smooth n-dimensional manifold **with boundary**.

Theorem 18.33 (Index sum and Gauss map on ε-neighborhood) Let X and $\bar{X}^\varepsilon$ be as above. Let $\mathbf{w}$ be a vector field on X with only finitely many zeros all of which are **nondegenerate**. Then the **index sum** of $\mathbf{w}$ is equal to the **degree of the Gauss map**

$$g : \partial \bar{X}^\varepsilon \to \mathbb{S}^{k-1}.$$

In particular, the index sum does not depend on the choice of vector field.

Proof: Given a point $y \in \bar{X}^\varepsilon$. We know from Exercise 13.2 that, if we choose ε small enough, we can find the point $\pi(y) \in X$ which is **closest** to y and such that the vector $y - \pi(y)$ is **perpendicular** to the tangent space of X at $\pi(y)$. Moreover, we can consider the assignment $y \mapsto \pi(y)$ as a **smooth** map $\pi : \bar{X}^\varepsilon \to X$. Since $y - \pi(y)$ is **perpendicular** to $T_{\pi(y)}X$, the tangent space $T_y\bar{X}^\varepsilon$ at y equals the tangent space at $\pi(y)$. For a point y on the boundary $\partial\bar{X}^\varepsilon$, we have $|y-\pi(y)| = \varepsilon$ and the vector $y-\pi(y)$ points **outward** from $\pi(y)$ to y. Thus the **outward unit normal vector** at $y \in \partial\bar{X}^\varepsilon$ is

$$g(y) = (y - \pi(y))/\varepsilon.$$

Thus the **Gauss map** on $\partial\bar{X}^\varepsilon$ is given by

$$g : \partial\bar{X}^\varepsilon \to \mathbb{S}^{n-1},\ y \mapsto (y - \pi(y))/\varepsilon.$$

Now we **extend the vector field** $\mathbf{v}$ to a vector field $\mathbf{w}$ on $\bar{X}^\varepsilon$ by setting

$$\mathbf{w}(y) = (y - \pi(y)) + \mathbf{v}(\pi(y)).$$

We will now show that $\bar{X}^\varepsilon$ and $\mathbf{w}$ satisfy the hypotheses of Lemma 18.25. First, since X is **compact**, $\bar{X}^\varepsilon$ is a bounded and closed subset of $\mathbb{R}^n$ and therefore a **compact** n-dimensional submanifold of $\mathbb{R}^n$ with boundary.

- **Claim:** $\mathbf{w}$ points **outward** along the boundary.

Since $g(y)$ points outward and the inner product of $\mathbf{w}(y)$ and $g(y)$, considered as vectors in $\mathbb{R}^n$, is given by

$$\begin{aligned}
\mathbf{w}(y) \cdot g(y) &= [(y - \pi(y)) + \mathbf{v}(\pi(y))] \cdot [(y - \pi(y))/\varepsilon] \\
&= [(y - \pi(y)) \cdot (y - \pi(y))]/\varepsilon + [\mathbf{v}(\pi(y) \cdot (y - \pi(y))]/\varepsilon \\
&= \varepsilon^2/\varepsilon + 0 \\
&= \varepsilon > 0
\end{aligned}$$

where we use that y is on the **boundary** of $\bar{X}^\varepsilon$ and the $y - \pi(y)$ is **perpendicular** to the tangent vector $\mathbf{v}(\pi(y) \in T_{\pi(y)}X$.

- **Claim:** $\mathbf{v}$ and $\mathbf{w}$ have the **same zeros**.

Since the vectors $y-\pi(y)$ and $\mathbf{v}(\pi(y))$ are **perpendicular**, they cannot cancel each others whenever they are nonzero. Thus, $\mathbf{w}(y)$ can only be zero if both $y-\pi(y)$ and $\mathbf{v}(\pi(y))$ are the zero vector. Since $y-\pi(y)=0$ only if $y \in X$, we see that $\mathbf{w}$ vanishes exactly when $\mathbf{v}$ does.

- **Claim:** $\operatorname{ind}_z(\mathbf{w}) = \operatorname{ind}_z(\mathbf{v})$ at **every zero** $z \in X$.

Let $z \in X$ be a zero of $\mathbf{w}$ and therefore of $\mathbf{v}$ by the previous claim. By Lemma 18.29 we can compute both $\operatorname{ind}_z(\mathbf{w})$ and $\operatorname{ind}_z(\mathbf{v})$ as the the derivatives $d\mathbf{w}_z$ and $d\mathbf{v}_z$, respectively, and we know that $d\mathbf{v}_z$ is a linear transformation of T_zX into itself, and $d\mathbf{w}_z$ is a linear transformation of $T_z\bar{X}^\varepsilon$ into itself. Since $T_zX \subset T_z\bar{X}^\varepsilon$ and hence $T_z\bar{X}^\varepsilon = T_zX \oplus T_zX^\perp$, we can determine the effect of $d\mathbf{w}_z$ by computing its effect on vectors in T_zX and the orthogonal complement $T_zX^\perp$ separately. Since the restriction of π to X is the identity, the derivative of $y \mapsto y-\pi(y)$ in the direction of T_zX acts trivially. Thus we get

$$d\mathbf{w}_z(h) = d\mathbf{v}_z(h) \text{ for all } h \in T_zX.$$

In the direction orthogonal to T_zX, however, $d\mathbf{v}_z$ acts trivially and π is constant along a fixed line perpendicular to X. Thus we get

$$d\mathbf{w}_z(h) = h \text{ for all } h \in T_zX^\perp.$$

Since T_zX and $T_zX^\perp$ are orthogonal to each other and together span all of $T_z\bar{X}^\varepsilon$, we can compute the determinant of $d\mathbf{w}_z$ as the product

$$\det d\mathbf{w}_z = \det(d\mathbf{w}_z)_{|T_zX} \cdot \det(d\mathbf{w}_z)_{|T_zX^\perp} = \det d\mathbf{v}_z \cdot 1 = \det d\mathbf{v}_z.$$

This proves the claim by Lemma 18.29. Hence, in particular, the index sums of $\mathbf{w}$ and of $\mathbf{v}$ are equal. Moreover, since $\det d\mathbf{w}_z = \det d\mathbf{v}_z \neq 0$, all zeros of $\mathbf{w}$ are **isolated** by Lemma 18.29. Thus we can apply Lemma 18.25 which shows that the index sum of $\mathbf{w}$ is equal to the degree of the **Gauss map** g. Thus the index sum of $\mathbf{v}$ is equal to the degree of the Gauss map g which proves the theorem. □

It remains to remove the assumption that the zeros of the vector field have to be nondegenerate.

Theorem 18.34 (Index sums of fields with degenerate zeros) Let $X \subset \mathbb{R}^n$ be a **compact** k-dimensional smooth manifold without boundary. Let $\mathbf{v}$ be a vector field on X with finitely many isolated zeros. Then there is a vector field $\mathbf{w}$ on X with finitely many zeros all of which are **nondegenerate** such that the **index sums** of $\mathbf{v}$ and $\mathbf{w}$ are **equal**.

Proof: First we assume that $\mathbf{v}$ is a vector field defined on an open subset $U \subset \mathbb{R}^k$, and assume that z is the only zero of $\mathbf{v}$. We choose a **smooth bump function** $\lambda\colon U \to [0,1]$ which has the value 1 on a small ball $\mathbb{B}_{\varepsilon'}$ around z and has the value 0 outside a ball $\mathbb{B}_\varepsilon$ around z with slightly larger radius ε. Since $\mathbf{v}$ is a smooth map and has no other zeros than z, we can choose a **regular value** y of $\mathbf{v}$. We then define a new vector field $\mathbf{w}$ on U by

$$\mathbf{w}(u) = \mathbf{v}(u) - \lambda(u)y.$$

Then $\mathbf{w}(u)$ can only be zero if $\mathbf{v}(u) = \lambda(u)y$. Since z is the only zero of $\mathbf{v}$, our choice of λ implies that $\mathbf{w}$ does not have any zeros outside of $\mathbb{B}_\varepsilon$. After possibly rescaling by multiplying with a constant, we can choose y **small enough** such that $|\mathbf{v}(u) - \lambda(u)y| > 0$ on $\mathbb{B}_\varepsilon \setminus \mathbb{B}_{\varepsilon'}$. Moreover, within $\mathbb{B}_{\varepsilon'}$, we have $\lambda(u) = 1$, i.e., $\mathbf{w}(u) = \mathbf{v}(u) - y$ for all $u \in \mathbb{B}_{\varepsilon'}$. Thus, **inside** $\mathbb{B}_{\varepsilon'}$ we can **only** have $\mathbf{w}(u)$ **if** $\mathbf{v}(u) = y$. Since y is a **regular value** of $\mathbf{v}$, this implies that $d\mathbf{w}_u = d\mathbf{v}_u : T_uU = \mathbb{R}^k \to \mathbb{R}^k$ is an **isomorphism** at every point u in the set $\mathbf{v}^{-1}(y)$. This proves that $\mathbf{w}$ only has **nondegenerate** zeros. Since the **closed** ball $\bar{\mathbb{B}}_{\varepsilon'}$ is **compact**, $\mathbf{w}$ can only be **finitely many** zeros on U.

Again, since $\bar{\mathbb{B}}_{\varepsilon'}$ is **compact**, we can apply Lemma 18.25 to see that the index sum of $\mathbf{v}$ and the index sum of $\mathbf{w}$ on $\bar{\mathbb{B}}_{\varepsilon'}$ equals the **degree** of the Gauss map $g : \partial\bar{\mathbb{B}}_{\varepsilon'} = \mathbb{S}^{k-1}_{\varepsilon'} \to \mathbb{S}^{k-1}$, i.e.,

$$\sum_{z\in\mathbf{v}^{-1}(0)} \mathrm{ind}_z(\mathbf{v}) = \deg(g) = \sum_{u\in\mathbf{w}^{-1}(0)} \mathrm{ind}_z(\mathbf{w}).$$

Thus, the **index sums** of $\mathbf{v}$ and $\mathbf{w}$ are **equal**.

Now we let $X \subset \mathbb{R}^n$ and $\mathbf{v}$ be as assumed in the theorem. Let z_i be one of the finitely many isolated zeros $z_1, \dots, z_m$ of $\mathbf{v}$. Then we can choose a local parametrization $\phi_i : U_i \to X$ around z. Let $\lambda_i : U_i \to [0, 1]$ smooth bump function and $y_i \in \mathbb{R}^k$ be a regular value of $\phi_i^*\mathbf{v}$ as in the first case above. Then we can form the new vector field $\mathbf{w}_i$ on X by setting

$$\mathbf{w}_i(x) := \mathbf{v}(x) - \lambda_i(\phi_i^{-1}(x))d\phi_{\phi_i^{-1}(x)}(y_i).$$

Now we can check as above that $\mathbf{w}_i$ has finitely many zeros and the zeros of $\mathbf{w}_i$ which lie in the image of ϕ_i are all **nondegenerate**. Moreover, the index sums of $\mathbf{v}$ and $\mathbf{w}_i$ are equal. Choosing the local parametrizations around all the z_i small enough so that they do not overlap, we can perform this process for each zero and define a vector field $\mathbf{w}$ on X by

$$\mathbf{w}(x) := \mathbf{v}(x) - \sum_{i=1}^{m} \lambda_i(\phi_i^{-1}(x))d\phi_{\phi_i^{-1}(x)}(y_i).$$

The vector field $\mathbf{w}$ satisfies the properties claimed in the theorem. □

Now we can conclude that Theorem 18.33 and Theorem 18.34 together imply Theorem 18.22.

18.6 Poincaré–Hopf Theorem - Euler characterstic

Theorem 18.22 reduces the proof of Theorem 18.16 to finding **at least one** vector field on a given manifold X with index sum equal to $\chi(X)$. The idea is to find a suitable **Morse function** $f : X \to \mathbb{R}$ such that the index sum of its gradient field equals the Euler characteristic of X. We will now sketch how this can be achieved.

Given a smooth real-valued function $f : X \to \mathbb{R}$. By Lemma 18.31 we know that the nondegenerate zeros of $\mathbf{grad}(f)$ are exactly the nondegenerate critical points of f. We now need to express the index of a zero of $\mathbf{grad}(f)$ in terms of the behavior of the critical points of f.

Definition 18.35 (Gradient vector field - index of nondegenerate critical points) Let $X \subset \mathbb{R}^n$ be a smooth manifold. Let $f: X \to \mathbb{R}$ be a real-valued function on X with a **nondegenerate critical point** $z \in X$. The **index** of the critical point z is defined to be the **number** of **negative eigenvalues** of the **Hessian** matrix of f at z computed in a local coordinate system of X around z. We denote the index by $\mathrm{ind}_z(f)$.

- One can show that the index is **well-defined**, i.e., the number of negative eigenvalues the Hessian of f at z is the same for every local coordinate system.

Remark 18.36 (Index of a critical point and bilinear forms) Recall that the **index** of a bilinear form B on a vector space V is defined to be **maximal dimension** of a vector subspace of V on which B is **negative definite**. The Hessian matrix $H_z(f)$ of a smooth function $f: X \to \mathbb{R}$ at z, computed in a local coordinate system around z and with respect to a suitable basis, is a symmetric matrix and hence defines a symmetric bilinear form $H_z(f)$ on T_zX by setting $w \mapsto w^t \cdot H_z(f) \cdot w$. It then follows that the **index** of z as a **critical** point **equals** the index of the Hessian at z as a **symmetric bilinear form**.

Now we can relate the index of a critical point to the index of a zero as follows:

Lemma 18.37 (Gradient vector field - index of a nondegenerate zero) Let $X \subset \mathbb{R}^n$ be a smooth k-dimensional manifold. Let $f: X \to \mathbb{R}$ be a real-valued function on X with a nondegenerate critical point $z \in X$. Let $s = \mathrm{ind}_z(f)$ be the index of f at z. Then the **index** of $\mathbf{grad}(f)$ at the zero z is $(-1)^s$, i.e., $\mathrm{ind}_z(\mathbf{grad}(f)) = (-1)^s$.

Proof: Let $\phi: U \to X$ be a local parametrization of X around z with $\phi(0) = z$. we may choose U small enough such that U does not contain any other critical points of f. By definition, the index of $\mathbf{grad}(f)$ at z is the index of $\phi^*\mathbf{grad}(f)$ at 0. Thus, we may assume that $X = U$ is an **open** subset of $\mathbb{R}^k$ and 0 is the only critical point of $f: U \to \mathbb{R}$. Moreover, by **Morse's Lemma 7.10** we can choose ϕ such that we may assume that f is of the form

$$f(x) = f(0) - x_1^2 - \cdots - x_s^2 + x_{s+1}^2 + \cdots + x_k^2 \tag{18.9}$$

for all $x \in X$ where s is the **number of negative eigenvalues** of the Hessian $H(f)_0$ of f at 0. From Equation 18.9 we deduce that

$$\mathbf{grad}(f)(x) = (-2x_1, \ldots, -2x_s, 2x_{s+1}, \ldots, 2x_k) \in \mathbb{R}^k.$$

Writing $\mathbf{v} = \mathbf{grad}(f)$ this shows that the induced map $\bar{\mathbf{v}}: \mathbb{S}^{k-1} \to \mathbb{S}^{k-1}$ is the composition of the **reflection** of each of the first s coordinate. Since the reflection of an individual coordinate has degree -1, this implies that $\bar{\mathbf{v}}$ has degree $(-1)^s$. □

Lemma 18.38 (Gradient vector field - index sum) Let $X \subset \mathbb{R}^n$ be a **compact** smooth k-dimensional manifold. Let $f: X \to \mathbb{R}$ be a **Morse function** on X, i.e, a smooth real-valued function with only nondegenerate critical points. For $\lambda \in \mathbb{Z}$, let c_λ denote the number of critical points with index λ. Then the **index sum** of $\mathbf{grad}(f)$ is equal to

$\sum_\lambda (-1)^\lambda c_\lambda$.

Proof: Let z be a zero of $\mathbf{grad}(f)$ and let λ be its index as a **critical point**. By Lemma 18.37, z contributes with $(-1)^\lambda$ to the index sum of $\mathbf{grad}(f)$. Hence, if there are c_λ many such zeros, they contribute with $(-1)^\lambda c_\lambda$ to the index sum. Since the zeros of $\mathbf{grad}(f)$ are the critical points of f, this shows that the sum $\sum_\lambda (-1)^\lambda c_\lambda$ is exactly the index sum of f. □

By **Morse's Theorem 7.13** every smooth manifold admits a Morse function. Hence we can always find a Morse function and we know that the corresponding sum $\sum_\lambda (-1)^\lambda c_\lambda$ is independent of our choice of function. It remains to relate this sum to the Euler characteristic, i.e., it remains to show $\sum_\lambda (-1)^\lambda c_\lambda = \chi(X)$.

The details of this identification are beyond the scope of these notes. The idea, however, is based on the discussion of **homotopy types** at the beginning of Section 7.2 where we sketched how a Morse function on X helps us finding a **cell complex** which is homotopy equivalent to X. We now hint briefly at the general procedure and refer to [12, Part I] for the details. Let X be a **compact** smooth k-dimensional manifold and let $f : X \to \mathbb{R}$ be a Morse function. For $a \in \mathbb{R}$, set

$$X^a := f^{-1}((-\infty, a]) = \{x \in X : f(x) \leq a\}.$$

Now let $a_1 < \cdots < a_n$ be real numbers such that X^{a_i} contains exactly i critical points, and $X^{a_n} = X$. We then have $\emptyset \subset X^{a_1} \subset \ldots \subset X^{a_n} = X$. Let $\bar{\mathbb{B}}^\lambda$ denote the unit ball in $\mathbb{R}^\lambda$. We think of $\bar{\mathbb{B}}^\lambda$ as a λ-dimensional **cell**. In X^{a_i} there is exactly one critical point which is not contained in $X^{a_{i-1}}$. Let λ_i denote the **index** of this critical point. We then get

$$\begin{aligned} H_j(X^{a_i}, X^{a_{i-1}}; \mathbb{Z}) &\cong H_j(X^{a_{i-1}} \cup \bar{\mathbb{B}}^{\lambda_i}, X^{a_{i-1}}; \mathbb{Z}) \\ &\cong H_j(\bar{\mathbb{B}}^{\lambda_i}, \mathbb{B}^{\lambda_i}; \mathbb{Z}) \\ &\cong \begin{cases} \mathbb{Z} & \text{if } j = \lambda_i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where the first isomorphism can be deduced from the idea we sketched at beginning of Section 7.2 and the second follows from **excision**.
Now let $b_\lambda(X^{a_i}, X^{a_{i-1}})$ denote the **relative Betti number**, i.e., the **rank** of the abelian group $H_\lambda(X^{a_i}, X^{a_{i-1}}; \mathbb{Z})$. We then get

$$\chi(X^{a_i}, X^{a_{i-1}}) := \sum_{i=1}^{k} b_\lambda(X^{a_i}, X^{a_{i-1}}) = c_\lambda$$

where c_λ denotes the number of critical points of f of index λ. Since the Euler characteristic is **additive** (see [12, Part I, §5]), this shows

$$\chi(X) = \sum_{i=1}^{k} \chi(X^{a_i}, X^{a_{i-1}}) = c_0 - c_1 + c_2 - + \cdots \pm c_k = \sum_{i=1}^{k} (-1)^\lambda c_\lambda.$$

Together with Lemma 18.38 this shows that the index sum of the gradient field equals the Euler characteristic and finishes the proof of Theorem 18.16.

18.7 Existence of vector fields with no zeros

Given a compact oriented smooth manifold X, it is a natural question whether we can find a nowhere vanishing vector field $\mathbf{v}$ on X. Theorem 18.16 provides a **necessary** condition: if $\mathbf{v}$ has no zeros, then the index sum is zero and hence the Euler characteristic of X must be zero too. Hopf showed in [8] that this is also a **sufficient** condition:

Theorem 18.39 (Hopf - Existence of nowhere vanishing vector fields) Let X be a **compact**, **connected** smooth manifold. Then X possesses a nowhere vanishing vector field if and only if the **Euler characteristic** of X is zero.

- The proof is based on the ideas we developed for the proof of the Hopf Degree Theorem 17.1. It shows once more the power of Brouwer degree as an invariant.

Since smooth manifolds of odd dimension have Euler characteristic zero by Corollary 18.19, we get the following consequence.

Corollary 18.40 (Odd dimensional manifolds have nowhere vanishing vector fields) Let X be a compact smooth manifold without boundary. If the dimension of X is **odd**, then X possesses a nowhere vanishing vector field.

Proof of Theorem 18.39: First we assume that $X = \mathbb{R}^k$ and that we have a vector field $\mathbf{v}$ on X with only **finitely many** zeros. Then the vector field is just a smooth map $\mathbf{v}: \mathbb{R}^k \to \mathbb{R}^k$. Since $\mathbf{v}$ has only finitely many zeros, there is a closed ball $\bar{\mathbb{B}} = \bar{\mathbb{B}}^k_r$ containing all the zeros of $\mathbf{v}$ in its interior. Using the technique of the proof of Theorem 18.34 we can assume that the zeros of $\mathbf{v}$ are **nondegenerate**. Moreover, we can assume that the **origin** in $\mathbb{R}^k$ is **not** a zero of $\mathbf{v}$. Since there are no zeros of $\mathbf{v}$ on the boundary of $\bar{\mathbb{B}}$, $\mathbf{v}$ points **either** inward or outward at **all** boundary points of $\bar{\mathbb{B}}$. Since the index sum of $\mathbf{v}$ is zero, we can replace $\mathbf{v}$ with $-\mathbf{v}$ if $\mathbf{v}$ points inwards. Hence we can assume that $\mathbf{v}$ points outward at every boundary point of $\bar{\mathbb{B}}$. Since $\bar{\mathbb{B}}$ is compact, we can apply Lemma 18.25 and obtain that the index sum of $\mathbf{v}$ equals the **degree of the Gauss map** $g: \partial\bar{\mathbb{B}} = \mathbb{S}^{k-1}_r \to \mathbb{S}^{k-1}$. By assumption, the index sum of $\mathbf{v}$ is zero, and hence the degree of g is zero. Since g and $\bar{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ are homotopic, this show that $\bar{\mathbf{v}}$ has **degree zero**. Hence the **winding number** of $\mathbf{v}_{|\bar{\mathbb{B}}}$ around the origin is zero. Thus, $\partial\mathbf{v}_{|\bar{\mathbb{B}}} = \mathbf{v}_{|\partial\bar{\mathbb{B}}}: \partial\bar{\mathbb{B}} \to \mathbb{R}^k \setminus \{0\}$ is homotopic to a constant map by Theorem 17.10. Hence

$$\mathbf{v}_{|\mathbb{R}^k\setminus\mathbb{B}}: \mathbb{R}^k \setminus \mathbb{B} \to \mathbb{R}^k \setminus \{0\}$$

is a map to which we can apply Lemma 17.8. This implies that $\mathbf{v}_{|\mathbb{R}^k\setminus\mathbb{B}}$ extends to a smooth map

$$\mathbf{w}: \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$$

with $\mathbf{w} = \mathbf{v}$ outside the compact space $\bar{\mathbb{B}}$. Hence we have constructed a vector field $\mathbf{w}$ on $\mathbb{R}^k$ **without zeros** which equals $\mathbf{v}$ outside a compact subset.

Now we let $X \subset \mathbb{R}^N$ be a **compact**, oriented, connected k-dimensional smooth manifold with $\chi(X) = 0$. For $k = 1$, the assertion follows from the classification of compact one-manifolds in Theorem 11.1. Hence we may assume $k \geq 2$. As pointed out in Remark 18.32

there exists a vector field $\mathbf{v}$ on X with only finitely many nondegenerate zeros. Since $\chi(X) = 0$, Theorem 18.16 implies the index sum of $\mathbf{v}$ is zero. Let $\phi\colon U \to V \subset X$ be a local parametrization of X. By the **Isotopy Theorem 12.13**, we can find a **diffeomorphism** of X to itself such that $f^*\mathbf{v}$ has finitely many nondegenerate zeros all of which lie in V. Since V is diffeomorphic to U via ϕ and U is diffeomorphic to $\mathbb{R}^k$, we may assume that the zeros of $\mathbf{v}$ all lie in an **open** subset which is **diffeomorphic** to $\mathbb{R}^k$. Thus, we can apply the case Euclidean space we considered above to replace $\mathbf{v}$ with a vector field that has no zeros. This proves the theorem. □

A. Solutions to exercises

A.2 Smooth manifolds

A.2.1 Smooth maps and manifolds

Solution (**Exercise 2.1**) **(a)** We have remarked in the main text that f is smooth, since each component of f is a polynomial. To get some more exercise, we could calculate all partial derivatives (in all degrees) and check that they exist and are continuous. So let us calculate the partial derivatives. We denote the two components of f by $f_1(x, y) = x^2 - y^2$ and $f_2(x, y) = 2xy$. The first partial derivatives are

$$\frac{\partial f_1}{\partial x} = 2x,\ \frac{\partial f_1}{\partial y} = -2y,\ \frac{\partial f_2}{\partial x} = 2y,\ \frac{\partial f_2}{\partial y} = 2x.$$

All these functions are differentiable. Hence we can calculate the second derivatives:

$$\begin{aligned}\frac{\partial^2 f_1}{\partial x \partial x} = 2, \frac{\partial^2 f_1}{\partial x \partial y} = 0, \frac{\partial^2 f_1}{\partial y \partial x} = 0, \frac{\partial^2 f_1}{\partial y \partial y} = -2,\\ \frac{\partial^2 f_2}{\partial x \partial x} = 0, \frac{\partial^2 f_2}{\partial x \partial y} = 2, \frac{\partial^2 f_2}{\partial y \partial x} = 2, \frac{\partial^2 f_2}{\partial y \partial y} = 0.\end{aligned}$$

The second derivatives are again all differentiable and we see that the next derivatives will all vanish. This implies that all further partial derivatives vanish and therefor are differentiable. This shows that f is smooth.

(b) We denote the two components of f by $f_1(x, y) = x^2 - y^2$ and $f_2(x, y) = 2xy$. We calculate the Jacobian $J(f)(x, y)$ of $f_{|U}$ at a point $p = (x, y) \in U$:

$$J(f)(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(p) & \frac{\partial f_1}{\partial y}(p) \\ \frac{\partial f_2}{\partial x}(p) & \frac{\partial f_2}{\partial y}(p) \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$

The determinant of the Jacobian at $p = (x, y)$ is $4x^2 + 4y^2$. This is a positive real number for every p in $U = \mathbb{R}^2 \setminus \{(0, 0)\}$. This implies that the Jacobian is invertible at every point in U.

(c) Even though $f_{|U}$ has an invertible derivative at every point, it is not invertible itself. For $f_{|U}$ is **not injective**. For example, $f(1, 1) = (0, 2) = f(-1, -1)$.

Solution (Exercise 2.2) **(a)** The assertion is true if X, Y and Z are **open** subsets of $\mathbb{R}^N$, $\mathbb{R}^M$, $\mathbb{R}^L$, respectively. For in this case this is just the Chain Rule from Calculus. Now let $x \in X$ and $y = f(x)$. Since g is smooth at y, there is an open subset $V \subset \mathbb{R}^M$ with $y \in V$ and a smooth map $G : V \to \mathbb{R}^L$ such that $G_{|V\cap Y} = g_{|V\cap Y}$. Since f is smooth at x, there is an open subset $U \subset \mathbb{R}^N$ with $x \in U$ and a smooth map $F : U \to \mathbb{R}^M$ such that $F_{|U\cap X} = f_{|U\cap X}$. After replacing U with $U \cap f^{-1}(V)$ if necessary (to make sure that the image of F lies in V), we have

$$G\circ F = g\circ f \text{ on } X \cap U, \text{ since } f(X \cap U) \subset Y \cap V.$$

By the Chain Rule, we know that $G\circ F : U \to \mathbb{R}^L$ is smooth, since U is open in $\mathbb{R}^N$. This shows that $g\circ f$ is smooth.

(b) By assumption f and g are smooth and have smooth inverses f^{-1} and g^{-1} respectively. Since f and g are bijective, so is $g\circ f$. By the previous point, both $g\circ f$ and $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$ are smooth.

Solution (Exercise 2.3) **(a)** The inverse is

$$g : \mathbb{R}^k \to B_r,\ y \mapsto \frac{ry}{\sqrt{r^2 + |y|^2}}.$$

For we can calculate

$$\begin{aligned} f(g(y)) &= \frac{rg(y)}{\sqrt{r^2 - |g(y)|^2}} = \frac{r\frac{ry}{\sqrt{r^2+|y|^2}}}{\sqrt{r^2 - \frac{|ry|^2}{r^2+|y|^2}}} = \frac{r^2 y}{\sqrt{r^2 + |y|^2}} \cdot \frac{\sqrt{r^2 + |y|^2}}{\sqrt{r^2(r^2 + |y|^2 - |y|^2)}} \\ &= y \end{aligned}$$

and

$$\begin{aligned} g(f(x)) &= \frac{rf(x)}{\sqrt{r^2 + f(x)|^2}} = \frac{r\frac{rx}{\sqrt{r^2-|x|^2}}}{\sqrt{r^2 + \frac{|rx|^2}{r^2-|x|^2}}} = \frac{r^2 x}{\sqrt{r^2 - |x|^2}} \cdot \frac{\sqrt{r^2 - |x|^2}}{\sqrt{r^2(r^2 - |x|^2 + |x|^2)}} \\ &= x. \end{aligned}$$

Both f and g are smooth, since they are the composite of several smooth maps. Hence the previous exercise shows that they are both smooth.

(b) Let $x \in X$. By definition of a smooth manifold, there is an open subset $V \subset X$ with $x \in V$, an open subset $U \subset \mathbb{R}^k$ and a diffeomorphism $\phi : U \to V$. Let $u \in U$ be the inverse $\phi^{-1}(x)$. If u is not the origin in $\mathbb{R}^k$, we compose ϕ with the translation $T_u : \mathbb{R}^k \to \mathbb{R}^k$ defined by $T_u(y) = y+u$ which satisfies $T_u(0) = u$. Note that T_u is a diffeomorphism, since it is invertible and both T_u and its inverse T_{-u} have the identity matrix as their Jacobian matrix at any point. Hence all higher partial derivatives vanish and exist.

Thus after composing ϕ with T_u, we can assume $\phi(0) = x$. Now it suffices to choose a small enough radius r such that $\phi(B_r^k(0)) \subset V$. Then $\phi_{|B_r^k(0)} : B_r^k(0) \to \phi(B_r^k(0)) \subset X$ is the desired local parametrization.

(c) By the previous point, for every $x \in X$ there is a diffeomorphism $\phi\colon B_r^k(0) \to V$ for some open subset $V \subset X$ with $x \in V$. Now it suffices to precompose ϕ with the diffeomorphism $g\colon \mathbb{R}^k \to B_r^k(0)$ of the first point in this exercise.

Solution (Exercise 2.4) Every linear map $\mathbb{R}^n \to \mathbb{R}^m$ is smooth, and the derivative is equal to the map itself. Hence, given a k-dimensional vector subspace V of $\mathbb{R}^N$, it suffices to choose a basis in V to get a linear isomorphism $\phi\colon V \to \mathbb{R}^k$. This map serves as a parametrization, since it is a diffeomorphism.

Now given a linear map $f\colon V \to \mathbb{R}^m$, the composite $\phi \circ f\colon \mathbb{R}^k \to \mathbb{R}^m$ is linear and therefore smooth. Since ϕ is a diffeomorphism, this implies that f must be smooth too.

Solution (Exercise 2.5) **(a)** Let $a > 0$ be a real number. We want to show that the subset

$$H_a = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = a\} \subset \mathbb{R}^3$$

is a 2-dimensional manifold. Hence we need to find local parametrizations. First, there is a diffeomorphism

$$\phi\colon \mathbb{R}^2 \setminus \overline{B_a((0,0))} \to H \cap \{z > 0\},\ (x, y) \mapsto \left(x, y, \sqrt{x^2 + y^2 - a}\right)$$

where $\overline{B_a((0,0))}$ denotes the closed ball of radius a around the origin, i.e.,

$$\overline{B_{\sqrt{a}}((0,0))} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le a\}.$$

Note that $H \cap \{z > 0\}$ is an open subset of H, because it is equal to the intersection of H with the open subset $\{z > 0\} \subset \mathbb{R}^3$. The inverse to ϕ is the projection map

$$\phi^{-1}\colon H \cap \{z > 0\} \to \mathbb{R}^2,\ (x, y, z) \mapsto (x, y).$$

This map is smooth, since it can be extended to a smooth map on the whole of $\mathbb{R}^3$ to $\mathbb{R}^2$. We know that ϕ is smooth, since its component functions are infinitely often differentiable functions in each variable on the open subset $\mathbb{R}^2 \setminus \overline{B_a((0,0))}$. We can for example calculate the Jacobian matrix in the standard basis at a point

$$d\phi_x\colon \mathbb{R}^3 \to \mathbb{R}^2,\ \begin{pmatrix} \partial\phi_1/\partial x & \partial\phi_2/\partial x & \partial\phi_3/\partial x \\ \partial\phi_1/\partial y & \partial\phi_2/\partial y & \partial\phi_3/\partial y \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & x(x^2 + y^2 - a)^{-1/2} \\ 0 & 1 & y(x^2 + y^2 - a)^{-1/2} \end{pmatrix}.$$

On the open set $\mathbb{R}^2 \setminus \overline{B_a((0,0))}$, the entries of this matrix are continuously differentiable functions.

The local parametrization for $H \cap \{z < 0\}$ is similarly given by

$$\phi\colon \mathbb{R}^2 \setminus \overline{B_a((0,0))} \to H \cap \{z < 0\},\ (x, y) \mapsto \left(x, y, -\sqrt{x^2 + y^2 - a}\right).$$

It remains to cover the points in $H \cap \{z = 0\}$. We are going to cover those points by the following four open sets together with local parametrizations:

$$B_{\sqrt{a}}((0,0)) \to H \cap \{x^2 + z^2 < a\},\ (x, z) \mapsto \left(x, \sqrt{z^2 - x^2 + a}, z\right)$$

$$B_{\sqrt{a}}((0,0)) \to H \cap \{x^2 + z^2 < a\},\ (x,z) \mapsto \left(x, -\sqrt{z^2 - x^2 + a}, z\right)$$

$$B_{\sqrt{a}}((0,0)) \to H \cap \{y^2 + z^2 < a\},\ (y,z) \mapsto \left(\sqrt{z^2 - y^2 + a}, y, z\right)$$

$$B_{\sqrt{a}}((0,0)) \to H \cap \{y^2 + z^2 < a\},\ (y,z) \mapsto \left(-\sqrt{z^2 - y^2 + a}, y, z\right).$$

(b) **If** $a = 0$, then the set

$$H_0 = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2 + y^2 - z^2 = 0\} \subset \mathbb{R}^3$$

is **not a manifold**, since there is no local parametrization around the point $(0,0,0) \in H$. For, assume there was such a local parametrization $\phi\colon U \to V$ with both $U \subseteq \mathbb{R}^2$ and $(0,0,0) \in V \subseteq H$ open. We can assume $\phi((0,0)) = (0,0,0)$. (Remember that V being open in H means that there is an open subset $\tilde{V} \subseteq \mathbb{R}^3$ with $V = H \cap \tilde{V}$. In particular, $\tilde{V}$ must contain a small open ball $B_r((0,0,0))$ around the origin.) Then removing the point $(0,0,0)$ from H splits H and therefore V into **two disjoint connected components**. But $U \setminus \{(0,0)\} \subset \mathbb{R}^2$ is connected. Hence ϕ cannot be a diffeomorphism (because if it was, $\phi_{|U\setminus\{(0,0)\}}$ would also be a diffeomorphism and that would imply that $V \setminus \{(0,0,0)\}$ were connected as well; a contradiction).

Solution (**Exercise 2.6**) For $0 < b < a$, let $T(a,b)$ denote the set of points in $\mathbb{R}^3$ at distance b from the circle of radius a in the xy-plane. We can parametrize these points as follows: First the points in the xy-plane which lie on the circle of radius a satisfy

$$(a\cos t, a\sin t, 0) \text{ for } t \in [0, 2\pi).$$

A point in the plane in the direction of a fixed point $(a\cos t, a\sin t, 0)$ which lies on the circle of radius b around the point $(a\cos t, a\sin t, 0)$ has coordinates $(a + b\cos s)\cos t, (a + b\cos s)\sin t, b\sin s)$ with $s \in [0, 2\pi)$. For we have

$$\begin{aligned}
&|\,((a + b\cos s)\cos t, (a + b\cos s)\sin t, b\sin s) - (a\cos t, a\sin t, 0)\,|^2 \\
=&|\,((b\cos s)\cos t, (b\cos s)\sin t, b\sin s)\,|^2 \\
=&b^2\cos^2 s(\cos^2 t + \sin^2 t) + b^2\sin^2 s = b^2.
\end{aligned}$$

Hence the points on $T(a,b) \subset \mathbb{R}^3$ are given by

$$T(a,b) = \{((a + b\cos s)\cos t, (a + b\cos s)\sin t, b\sin s) \,:\, s,t \in [0, 2\pi)\}.$$

The points on $S^1 \times S^1 \subset \mathbb{R}^4$ are given by

$$\begin{aligned}
S^1 \times S^1 &= \{\left(x_1, x_2, x_3, x_4\right) \in \mathbb{R}^4 \,:\, x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\} \\
&= \{(\cos t, \sin t, \cos s, \sin s) \in \mathbb{R}^4 \,:\, t \in [0, 2\pi), s \in [0, 2\pi)\}.
\end{aligned}$$

Now it is clear how we can define a continuous map

$$\begin{aligned}
\phi\colon\ S^1 \times S^1 &\to T(a,b), \\
(\cos t, \sin t, \cos s, \sin s) &\mapsto ((a + b\cos s)\cos t, (a + b\cos s)\sin t, b\sin s).
\end{aligned}$$

In order to check that ϕ is smooth, we use the coordinates of $\mathbb{R}^4$ again. Then ϕ is given by

$$\left(x_1, x_2, x_3, x_4\right) \mapsto \left((a+bx_3)x_1, (a+bx_3)x_2, bx_4\right).$$

Its derivative in the standard basis at a point $p = (x_1, x_2, x_3, x_4)$ is then given by

$$d\phi_p : \mathbb{R}^4 \to \mathbb{R}^3,\ d\phi_p = \begin{pmatrix} a+bx_3 & 0 & bx_1 & 0 \\ 0 & a+bx_3 & bx_2 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$$

Since all partial derivatives are smooth maps, ϕ is a smooth map.
Its inverse is the map $\psi : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$\left(y_1, y_2, y_3,\right) \mapsto \left(\frac{y_1}{\sqrt{y_1^2+y_2^2}}, \frac{y_2}{\sqrt{y_1^2+y_2^2}}, \frac{\sqrt{y_1^2+y_2^2}-a}{b}, y_3/b \right).$$

The image of ψ lies in $S^1 \times S^1$ since

$$\left(\frac{y_1}{\sqrt{y_1^2+y_2^2}}\right)^2 + \left(\frac{y_2}{\sqrt{y_1^2+y_2^2}}\right)^2 = \frac{y_1^2+y_2^2}{y_1^2+y_2^2} = 1.$$

To check the other equation, we recall that y_1, y_2 and y_3 are connected by the condition of being on $T(a,b)$ which means

$$\begin{aligned} &\left(y_1 - \frac{ay_1}{\sqrt{y_1^2+y_2^2}}\right)^2 + \left(y_2 - \frac{ay_2}{\sqrt{y_1^2+y_2^2}}\right)^2 + y_3^2 = b^2 \\ \iff &\left(\sqrt{y_1^2+y_2^2} - a\right)^2 + y_3^2 = b^2 \\ \iff &y_1^2+y_2^2+y_3^2+a^2-2a\sqrt{y_1^2+y_2^2} = b^2. \end{aligned}$$

Now we can calculate

$$\left(\frac{\sqrt{y_1^2+y_2^2}-a}{b}\right)^2 + (y_3/b)^2 = \frac{y_1^2+y_2^2+y_3^2+a^2-2a\sqrt{y_1^2+y_2^2}}{b^2} = 1.$$

Hence the image of (y_1, y_2, y_3) does lie on $S^1 \times S^1$.

We can easily check that $\psi \circ \phi$ is the identity. For example, using $y_1^2 + y_2^2 = (a+bx_3)^2$, we get

$$\psi(\phi(x)) = \left(\frac{(a+bx_3)x_1}{\sqrt{(a+bx_3)^2}}, \frac{(a+bx_3)x_2}{\sqrt{(a+bx_3)^2}}, \frac{(a+bx_3)-a}{b}, \frac{bx_4}{b} \right) = (x_1, x_2, x_3, x_4),$$

and, setting $\sqrt{y_{12}} := \sqrt{y_1^2 + y_2^2}$,

$$\phi(\psi(y)) = \left(\left(a + b\frac{\sqrt{y_{12}} - a}{b}\right)\frac{y_1}{\sqrt{y_{12}}}, \left(a + b\frac{\sqrt{y_{12}} - a}{b}\right)\frac{y_2}{\sqrt{y_{12}}}, \frac{by_3}{b}\right)$$
$$= (y_1, y_2, y_3),$$

It remains to check that ψ is smooth. The derivative of ψ in the standard basis at a point $q = (y_1, y_2, y_3)$ is then given by

$$d\psi_q : \mathbb{R}^3 \to \mathbb{R}^4, \; d\psi_q = \begin{pmatrix} \frac{y_2^2}{(y_1^2+y_2^2)^{3/2}} & -\frac{y_1 y_2}{(y_1^2+y_2^2)^{3/2}} & 0 \\ -\frac{y_1 y_2}{(y_1^2+y_2^2)^{3/2}} & \frac{y_1^2}{(y_1^2+y_2^2)^{3/2}} & 0 \\ \frac{y_1}{b(y_1^2+y_2^2)^{1/2}} & \frac{y_2}{b(y_1^2+y_2^2)^{1/2}} & 0 \\ 0 & 0 & 1/b \end{pmatrix}.$$

Since $b < a$ we know $y_1^2 + y_2^2 \neq 0$ and all partial derivatives are continuous smooth functions. Hence ψ is smooth. This proves that ϕ is a global diffeomorphism

$$\phi : S^1 \times S^1 \to T(a, b) \text{ for all } 0 < b < a.$$

Solution (Exercise 2.7) **(a)** Let $N = (0, \dots, 0, 1) \in S^k$ be the north pole on the k-dimensional sphere. The stereographic projection ϕ_N^{-1} from $S^k \setminus \{N\}$ onto $\mathbb{R}^k$ is the map which sends a point p to the point at which the line through N and p intersects the subspace in $\mathbb{R}^{k+1}$ defined by $x_{k+1} = 0$. In order to get from N to p, we walk in the direction of the vector

$$v := p - N = (x_1, \dots, x_{k+1} - 1).$$

To find $\phi_N^{-1}(p)$ we need to find the real number λ such that the $(k+1)$-st coordinate of $N + \lambda \cdot v$ is 0. Hence we need to solve

$$1 + \lambda(x_{k+1} - 1) = 0 \iff \lambda = \frac{1}{1 - x_{k+1}}.$$

Thus

$$\phi_N^{-1}(x_1, \dots, x_{k+1}) = \frac{1}{1 - x_{k+1}}(x_1, \dots, x_k).$$

(b) We calculate the inverse ϕ_N: Given the point $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. To find its image under ϕ_N, we walk from N in the direction of the vector $w = x - N$ until we reach the sphere, i.e. we need to find $\lambda \in \mathbb{R}$ such that

$$N + \lambda \cdot w = (\lambda x_1, \dots, \lambda x_k, 1 - \lambda)$$

lies on $\mathbb{S}^k$. Hence we need to solve

$$\begin{aligned} &\lambda^2(x_1^2 + \cdots + x_k^2) + (1-\lambda)^2 = 1 \\ \iff &\lambda^2|x|^2 + 1 - 2\lambda + \lambda^2 = 1 \\ \iff &\lambda^2(1+|x|^2) - 2\lambda = 0 \\ \Rightarrow &\lambda = \frac{2}{1+|x|^2} \text{ (since 0 is not a valid solution).} \end{aligned}$$

Thus

$$\phi_N(x) = \frac{1}{1+|x|^2}\left(2x_1, \dots, 2x_k, |x|^2 - 1\right).$$

We need to check that both ϕ_S and ϕ_S^{-1} are smooth. All entries in ϕ_N are smooth functions (they are polynomials on the domain and these are infinitely often differentiable functions in each variable). This implies that ϕ_N is smooth. But let us calculate the derivative anyway since we are going to use it later:

$$d(\phi_N)_x = \frac{2}{(1+|x|^2)^2}\begin{pmatrix} 1+|x|^2-x_1^2 & -2x_1x_2 & -2x_1x_3 & \dots & -2x_1x_k \\ -2x_2x_1 & 1+|x|^2-x_2^2 & -2x_2x_3 & \dots & -2x_2x_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -2x_kx_1 & \dots & \dots & \dots & 1+|x|^2-x_k^2 \\ 2x_1 & 2x_2 & \dots & \dots & 2x_k \end{pmatrix}.$$

All entries are smooth functions on the domain and ϕ_N is smooth. Similarly for ϕ_N^{-1}, all entries are smooth functions, since $x_{k+1} \neq 1$. Its derivative is given by

$$d(\phi_N^{-1})_x = \begin{pmatrix} \frac{1}{1-x_{k+1}} & 0 & \dots & 0 & \frac{x_1}{(1-x_{k+1})^2} \\ 0 & \frac{1}{1-x_{k+1}} & \dots & 0 & \frac{x_2}{(1-x_{k+1})^2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{1-x_{k+1}} & \frac{x_k}{(1-x_{k+1})^2} \end{pmatrix}.$$

(c) The formulae for the projection from the south pole $S = (0, \dots, 0, -1) \in \mathbb{S}^k$ are similar:

$$\phi_S^{-1}(x_1, \dots, x_{k+1}) = \frac{1}{1+x_{k+1}}(x_1, \dots, x_k).$$

and, while we get the same λ,

$$\phi_S(x) = S + \lambda(x - S) = \frac{1}{1+|x|^2}\left(2x_1, \dots, 2x_k, 1 - |x|^2\right).$$

To check that both ϕ_S and ϕ_S^{-1} are both smooth is completely analogous. Since all points are covered by these two parametrizations, $\mathbb{S}^k$ is a smooth manifold. We note that the formula for the derivative of ϕ_S^{-1} is given by

$$d(\phi_S^{-1})_x = \begin{pmatrix} \frac{1}{1+x_{k+1}} & 0 & \dots & 0 & \frac{-x_1}{(1+x_{k+1})^2} \\ 0 & \frac{1}{1+x_{k+1}} & \dots & 0 & \frac{-x_2}{(1+x_{k+1})^2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{1+x_{k+1}} & \frac{-x_k}{(1+x_{k+1})^2} \end{pmatrix}.$$

Solution (Exercise 2.8) **(a)** For $(z_0, z_1) \in \mathbb{S}^3$, we need to check $\pi(z_0, z_1) \in \mathbb{S}^2$. To do this, we recall that $|z| = z\bar{z}$ and $\bar{\bar{z}} = z$ for any complex number z. Now we compute:

$$\begin{aligned}
&(2z_0\bar{z}_1) \cdot (2\bar{z}_0 z_1) + \left(|z_0|^2 - |z_1|^2\right)^2 \\
&= 4|z_0|^2|z_1|^2 + |z_0|^4 - 2|z_0|^2|z_1|^2 + |z_1|^4 \\
&= \left(|z_0|^2 + |z_1|^2\right)^2 \\
&= 1
\end{aligned}$$

where the final step uses that $(z_0, z_1) \in \mathbb{S}^3$.

(b) First we assume $\pi(z_0, z_1) = \pi(w_0, w_1)$: Then we get

$$\begin{aligned}
&\left(2w_0\bar{w}_1, |w_0|^2 - |w_1|^2\right) = \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right) \\
\iff & w_0\bar{w}_1 = z_0\bar{z}_1 \text{ and } |w_0|^2 - |w_1|^2 = |z_0|^2 - |z_1|^2.
\end{aligned}$$

We have $|w_0|^2 - |w_1|^2 = |z_0|^2 - |z_1|^2$ and $|w_0|^2 + |w_1|^2 = 1 = |z_0|^2 + |z_1|^2$. Putting these together implies

$$z_0\bar{z}_0 = |z_0|^2 = |w_0|^2 = w_0\bar{w}_0 \text{ and } z_1\bar{z}_1 = z_1^2 = w_1^2 = w_1\bar{w}_1.$$

Remembering that neither of the numbers can be zero, by rewriting these equations we get

$$\frac{w_0}{z_0} = \frac{\bar{z}_0}{\bar{w}_0} \text{ and } \frac{w_1}{z_1} = \frac{\bar{z}_1}{\bar{w}_1}.$$

Hence, looking at the left-hand side, there is a complex number $\alpha \neq 0$ such that

$$\alpha = \frac{w_0}{z_0} = \frac{\bar{z}_0}{\bar{w}_0} = \frac{1}{\bar{\alpha}}, \text{ and thus } w_0 = \alpha z_0 \text{ with } \alpha\bar{\alpha} = 1.$$

On the other hand, we also know $\frac{w_0}{z_0} = \frac{\bar{z}_1}{\bar{w}_1}$. Combining these equations yields

$$\alpha = \frac{w_0}{z_0} = \frac{\bar{z}_1}{\bar{w}_1} = \frac{w_1}{z_1}, \text{ and thus } w_1 = \alpha z_1.$$

This shows that the desired α with $\alpha\bar{\alpha} = 1$ exists.

Now we assume that $(w_0, w_1) = (\alpha z_0, \alpha z_1)$ with $|\alpha|^2 = \alpha\bar{\alpha} = 1$: Then we compute

$$\begin{aligned}
\pi(w_0, w_1) &= \left(2w_0\bar{w}_1, |w_0|^2 - |w_1|^2\right) \\
&= \left(2\alpha z_0\bar{\alpha}\bar{z}_1, |\alpha|^2|z_0|^2 - |\alpha|^2|z_1|^2\right) \\
&= \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right) \\
&= \pi(z_0, z_1).
\end{aligned}$$

(c) Let $p \in \mathbb{S}^2$ be a fixed point and fix a point (z_0, z_1) in $\mathbb{S}^3$ with $\pi(z_0, z_1) = p$. By the previous point, we know that the points in $\pi^{-1}(p)$ are parametrized by the complex numbers α with $|\alpha|^2 = 1$. The latter condition means $\alpha \in \mathbb{S}^1 \subset \mathbb{C}$. Hence we get a bijective map

$$f: \mathbb{S}^1 \to \pi^{-1}(p),\ \alpha \mapsto (\alpha z_0, \alpha z_1).$$

Since this map just consists of multiplication with nonzero complex numbers, we can conclude that it is a smooth map where we consider $\mathbb{S}^1 \subset \mathbb{C} = \mathbb{R}^2$ and $\pi^{-1}(p) \subset \mathbb{R}^4$ as subsets in real Euclidean space. The previous point implies that f is onto. Since not both of z_0 and z_1 can be zero at the same time, we see that f is one-to-one. For, say $z_0 \neq 0$, then $\alpha z_0 = \beta z_0$ implies $\alpha = \beta$. The inverse is given by sending $(w_0, w_1) \in \pi^{-1}(p)$ to $\alpha \in \mathbb{S}^1$ such that $w_0 = \alpha z_0$ and $w_1 = \alpha z_1$. In every open subset of $\pi^{-1}(p) \subset \mathbb{S}^3$ where $z_0 \neq 0$, the map $w_0 \mapsto w_0 z_0 = \alpha$ is smooth. Similarly, in every open subset of $\pi^{-1}(p) \subset \mathbb{S}^3$ where $z_1 \neq 0$, the map $w_1 \mapsto w_1 z_1 = \alpha$ is smooth. Hence f^{-1} is a smooth map as well, and f is a diffeomorphism.

A.2.2 Tangent spaces

Solution (**Exercise 2.9**) We can choose any basis of V to define a linear isomorphism $\phi\colon \mathbb{R}^k \to V$ which is a diffeomorphism. Given a point $x \in V$, we modify ϕ by adding x and get a new diffeomorphism (not linear anymore!)

$$\psi: \mathbb{R}^k \to V,\ w \mapsto \phi(w) + x.$$

The derivative $d\psi_0$ of ψ at 0 is just $\phi\colon \mathbb{R}^k \to \mathbb{R}^N$ (independent of x). The tangent space of V at x is by our definition the image of $d\psi_0$ in $\mathbb{R}^N$, which is equal to the image of ϕ in $\mathbb{R}^N$ which is by definition of ϕ equal to V.

Solution (**Exercise 2.10**) We only answer the question about the tangent space of $\mathbb{T}(a,b)$ in $\mathbb{R}^3$. For all points apart from $(a+b,0,0)$ we can parametrize $\mathbb{T}(a,b) \subset \mathbb{R}^3$ by

$$\begin{aligned}\phi\colon (0,2\pi)\times(0,2\pi) &\to T(a,b)\\ (s,t) &\mapsto ((a+b\cos s)\cos t, (a+b\cos s)\sin t, b\sin s).\end{aligned}$$

The derivative of ϕ at (s,t) is

$$d\phi_{(s,t)}\colon \mathbb{R}^2 \to \mathbb{R}^3,\ d\phi_{(s,t)} = \begin{pmatrix} -b\sin s\cos t & -(a+b\cos s)\sin t \\ -b\sin s\sin t & (a+b\cos s)\cos t \\ b\cos s & 0 \end{pmatrix}.$$

The tangent space to $T(a,b)$ at the point $\phi(s,t)$ is $d\phi_{(s,t)}(\mathbb{R}^2) \subset \mathbb{R}^3$.

Let us check that the column vectors of the matrix $d\phi_{(s,t)}$, and hence the whole tangent space, is orthogonal to the vector pointing from the center of the circle with radius b to $\phi(s,t)$. The center point is $(a\cos t, a\sin t, 0$, and hence the vector we need to look at is

$(b\cos s\cos t, b\cos s\sin t, b\sin s)$. We calculate the two scalar products:

$$\begin{aligned}
&(b\cos s\cos t, b\cos s\sin t, b\sin s)\cdot\begin{pmatrix}-b\sin s\cos t\\ -b\sin s\sin t\\ b\cos s\end{pmatrix}\\
=&-b^2\cos s\sin s\cos^2 t - b^2\cos s\sin s\sin^2 t + b^2\sin s\cos s\\
=&b^2\cos s\sin s(-\cos^2 t-\sin^2 t+1)\\
=&b^2\cos s\sin s(-1+1)=0
\end{aligned}$$

and

$$\begin{aligned}
&(b\cos s\cos t, b\cos s\sin t, b\sin s)\cdot\begin{pmatrix}-(a+b\cos s)\sin t\\ (a+b\cos s)\cos t\\ 0\end{pmatrix}\\
=&-b(a+b\cos s)\cos s\sin t\cos t b(a+b\cos s)\cos s\sin t\cos t+0\\
=&0.
\end{aligned}$$

In order to cover also the point $(a+b,0,0)$ it suffices to rotate our parametrization by the angle π in the $xy-$plane and use the diffeomorphism

$$\begin{aligned}
\phi\colon\ (0,2\pi)\times(0,2\pi) &\to \mathbb{T}(a,b)\\
(s,t) &\mapsto ((-a+b\cos s)\cos t, (-a+b\cos s)\sin t, b\sin s)
\end{aligned}$$

which covers $(a+b,0,0)$ for $(s,t)=(\pi,\pi)$.

Solution (**Exercise 2.11**) Around the point $(\sqrt{a},0,0)$ on H_a we can choose the local parametrization

$$\phi\colon\ B_{\sqrt{a}}((0,0))\to H\cap\{y^2-z^2<a\},\ (y,z)\mapsto\left(\sqrt{z^2-y^2+a},y,z\right).$$

The derivative in the standard basis at a point (y,z) is the linear map

$$d\phi_{(y,z)}\colon\ \mathbb{R}^2\to\mathbb{R}^3,\ d\phi_{(y,z)}=\begin{pmatrix}-\frac{y}{\sqrt{z^2-y^2+a}} & \frac{z}{\sqrt{z^2-y^2+a}}\\ 1 & 0\\ 0 & 1\end{pmatrix}.$$

Hence at a point $(u,v,w)\in H_a$ the image of the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix}-v/u\\ 1\\ 0\end{pmatrix}\text{ and }\begin{pmatrix}w/u\\ 0\\ 1\end{pmatrix}$$

where we write $u=\sqrt{w^2-v^2+a}$). Hence the tangent space at $T_{(u,v,w)}(H_a)$ is spanned by these two vectors.

For $(u,v,w)=(\sqrt{a},0,0)$ we get that $T_{(\sqrt{a},0,0)}(H_a)$ is simply spanned by

$$\begin{pmatrix}0\\1\\0\end{pmatrix}\text{ and }\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Solution (Exercise 2.12) **(a)** The inverse map to F is the projection π onto the first factor, for we obviously have $\pi \circ F = \mathrm{Id}_X$ and $F \circ \pi = \mathrm{Id}_{\Gamma(f)}$. Let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$, and let f be smooth. Then for any point $x \in X$, there is an open subset $U \subset \mathbb{R}^N$ and a smooth map $\tilde{f}\colon U \to \mathbb{R}^M$ with $\tilde{f}_{X\cap U} = f_{X\cap U}$. Then $(\tilde{f} \times \mathrm{Id})\colon U \to \mathbb{R}^N \times \mathbb{R}^M$ is a smooth extension of $F_{X\cap U}$. Hence F is smooth. The inverse map π is a smooth map, since it extends to the smooth projection on all of $\mathbb{R}^N \times \mathbb{R}^M$. Hence F is a diffeomorphism when f is smooth. Hence any local parametrization $\phi\colon V \to X$ can be extended to a local parametrization $F \circ \phi\colon V \to \Gamma(f)$. Thus the graph $\Gamma(f)$ is a manifold if X is.

(b) For $\in X$, let $\phi\colon U \to X$ be a local parametrization around x with $\phi(0) = x$ and open $U \subseteq \mathbb{R}^k$. Let $\psi\colon W \to Y$ be a local parametrization around $f(x)$ with $\psi(0) = f(x)$ and open $W \subseteq \mathbb{R}^l$. Then $\phi \times \psi\colon U \times W \to X \times Y$ is a local parametrization around $(x, f(x))$ with $U \times W \subset \mathbb{R}^{k+l}$ open. Then we can construct a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F"] & X \times Y \\ U \arrow[u, "\phi"] \arrow[r, "G=\psi^{-1}\circ F\circ\phi"'] & U \times W \arrow[u, "\phi\times\psi"'] \end{tikzcd}$$

where G is the map defined by $v \mapsto (v, \psi^{-1}(f(\phi(v)))$. Thus G is the map $\mathrm{Id}_V \times (\psi^{-1} \circ f\phi)$. Hence $dG_0\colon \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^l$ is the linear map

$$dG_0 = \mathrm{Id}_{\mathbb{R}^k} \times (d\psi^{-1}_{f(x)} \circ df_x \circ d\phi_0).$$

Thus in the commutative diagram below, dF_x has to be defined as $\mathrm{Id}_{T_x(X)} \times df_x$:

$$\begin{tikzcd} T_x(X) \arrow[r, dotted, "dF_x=\mathrm{Id}_{T_x(X)}\times df_x"] & T_x(X) \times T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dG_0=\mathrm{Id}_{\mathbb{R}^k}\times(d\psi^{-1}_{f(x)}\circ df_x\circ d\phi_0)"'] & \mathbb{R}^k \times \mathbb{R}^l. \arrow[u, "d\phi_0\times d\psi_0"'] \end{tikzcd}$$

Hence $dF_x(v) = (v, df_x(v))$.

(c) For $\in X$, let $\phi\colon U \to X$ be a local parametrization around x with $\phi(0) = x$ and open $U \subseteq \mathbb{R}^k$. Since F is a diffeomorphism, $\psi = F \circ \phi\colon U \to \Gamma(f)$ is then a local parametrization around $(x, f(x))$ with $\psi(0) = F(\phi(0)) = (x, f(x))$. The tangent space of $\Gamma(f)$ at $(x, f(x))$ is by definition the image of $d\psi_0\colon \mathbb{R}^k \to \mathbb{R}^{N+M}$. By the chain rule we have

$$d\psi_0 = dF_x \circ d\phi_0\colon \mathbb{R}^k \xrightarrow{d\phi_0} \mathbb{R}^N \xrightarrow{dF_x} \mathbb{R}^{N+M}.$$

Hence by our definition of tangent spaces:

$$T_{(x,f(x))}(\Gamma(f)) = d\psi_0(\mathbb{R}^k) = dF_x(d\phi_0(\mathbb{R}^k)) = dF_x(T_x(X)).$$

Finally, by the previous point, we know $dF_x = \mathrm{Id}_{T_x(X)} \times df_x$ and get

$$T_{(x,f(x))}(\Gamma(f)) = (\mathrm{Id}_{T_x(X)} \times df_x)(T_x(X)) = \Gamma(df_x) \subset T_x(X) \times T_{f(x)}(Y)$$

which is the graph of df_x in $T_x(X) \times T_{f(x)}(Y)$.

Solution (Exercise 2.13) **(a)** Given a smooth map $c\colon I \to \mathbb{R}^k$ with $c = (c_1, \dots, c_k)$ and $c_i\colon I \to \mathbb{R}$ all smooth. The derivative of c at $t_0 \in I$ is a linear map $dc_{t_0}\colon T_{t_0}I = \mathbb{R} \to \mathbb{R}^k = T_{x_0}(\mathbb{R}^k)$:

$$dc_{t_0}(v) = (c_1'(t_0), \dots, c_k'(t_0)) \cdot v.$$

Since $v \in \mathbb{R}$ is just a real number, we get

$$dc_{t_0}(1) = (c_1'(t_0), \dots, c_k'(t_0)) \in \mathbb{R}^k.$$

(b) First, assume $X = \mathbb{R}^k$ and let $w = (w_1, \dots, w_k)$ be a vector in $T_xX = \mathbb{R}^k$. Then define the curve $c_w\colon \mathbb{R} \to \mathbb{R}^k$ by $t \mapsto t \cdot w$. The derivative of c_w at any t_0 is

$$d(c_w)_{t_0}\colon \mathbb{R} \to \mathbb{R}^k,\ t \mapsto (w_1, \dots, w_k) \cdot t.$$

Thus we have $d(c_w)_{t_0}(1) = w$.

Now let X be an arbitrary k-dimensional smooth manifold, $x \in X$, and let v be a vector in $T_x(X)$. Let $\phi\colon U \to X$ be a local parametrization around x with $\phi(0) = x$. By definition, $T_x(X) = d\phi_0(\mathbb{R}^k)$ and there is a unique vector $w \in \mathbb{R}^k$ with $d\phi_0(w) = v$. Since any open ball around the origin in $\mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^k$, we can assume $U = \mathbb{R}^k$ and have $w \in U$. Let $c_w\colon \mathbb{R} \to \mathbb{R}^k$ be the linear curve in $\mathbb{R}^k$ defined in the previous point. Then we define $c\colon \mathbb{R} \to X$ by $c = \phi \circ c_w$, i.e., $c(t) = \phi(t \cdot w)$. The derivative of c at $t_0 = 0 \in \mathbb{R}$ is

$$d(c)_{t_0} = d\phi_0 \circ d(c_w)_{t_0}.$$

Thus

$$d(c)_{t_0}(1) = d\phi_0(d(c_w)_{t_0}(1)) = d\phi_0(w) = v.$$

A.3 The Inverse Function Theorem, immersions and embeddings

A.3.1 Diffeomorphisms, immersions and embeddings

Solution (**Exercise 3.1**) The derivative of f at $x \in \mathbb{R}^n$ is just given by the linear map $df_x : \mathbb{R}^n \to \mathbb{R}^n$, $v \mapsto A \cdot v$. Hence df_x is an isomorphism if and only if A is invertible. So f is a local diffeomorphism if and only if A is invertible. Now f is a diffeomorphism if and only if the map $x \mapsto f(x) - b = Ax$ is a diffeomorphism which is the case if and only if A is invertible. For, if this is the case then $y \mapsto A^{-1}y + A^{-1}b$ is the inverse of f. Hence f is a diffeomorphism if and only if A is invertible.

Solution (**Exercise 3.2**) Since f is bijective there is an inverse map $f^{-1} : Y \to X$. Since f is a local diffeomorphism we can find, for every $x \in X$, an open subset $U \subset$ such that $f_{|U} : U \to f(U)$ is a diffeomorphism and $f(U)$ is open in Y. Hence there is a smooth inverse $(f_{|U})^{-1}$ of $f_{|U}$. Since the inverse of a map is unique, we must have $(f_{|U})^{-1} = (f^{-1})_{|f(U)}$. This shows that f^{-1} is smooth on the open subset $f(U) \subset Y$. Since f is bijective, the open subsets $f(U)$, for all $x \in X$, cover Y. Hence, at every point $y \in Y$, we can find an open subset on which the restriction of f^{-1} is smooth. This shows that f^{-1} is a smooth map. Thus, f is a diffeomorphism.

Solution (**Exercise 3.3**) By definition of embeddings, we need to show that f is an injective, proper immersion.

- f is injective: If $f(t) = f(s)$, then $\frac{e^t+e^{-t}}{2} = \frac{e^s+e^{-s}}{2}$ and $\frac{e^t-e^{-t}}{2} = \frac{e^s-e^{-s}}{2}$. Adding these two equations, implies $e^t = e^s$. Since the exponential function is injective, this shows $t = s$.

- f is proper: Let K be a compact subset of $\mathbb{R}^2$. That means that K is both closed and bounded in $\mathbb{R}^2$. Since f is continuous, $f^{-1}(K)$ is closed in $\mathbb{R}$. Since both coordinates of $f(t)$ are unbounded when t varies in all of $\mathbb{R}$, $f^{-1}(K)$ must be bounded as well. Thus $f^{-1}(K)$ is both closed and bounded in $\mathbb{R}$ and therefore compact.

- f is an immersion: The derivative of f at any $t \in \mathbb{R}$ is given in the standard basis by the 2×1-matrix

$$df_t = \begin{pmatrix} \frac{e^t-e^{-t}}{2} \\ \frac{e^t+e^{-t}}{2} \end{pmatrix}.$$

  For each t, df_t is a linear map $\mathbb{R} \to \mathbb{R}^2$. Since $\mathrm{Ker}\,(df_t)$ is a vector subspace of $\mathbb{R}$, it is either $\{0\}$ or $\mathbb{R}$ itself. Since df_t is not the zero matrix for any t, df_t must be injective for all $t \in \mathbb{R}$.

Solution (**Exercise 3.4**) The map f is not an embedding, since it is not injective. But we can check it is an immersion by showing that the derivative is injective everywhere.

The derivative of f at (s,t) is

$$df_{(s,t)} : \mathbb{R}^2 \to \mathbb{R}^3,\ df_{(s,t)} = \begin{pmatrix} -\sin s \cos t & -(2+\cos s)\sin t \\ -\sin s \sin t & (2+\cos s)\cos t \\ \cos s & 0 \end{pmatrix}.$$

In order to show that $df_{(s,t)}$ is injective, we need to check that it has full or maximal rank, i.e. rank 2. Hence we need to check that the two column vectors are always linearly independent. To simplify notation, we set $x = \sin s$, $y = \cos s$, $u = \sin t$, and $v = \cos t$. Now assume there are two real numbers λ and μ such that

$$\begin{aligned} \lambda(-xv) + \mu(-(2+y)u) &= 0 \\ \lambda(-xu) + \mu((2+y)u) &= 0 \\ \lambda y &= 0. \end{aligned}$$

We distinguish two cases: $y = \cos s = 0$ and $y = \cos s \neq 0$. If $y \neq 0$, then we must have $\lambda = 0$ and

$$\begin{aligned} \mu(2+y)u &= 0 \\ \mu(2+y)v &= 0. \end{aligned}$$

Since $|y| \leq 1$, we know $2 + y \neq 0$. Hence we can divide by $2 + y$. Moreover, we know that not both u and v can be 0 at the same time. This implies $\mu = 0$.

If $y = 0$, then $x = \sin s = \pm 1$ and still $2 + y \neq 0$. Hence we get the system

$$\begin{aligned} \pm\lambda v - \mu(2+y)u &= 0 \\ \pm\lambda u + \mu(2+y)v &= 0. \end{aligned}$$

But since u and v are never both 0, we know that the vectors (v,u) and (u,v) are linearly independent. Hence we must have $\lambda = 0$ and $\mu(2+y) = 0$. The latter implies $\mu = 0$, since $2 + y \neq 0$.

Solution (**Exercise 3.5**) Let a and b be two relatively prime integers with $a \neq 0$. Recall that, for $t_1, t_2 \in R$, we have

$$e^{2\pi i t_1} = e^{2\pi i t_2} \iff t_1 - t_2 \in \mathbb{Z}.$$

Since a and b are integers, $t_1 - t_2 \in \mathbb{Z}$ implies $at_1 - at_2 \in \mathbb{Z}$ and $bt_1 - bt_2 \in \mathbb{Z}$. Hence we have

$$e^{2\pi i t_1} = e^{2\pi i t_2} \Rightarrow e^{2\pi i a t_1} = e^{2\pi i a t_2} \text{ and } e^{2\pi i b t_1} = e^{2\pi i b t_2}.$$

This shows that we have a well-defined map

$$g_{a,b} : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1,\ e^{2\pi i t} \mapsto (e^{2\pi i a t}, e^{2\pi i b t}).$$

Hence, for $f : \mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$, we get a commutative diagram

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "\gamma_{a,b}"] \arrow[dr, "f"'] & & \mathbb{S}^1 \times \mathbb{S}^1 \\ & \mathbb{S}^1. \arrow[ur, "g_{a,b}"'] & \end{tikzcd}$$

Since $\mathbb{S}^1$ is compact and $g_{a,b}$ a one-to-one immersion, $g_{a,b}$ is an embedding.

Solution (Exercise 3.6) **(a)** The derivative of f at t is the map $df_t : \mathbb{R} \to \mathbb{R}^2$ given by the Jacobian matrix

$$
\begin{aligned}
&(2\cos(2t)\cos t + \sin(2t)(-\sin t), 2\cos(2t)\sin t + \sin(2t)\cos t) \\
&= (2(\cos^2 t - \sin^2 t)\cos t - 2\sin t\cos t\sin t, 2(\cos^2 t - \sin^2 t)\sin t + 2\sin t\cos t\cos t) \\
&= 2(\cos^3 t - 2\sin^2 t\cos t, 2\cos^2 t\sin t - \sin^3 t) \\
&= 2(\cos t(\cos^2 t - 2\sin^2 t), \sin t(2\cos^2 t - \sin^2 t)) \\
&= 2(\cos t(1 - 3\sin^2 t), \sin t(3\cos^2 t - 1))
\end{aligned}
$$

where we have used several trigonometric identities. Since the derivative df_t is nontrivial for all t, it is always injective as a linear map $\mathbb{R} \to \mathbb{R}^2$. Hence f is an immersion.

(b) But f is not a homeomorphism onto $\operatorname{Im}(f)$. See Figure A.1. For, consider the open subset $(\pi/4, 3\pi/4)$ in $(0, 3\pi/4)$. If f was a homeomorphism, then $f((\pi/4, 3\pi/4))$ had to be open in $\operatorname{Im}(f)$ as well. That means that around any point, for example the point $f(\pi/2) = (0, 0)$, there had to an open neighborhood contained in $f((\pi/4, 3\pi/4))$. By the definition of the open sets in $\operatorname{Im}(f)$ as a subspace of $\mathbb{R}^2$, there had to be an open ball $B_\varepsilon(0, 0) \in \mathbb{R}^2$ with

$$B_\varepsilon(0,0) \cap \operatorname{Im}(f) \subset f((\pi/4, 3\pi/4)).$$

But for every $\epsilon > 0$, we have

$$B_\varepsilon(0,0) \cap f((0, \pi/4)) \neq \emptyset,$$

since $|\sin(2t)(\cos t, \sin t)| < \varepsilon$ for all $t < \epsilon/2$ (where we use $\sin x \le x$ and $|(\cos t, \sin t)| = 1$). Hence f cannot be an open map and therefore not a homeomorphism.

(c)

- What is the difference between $\operatorname{Im}(f)$ and the graph $\Gamma(f)$?
  Answer: The graph of a map $X \to Y$ is a subspace of $X \times Y$. In this case, $\Gamma(f)$ is a subspace of $(0, 3\pi/4) \times \mathbb{R}^2$, whereas $\operatorname{Im}(f)$ is a subspace of $\mathbb{R}^2$.
- Is the map $F : (0, 3\pi/4) \to (0, 3\pi/4) \times \mathbb{R}^2$ an embedding?
  Answer: Yes, because F is a diffeomorphism $(0, 3\pi/4) \to \Gamma(f)$ (since f is smooth, see previous exercise set). Hence it is in particular, a one-to-one immersion and proper.
- Would f be an embedding if it was defined on the closed interval $[0, 3\pi/4]$?
  Answer: No, because f would not be injective anymore: $f(0) = (0, 0) = f(\pi/2)$.
- Is the map $g : (0, 3\pi/4) \to \mathbb{R}^3,\ t \mapsto \sin(2t)(\cos t, \sin t, t)$ an embedding?
  Answer: No, this map is still just an immersion and it is even one-to-one, but it is not a homeomorphism onto its image in $\mathbb{R}^3$. The same argument as in the previous point.

- Is the map $h\colon [0, 3\pi/4] \to \mathbb{R}^3$, $t \mapsto (\sin(2t)\cos t, \sin(2t)\sin t, 2t)$ an embedding?

  Answer: Yes, this time we have a map which is an immersion, it is one-to-one this time $f(0) = (0,0,0) \neq (0,0,\pi) = f(\pi/2)$, and it is defined on a compact space and is therefore a proper map.

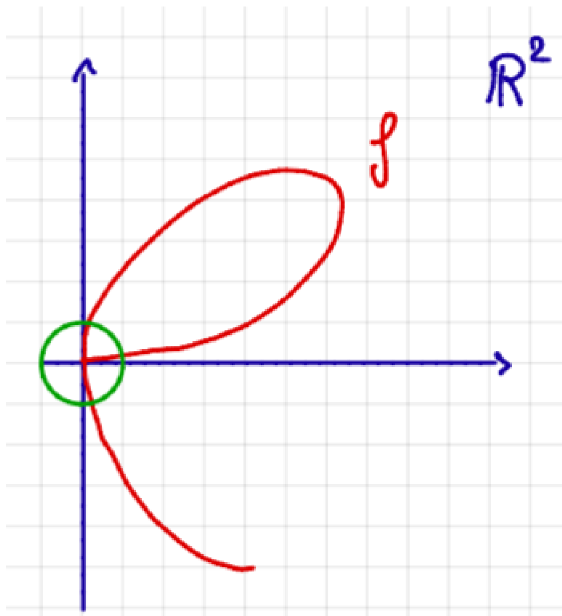


Figure A.1: The origin is the critical point. We cannot separate the two branches of the graph with open subsets.

Solution (Exercise 3.7) By the Local Immersion Theorem, we can choose local parametrizations $\phi\colon V \to Z$ and $\psi\colon W \to X$ around z with $V \subset \mathbb{R}^k$ and $W = V \oplus V' \subset \mathbb{R}^n$ such that

$$\begin{tikzcd} Z \arrow[r, "\text{inclusion}"] & X \\ \mathbb{R}^k \supseteq V \arrow[u, "\phi"] \arrow[r, "\text{canonical}", "\text{immersion}"'] & W \subseteq \mathbb{R}^n \arrow[u, "\psi"'] \end{tikzcd}$$

commutes. The map ψ is a diffeomorphism onto its image $\psi(W) \subset X$. The inverse map $\psi^{-1}\colon \psi(W) \to W$ is a local coordinate system on the open neighborhood $\psi(W)$ around $z \in X$. We write $x_i\colon \psi(W) \to \mathbb{R}$ for the ith component of ψ^{-1}, i.e. a point $p \in \psi(W)$ has the local coordinates $(x_1(p), \dots, x_n(p)) = (\psi_1^{-1}(p), \dots, \psi_n^{-1}(p))$. Since the above diagram commutes and ϕ is a diffeomorphism onto its image, we have

$$\phi(V) = Z \cap \psi(W).$$

Hence, since the lower horizontal map is the canonical immersion, the points in $Z \cap \psi(W)$ are exactly those on which the coordinate functions $x_{k+1}, \dots, x_n$ vanish. Relabelling the open subset $\psi(W)$ as U we have

$$Z \cap U = \{p \in U \text{ such that } x_{k+1}(p) = \cdots = x_n(p) = 0\}.$$

A.4 Submersions and regular values

A.4.1 Submersions and regular values

Solution (**Exercise 4.1**) It suffices to show that $f(X) \subset Y$ is open in Y since for an arbitrary open subset $U \subset X$ we my consider the map $U \subseteq X \xrightarrow{f} Y$. Let y be any point in $f(X)$. We need to show that there is an open neighborhood W around y which is contained in $f(X)$. Let x be a point in X with $f(x) = y$ which exists since $y \in f(X)$. By the Local Submersion Theorem 4.2, we can choose local parametrizations $\phi : V \to X$ around x with $V \subset \mathbb{R}^n$ open and $\psi : V' \to Y$ around y with $V' \subset \mathbb{R}^m$ open such that the induced map $V \to V'$ is the restriction of the canonical submersion:

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
V \arrow[u, "\phi"] \arrow[r, "\text{canonical}", "\text{submersion}"'] & V' \arrow[u, "\psi"']
\end{tikzcd}
$$

By possibly shrinking V and V', we can assume that $V = B_\varepsilon(0) \subset \mathbb{R}^n$ and $V' = B_\varepsilon(0) \subset \mathbb{R}^m$. Then the canonical submersion maps V **onto** V'. Since the above diagram commutes, we see that $\psi(V')$ is contained in $f(X)$. Since ψ is a local parametrization, $W := \psi(V')$ is open in Y as required.

Solution (**Exercise 4.2**) The derivative of g at a point (x, y) is given by the 1×2-matrix

$$dg_{(x,y)} = \begin{pmatrix} 2x & -2y \end{pmatrix}.$$

As a linear map from $\mathbb{R}^2$ to $\mathbb{R}$, $dg_{(x,y)}$ is surjective whenever it is not the zero map. Hence $dg_{(x,y)}$ is surjective for all $(x, y) \neq (0, 0)$. Thus the set of regular values of g is the subset $\mathbb{R} \setminus \{0\}$. Since $g(0, 0) = 0$, the only critical value is 0. Since the derivative of g is not surjective at all points, g is not a submersion.

Solution (**Exercise 4.3**) **(a)** Let $f : X \to Y$ be a submersion. We have $Y = f(X) \cup (Y \setminus f(X))$. Since X is compact and f is continuous, we know $f(X)$ is compact and therefore closed in Y. Since X is open in X and f is a submersion, the previous exercise shows that $f(X)$ is open in Y. Hence $f(X)$ is both open and closed in Y. Hence $f(X)$ must be either Y or $\emptyset$. Assuming f is nontrivial, $f(X)$ must be all of Y.

(b) Given a compact smooth manifold X. Assume we had a submersion $f : X \to \mathbb{R}^n$ with for some n. By the previous point, we would have $f(X) = \mathbb{R}^n$. But since X is compact, $f(X)$ is compact too. But $\mathbb{R}^n$ is not compact. Hence such a submersion cannot exist.

Solution (Exercise 4.4) Given $A = (a_{ij}) \in O(n)$. Unfolding the matrix-multiplication, we see that $\sum_j a_{ij}^2$ is the ith diagonal entry in AA^T. Moreover, $\sum_j a_{ij}^2$ is also the square of the norm of the ith row vector of A. Since $A \in O(n)$, we have $AA^T = I$ and the ith diagonal entry in AA^T is equal 1. This shows that $O(n)$ is contained in the product of n spheres $\prod \mathbb{S}^{n-1}$ in $\prod \mathbb{R}^n = \mathbb{R}^{n^2} = M(n)$. Hence $O(n)$ is bounded. But $O(n)$ is also closed in $\mathbb{R}^{n^2}$, since we can define it as the inverse image of the closed point $I \in S(n)$ under the map $M(n) \to S(n)$ sending A to AA^T. Thus $O(n)$ is closed and bounded in $\mathbb{R}^{n^2}$ and therefore compact.

Solution (Exercise 4.5) We consider $O(n)$ as a subspace in $M(n) = \mathbb{R}^{n^2}$. We defined $O(n)$ as $f^{-1}(I)$ under the map $f\colon M(n) \to S(n)$, $f(A) = AA^T$. We have checked in the proof of Theorem 4.14 that I is a regular value for f. As a consequence of the Preimage Theorem 4.7 we saw that this implies that $T_I(O(n))$ equals the kernel of $df_I\colon M(n) = T_I(M(n)) \to T_I(S(n)) = S(n)$. We calculated the derivative df_A for any $A \in O(n)$ in the proof of Theorem 4.14: it is given by $df_A(B) = BA^t + AB^t$. For $A = I$, this gives $df_I(B) = B + B^t$. Hence the kernel of df_I is the space of matrices satisfying $B + B^t = 0$, i.e., $B^t = -B$.

Solution (Exercise 4.6) **(a)** Since $d(\det)_A$ is a linear map $\mathbb{R}^4 \to \mathbb{R}$, it suffices to show $d(\det)_A$ is nonzero. Therefore, it suffices to show that $d(\det)_A(B) \neq 0$ for some matrix B.

Since $A \neq 0$, there is at least one entry in A which is nonzero. Assume that $a_{11} \neq 0$, and for the other cases the argument is similar. Then we take the matrix $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{aligned}
d(\det)_A(E_{22}) &= \lim_{s\to 0} \frac{\det(A + sE_{22}) - \det A}{s} \\
&= \lim_{s\to 0} \frac{\det(A + sE_{22}) - \det A}{s} \\
&= \lim_{s\to 0} \frac{(a_{11})(a_{22} + s) - a_{12}a_{21} - \det A}{s} \\
&= \lim_{s\to 0} \frac{a_{11}s + a_{11}a_{22} - a_{12}a_{21} - \det A}{s} \\
&= \lim_{s\to 0} \frac{a_{11}s + \det A - \det A}{s} \\
&= \lim_{s\to 0} \frac{a_{11}s}{s} = a_{11} \neq 0.
\end{aligned}$$

(b) A 2×2-matrix A has rank 0 if and only if it is the zero matrix. Thus $A \in M(2)\setminus\{0\}$ has rank 1 if and only if it does not have rank 2, i.e., if and only if it is not invertible. Hence $A \in M(2) \setminus \{0\}$ has rank 1 if and only if $\det A = 0$. By the previous point, the determinant function is a submersion $M(2) \setminus \{0\} \to \mathbb{R}$. Hence $R_1 = \det^{-1}(0)$ is a submanifold of dimension $4 - 1 = 3$ by the Preimage Theorem 4.7.

Solution (**Exercise 4.7**) We consider the function Q defined in the hint. Since P is homogeneous, we know Q is always 0. Hence its derivative with respect to t is zero as well. Hence we get

$$0 = \partial Q/\partial t = \sum_i x_i \partial P/\partial x_i(tx_1, \dots, tx_k) - mt^{m-1} P(tx_1, \dots, tx_k) \tag{A.1}$$

where we apply the chain rule to the first summand of Q which is the composite $t \mapsto tx \mapsto P(tx)$. Setting $t = 1$ in (A.1) yields Euler's identity (4.4).

Solution (**Exercise 4.8**) **(a)** The derivative of P at a point $(x_1, \dots, x_k)$ is

$$\begin{aligned} dP_x \colon \mathbb{R}^k \to \mathbb{R}, \ (z_1, \dots, z_k) &\mapsto (\partial P/\partial x_1(x) \dots \partial P/\partial x_k(x)) \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \\ &= \sum_i z_i \partial P/\partial x_i(x). \end{aligned}$$

To show that dP_x is nonsingular, i.e. surjective, it suffices to show that dP_x is nontrivial. But applying dP_x to x and using Euler's identity yields

$$dP_x(x) = \sum_i x_i \partial P/\partial x_i(x_1, \dots, x_k) = mP(x_1, \dots, x_k).$$

Hence if $x = (x_1, \dots, x_k)$ is not a zero of P, then $dP_x(x)$ is nonzero. Hence all nonzero real numbers are regular values of P. The Preimage Theorem now implies that $P^{-1}(a)$ is a $k-1$-dimensional submanifold of $\mathbb{R}^k$ for all $a \neq 0$.

(b) Given two real numbers $a, b > 0$, then $(b/a)^{1/m}$ exists and we if $P(x) = a$, we have

$$P((b/a)^{1/m} x_1, \dots, (b/a)^{1/m} x_k) = b/a P(x_1, \dots, x_k) = b.$$

Multiplying each coordinate with $(b/a)^{1/m}$ corresponds to multiplication with the diagonal matrix with $(b/a)^{1/m}$ on the diagonal. This map is a linear isomorphism of $\mathbb{R}^k$ to itself. Hence we have the diffeomorphism

$$P^{-1}(a) \to P^{-1}(b), \ (x_1, \dots, x_k) \mapsto ((b/a)^{1/m} x_1, \dots, (b/a)^{1/m} x_k).$$

Similarly, if both $a, b < 0$ are negative, then $(b/a)^{1/m}$ exists and the same argument shows that $P^{-1}(a)$ and $P^{-1}(b)$ are diffeomorphic.

Solution (**Exercise 4.9**) **(a)** If we think of the entries in an $n \times n$-matrix A as variables, then $\det A$ is a homogeneous polynomial of degree n given by Leibniz' formula. Hence we can apply a previous exercise to

$$P = \det \colon M(n) = \mathbb{R}^{n^2} \to \mathbb{R}$$

and conclude that 0 is the only critical value of det.

Alternatively: Using the formula $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$ we can compute the partial derivatives of det with respect to each variable x_{ij}. Since we remove the jth column from A for each of the A_{ij}, the entry a_{ij} occurs in this formula exactly once. Hence the partial derivative with respect to x_{ij} is just the factor of a_{ij}:

$$\frac{\partial \det}{\partial x_{ij}}(A) = (-1)^{i+j} \det A_{ij}.$$

The total derivative of det as a function of its entries can be represented by the $n \times 1$-matrix with these partial derivatives as entries. This linear map is then not surjective if and only if all entries are zero, i.e., if and only if $\det A_{ij} = 0$ for all i, j. The latter happens if and only if the rank of A is $< n$, i.e., if and only if $\det A = 0$. Hence 0 the only critical value for det.

(b) By the previous point, 1 is a regular value for det. Hence, by the Preimage Theorem 4.7, $SL(n) = \det^{-1}(1)$ is a smooth manifold of dimension $\dim M(n) - \dim \mathbb{R} = n^2 - 1$.

(c) According to the Preimage Theorem 4.7 and the previous point, we can determine the tangent space as the kernel of the derivative of det at the identity matrix. Hence we need to calculate the derivative of det at the identity. To do this we are going to use Leibniz' formula (4.5).

Given a matrix A, in Leibniz' formula for the determinant of $B := I + sA$, every summand contains at least a factor s^2 unless it is the product of at least $n-1$ diagonal entries $b_{ii} = 1 + sa_{ii}$. For we need $n-1$ factors not containing s which is only possible when we multiply $n-1$ times 1. But if a permutation $\{1, \dots, n\}$ leaves $n-1$ numbers fixed, it also has to leave the remaining one fixed. Hence the only summand in (4.5) which does not contain a factor s^2 is the summand

$$\prod_{i=1}^{n} (1 + sa_{ii}) = (1 + sa_{11}) \cdots (1 + sa_{nn}) = 1 + s \cdot \operatorname{tr}(A) + O(s^2).$$

The derivative of the determinant at the identity

$$d(\det)_I : T_I(M(n)) = M(n) \to T_1(\mathbb{R}) = \mathbb{R}$$

is then given by

$$\begin{aligned} d(\det)_I(A) &= \lim_{s\to 0} \frac{\det(I + sA) - \det I}{s} \\ &= \lim_{s\to 0} \frac{1 + s \cdot \operatorname{tr}(A) + O(s^2) - 1}{s} \\ &= \lim_{s\to 0} \frac{s \cdot \operatorname{tr}(A) + O(s^2)}{s} \\ &= \lim_{s\to 0} \operatorname{tr}(A) + O(s) \\ &= \operatorname{tr}(A). \end{aligned}$$

Hence we get

$$T_I(SL(n)) = \operatorname{Ker}(d(\det)_I) = \{A \in M(n) : \operatorname{tr}(A) = 0\}.$$

In other words, the tangent space to $SL(n)$ at the identity is the space of matrices whose trace vanishes.

Solution (Exercise 4.10) **(a)** We write $(z_0, z_1) = (x_0 + iy_0, x_1 = iy_1)$ for real coordinates x_0, y_0, x_1, y_1. First we get

$$\begin{aligned}\tilde{\pi}(z_0, z_1) &= \left(2z_0\bar{z}_1, |z_0|^2 - |z_1|^2\right)\\ &= \left(2(x_0x_1 + y_0y_1) + i2(-x_0y_1 + y_0x_1), (x_0^2 + y_0^2) - (x_1^2 + y_1^2)\right).\end{aligned}$$

Then we can compute

$$d\tilde{\pi}_q = 2 \cdot \begin{pmatrix} x_1 & y_1 & x_0 & y_0 \\ -y_1 & x_1 & y_0 & -x_0 \\ x_0 & y_0 & -x_1 & -y_1 \end{pmatrix}.$$

(b) Let $g_4 : \mathbb{R}^4 \to \mathbb{R}$ and $g_3 : \mathbb{R}^3 \to \mathbb{R}$ be the usual smooth maps such that $\mathbb{S}^3 = g_4^{-1}(1)$ and $\mathbb{S}^2 = g_3^{-1}(1)$ respectively. Then we have $T_q\mathbb{S}^3 = \operatorname{Ker}(d(g_4)_q) \subset T_q\mathbb{R}^4 = \mathbb{R}^4$ and $T_p\mathbb{S}^2 = \operatorname{Ker}(d(g_3)_p) \subset T_p\mathbb{R}^3 = \mathbb{R}^3$. We know that $\tilde{\pi}(q) \in \mathbb{S}^2$ if $q \in \mathbb{S}^3$. Actually, our calculation above shows that $\tilde{\pi}(q) \in \mathbb{S}^2$ if and only if $q \in \mathbb{S}^3$. This implies $\mathbb{S}^3 = \tilde{\pi}^{-1}(g_3^{-1}(1)) = (g_3 \circ \tilde{\pi})^{-1}(1)$. In particular, $g_3(\tilde{\pi}(q)) = 1$ is constant on $\mathbb{S}^3$. Hence, for every $q \in \mathbb{S}^3$, the image of the restriction $(d\tilde{\pi}_q)_{|T_q\mathbb{S}^3}$ is contained in the kernel of $d(g_3)_{\tilde{\pi}(q)}$ which is $T_{\tilde{\pi}(q)}\mathbb{S}^2$.

(c)

- The fiber over a is

$$\pi^{-1}(a) = \{(z_0, 0) \in \mathbb{S}^3 \subset \mathbb{C}^2 : |z_0|^2 = 1\}.$$

Let $q = (x_0, y_0, 0, 0) \in \pi^{-1}(a)$ be a point in the fiber over a. The tangent space $T_q\mathbb{S}^3$ is the vector space

$$\begin{aligned}T_q\mathbb{S}^3 &= \{\mathbf{u} \in \mathbb{R}^4 : \mathbf{u} \perp q\}\\ &= \operatorname{span}\left\{q^\perp = \begin{pmatrix} -y_0 \\ x_0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_3^4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_4^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.\end{aligned}$$

The vectors $q^\perp$, $\mathbf{e}_3^4$ and $\mathbf{e}_4^4$ are linearly independent and hence form a basis of $T_q\mathbb{S}^3$.

Now we consider the map $d\tilde{\pi}_q$. We computed this map as represented by the matrix

$$d\tilde{\pi}_q = 2 \cdot \begin{pmatrix} 0 & 0 & x_0 & y_0 \\ 0 & 0 & y_0 & -x_0 \\ x_0 & y_0 & 0 & 0 \end{pmatrix}.$$

The kernel of this map is the span of the vector $q^\perp$ that we have just seen. This implies that $d\pi_q$ is surjective onto $T_a\mathbb{S}^2$.

More concretely, the tangent space $T_a\mathbb{S}^2$ consists of the vectors which are orthogonal to a in $\mathbb{R}^3$. Hence it has a basis $(\mathbf{e}_1^3, \mathbf{e}_2^3)$ as a subspace in $\mathbb{R}^3$. The map $d\tilde{\pi}_q : T_q\mathbb{R}^4 \to T_a\mathbb{R}^3$ sends

$$q^\perp \mapsto \mathbf{0}, \mathbf{e}_3^4 \mapsto \begin{pmatrix} 2x_0 \\ 2y_0 \\ 0 \end{pmatrix}, \mathbf{e}_4^4 \mapsto \begin{pmatrix} 2y_0 \\ -2x_0 \\ 0 \end{pmatrix}.$$

Hence the map $d\pi_q : T_q\mathbb{S}^3 \to T_a\mathbb{S}^2$ can be represented in the chosen bases by the matrix

$$d\pi_q = 2 \cdot \begin{pmatrix} 0 & x_0 & y_0 \\ 0 & y_0 & -x_0 \end{pmatrix}.$$

This map is surjective, since $-x_0^2 - y_0^2 = -1 \neq 0$. Hence q is a regular point for π. Since q was any point in the fiber over a, we have shown that a is a regular value for π.

- To determine the fiber over b, we write $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$. Then we get

$$\begin{aligned} \pi(z_0, z_1) = (0, 1, 0) &\Rightarrow 2z_0\bar{z}_1 = i \text{ and } |z_0|^2 = |z_1|^2 = \frac{1}{2} \\ &\Rightarrow y_0 = x_1,\ y_1 = -x_0 \text{ and } x_0^2 + x_1^2 = \frac{1}{2}. \end{aligned}$$

Thus the fiber over b has the form

$$\begin{aligned} \pi^{-1}(b) &= \{(z_0, z_1) \in \mathbb{S}^3 \,:\, \bar{z}_1 = \frac{i}{2z_0}\} \\ &= \{(x_0, y_0, x_1, y_1) \in \mathbb{S}^3 \,:\, y_0 = x_1, y_1 = -x_0\}. \end{aligned}$$

Let $q = (x_0, x_1, x_1, -x_0) \in \pi^{-1}(b)$ be a point in the fiber over b. Since not both x_0 and x_1 can be zero, we assume that $x_0 \neq 0$. The tangent space $T_q\mathbb{S}^3$ is the vector space

$$\begin{aligned} T_q\mathbb{S}^3 &= \{\mathbf{u} \in \mathbb{R}^4 \,:\, \mathbf{u} \perp q\} \\ &= \operatorname{span}\left\{ q_1^\perp = \begin{pmatrix} -x_1 \\ x_0 \\ 0 \\ 0 \end{pmatrix}, q_2^\perp = \begin{pmatrix} -x_1 \\ 0 \\ x_0 \\ 0 \end{pmatrix}, q_3^\perp = \begin{pmatrix} x_0 \\ 0 \\ 0 \\ x_0 \end{pmatrix} \right\}. \end{aligned}$$

Now we consider the map $d\tilde{\pi}_q$. We computed this map as represented by the matrix

$$d\tilde{\pi}_q = 2 \cdot \begin{pmatrix} x_1 & -x_0 & x_0 & x_1 \\ x_0 & x_1 & x_1 & -x_0 \\ x_0 & x_1 & -x_1 & x_0 \end{pmatrix}.$$

The kernel of this map is the span of the vector $q_0^\perp = \begin{pmatrix} -x_1 \\ x_0 \\ x_0 \\ x_1 \end{pmatrix}$. This implies that $d\pi_q$ is surjective onto $T_a\mathbb{S}^2$.

More concretely, the tangent space $T_b\mathbb{S}^2$ consists of the vectors which are orthogonal to b in $\mathbb{R}^3$. Hence it has a basis $(\mathbf{e}_1^3, \mathbf{e}_3^3)$ as a subspace in $\mathbb{R}^3$. The map $d\tilde{\pi}_q : T_q\mathbb{R}^4 \to T_b\mathbb{R}^3$ sends

$$q_1^\perp \mapsto \begin{pmatrix} -2(x_0^2 + x_1^2) \\ 0 \\ 0 \end{pmatrix}, q_2^\perp \mapsto \begin{pmatrix} 2(x_0^2 - x_1^2) \\ 0 \\ -2x_0x_1 \end{pmatrix}, q_3^\perp \mapsto \begin{pmatrix} 2x_0x_1 \\ 0 \\ 2x_0^2 \end{pmatrix}.$$

Hence the map $d\pi_q : T_q\mathbb{S}^3 \to T_b\mathbb{S}^2$ can be represented in the chosen bases by the matrix

$$d\pi_q = 2 \cdot \begin{pmatrix} -x_0^2 - x_1^2 & x_0^2 - x_1^2 & x_0x_1 \\ 0 & -x_0x_1 & x_0^2 \end{pmatrix}.$$

This map is surjective, as one can check by using the conditions we have on x_0 and x_1. Hence q is a regular point for π. Since q was any point in the fiber over a, we have shown that b is a regular value for π.

(d) By a previous point, we need to show that $d\tilde{\pi}_q$ restricted to $T_q\mathbb{S}^3$ is surjective onto $T_{\pi(q)}\mathbb{S}^2$ at every $q \in \mathbb{S}^3$. Since the tangent space $T_{\pi(q)}\mathbb{S}^2$ of $\mathbb{S}^2$ is two-dimensional, we need to check that the image of $d\tilde{\pi}_q$ restricted to $\operatorname{Ker}(d(g_4)_q)$ spans a two-dimensional subspace. Since $\operatorname{Ker}(d(g_4)_q)$ is of dimension 3, it suffices to show that $d\tilde{\pi}_q$ has rank 3, which implies that the kernel of $d\tilde{\pi}_q$ has dimension 1. Hence we need to show that $d\tilde{\pi}_q$ always has 3 linear independent columns.

We can show this for example by calculating the determinants of appropriate 3×3-minors. Ignoring the factor 2 in our formula for $d\tilde{\pi}_q$ we look at the minors A_j of the remaining matrix where we omit the jth column:

- The determinant of A_4 is $-x_1(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -x_1$.
- The determinant of A_3 is $-y_0(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -y_0$.
- The determinant of A_2 is $-x_0(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -x_0$.
- The determinant of A_1 is $-y_1(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -y_1$.

For every point $q \in \mathbb{S}^3$, at least one of the coordinates x_0, y_0, x_1, y_1 is nonzero. Hence the matrix always has three linear independent columns and $d\tilde{\pi}_q$ has rank 3. This shows that each point in $\mathbb{S}^3$ is a regular point for π, and hence every point in $\mathbb{S}^2$ is a regular value for π.

A.5 Lie groups

A.5.1 Lie groups

Solution (Exercise 5.1) **(a)** Every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2)$ satisfies

$$A^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence A corresponds to two points (a, c) and (b, d) on $\mathbb{S}^1 \subset \mathbb{R}^2$ whose corresponding vectors are orthogonal to each other. Since we also know $\det A = ad - bc = 1$, one of these points uniquely determines the other. See Figure A.2. Hence we can write A as $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for some real number t. Now one can check that the map

$$\mathbb{S}^1 \to SO(2),\ (\cos t, \sin t) \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a diffeomorphism and Lie group isomorphism.

(b) Every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ satisfies

$$\bar{A}^T A = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a}a + \bar{c}c & \bar{a}b + \bar{c}d \\ \bar{b}a + \bar{d}c & \bar{b}b + \bar{d}d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with $\det A = ad - bc = 1$ we get four linear equations for the complex numbers a, b, c, d, and their complex conjugates. Unraveling these equations shows that we can write A as

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ with } a\bar{a} + b\bar{b} = 1.$$

Hence A corresponds uniquely to a pair of complex numbers (a, b) which satisfies $a\bar{a} + b\bar{b} = 1$. Since this is exactly the defining condition for elements of $\mathbb{S}^3 \subset \mathbb{C}^2$, we see that

$$\mathbb{S}^3 \to SU(2), (a, b) \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

is a diffeomorphism.

Solution (Exercise 5.2) **(a)** Since f is bijective, it has an inverse $f^{-1}: Y \to X$. We need to show that f^{-1} is smooth. Let y be a point in Y. Since f is a local diffeomorphism, there is an open neighborhood $U \subset X$ around the point $f^{-1}(y)$ and an open neighborhood $V \subset Y$ around y such that $f_{|U}: U \to V$ is a diffeomorphism. Hence there is a smooth inverse $(f_{|U})^{-1}: V \to U$. Since

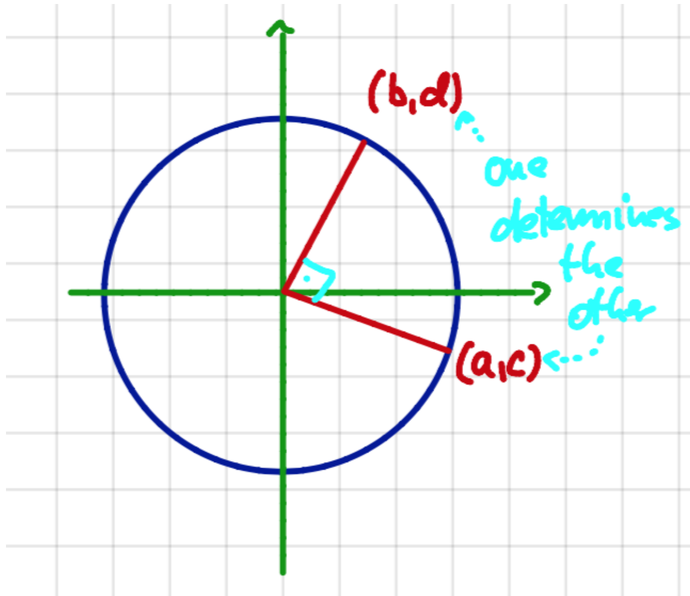


Figure A.2: The column vectors are orthogonal to each other and they determine each other.

inverses are unique (as maps of sets), $(f^{-1})_{|V}$ must agree with $(f_{|U})^{-1}$. Hence f^{-1} is a smooth map on an open neighborhood of y. Since y was arbitrary, we see that f^{-1} is smooth at every point and therefore smooth.

(b) Since f is one-to-one, it is a bijection from X onto its image $\mathrm{Im}(f) \subseteq Y$. Since it is a local diffeomorphism, $f : X \to \mathrm{Im}(f)$ is a bijective local diffeomorphism. By the previous point, it is a diffeomorphism.

(c) We would like to show $\dim X = \mathrm{rank}(f) = \dim Y$. Because then the Inverse Function Theorem implies that f is a local diffeomorphism, and, since f is also bijective, f would be a diffeomorphism by the first point and we were done.

Assume $X \subseteq \mathbb{R}^M$ and $Y \subseteq R^N$, $\dim X = m$, $\dim Y = n$, and set $r := \mathrm{rank}(f)$. By definition of the rank, we have $m \geq r$ and $n \geq r$. We want to show $m = r = n$.

For any point $x \in X$, the linear map df_x has rank r. Recall that for a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^n$ of rank r, we can choose a basis of $\mathbb{R}^n$ such that the first r basis vectors $b_1, \dots, b_r$ span the image of L and the remaining $n - r$ basis vectors $b_{r+1}, \dots, b_n$ span the orthogonal complement of L in $\mathbb{R}^n$. Then we choose a basis of $\mathbb{R}^m$ such that the ith basis vector is sent to b_i. The matrix representing L in these bases has the $r \times r$-identity matrix sitting in the upper left corner and zeros elsewhere. Then, as in the proof of the Local Immersion (or Submersion) Theorem, we can choose local parametrizations $\phi\colon U \to X$ around x and $\psi : V \to Y$ around y such that the map $\theta : U \to V$ in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\theta=\psi^{-1}\circ f\circ\phi"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

has the form $\theta(x_1, \dots, x_m) = (x_1, \dots, x_r, 0) \in \mathbb{R}^n$. (Note that the 0 at the end of $\theta(x)$ only occur if $r < n$.)

If $m > r$, then for a sufficiently small $\epsilon > 0$, $\theta(x_1, \dots, x_r, \epsilon, 0) = (x_1, \dots, x_r, 0)$ and θ is not injective. Since ϕ and ψ are diffeomorphisms, this would imply that f is not injective which contradicts that f is bijective. Hence we can assume $m = r$ and f is an immersion.

Assume we had $r < n$. Then, after possibly shrinking U, we can assume that U is a small open $B_\epsilon(0)$ around 0 in $\mathbb{R}^m$ and that $\theta(\bar{B}_\epsilon(0)) \subseteq V$ (where $\bar{B}_\epsilon(0)$ denotes the closed ball of radius ϵ: $\bar{B}_\epsilon(0) = \{x \in \mathbb{R}^m : |x| \leq \epsilon\}$). Since $\bar{B}_\epsilon(0)$ is compact, so is $\theta(\bar{B}_\epsilon(0))$. Hence $\theta(\bar{B}_\epsilon(0))$ is closed in V and is contained in $V \cap (\mathbb{R}^r \times \{0\})$. Hence $\theta(\bar{B}_\epsilon(0))$ does not contain any open subsets of V. Since ϕ and ψ are diffeomorphisms, this implies that $f(\phi(\bar{B}_\epsilon(0)))$ is closed and does not contain any nonempty open subset of Y. Since we can cover X by such local parametrizations, we see that $f(X)$ is the union of subsets which do not contain any nonempty open subset of Y.

Now if X could be assumed to be compact, then $f(X)$ is compact, and $f(X)$ can be covered by finitely many closed subsets which do not contain any nonempty open subset of Y. That would imply that $f(X)$ is itself a closed subset which does not contain any nonempty open subset of Y. Hence $f(X)$ cannot be all of Y, and f would not be surjective.

In general, for any open cover of a subspace in $\mathbb{R}^M$, we can always choose a countable subcover. This implies that $f(X)$ is the countable union of subsets which do not contain any nonempty open subset of Y. By Baire's Category Theorem, this implies that $f(X)$ does not contain any nonempty open subset of Y. Hence $f(X)$ cannot be equal Y and f would not be surjective.

(d) We learned in Theorem 5.3 that a Lie group homomorphism has constant rank. Hence we just need to apply the previous point.

Solution (Exercise 5.3) For any $g \in G$, left multiplication $L_g : G \to G$ by g maps the subgroup H to the left coset $gH = \{gh : h \in H\}$. Since H is open and L_g is a diffeomorphism, the coset gH is open. Thus, G can be written as the union of the open subsets gH where g ranges over all elements in G. But since cosets are pairwise disjoint, this would give us a way to write G as the union of nonempty disjoint open subsets. Since G is connected, there can be only one coset. Therefore, $H = G$.

Solution (Exercise 5.4) **(a)** Let $g, h \in G$ be any fixed elements. Let $j : G \to G \times G$ be the map $j(g) = (g, h)$. Note that the composite $\mu \circ j = R_h$ is right translation by h.

For $x \in G$, let $\phi_x : U_x \to G$ be a local parametrization around x with $\phi(0) = x$.

Then we get the diagram

$$\begin{tikzcd} G \arrow[r, "j"] & G \times G \arrow[r, "\mu"] & G \\ U_g \arrow[u, "\phi_g"] \arrow[r, "\gamma"'] & U_g \times U_h \arrow[u, "\phi_g \times \phi_h"'] \arrow[r, "\theta"'] & U_{gh}. \arrow[u, "\phi_{gh}"'] \end{tikzcd}$$

where we define the maps γ and θ such that the diagram commutes. Since $\phi_g(0) = g$ and $\phi_h(0) = h$, we must have $\gamma(u) = (u, 0) \in U_g \times U_h$ to make the left hand diagram commute. Moreover, we must have $\theta(0,0) = 0 \in U_{gh}$.

Taking derivatives at g and using $T_{(g,h)}(G \times G) = T_g(G) \times T_h(G)$ gives

$$\begin{tikzcd} T_g(G) \arrow[r, "dj_g"] & T_g(G) \times T_h(G) \arrow[r, "d\mu_{(g,h)}"] & T_{gh}(G) \\ \mathbb{R}^n \arrow[u, "d(\phi_g)_0"] \arrow[r, "d\gamma_0"'] & \mathbb{R}^n \times \mathbb{R}^n \arrow[u, "d(\phi_g)_0 \times d(\phi_h)_0"'] \arrow[r, "d\theta_0"'] & \mathbb{R}^n. \arrow[u, "d(\phi_{gh})_0"'] \end{tikzcd}$$

Since $\gamma(u) = (u, 0)$, we have $d\gamma_0(v) = (v, 0)$ and hence $dj_g(X) = (X, 0)$. Since $\mu \circ j = R_h$, we have $d\mu_{(g,h)} \circ dj_g = d(R_h)_g$. Thus

$$d\mu_{(g,h)}(X, 0) = d\mu_{(g,h)}(dj_g(X)) = d(R_h)_g(X).$$

Repeating this argument with j replaced with $j\colon\ h \mapsto (g, h)$ yields

$$d\mu_{(g,h)}(0, Y) = d(L_g)_h(Y).$$

Since $d\mu_{(g,h)}$ is linear, it satisfies

$$d\mu_{(g,h)}(X, Y) = d\mu_{(g,h)}(X, 0) + d\mu_{(g,h)}(0, Y) = d(R_h)_g(X) + d(L_g)_h(Y).$$

(b) Let $\iota\colon\ G \to G$ denote the inversion map. Show that

$$d\iota_e\colon\ T_e(G) \to T_e(G)$$

is given by $d\iota_e(X) = -X$.

Solution:

Consider the map

$$G \xrightarrow{(\mathrm{Id}, \iota)} G \times G \xrightarrow{\mu} G,\ g \mapsto (g, g^{-1}) \mapsto gg^{-1} = e.$$

Since this map is constant, its derivative at e vanishes. Hence we get

$$T_e(G) \xrightarrow{(d\mathrm{Id}_e, d\iota_e)} T_e(G) \times T_e(G) \xrightarrow{d\mu_{(e,e)}} T_e(G),\ X \mapsto (X, d\iota_e(X)) \mapsto 0.$$

As we have just learned $d\mu_{(e,e)}(X, d\iota_e(X)) = X + d\iota_e(X) = 0$, and hence $d\iota_e(X) = -X$.

(c) Given $g \in G$, we consider the diagram

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[d, "L_{g^{-1}}"'] & G \\ G \arrow[r, "\iota"'] & G. \arrow[u, "R_{g^{-1}}"'] \end{tikzcd}$$

One easily checks that it commutes. Taking the derivative at g of the top map yields a commutative diagram of derivatives

$$\begin{tikzcd} T_g(G) \arrow[r, "d\iota_g"] \arrow[d, "d(L_{g^{-1}})_g"'] & T_{g^{-1}}(G) \\ T_e(G) \arrow[r, "d\iota_e"'] & T_e(G). \arrow[u, "d(R_{g^{-1}})_e"'] \end{tikzcd}$$

We just calculated the effect of the map $d\iota_e : T_e(G) \to T_e(G)$ as $X \mapsto -X$. Hence, since all maps in the above diagram are linear, we get

$$d\iota_g : T_g(G) \to T_{g^{-1}}, \ Y \mapsto -d(R_{g^{-1}})_e(d(L_{g^{-1}})_g(Y)).$$

Solution (Exercise 5.5) Given elements $g, h \in G$. Let $R_{h^{-1}}$ denote the right translation with h^{-1}. We define the smooth map j_h by

$$j_h : G \to G \times G, \ x \mapsto (R_{h^{-1}}(x), h).$$

Note that $j_h(gh) = (ghh^{-1}, h) = (g, h) \in G \times G$. For the tangent spaces we get

$$T_{j_h(gh)}(G \times G) = T_{(g,h)}(G \times G) \cong T_g(G) \times T_h(G).$$

The composite of the map

$$G \xrightarrow{j_h} G \times G \xrightarrow{\mu} G, \ x \mapsto (R_{h^{-1}}(x), h) \mapsto \mu(xh^{-1}, h) = x$$

is the identity of G. Taking derivatives at gh yields

$$T_{gh}(G) \xrightarrow{d(j_h)_{gh}} T_g(G) \times T_h(G) \xrightarrow{d\mu_{(g,h)}} T_{gh}(G).$$

Since $\mu \circ j_h = \mathrm{Id}_G$, we also have $d\mu_{(g,h)} \circ d(j_h)_{gh} = \mathrm{Id}_{T_{gh}(G)}$. In particular,

$$d\mu_{(g,h)} : T_{(g,h)}(G \times G) \to T_{gh}(G)$$

is surjective. Since we started with arbitrary elements g and h, this shows that μ is a submersion.

Solution (Exercise 5.6) For matrices $A \in GL(n)$ and $B \in M(n)$, we have

$$\det(B) = (\det A) \cdot \det(A^{-1}B) = (\det A) \cdot \det(L_{A^{-1}}B).$$

Taking the derivative at A and remembering the chain rule yields

$$\begin{aligned} d(\det)_A(B) &= d((\det A)\cdot \det \circ L_{A^{-1}})(B) \\ &= (\det A)\cdot d(\det \circ L_{A^{-1}})B \\ &= (\det A)\cdot d(\det)_{(L_{A^{-1}}A)} \circ d(L_{A^{-1}})_A(B) \\ &= (\det A)\cdot d(\det)_I(A^{-1}B) \\ &= (\det A)\cdot \operatorname{tr}(A^{-1}B), \end{aligned}$$

where we have used $d(L_{A^{-1}})_A(B) = A^{-1}B$ which can be easily checked, since matrix multiplication is linear. Hence, after replacing B with AB, we get

$$d(\det)_A(AB) = (\det A)\cdot(\operatorname{tr} B) \text{ for all } B \in M(n).$$

A.6 Transversality

A.6.1 Transversality

Solution (Exercise 6.1) **(a)** The tangent space to $\mathbb{S}^1$ at z is $T_z(\mathbb{S}^1) = \{v \in \mathbb{R}^2 : v \cdot z = 0\}$. The tangent space to N_z at z is $T_z(N_z) = \{(0, y) : y \in \mathbb{R}\} \subset \mathbb{R}^2$. Since $T_z(\mathbb{R}^2) = \mathbb{R}^2$, $T_z(\mathbb{S}^1)$ and $T_z(N_z)$ span all of $T_z(\mathbb{R}^2)$ for all $z \neq (\pm 1, 0)$. Hence $\mathbb{S}^1 \pitchfork N_z$ if and only if $z \neq (\pm 1, 0)$. (Drawing a picture explains everything.)

(b) Which of the following linear spaces intersect transversally?

- The xy-plane and the z-axis.
  Answer: Transverse, since the two spaces span all of $\mathbb{R}^3$.
- The xy-plane and the plane spanned by $\{(3, 2, 0), (0, 4, -1)\}$.
  Answer: Transverse, since the two planes span all of $\mathbb{R}^3$.
- The plane spanned by $\{(1, 0, 0), (2, 1, 0)\}$ and the y-axis in $\mathbb{R}^3$.
  Answer: Not transverse, since the y-axis and lies in the span of $\{(1, 0, 0), (2, 1, 0)\}$.
- $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (The answer depends on k, l, and n.)
  Answer: Transverse, if $k + l \geq n$.
- $V \times \{0\}$ and the diagonal in $V \times V$, for a real vector space V.
  Answer: Transverse, since they span all of $V \times V$: any $(v, w) \in V \times V$ is equal to the sum of $(w, w) \in \Delta_V$ and $(v - w, 0) \in V \times \{0\}$.
- The spaces of symmetric ($A^t = A$) and skew symmetric ($A^t = -A$) matrices in $M(n)$.
  Answer: Transverse, since every matrix in $M(n)$ can be written as a sum of a symmetric and an antisymmetric matrix:

$$C = \frac{1}{2}(C + C^t) + \frac{1}{2}(C - C^t) \text{ for any } C \in M(n).$$

(c) Yes, $SL(n)$ and $O(n)$ do not meet transversally in $M(n)$, since $SO(n)$ is contained in $SL(n)$. Hence we also have $T_A(O(n)) = T_A(SO(n)) \subseteq T_A(SL(n))$, and these tangent spaces do not span all of $T_A(M(n)) = M(n)$.

Solution (Exercise 6.2) The image of f is a submanifold of $\mathbb{R}^2$. This follows, for example, from the fact that f is an embedding. We could also observe that $\mathrm{Im}\,(f) = g^{-1}(1)$ and remark that 1 is a regular value of g. The composition $g \circ f$ is the constant map $\mathbb{R} \to \mathbb{R}$ with value 1. Hence $(g \circ f)^{-1}(1) = \mathbb{R}$ is a manifold.

Solution (Exercise 6.3) By assumption, g is transverse to W, i.e.,

$$\mathrm{Im}\,(dg_y) + T_{g(y)}(W) = T_{g(y)}(Z) \text{ for all } y \in Y \text{ with } g(y) \in W. \tag{A.2}$$

Now we assume $f \pitchfork g^{-1}(W)$, i.e.,

$$\operatorname{Im}(df_x) + T_{f(x)}(g^{-1}(W)) = T_{f(x)}(Y) \text{ for all } x \in X \text{ with } f(x) \in g^{-1}(W). \tag{A.3}$$

We need to show $(g \circ f) \pitchfork W$. So let $x \in X$ be a point such that $g(f(x)) \in W$ and let $c \in T_{g(f(x))}(Z)$. By (A.2), there are vectors $b_1 \in T_{f(x)}(Y)$ and $b_2 \in T_{g(f(x))}(W)$ such that

$$dg_{f(x)}(b_1) + b_2 = c.$$

By (A.3), there are vectors $a \in T_x(X)$ and $b_3 \in T_{f(x)}(g^{-1}(W))$ such that

$$df_x(a) + b_3 = b_1.$$

Putting these two equations together we get

$$c = dg_{f(x)}(df_x(a) + b_3) + b_2 = dg_{f(x)}(df_x(a_1)) + dg_{f(x)}(b_3) + b_2.$$

By the chain rule, we have $d(g \circ f)_x = dg_{f(x)} \circ df_x$. By a previous exercise, we know $T_{f(x)}(g^{-1}(W)) = (dg_{f(x)})^{-1}(T_{g(f(x))}(W)$ in $T_{f(x)}(Y)$. In particular, $dg_{f(x)}(b_3) \in T_{g(f(x))}(W)$ and thus

$$dg_{f(x)}(b_3) + b_2 \in T_{g(f(x))}(W).$$

Hence we have

$$c = d(g \circ f)_x(a) + dg_{f(x)}(b_3) + b_2 \in [\operatorname{Im}(d(g \circ f)_x) + T_{g(f(x))}(W)] \subset T_{g(f(x))}(Z).$$

Since c was an arbitrary element in $T_{g(f(x))}(Y)$, we have proven

$$\operatorname{Im}(d(g \circ f)_x) + T_{g(f(x))}(W) = T_{g(f(x))}(Z) \text{ for all } x \in X \text{ with } g(f(x)) \in W.$$

In other words, $(g \circ f) \pitchfork W$.

Now we assume $(g \circ f) \pitchfork W$, i.e.,

$$\operatorname{Im}(d(g \circ f)_x) + T_{g(f(x))}(W) = T_{g(f(x))}(Z) \text{ for all } x \in X \text{ with } g(f(x)) \in W. \tag{A.4}$$

Let $x \in X$ be a point such that $f(x) \in g^{-1}(W)$ and let $b \in T_{f(x)}(Y)$. Since $g(f(x)) \in W$, we can use (A.4) to find vectors $a \in T_x(X)$ and $c \in T_{g(f(x))}(W)$ such that

$$d(g \circ f)_x(a) + c = dg_{f(x)}(b).$$

By the chain rule, we have $d(g \circ f)_x(a) = dg_{f(x)}(df_x(a))$. Thus

$$dg_{f(x)}(b - df_x(a)) = c \in T_{g(f(x))}(W),$$

In other words,

$$b - df_x(a) \in (dg_{f(x)})^{-1}(T_{g(f(x))}(W)) = T_{f(x)}(g^{-1}(W)).$$

Since b was an arbitrary element, we have proven $\operatorname{Im}(df_x) + T_{f(x)}(g^{-1}(W)) = T_{f(x)}(Y)$ for all $x \in X$ with $f(x) \in g^{-1}(W)$. In other words, $f \pitchfork g^{-1}(W)$.

Solution (Exercise 6.4) By definition, $\Gamma(A) \pitchfork \Delta$ if and only if $\Gamma(A)+\Delta(V) = V\times V$. Let (v_1, v_2) be an arbitrary element of $V \times V$. We need to check under which conditions we can find $v, w \in V$ such that

$$(v_1, v_2) = (v, v) + (w, Aw) = (v + w, v + Aw),\ \text{i.e. } v_1 = v + w \text{ and } v_2 = v + Aw.$$

If we can find a suitable w, then we just set $v := v_1 - w$. Hence, by taking the difference of the two equations, we see that the question is reduced to checking whether we can find a w such that $v_2 - v_1 = Aw - w = (A - I)w$ where I is the identity map of V. But such a w exists for any choice of v_1 and v_2 if and only if $A - I$ is invertible, i.e. if and only if $\det(A - I) \neq 0$ which happens if and only if $+1$ is not an eigenvalue of A (because the eigenvalues are the λ such that $\det(A - \lambda I) = 0$).

Solution (Exercise 6.5) Let $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ be the diagonal of X and $\Gamma(f) = \{(x, f(x)) : x \in X\} \subseteq X \times X$ be the graph of f. Then the set of fixed points of f in X is the intersection $\Delta_X \cap \Gamma(f)$. For

$$x = f(x) \iff (x, x) = (x, f(x)) \iff (x, x) \in \Gamma(f).$$

Recall that a 0-dimensional manifold is just a discrete set of points. Since X is compact, $X \times X$ is also compact. Hence a 0-dimensional submanifold is a discrete subset of the compact space $X \times X$ and is therefore finite. Note that we have used this before: discrete closed subspaces of compact spaces are finite. Thus, in order to prove that f has only finitely many fixed points, it suffices to show that $\Delta_X \cap \Gamma(f)$ is a 0-dimensional submanifold of $X \times X$.

Hence we would like to show that Δ_X and $\Gamma(f)$ meet transversally in $X \times X$. Since both have codimension equal to $\dim X$ in $X \times X$ the Transversality Theorem 6.2 then implies that $\Delta_X \cap \Gamma(f)$ is a 0-dimensional submanifold of $X \times X$. By definition, $\Delta_X \pitchfork \Gamma(f)$ means

$$T_{(x,x)}(\Gamma(f)) + T_{(x,x)}(\Delta_X) = T_{(x,x)}(X \times X)$$

for every point $(x, x) \in \Delta_X \cap \Gamma(f)$. We know $T_{(x,x)}(\Gamma(f)) = \Gamma(df_x)$ and $T_{(x,x)}(\Delta_X) = \Delta_{T_x(X)}$ by a previous exercise. Moreover, we know $T_{(x,x)}(X \times X) = T_x(X) \times T_x(X)$. Hence we need to show

$$\Gamma(df_x) + \Delta_{T_x(X)} = T_x(X) \times T_x(X) \tag{A.5}$$

for every point $(x, x) \in \Delta_X \cap \Gamma(f)$. This means exactly $\Gamma(df_x) \pitchfork \Delta_{T_x(X)}$ which we have shown to be true if $+1$ is not an eigenvalue of df_x in the previous exercise. Thus we can stop here, since f is Lefschetz by assumption.

But we could also just continue and give another proof as follows: Since we know

$$\dim \Delta_{T_x(X)} = \dim \Gamma(df_x) = \dim T_x(X),$$

equality (A.5) will follow once we show $\Gamma(df_x) \cap \Delta_{T_x(X)} = \{0\}$. For then we have shown that $\Gamma(df_x) + \Delta_{T_x(X)}$ is a subspace of $T_x(X) \times T_x(X)$ of the same dimension as $T_x(X)\times T_x(X)$. (Recall that in a finite dimensional vector space V with subspaces U and W, the following dimension formula holds:

$$\dim U + \dim W = \dim(U + W) - \dim(U \cap W)$$

where $U + W \subseteq V$ is the subspace of V generated by U and W and $U \cap W$ is their intersection.)

Since f is a Lefschetz map, we know that for every fixed point x of f, $+1$ is not an eigenvalue of df_x. This means that df_x does not have any fixed points, for there is no $v \in T_x(X) \setminus \{0\}$ with $df_x(v) = 1 \cdot v$. As we observed for f above, this is equivalent to

$$\Gamma(df_x) \cap \Delta_{T_x(X)} = \{0\}.$$

Solution (**Exercise 6.6**) We define the map

$$\begin{aligned} f : \mathbb{C}^5 \setminus \{0\} &\to \mathbb{C}, \\ (z_1, \dots, z_5) &\mapsto z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1}. \end{aligned}$$

Setting $Z = f^{-1}(0)$, we need to show that Z and $\mathbb{S}^9$ meet transversally. The tangent space to Z in a point $z \in Z$ is the kernel of the derivative df_z. Since f is a polynomial in the variables $z_1, \dots, z_5$, we can use our usual rules for partial differentiation to get the following matrix for df_z (in the standard basis):

$$df_z : \mathbb{C}^5 \to \mathbb{C},\ df_z = \left(2z_1, 2z_2, 2z_3, 3z_4^2, (6k-1)z_5^{6k-2}\right).$$

Recall that we can represent every element $x+iy \in \mathbb{C}$ by the real 2×2-matrix $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. Then we see that df_z is a real 2×10-matrix. Its maximal rank (as a matrix with entries in $\mathbb{R}$) is therefore 2. And, in fact, for every $z \neq 0$, df_z has rank 2, since it maps surjectively onto $\mathbb{C} \cong \mathbb{R}^2$. Thus 0 is a regular value for f and the tangent space $T_z(Z)$ is the kernel of df_z.

Writing a complex number $z = x + iy$, we can express $\mathbb{S}^9$ as the fiber of the smooth map

$$\begin{aligned} g : \mathbb{C}^5 \cong \mathbb{R}^{10} &\to \mathbb{R}, \\ (z_1, \dots, z_5) &\mapsto x_1^2 + y_1^2 + x_2^2 + y_2^2 + \cdots + x_5^2 + y_5^2 - 1 \end{aligned}$$

at the regular value 0, i.e., $\mathbb{S}^9 = g^{-1}(0) \subset \mathbb{C}^5 \cong \mathbb{R}^{10}$. The tangent space to $\mathbb{S}^9$ at z is then given by the kernel of the derivative (in standard bases)

$$dg_z : \mathbb{C}^5 = \mathbb{R}^{10} \to \mathbb{R},\ dg_z = \left(2x_1, 2y_1, 2x_2, 2y_2, \dots, 2x_5, 2y_5\right).$$

Thus, as expected, the tangent space $T_z(\mathbb{S}^9)$ consists of all vectors w in $\mathbb{R}^{10}$ which are orthogonal to z, i.e., which satisfy $w \cdot z = 0$.

The tangent space of $\mathbb{S}^9$ is of dimension 9 and the tangent space of $\mathbb{R}^{10} \setminus \{0\}$ is of dimension 10. Hence in order to show that Z and $\mathbb{S}^9$ meet transversally in $\mathbb{R}^{10} \setminus \{0\}$ we need to show: For every $z \in Z \cap \mathbb{S}^9$, there is at least one vector w in $T_z(Z)$ which does not belong to $T_z\mathbb{S}^9$. Then we have $T_z(Z) + T_z(\mathbb{S}^9) \subseteq T_z(\mathbb{R}^{10} \setminus \{0\})$ is a vector subspace of the same dimension as $T_z(\mathbb{R}^{10} \setminus \{0\})$ and therefore equal $T_z(\mathbb{R}^{10} \setminus \{0\})$.

So let $z = (z_1, \dots, z_5)$ be a fixed point in $Z \cap \mathbb{S}^9$. The tangent space $T_z(Z)$ is the kernel of df_z. Hence we need to find at least one vector $w \in \mathbb{C}^5 = \mathbb{R}^{10}$ with $df_z(w) = 0$ and $w \cdot z \neq 0$.

Set $m := 2 \cdot 3 \cdot (6k-1)$ and $w := (\frac{m}{2}z_1, \frac{m}{2}z_2, \frac{m}{2}z_3, \frac{m}{3}z_4, \frac{m}{6k-1}z_5)$. Then we have

$$\begin{aligned} df_z(w) &= \left(2z_1, 2z_2, 2z_3, 3z_4^2, (6k-1)z_5^{6k-2}\right) \cdot \begin{pmatrix} \frac{m}{2}z_1 \\ \frac{m}{2}z_2 \\ \frac{m}{2}z_3 \\ \frac{m}{3}z_4 \\ \frac{m}{6k-1}z_5 \end{pmatrix} \\ &= m\left(z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1}\right) \\ &= 0 \end{aligned}$$

since z is by assumption a point on Z. Hence $w \in T_z(Z)$.

On the other hand, we can calculate the inner product of w and z as vectors in $\mathbb{R}^{10}$, using (x_i, y_i) for the coordinates of z_i in $\mathbb{R}^2 \cong \mathbb{C}$, and get

$$\begin{aligned} w \cdot z &= \left(\frac{m}{2}x_1, \frac{m}{2}y_1, \frac{m}{2}x_2, \frac{m}{2}y_2, \frac{m}{2}x_3, \frac{m}{2}y_3, \frac{m}{3}x_4, \frac{m}{3}y_4, \frac{m}{6k-1}x_5, \frac{m}{6k-1}y_5\right) \cdot \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ \vdots \\ x_5 \\ y_5 \end{pmatrix} \\ &= \frac{m}{2}|z_1|^2 + \frac{m}{2}|z_2|^2 + \frac{m}{2}|z_3|^2 + \frac{m}{3}|z_4|^2 + \frac{m}{6k-1}|z_5|^2 \\ &> 0 \end{aligned}$$

which is bigger than zero, since z is a point on $\mathbb{S}^9$. Thus w is a vector in $T_z(\mathbb{R}^{10})$ which is in $T_z(Z)$, but not in $T_z(\mathbb{S}^9)$, and we have shown

$$T_z(Z) + T_z(\mathbb{S}^9) = T_z(\mathbb{R}^{10} \setminus \{0\}).$$

Hence Z and $\mathbb{S}^9$ meet transversally in $\mathbb{R}^{10} \setminus \{0\}$. The codimension of $Z \cap \mathbb{S}^9$ in $\mathbb{R}^{10} \setminus \{0\}$ is $2+1$ by the codimension formula. Thus $\dim(Z \cap \mathbb{S}^9) = 10 - 3 = 7$.

A.8 Smooth Homotopy

Solution (Exercise 8.1) **(a)** Let f and g be two smooth maps $Y \to X$. Let F be a homotopy from the identity map of X to the constant map $X \to \{x_0\}$ for some $x_0 \in X$. Then we can use F to define a homotopy from f to $Y \to \{x_0\}$ and a homotopy from $Y \to \{x_0\}$ to g. Setting these two homotopies together yields a homotopy H from f to g. It only remains to make sure that H is smooth. To achieve this we apply the technique used in the main text. After composing with a smooth bump function, we can assume $F(x,t) = x$ for all $(x,t) \in X \times [0,1/4]$ and $F(x,t) = x_0$ for all $(x,t) \in X \times [3/4,1]$. Then we can define H by

$$H: Y \times [0,1] \to X,\ (y,t) \mapsto \begin{cases} F(f(y), 2t) & t \in [0,1/2] \\ F(g(y), 2(1-t)) & t \in [1/2,1]. \end{cases}$$

(b) Let $Y = X$ and let $f: X \to X$ be the identity and $g: X \to \{x_0\} \subset X$ be the constant map for some point $x_0 \in X$. By the assumption, f and g are homotopic. Hence X is contractible.

(c) The map

$$F: \mathbb{R}^k \times [0,1] \to \mathbb{R}^k,\ (x,t) \mapsto (1-t)x$$

is a smooth homotopy from the identity map to the constant map $\mathbb{R}^k \to \{0\}$.

Solution (Exercise 8.2) By Sard's Theorem 7.1, there is a regular value $y \in \mathbb{S}^n$ for f. Assume there is a point $x \in f^{-1}(y)$. Since $\dim T_x X = k$ and $\dim T_y \mathbb{S}^n = n$, df_x cannot be surjective if $k < n$. Thus, if $k < n$, then $f^{-1}(y)$ must be empty. Hence we can assume that the image of f is contained in $U := \mathbb{S}^n \setminus \{y\}$. Now we can use stereographic projection from y to define a diffeomorphism $\psi: U \to \mathbb{R}^n$. Thus, $\psi \circ f$ is homotopic to a constant map. Composing the homotopy with the inverse of ψ defines a homotopy from f to a constant map.

Solution (Exercise 8.3) For $k = 1$, the antipodal map is $(x,y) \mapsto (-x,-y)$. The map

$$F_1: \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1,\ ((x,y),t) \mapsto \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

is a smooth homotopy from the identity of $\mathbb{S}^1$ to the antipodal map. To convince ourselves that $F_1(x,y,t)$ is an element in $\mathbb{S}^1$, we can either just calculate its norm or observe that the matrix $\begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}$ is an element in $O(2)$ for every t. Elements in $O(2)$ preserve the scalar product and hence the norm of vectors in $\mathbb{R}^2$.

For an arbitrary odd k, we have $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ and $k+1$ is even. Then we define a smooth homotopy from the identity in $\mathbb{S}^k$ to the antipodal map by

$$F_k: \mathbb{S}^k \times [0,1] \to \mathbb{S}^k,$$
$$((x_1,y_1),\dots,(x_{(k+1)/2}, y_{(k+1)/2}), t) \mapsto (F_1(x_1,y_1,t),\dots,F_1((x_{(k+1)/2}, y_{(k+1)/2}, t)$$

Again, for every t, $F_k(-,-,t)\colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is an element in $O(k+1)$ and preserves the norm on $\mathbb{R}^{k+1}$.

Solution (**Exercise 8.4**) Let $f\colon \mathbb{S}^1 \to X$ be a smooth map. Since X is contractible, there is a smooth homotopy F from the identity on X and a constant map $\{x_0\}$. The composition

$$\mathbb{S}^1 \times [0,1] \to X,\ (x,t) \mapsto F(f(x),t)$$

defines a smooth homotopy from f to the constant map $\mathbb{S}^1 \to \{x_0\}$.

Solution (**Exercise 8.5**) Given two points $x, y \in X$, we define the relation $x \sim y$, and say x and y are path-connected, by: $x \sim y$ if and only if there is a smooth path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We would like to show that $\sim$ is an equivalence relation. Therefor, we are going to link it to the homotopy relation.

Let $f\colon \{x\} \to X$, $f(x) = x$, and $g\colon \{x\} \to X$, $g(x) = y$. If $F\colon \{x\} \times [0,1] \to X$ is a smooth homotopy from f to g, then $\gamma(t) := F(x,t)$ is a smooth path from x to y. Conversely, if γ is a smooth path from x to y, then $F(x,t) := \gamma(t)$ is a smooth homotopy from f to g. Thus $x \sim y$ if and onlf if $f \sim g$.

Since homotopy is an equivalence relation, we see that path-connectedness is also an equivalence relation. Recall that the equivalence class $[x]$ of a point $x \in X$ is the set

$$[x] = \{y \in X : x \sim y\}.$$

A crucial feature of equivalence relations is that equivalence classes are either equal or disjoint, i.e. for any x and y in X we have either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. We are going to use this fact in the following way: If we can show that every equivalence class $[x]$ is an open subset of X, then we know that every $[x]$ is also a closed subset. For, $X \setminus [x]$ is the union of all the other open classes and is therefore open itself (arbitrary unions of open sets are open).

So, given an arbitrary point $x_0 \in X$, we would like to show that $[x_0]$ is open. Let $x \in X$ be a point in $[x]$. Since X is a smooth manifold, there is a local parametrization $\phi\colon B_\epsilon(0) \to U$ with $\phi(0) = x$, where U is open in X and $B_\epsilon(0)$ is the open ball of radius ϵ in $\mathbb{R}^{\dim X}$. Given any $y \in U$, let $\phi^{-1}(y)$ be its preimage in $B_\epsilon(0)$. In $B_\epsilon(0)$, all points are path-connected to 0. Hence there is a smooth path

$$\gamma\colon [0,1] \to B_\epsilon(0),\ t \mapsto t \cdot \phi^{-1}(y)$$

with $\gamma(0) = 0$ and $\gamma(1) = \phi^{-1}(y)$. Since ϕ is a diffeomorphism, the composite $\phi \circ \gamma$ is a smooth path from x to y in X, i.e. $x \sim y$.

This shows that U is contained in $[x_0]$. Thus $[x_0]$ is an open subset in X, since every point $x \in [x_0]$ has an open neighborhood in X which is completely contained in $[x_0]$.

Thus $[x_0]$ is a nonempty, open and closed subset of X. Since X is connected, this implies $[x_0] = X$. Thus X is path-connected.

Solution (**Exercise 8.6**) **(a)** The assumption that $|f(x) - g(x)| < 2$ implies that $f(x)$ and $g(x)$ are never antipodal points. In particular, the straight line segment from $f(x)$ to $g(x)$ in $\mathbb{R}^{k+1}$ does not go through the origin for all $x \in X$. Hence the vector $(1-t)f(x)+tg(x)$ is nonzero for all $x \in X$ and all $t \in [0,1]$. Thus, we can form the well-defined map

$$H: X \times [0,1] \to \mathbb{S}^k,\ x \mapsto \frac{(1-t)f(x)+tg(x)}{|(1-t)f(x)+tg(x)|}.$$

Since f and g are continuous, H is continuous and defines a homotopy from f to g.

(b) By the previous point, there is a continuous homotopy H from f to g. Now it suffices to compose H with a smooth bump function to turn it into a smooth homotopy.

Solution (Exercise 8.7) **(a)** If k is odd, then $k+1$ is even and we can define the map

$$s: \mathbb{S}^k \to \mathbb{R}^{k+1}, (x_1, \dots, x_{k+1}) \mapsto (-x_2, x_1, -x_3, x_4, \dots, -x_{k+1}, x_k).$$

This map can be extended to a linear map $\mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ and therefore s is smooth. For each $x \in \mathbb{S}^k$, $s(x)$ is nonzero and satisfies $x \perp s(x)$. Thus $s(x)$ is a tangent vector at x, i.e. $s(x) \in T_x(\mathbb{S}^k) \setminus \{0\}$. Hence

$$\sigma: \mathbb{S}^k \to T(\mathbb{S}^k),\ \sigma(x) := (x, s(x))$$

is the desired non-vanishing vector field on $\mathbb{S}^k$.

(b) Given a vector field $\sigma: \mathbb{S}^k \to T(S^k)$ which has no zeros. Let $\sigma(x) = (x, s(x))$. Since $s(x) \neq 0$ for every $x \in \mathbb{S}^k$, we can define a new vector field by

$$x \mapsto \frac{s(x)}{|s(x)|}.$$

By replacing s with this new non-vanishing vector field, we can assume $|s(x)| = 1$. Hence we can assume $s(x) \in \mathbb{S}^k$ and $s(x) \cdot x = 0$ for every $x \in \mathbb{S}^k$.

Now we define the map

$$F: \mathbb{S}^k \times [0,1] \to \mathbb{S}^k,\ (x,t) \mapsto \cos(\pi t)x + \sin(\pi t)s(x).$$

We need to check that $F(x,t)$ is in fact an element in $\mathbb{S}^k$ for every $x \in \mathbb{S}^k$:

$$\begin{aligned} F(x,t)\cdot F(x,t) &= (\cos(\pi t)x + \sin(\pi t)s(x)) \cdot (\cos(\pi t)x + \sin(\pi t)s(x)) \\ &= \cos^2(\pi t)(x\cdot x) + 2\cos(\pi t)\sin(\pi t)(x \cdot s(x)) + \sin^2(\pi t)(s(x)\cdot s(x)) \\ &= \cos^2(\pi t) + \sin^2(\pi t) \\ &= 1 \end{aligned}$$

where we use $x \cdot x = 1 = s(x)\cdot s(x)$ and $x \cdot s(x) = 0$. Thus $F(x,t)$ is a vector of norm 1 for every x and every t. Moreover, F is a smooth map with $F(x,0) = x$ and $F(x,1) = -x$, i.e. F is a smooth homotopy from the identity to the antipodal map on $\mathbb{S}^k$.

(c) For $1 \leq i \leq k+1$, let r_i be the reflection map on the ith coordinate:

$$r_i : \mathbb{S}^k \to \mathbb{S}^k, (x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, -x_i, \dots, x_{k+1}).$$

Then the map $\mathbb{S}^k \times [0,1] \to \mathbb{S}^k$ defined by sending $(x_1, \dots, x_{k+1}, t)$ to

$$(x_1, \dots, x_{i-1}, \cos(\pi t)x_i - \sin(\pi t)x_{i+1}, \sin(\pi t)x_i + \cos(\pi t)x_{i+1}, x_{i+2}, \dots, x_{k+1})$$

is a homotopy from the identity on $\mathbb{S}^k$ to the map $r_i \circ r_{i+1} : \mathbb{S}^k \to \mathbb{S}^k$.

The antipodal map is equal to the composition of reflections $r_1 \circ r_2 \circ \cdots \circ r_{k+1}$. Since k is even $r_2 \circ \cdots \circ r_{k+1}$ is homotopic to the identity. Thus the antipodal map is homotopic to the reflection r_1.

A.9 Abstract Smooth Manifolds

Solution (**Exercise 9.1**) **(a)** A line in $\mathbb{R}^2$ is determined by an equation of the form $ax+by+c = 0$ with fixed $(a, b, c) \in \mathbb{R}^3$. Since an equation of the form $ax+by+c = 0$ with $a = b = 0$ does not define a line, we have to exclude triples of the form $(0, 0, c)$. Moreover, the equations $ax + by + c = 0$ and $(\lambda a)x_(\lambda b)y + (\lambda c) = 0$ with $\lambda \neq 0$ determine the same line. Hence X can be identified with the set of equivalence classes

$$X = (\mathbb{R}^3 \setminus \{(0, 0, 0), (0, 0, 1)\})/ \sim$$

where $\sim$ is the relation defined by

$$(a, b, c) \sim (\lambda a, \lambda b, \lambda c) \text{ if there is a } \lambda \neq 0.$$

But this is the subspace of $\mathbb{R}\mathrm{P}^2$ given by removing the point $[0 : 0 : 1]$. Since any subspace consisting of just one point is closed in $\mathbb{R}\mathrm{P}^2$, we have shown that X can be identified with an open subset of $\mathbb{R}\mathrm{P}^2$:

$$X = \mathbb{R}\mathrm{P}^2 \setminus \{[0 : 0 : 1]\}.$$

(b) Every line in $\mathbb{R}^2$ is determined by the point where it crosses the x-axis and a direction which can be expressed by an angle $\in [0, 2\pi]$. Since we have not specified a direction for the line, two angles which differ by adding π determine the same line. Any angle between 0 and 2π can be described by a point on the unit circle, where the points s and $-s$ on $\mathbb{S}^1$ correspond to angles which differ by adding π. Hence any line in $\mathbb{R}^2$ is determined by a $(s, x) \in \mathbb{S}^1 \times \mathbb{R}$ where s is uniquely determined up to multiplying with ± 1.

Solution (**Exercise 9.2**) **(a)** We define

$$V_1 := \{[z : w] \in \mathbb{C}\mathrm{P}^1 : z \neq 0\} \text{ and } V_2 := \{[z : w] \in \mathbb{C}\mathrm{P}^1 : w \neq 0\}.$$

The preimage of V_1 in $\mathbb{C}^2$ is the open subset $\{(z, w) \in \mathbb{C}^2 : z \neq 0\}$ and preimage of V_2 in $\mathbb{C}^2$ is the open subset $\{(z, w) \in \mathbb{C}^2 : w \neq 0\}$. Hence V_1 and V_2 are open in $\mathbb{C}\mathrm{P}^1$. Since either z or w must be nonzero for every point $[z : w] \in \mathbb{C}\mathrm{P}^1$, $\{V_1, V_2\}$ provides an open cover of $\mathbb{C}\mathrm{P}^1$.

We define maps $\phi_1 : \mathbb{R}^2 \to V_1$ and $\phi_2 : \mathbb{R}^2 \to V_2$

$$\phi_1 : (x, y) \mapsto [1 : (x + iy)] \text{ and } \phi_2 : (x, y) \mapsto [(x + iy) : 1].$$

The inverses are $\phi_1^{-1} : V_1 \to \mathbb{R}^2$ and $\phi_2^{-1} : V_2 \to \mathbb{R}^2$ defined by

$$\phi_1^{-1} : [(x_1 + iy_1) : (x_2 + iy_2)] \mapsto \frac{1}{x_1^2 + y_1^2}(x_1x_2 + y_1y_2, x_1y_2 - x_2y_1)$$

and

$$\phi_2^{-1} : [(x_1 + iy_1) : (x_2 + iy_2)] \mapsto \frac{1}{x_2^2 + y_2^2}(x_1x_2 + y_1y_2, x_2y_1 - x_1y_2)$$

where the right hand sides arise from calculating the quotients of complex numbers.

As in the proof of Theorem 9.9 we can check that these maps do not depend on the chosen representatives. It is also easy to see that ϕ and ϕ_i^{-1} are mutual inverses which are both continuous. We check that the change-of-coordinate maps are smooth: Both composites

$$\phi_1^{-1}(V_1 \cap V_2) \xrightarrow{\phi_1} V_1 \cap V_2 \xrightarrow{\phi_2^{-1}} \phi_2^{-1}(V_1 \cap V_2)$$

and

$$\phi_2^{-1}(V_1 \cap V_2) \xrightarrow{\phi_2} V_1 \cap V_2 \xrightarrow{\phi_1^{-1}} \phi_2^{-1}(V_1 \cap V_2)$$

are given by

$$(x, y) \mapsto \frac{1}{x^2 + y^2}(x, -y)$$

and are therefore smooth maps.

(b) We copy the relation we used before and just add a condition to make sure that the norms are respected. We define $\sim_s$ by

$$\begin{aligned} &(z_0, w_0) \sim_s (z_1, w_1) \\ \iff\ & z_1 = \lambda z_0 \text{ and } w_1 = \lambda w_0 \text{ for some } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } |\lambda| = 1. \end{aligned}$$

Then it is easy to check that $\mathbb{S}^3 \sim_s \cong (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$.

(c) For $w \neq 0$, we consider the points $[z : w] = [z/w : 1]$ in $\mathbb{C}\mathrm{P}^1$ as points in $\mathbb{C}$ by identifying them with z/w. This misses only one point in $\mathbb{C}\mathrm{P}^1$, the point $[1 : 0]$ 'at infinity'. Then we mimic the stereographic projection and define $h\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{S}^2$ by

$$[z : w] \mapsto \begin{cases} \frac{1}{|z/w|^2+1}(2z/w, |z/w|^2 - 1) & \text{if } w \neq 0 \\ (0, 1) & \text{if } w = 0 \end{cases}$$

where we consider $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$ as before. We need to check that this map has the desired properties:

- We need to check that $h([z : w])$ actually is a point on $\mathbb{S}^2$. This is true for $h([z : 0]) = (0, 1)$ and for the other points we check

$$\begin{aligned} |h([z : w])|^2 &= \frac{2z/w}{|z/w|^2 + 1} \cdot \frac{2\bar{z}/\bar{w}}{|z/w|^2 + 1} + \frac{(|z/w|^2 - 1)^2}{(|z/w|^2 + 1)^2} \\ &= \frac{4|z/w|^2 + |z/w|^4 - 2|z/w|^2 + 1}{(|z/w|^2 + 1)^2} \\ &= \frac{|z/w|^4 + 2|z/w|^2 + 1}{(|z/w|^2 + 1)^2} \\ &= 1. \end{aligned}$$

- We need to check that this map is well-defined, i.e., if $\lambda \neq 0 \in \mathbb{C}$, we need to check that h sends $[z : w]$ and $[\lambda z : \lambda w]$ to the same point. This is true, since we have $|z/w| = |(\lambda z)/(\lambda w)|$.
- For $(z, 0) \in \mathbb{S}^3$, we compute

$$h(\varphi([z : 0])) = (0, 1) = \pi(z, 0)$$

where we use that $|z|^2 = 1$ as z is a point on $\mathbb{S}^3$. And for $(z, w) \in \mathbb{S}^3$ with $w \neq 0$, we get

$$\begin{aligned} h(\varphi([z : w])) &= \frac{1}{|z/w|^2 + 1}(2z/w, |z/w|^2 - 1) \\ &= \frac{|w|^2}{|z|^2 + |w|^2}(2z/w, |z/w|^2 - 1) \\ &= (2z/w \cdot (w\bar{w}), |z|^2 - |w|^2) \\ &= (2z \cdot \bar{w}, |z|^2 - |w|^2) \\ &= \pi(z, w) \end{aligned}$$

where we used $|z|^2 + |w|^2 = 1$. Hence we have $h \circ \varphi = \pi$.

(d) The fiber consists of all points (z, w) in $\mathbb{S}^3$ with $[z : w] = [z_0 : w_0]$. By the definition of $\mathbb{C}\mathrm{P}^1$, that means (z, w) is in the fiber if and only if there is a $\lambda \in \mathbb{C}\backslash\{0\}$ such that $(z, w) = (\lambda z_0, \lambda w_0)$. The additional feature we need to remember is that (z, w) is in $\mathbb{S}^3$ just as we did for the relation $\sim_s$. Hence λ needs to satisfy $|\lambda| = 1$. Summarising, we have

$$\begin{aligned} & (z, w) \in \varphi^{-1}([z_0 : w_0]) \\ \iff & (z, w) = (\lambda z_0, \lambda w_0) \text{ for some } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } |\lambda| = 1. \end{aligned}$$

In other words, the points in the fiber $\varphi^{-1}([z_0 : w_0])$ are in one-to-one correspondence to points λ on the circle $\mathbb{S}^1 \subset \mathbb{C}$. This shows again that the fiber of the Hopf map at any point is diffeomorphic to $\mathbb{S}^1$.

Solution (Exercise 9.3) **(a)** We write down local coordinate charts. For $z \in \mathbb{C}\backslash\{0\}$, let $[z]$ be its equivalence class in H^2. We pick a point $[z_0] \in H^2$. Choosing $\varepsilon > 0$ small enough, i.e., in our case $0 < \varepsilon < \lambda/2$ is enough, the open ball $B^2_\varepsilon(z_0) \subset \mathbb{C}\backslash\{0\}$ does not contain any point z with $[z] = [z_0]$. Hence the restriction $B^2_\varepsilon(z_0) \to H^2$ of the quotient map to $B^2_\varepsilon(z_0)$ is a homeomorphism onto its image $\bar{B}^2_\varepsilon([z_0]) \subset H^2$. The inverse is given by sending a point $[z] \in \bar{B}^2_\varepsilon([z_0])$ to the point $z \in B^2_\varepsilon(z_0)$. Since ε is small enough, there is a unique such lift in $B^2_\varepsilon(z_0)$. This defines a homeomorphism

$$\phi\colon\ B^2_\varepsilon(z_0) \to \bar{B}^2_\varepsilon([z_0])$$

whose inverse ψ a local chart around $[z_0] \in H^2$.

If ψ_1 and ψ_1 are charts around $[z_1]$ and $[z_2]$, respectively, then, since we chose ε

small enough, the change of coordinate map

$$\psi_2 \circ \psi_1^{-1} : \; B_\varepsilon^2(z_1) \cap B_\varepsilon^2(z_2) \to B_\varepsilon^2(z_1) \cap B_\varepsilon^2(z_2)$$

is just the identity. Since this is a smooth map, we have shown that H^2 is an abstract smooth 2-manifold.

(b) We write $[z]$ for the image of $z \in A$ in $A/\mathbb{Z}$. We claim that the map

$$h : \; A/\mathbb{Z} \to H^2, [z]_A \mapsto [z]_H$$

where $[z]_A$ and $[z]_H$ denote the equivalence classes of $z \in \mathbb{C}$ in $A/\mathbb{Z}$ and H^2, respectively. We claim that h is a homeomorphism:

- First we check that h is injective. Assume $w, z \in A$ with $h([w]_A) = h([z)_A]$. After possibly switching w and z we can assume $|w| \leq |z|$. Then there is a $k \in \mathbb{Z}$, $k \leq 0$, such that $w = \lambda^k z$. If $k = 0$, then $w = z$, and if $k < 0$, then not both z and w can be in A, unless one of w and z has absolute value 1 and the other one has absolute value $1/\lambda$. So let us say $|w| = 1$ and $|z| = 1/\lambda$. But in this case their images in the quotient $A/\mathbb{Z}$ are the same. Hence h is injective.
- Next we check that f is surjective. Let $z \in \mathbb{C} \setminus \{0\}$. Then there is a $k \in \mathbb{Z}$ such that $\lambda^k z \in A$. Hence $h([\lambda^k z]_A) = [z]_H \in H^2$, and h is surjective.
- Since A is a closed subset of $\mathbb{C}$, the map h is continuous. Since A is compact and the quotient map $A \to A/\mathbb{Z}$ is continuous, $A/\mathbb{Z}$ is compact. Hence h is a continuous bijection with compact domain and Hausdorff codomain. This implies, by general topology arguments, that h is a homeomorphism.

This shows that H^2 is compact, since it is the image of a compact space under a continuous map.

(c) For $(z, w) \in \mathbb{C}^2 \setminus \{0\}$, let $[z, w]$ be its equivalence class in H^4. We pick a point $[z_0, w_0] \in H^4$. Choosing $\varepsilon > 0$ small enough, the open ball $B_\varepsilon^4(z_0) \subset \mathbb{C}^2 \setminus \{0\}$ does not contain any point (z, w) with $[z, w] = [z_0, w_0]$. Hence the restriction $B_\varepsilon^4(z_0, w_0) \to H^4$ to $B_\varepsilon^2(z_0)$ is a homeomorphism onto its image $\bar{B}_\varepsilon^4([z_0, w_0]) \subset H^4$. The inverse is given by sending a point $[z, w] \in \bar{B}_\varepsilon^4([z_0, w_0])$ to the point $(z, w) \in B_\varepsilon^4(z_0, w_0)$. Since ε is small enough, there is a unique such lift in $B_\varepsilon^4(z_0, w_0)$. This defines a homeomorphism

$$\phi : \; B_\varepsilon^4(z_0, w_0) \to \bar{B}_\varepsilon^4([z_0, w_0])$$

whose inverse ψ a local chart around $[z_0, w_0] \in H^4$.

If ψ_1 and ψ_1 are charts around $[z_1, w_1]$ and $[z_2, w_2]$, respectively, then, by chosing ε small enough, the change of coordinate map

$$\psi_2 \circ \psi_1^{-1} : \; B_\varepsilon^4(z_1, w_1) \cap B_\varepsilon^4(z_2, w_2) \to B_\varepsilon^4(z_1, w_1) \cap B_\varepsilon^4(z_2, w_2)$$

is just the identity. Since this is a smooth map, we have shown that H^4 is an abstract smooth 4-manifold.

(d) We consider $\mathbb{S}^3$ as a subset in $\mathbb{C}^2 \setminus \{0\}$ and $\mathbb{S}^1$ as the quotient $[1, 1/\lambda]/(1 \sim 1/\lambda)$, i.e., the closed interval $[1, 1/\lambda] \subset \mathbb{R}$ where we identify the endpoints. We define a map

$$f : \mathbb{S}^3 \times \mathbb{S}^1 = \mathbb{S}^3 \times [1, 1/\lambda]/(1 \sim 1/\lambda) \to H^4, (z, w, s) \mapsto [sz, sw].$$

We need to check that this is well-defined and compute $f(z, w, 1) = [z, w] = [1/\lambda z, 1/\lambda w] = f(z, w, 1/\lambda)$.

- To show that f is a homeomorphism, we recall that every point in $\mathbb{C}^2 \setminus \{0\}$ is determined by a direction, i.e., a point in $\mathbb{S}^3$, and the distance from the origin. Since we identify points p_1 and p_2 in H^4 if $p_2 = \lambda^k p_1$ for some $k \in \mathbb{Z}$, it suffices to specify the distance from the origin up to a multiple of λ^k. This shows that f is injective. But f is also surjective, since every point p in $\mathbb{C}^2 \setminus \{0\}$ determines a point in $\mathbb{S}^3$, that is the point (z, w) where the line from the origin to p meets $\mathbb{S}^3$. Then there is a unique $k \in \mathbb{Z}$ such that $1 \leq \lambda^k |p| < 1/\lambda$. This real number $\lambda^k |p|$ is the coordinate s.

  By general topology, that facts that f is continuous and has a compact domain and a Hausdorff codomain, imply that f is a homeomorphism.

- Our copy of $\mathbb{S}^1$ equals the quotient $\mathbb{R}/\mathbb{Z}$ where we identify two real numbers s_1 and s_2 if $s_1 - s_2 \in \mathbb{Z}$. This makes it easy to provide local charts. Around any $x \in \mathbb{R}$ we can look at the open neighborhood $(-\varepsilon + x, x + \varepsilon)$ which maps homeomorphically onto its image in $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ under the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. On points in $\mathbb{C}^2 \setminus \{0\}$ and $\mathbb{S}^3$, f and its inverse are just scaling by a real number, some integer power of λ. So composition with local charts results in a smooth map, since the local charts on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ are just identity maps and the local charts on $\mathbb{S}^3$ are diffeomorphisms as we have seen many times before. Hence f is a diffeomorphism as it is a smooth homeomorphism with a smooth inverse.

Solution **(??)** Recall the subsets $V_i := \{[x_0 : \ldots : x_n] \in \mathbb{R}\mathrm{P}^n : x_i \neq 0\}$ which are open in $\mathbb{R}\mathrm{P}^n$. The product spaces $V_i \times V_j$ are open in $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^n$ and hence the subsets $V_{ij} = H(m, n) \cap (V_i \times V_j)$ are open in $H(m, n)$. The union of all $V_i \times V_j$ for all i, j covers $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^n$. Hence the union of all $V_{ij} = H(m, n) \cap (V_i \times V_j)$ for all i, j covers $H(m, n)$. In fact, it suffices to consider all pairs (i, j) with $i \neq j$, since if x_i is the only coordinate in $[x]$ with $x_i \neq 0$, then we must have $y_i = 0$ in order to satisfy the condition $\sum_{i=0}^m x_i y_i = 0$. Thus we cannot have that x_i and y_i are the only non-zero coordinates in $[x[$ and $[y]$, respectively.

Now we define maps $\phi_{ij} : \mathbb{R}^{m+n-1} \to V_{ij}$ by sending the $(m + n - 1)$-tuple $((x_1, \ldots, x_m), (y_1, \ldots, y_{n-1}))$ to

$$\left([x_1 : \ldots : 1 : \ldots : x_m), [y_1 : \ldots : -\sum_{s=1, s\neq i}^{m} x_s y_s : \ldots : 1 : \ldots : y_{n-1}) \right).$$

where the first 1 and the sum are at position $i + 1$, and the second 1 is at position $j + 1$. Note that this actually yields an element in $H(m, n)$, and not just $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^n$, since the defining condition $H(m,n)$ is satisfied. Their inverses $\phi_{ij}^{-1} : V_{ij} \to \mathbb{R}^{m+n-1}$ are given by sending $([x_0 : \ldots : x_m], [y_0 : \ldots : y_n])$ to

$$\left(\frac{1}{x_i}(x_0, \ldots, \widehat{x}_i, \ldots, x_m), \frac{1}{y_j}(y_0, \ldots, \widehat{y}_j, \ldots, y_n)\right).$$

We can check that the change of coordinate maps are smooth just as we did for real projective space, since the sum $\sum_{i=0}^{m} x_i y_i$ is a polynomial and hence smooth.

A.10 Manifolds with Boundary

A.10.1 Manifolds with boundary

Solution (**Exercise 10.1**) Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^k$ be open neighborhoods of 0. Suppose there was a diffeomorphism $\theta : U \to V$. We can assume that 0 is sent to a boundary point of V. In fact, we can assume that $\theta(0) = 0$. Otherwise we just another pick point $u \in U$ with $\theta(u) \in \partial V$. Then $d\theta_0 : \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism. By the Inverse Function Theorem, there are subsets W_1 and W_2 in $\mathbb{R}^k$ containing 0 which are both **open in** $\mathbb{R}^k$ such that θ maps W_1 diffeomorphically onto W_2. Since W_2 is open in $\mathbb{R}^k$ and contained in the image of θ, we get that V must be **open in** $\mathbb{R}^k$. But since V contains 0, it satisfies $V \cap \partial\mathbb{H}^k \neq \emptyset$ and cannot be open **in** $\mathbb{R}^k$.

Solution (**Exercise 10.2**) Let $f : X \to Y$ be a diffeomorphism of manifolds with boundary. Let $\phi : U \to X$ and $\psi : V \to Y$ be local parametrizations, where U and V are open subsets of $\mathbb{H}^k$ (check that you know why the dimensions of X and Y must be equal). Let $\theta : U \to V$ be the induced map. By shrinking U and V if necessary, we can assume that θ is a diffeomorphism with

$$f \circ \phi = \psi \circ \theta.$$

Boundary points of X are those which are in the image $\phi(\partial U) = \phi(U \cap \partial\mathbb{H}^k)$. Similarly, boundary points of X are those which are in the image $\psi(\partial V) = \psi(V \cap \partial\mathbb{H}^k)$. Hence we need to show $\theta(\partial U) \subset \partial V$, for then

$$f(\phi(\partial U)) = \psi(\theta(\partial U)) \subset \psi(\partial V) \subset \partial Y.$$

The argument is again based on the Inverse Function Theorem. Suppose there is a point $u \in \partial U$ which is mapped to an interior point $v = \theta(u)$ in V. Since θ is a diffeomorphism, the derivative $d(\theta^{-1})_v : \mathbb{R}^k \to \mathbb{R}^k$ of its inverse is an isomorphism. But, since $v \in \mathrm{Int}(V)$, V contains a neighborhood W of v that is open in $\mathbb{R}^k$. Thus the Inverse Function Theorem implies that $\theta^{-1}(W)$ contains a neighborhood of u that is open in $\mathbb{R}^k$. Hence u is also an interior point in U which contradicts the assumption $u \in \partial U$.

Solution (**Exercise 10.3**) **(a)** The image of F is the product $\mathbb{S}^1 \times [-1/2, 1/2]$. This is a product of a manifold without a boundary $\mathbb{S}^1$ and the manifold $[-1/2, 1/2]$ with boundary. The boundary of $[-1/2, 1/2]$ consists of the disjoint union of $\{-1/2\}$ and $\{1/2\}$. By Lemma 10.9, we get

$$\partial X = \partial(\mathbb{S}^1 \times [-1/2, 1/2]) = \mathbb{S}^1 \times \{-1/2\} \cup \mathbb{S}^1 \times \{1/2\}.$$

(b) We define the maps

$$\begin{aligned}
\phi_+ :\ (-\pi,\pi)\times[0,3/4) \to & Y,\\
(t,s) \mapsto & ((1+(-1/2+s)\cos(t/2))\cos t,\\
& (1+(-1/2+s)\cos(t/2))\sin t, (-1/2+s)\sin(t/2))\\
\phi_- :\ (-\pi,\pi)\times[0,3/4) \to & Y,\\
(t,s) \mapsto & ((1+(1/2-s)\cos(t/2))\cos t,\\
& (1+(1/2-s)\cos(t/2))\sin t, (1/2-s)\sin(t/2))\\
\psi_+ :\ (0,2\pi)\times[0,3/4) \to & Y,\\
(t,s) \mapsto & ((1+(-1/2+s)\cos(t/2))\cos t,\\
& (1+(-1/2+s)\cos(t/2))\sin t, (-1/2+s)\sin(t/2))\\
\psi_- :\ (0,2\pi)\times[0,3/4) \to & Y,\\
(t,s) \mapsto & ((1+(1/2-s)\cos(t/2))\cos t,\\
& (1+(1/2-s)\cos(t/2))\sin t, (1/2-s)\sin(t/2)).
\end{aligned}$$

As one can check by calculating the partial derivatives, each of these maps are diffeomorphisms, and the union of their images covers Y. Hence we can use these four maps as local parametrizations of Y.

The boundary of Y is then given by the union of the points

$$\begin{aligned}
\partial Y = & \phi_+((-\pi,\pi)\times\{0\}) \cup \phi_-((-\pi,\pi)\times\{0\})\\
& \cup \psi_+((0,2\pi)\times\{0\}) \cup \psi_-((0,2\pi)\times\{0\}).
\end{aligned}$$

Setting $s=0$ in the fomulae for those maps gives

$$\begin{aligned}
\partial Y = & \{((1-\frac{1}{2}\cos(t/2))\cos t, (1-\frac{1}{2}\cos(t/2))\sin t, -\frac{1}{2}\sin(t/2)) \in \mathbb{R}^3 \,:\, t\in\mathbb{R}\}\\
& \cup \{((1+\frac{1}{2}\cos(t/2))\cos t, (1+\frac{1}{2}\cos(t/2))\sin t, \frac{1}{2}\sin(t/2)) \in \mathbb{R}^3 \,:\, t\in\mathbb{R}\}.
\end{aligned}$$

But, in fact, the two sets describing ∂Y are the same which we see when we replace t with $t+2\pi$ and use some simple trigonometric identities:

$$\begin{cases}
\Big(1-\frac{1}{2}\cos(\frac{t+2\pi}{2})\Big)\cos(t+2\pi) & = \Big(1+\frac{1}{2}\cos(t/2)\Big)\cos t,\\
\Big(1-\frac{1}{2}\cos(\frac{t+2\pi}{2})\Big)\sin(t+2\pi) & = \Big(1+\frac{1}{2}\cos(t/2)\Big)\sin t\\
-\frac{1}{2}\sin(\frac{t+2\pi}{2})) & = \frac{1}{2}\sin(t/2).
\end{cases}$$

Hence

$$\partial Y = \left\{\left(\left(1+\frac{1}{2}\cos(t/2)\right)\cos t, \left(1+\frac{1}{2}\cos(t/2)\right)\sin t, \frac{1}{2}\sin(t/2)\right) \in \mathbb{R}^3 \,:\, t\in\mathbb{R}\right\}.$$

Now we would like to show that ∂Y is diffeomorphic to $\mathbb{S}^1$. Remembering the trigonometric identities

$$\sin t = 2\sin(t/2)\cos(t/2) \text{ and } \cos t = \cos^2(t/2) - \sin^2(t/2)$$

we see that the map

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^3,\ (x, y) \mapsto \left(\left(1 + \frac{1}{2}x\right)\left(x^2 - y^2\right), \left(1 + \frac{1}{2}x\right)2xy, \frac{1}{2}y\right)$$

restricts to a bijection from $\mathbb{S}^1$ onto ∂Y (for injectivity, note that the last coordinate determines y uniquely, then the circle equation determines x up to sign, and the first and/or second coordinate determine the sign of x).

It remains to check that $\varphi_{|\mathbb{S}^1}$ is a local diffeomorphism. Since $\varphi_{|\mathbb{S}^1}$ is a bijection onto its image, this will show that it is a diffeomorphism.

First we observe that φ is smooth, since the three functions in each coordinate are just polynomials and hence smooth. To see that $\varphi_{|\mathbb{S}^1}$ is a local diffeomorphism, we use the maps

$$\phi : U \to \mathbb{S}^1, t \mapsto (\cos(t/2), \sin(t/2))$$

and

$$\psi : U \to \partial Y, t \mapsto \left(\left(1 + \frac{1}{2}\cos(t/2)\right)\cos t, \left(1 + \frac{1}{2}\cos(t/2)\right)\sin t, \frac{1}{2}\sin(t/2)\right)$$

where $U \subset \mathbb{R}$ is some sufficiently small open subset. Then ϕ and ψ serve as local parametrizations of $\mathbb{S}^1$ and ∂Y, respectively, for suitable choices of U. But the induced map $\theta : U \to U$ which arises as the composite $\psi^{-1} \circ \varphi_{|\mathbb{S}^1} \circ \phi$ is just the identity $t \mapsto t$. Hence $\varphi_{|\mathbb{S}^1}$ is a local diffeomorphism.

Solution (Exercise 10.4) **(a)** We proceed as before when we showed that tangent spaces are well-defined.

Let $\psi : V \to X$ be another local parametrization around x with $\psi(0) = x$, where V is an open subset of $\mathbb{H}^k$. By shrinking both U and V, we can assume $\phi(U) = \psi(V)$ (replace U by $\phi^{-1}(\phi(U) \cap \psi(V)) \subset U$ and V by $\psi^{-1}(\phi(U) \cap \psi(V)) \subset V$).

Then the map

$$\theta := \psi^{-1} \circ \phi : U \to V$$

is a diffeomorphism (its the composite of two diffeomorphisms). By definition of θ, we have $\phi = \psi \circ \theta$. Differentiating yields

$$d\phi_0 = d\psi_0 \circ d\theta_0$$

(where we have used the chain rule). This implies that the image of $d\phi_0$ is contained in the image of $d\psi_0$:

$$d\phi_0(\mathbb{R}^k) \subseteq d\psi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N.$$

By switching the roles of ϕ and ψ in the argument, we also get:

$$d\psi_0(\mathbb{R}^k) \subseteq d\phi_0(\mathbb{R}^k) \text{ in } \mathbb{R}^N.$$

Hence $T_x(X) = d\phi_0(\mathbb{R}^k) = d\psi_0(\mathbb{R}^k)$ is well-defined in $\mathbb{R}^N$.

In particular, the image of the upper halfplane $\mathbb{H}^k \subset \mathbb{R}^k$ is well-defined:

$$H_x(X) = d\phi_0(\mathbb{H}^k) = d\psi_0(\mathbb{H}^k) \text{ in } \mathbb{R}^N.$$

(b) The codimension of $T_x(\partial X)$ in $T_x(X)$ is one. Thus the orthogonal complement of $T_x(\partial X)$ is one-dimensional and is spanned by one vector. By definition of ∂X as the image of the points in ∂HH^k under local parametrizations, we know that $d\phi_0(e_k)$ spans the complement of $T_x(\partial X)$ in $T_x(X)$, since $d\phi_0$ is an isomorphism and $e_k = (0, \dots, 0, 1)$ is nonzero and not contained in $d\phi_0(\mathbb{H}^k)$. We also know by the definition of $H_x(X)$ that $d\phi_0(e_k) \in H_x(X)$, and therefore $d\phi_0(-e_k) \notin H_x(X)$. But we do not know whether $d\phi_0(-e_k)$ is orthogonal to $T_x(\partial X)$ in $T_x(X)$. To make $d\phi_0(-e_k)$ into a vector which is orthogonal to $T_x(\partial X)$, we apply the Gram-Schmidt process. It produces a unit vector which is orthogonal to $T_x(\partial X)$. We denote this vector $n(x)$, this is the outward unit normal vector to ∂X. Note that $-n(x)$ is a unit vector contained in $H_x(X)$ and orthogonal to $T_x(\partial X)$, this is the inward unit normal vector to ∂X.

(c) From what we have learned in the previous point, we can construct $n(x)$ by applying the Gram-Schmidt orthonormalization process to $d\phi_0(-e_k)$. This process depends smoothly on the coefficients in the matrix representing $d\phi_0$. Since the derivative $d\phi_u$ depends smoothly on u, $d\phi_u(-e_k)$ depends smoothly on u. By the independence of the choice of local parametrization, we see that $n(y) = d\phi_u(-e_k)$ for all $y \in \phi(\partial U)$ which is an open neighborhood of x in ∂X, where $\phi(u) = y$. Thus, in total we see that $n(x)$ depends smoothly on x in ∂X.

Solution (Exercise 10.5) **(a)** The boundary of X is $\partial X = \{(x, y) \in \mathbb{R}^2 : x = -1\}$. The derivative of f is given by the 1×2-matrix $df_{(x,y)} = \begin{pmatrix} 2x & 2y \end{pmatrix}$. Hence $df_{(x,y)}$ is a surjective linear map for all $(x, y) \neq (0, 0)$. Since $f(0, 0) = 0 \neq 1$, $df_{(x,y)}$ is surjective for all $(x, y) \in f^{-1}(1)$ and 1 is a regular value of f.

The restriction of f to the boundary of X is

$$\partial f : \partial X \to Y, \ (-1, y) \mapsto 1 + y^2.$$

Hence the derivative of ∂f is given by the 1×1-matrix $(\partial f)_{(-1,y)} = 2y$. This is a linear map which is surjective if and only if $y \neq 0$. Since $(-1, 0) \in \partial X$ and $\partial f(-1, 0) = 1$, we see that 1 is not a regular value of ∂f.

(b) The preimage $f^{-1}(1)$ is just the unit sphere $\mathbb{S}^1$. Hence the boundary $\partial(f^{-1}(1))$ is empty. However,

$$f^{-1}(1)\cap\partial X = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 1\}\cap\{(x, y) \in \mathbb{R}^2 : x = -1\} = \{(-1, 0)\} \neq \emptyset.$$

In particular, $\partial(f^{-1}(1)) \neq f^{-1}(1) \cap \partial X$.

This is not a contradiction to the Preimage Theorem for manifolds with boundary, since the conclusion of the theorem required that 1 was a regular of both f and ∂f. But we showed in the first part that 1 is not a regular value of ∂f.

A.11 Brouwer Fixed Point Theorem

A.11.1 Brouwer Fixed Point Theorem

Solution (**Exercise 11.1**) If A is not invertible, then 0 is an eigenvalue, and we are done. So assume A is nonsingular. For any vector $v \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, the vector $Av/|Av|$ has norm one and lies on $\mathbb{S}^{n-1}$. (Note that this map is not defined for $v = 0$ and we cannot continuously extend it on 0. Hence we cannot just consider this as a map $\mathbb{D}^n \to \mathbb{D}^n$!)

Let $g\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be the map $v \mapsto Av/|Av|$. Now we use the assumption on A: if

$$v \in Q = \{(x_l, \dots, x_n) \in \mathbb{S}^{n-1} \ : \ \text{all } x_i \geq 0\}$$

then Av has only nonnegative entries, since the entries in v are all nonnegative and all the entries in A are by assumption nonnegative. Since $|Av| > 0$ is nonnegative as well, we know that $g(v)$ is an element in Q. Thus we can restrict g to a map $g\colon Q \to Q$.

Now we can compose with a homeomorphism $\varphi\colon Q \xrightarrow{\cong} \mathbb{D}^{n-1}$ to get a continuous map

$$f\colon \mathbb{D}^{n-1} \xrightarrow{\varphi} Q \xrightarrow{g} Q \xrightarrow{\varphi^{-1}} \mathbb{D}^{n-1}.$$

By the Brouwer Fixed Point Theorem 11.10 for continuous maps, f must have a fixed point $y \in \mathbb{D}^{n-1}$ with $f(y) = y$. Hence the image of $w := \varphi^{-1}(y)$ is a vector in $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ with

$$Aw/|Aw| = w, \text{ i.e., } Aw = |Aw| \cdot w.$$

Since w is nonzero being a point on $\mathbb{S}^{n-1}$, w is an eigenvector with real nonnegative eigenvalue $|Aw|$.

Solution (**Exercise 11.2**) **(a)** Assuming that X is simply-connected, f can be extended to a map $F\colon \mathbb{D}^2 \to X$. Let $r\colon X \to \mathbb{S}^1$ be the map defined by $p \mapsto \frac{p}{|p|}$. We can compose these maps to get

$$g\colon \mathbb{D}^2 \xrightarrow{F} X \xrightarrow{r} \mathbb{S}^1 \hookrightarrow \mathbb{D}^2.$$

(b) The image of g is contained in $\mathbb{S}^1$ by definition of g as a composition with the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$. Hence the only possible fixed points of g are contained in $\mathbb{S}^1$. But g sends points in $\mathbb{S}^1 \subset \mathbb{D}^2$, to their antipodal points. Hence g does not have any fixed points.

(c) If $X = \mathbb{R}^2 \setminus \{(0,0)\}$ was simply-connected we could construct a continuous map $g\colon \mathbb{D}^2 \to \mathbb{D}^2$ without a fixed point. This contradicts Brouwer's Fixed Point Theorem 11.10. Hence $X = \mathbb{R}^2 \setminus \{(0,0)\}$ cannot be simply-connected.

Solution (**Exercise 11.3**) Assume there was such a homotopy $F\colon \mathbb{S}^1 \times [0,1] \to X = \mathbb{R}^2 \setminus \{(0,0)\}$. Then we get a homotopy H between the identity map of $\mathbb{R}^2 \setminus \{(0,0)\}$ and a constant map as follows: Let $G\colon X \times [0,1] \to X$ be a homotopy between the identity of X, i.e., $G(p, 0) = p$, and the map r, i.e., $G(p, 1) = p/|p|$. Then we define a homotopy by

$$H : \mathbb{R}^2 \setminus \{(0,0)\} \times [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$$
$$(p, t) \mapsto \begin{cases} G(p, 2t) & \text{for } 0 \le t \le 1/2 \\ F(r(p), 1-2t) & \text{for } 1/2 \le t \le 1. \end{cases}$$

This would imply that $\mathbb{R}^2 \setminus \{(0,0)\}$ is contractible. This contradicts our result that $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply-connected.

A.12 Brouwer Degree mod 2 and Borsuk–Ulam Theorem

A.12.1 Degree modulo 2

Solution (**Exercise 12.1**) Let z be a regular value for g. If z is not in the image of g, then $\deg_2(g)$ and $\deg_2(g\circ f)$ both vanish and the claim is true. So let us assume that z is in the image of g. We showed in a previous exercise that z then is a regular value for $g\circ f$ if and only if every $y \in g^{-1}(z)$ is a regular value for f. Hence we can use z to compute $\deg_2(g\circ f)$ if and only if we can use $y \in g^{-1}(z)$ to compute $\deg_2(f)$. If z is not in the image of $g\circ f$, then y is not in the image of f. In this case both $\deg_2(g\circ f)$ and $\deg_2(f)$ vanish and the claim is true. So assume there are points $x \in X$ with $g(f(x)) = z$. Then it remains to apply the definition of $\deg_2$. By our assumptions, we have

$$\begin{aligned}\deg_2(g\circ f) &= \#(g\circ f)^{-1}(z)\\ &= \# f^{-1}(g^{-1}(z))\\ &= (\# g^{-1}(z)) \cdot (\# f^{-1}(y))\\ &= \deg_2(g)\cdot \deg_2(f) \mod 2\end{aligned}$$

where we use that the number of points in the fiber $f^{-1}(y)$ is the same for every $y \in g^{-1}(z)$ which is a regular value for f.

Solution (**Exercise 12.2**) **(a)** Assume f is not surjective. Then there is a $y \in Y$ which is not in the image of f. Hence y is a regular value for f and we can use it to compute the degree of f as

$$\deg_2(f) = \# f^{-1}(y) = 0.$$

This contradicts the assumption that $\deg_2(f) \neq 0$. Hence f must be surjective.

Alternative argument: Since $\deg_2(f) \neq 0$, we must have $\# f^{-1}(y) \neq 0 \mod 2$ for some $y \in Y$. But, since Y is connected, the function

$$\# f^{-1}(-)\colon Y \to \mathbb{Z}/2, y \mapsto \# f^{-1}(y) \mod 2$$

is constant. Thus we must have $\# f^{-1}(y) \neq 0 \mod 2$ for all $y \in Y$. Hence $f^{-1}(y) \neq \emptyset$ for all $y \in Y$ and f is surjective.

(b) Let us assume $\deg_2(f) \neq 0$ and derive a contradiction. By the previous point, if $\deg_2(f) \neq 0$, then f is surjective. But that means $Y = f(X)$. Since f is continuous and X is compact, the image of X under f is compact. Hence Y would be compact as the continuous image of a compact space. This contradicts the assumption. Hence we must have $\deg_2(f) = 0$.

(c) Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a smooth map without fixed points. We define the map

$$G(x,t)\colon \mathbb{S}^1 \times [0,1] \to \mathbb{R}^2, (x,t) \mapsto f(x)(1-t) - tx.$$

We would like to turn G into a homotopy between f and α. Hence we need to manipulate G such that its image is contained in $\mathbb{S}^1 \subset \mathbb{R}^2$. We can arrange this if $G(x,t) \neq 0$. For then $\frac{G(x,t)}{|G(x,t)|}$ is in $\mathbb{S}^1$. Hence we need to check $G(x,t) \neq 0$ for all $(x,t) \in \mathbb{S}^1 \times [0,1]$.

For a fixed x and varying t, $f(x)(1-t) - tx$ describes the line segment in $\mathbb{R}^2$ between the two points $f(x)$ and $-x$ on $\mathbb{S}^1$. The only way, this line segment can pass $0 \in \mathbb{R}^2$, is when $f(x) = x$ is the antipodal point to $-x$. But, by the assumption on f, $f(x) \neq x$ for all $x \in \mathbb{S}^1$.

Thus the smooth map

$$F(x,t)\colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1, (x,t) \mapsto \frac{f(x)(1-t) - tx}{|f(x)(1-t) - tx|}$$

is a homotopy between f and α.

Since $\alpha^{-1}(x) = -x$ for all $x \in \mathbb{S}^1$, there is exactly one preimage point for each x. Hence $\deg_2(\alpha) = 1$. Since f and α are homotopic, the invariance of $\deg_2$ under homotopy implies $\deg_2(f) = 1$. By the first point, $\deg_2(f) = 1$ implies that f is surjective.

Solution (Exercise 12.3) We define

$$f\colon \mathbb{C} \to \mathbb{C},\ z \mapsto z^7 + \cos(|z|^2)(1 + 93z^4).$$

We can define a homotopy from $f_0(z) = z^7$ to $f_1(z) = f(z)$ by

$$f_t(z) = tf(z) + (1-t)z^7 = z^7 + t\cos(|z|^2)(1 + 93z^4).$$

Since $|z|^7$ dominates the absolute value of $f_t(z)$ for all $t \in [0,1]$, or in other words, since the second summand in the parantheses goes to 0 when $z \to \infty$ in

$$\frac{f_t(z)}{z^7} = 1 + t\frac{\cos(|z|^2)}{z^7}(1 + 93z^4),$$

$f_t(z)$ has no zero on the boundary of a closed ball $W \subset \mathbb{C}$ of large enough radius. Thus the homotopy

$$\frac{f_t(z)}{|f_t(z)|}\colon \partial W \to \mathbb{S}^1$$

is defined for all t. By the homotopy invariance of $\deg_2$, we have

$$\deg_2\left(\frac{f(z)}{|f(z)|}\right) = \deg_2\left(\frac{f_0(z)}{|f_0(z)|}\right) = \deg_2(z^7) = 1 \quad \mod 2.$$

Thus, by the Boundary Theorem 12.10 for $\deg_2$, $f(z)$ must have a zero inside W.

Solution (Exercise 12.4) We define a **homotopy** from $p_0(z) = z^m$ to $p_1(z) = p(z)$ by

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

For $z \neq 0$, we consider

$$\frac{p_t(z)}{z^m} = 1 + t \cdot \left(\frac{a_1}{z} + \cdots + \frac{a_m}{z^m}\right).$$

As $z \to \infty$ moves towards **infinity**, the term $\frac{a_1}{z} + \cdots + \frac{a_m}{z^m} \to 0$ moves towards **zero**. Hence, if **W is a closed ball** around the origin in $\mathbb{C}$ with sufficiently **large radius**, **none** of the p_t has a **zero on ∂W**.

Thus the **homotopy**

$$\frac{p_t}{|p_t|} : \partial W \to \mathbb{S}^1$$

is **defined for all** $t \in [0, 1]$. This implies

$$\deg_2\left(\frac{p}{|p|}\right) = \deg_2\left(\frac{p_0}{|p_0|}\right).$$

Since $p_0(z) = z^m$ and $\#\{z \in \partial W : \frac{z^m}{|z|^m} = 1\} = m$, we have

$$\deg_2\left(\frac{p_0}{|p_0|}\right) = m \quad \mod 2.$$

Hence, if m is **odd**, then $\deg_2\left(\frac{p}{|p|}\right) \neq 0$, and **there must be** $w \in W$ with $p(w) = 0$ by Theorem 12.12.

A.12.2 Borsuk–Ulam Theorem

Solution (Exercise 12.5) Assume that $f_1, \dots, f_k$ did not have a common zero. Then we can form the smooth odd map

$$f := (f_1, \dots, f_k, 0) : \mathbb{S}^k \to \mathbb{R}^{k+1} \setminus \{0\}.$$

Now we can apply Theorem 12.21 to f and L being the x_{k+1}-axis. Hence f intersects L at least once. But x with $f(x) \in L$ is a common zero of the $f_1, \dots, f_k$. Hence the $f_1, \dots, f_k$ must have had a common zero after all.

Solution (Exercise 12.6) We define functions $f_1, \dots, f_k$ on $\mathbb{S}^k$ by

$$f_i(x) := g_i(x) - g_i(-x).$$

Then each f_i is smooth and odd. The functions $f_1, \dots, f_k$ satisfy the assumption of the previous exercise. Hence there is a common zero of the $f_1, \dots, f_k$ which is the desired point $p \in \mathbb{S}^k$.

Solution (**Exercise 12.7**) Since the p_i's are all homogeneous of odd order, they satisfy

$$p_i(-x) = (-1)^{m_i} p_i(x) = -p(x_i).$$

Moreover, for any $x \in \mathbb{R}^{n+1} \setminus \{0\}$, we can consider the associated map

$$q_i : \mathbb{S}^n \to \mathbb{R}, \; x \mapsto p_i\left(\frac{x}{|x|}\right).$$

Since $n+1 \geq 2$, this is a smooth map. Hence we have n smooth real-valued functions $q_1, \dots, q_n$ on $\mathbb{S}^n$, all satisfying the symmetry condition

$$q_i(-x) = p_i(-x/|x|) = -p_i(x/|x|) = -q_i(x).$$

By the first exercise, we know that these n maps $q_1, \dots, q_n$ must have a common zero on $\mathbb{S}^n$, say $x_0 \in \mathbb{S}^n$.

Since the p_i are homogeneous, $q_i(x_0) = p_i(x_0) = 0$ implies

$$p_1(x) = \cdots = p_n(x) = 0 \text{ for all } x = \lambda x_0 \text{ for some } \lambda \in \mathbb{R}.$$

Hence the line spanned by x_0 in $\mathbb{R}^{n+1}$ is the desired line on which all p_i vanish simultaneously.

Solution (**Exercise 12.8**) Assume such a map f existed. Then we could define the continuous map

$$g : \mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \to \mathbb{S}^1, \; g(x, y) := f(x, y, \sqrt{1 - x^2 - y^2}).$$

Note that g is continuous, since g is the composite of f with the inverse of the projection the projection from the upper hemisphere of the sphere to $\mathbb{B}^2$. Then, by the assumption on f, we have

$$g(-x, -y) = f(-x, -y, 0) = -f(x, y, 0) = -g(x, y)$$

for any $(x, y) \in \mathbb{S}^1 = \partial\mathbb{B}^2$, i.e., for (x, y) such that $1 - x^2 - y^2 = 0$. In particular, $g_{|\partial\mathbb{B}^2} : \mathbb{S}^1 \to \mathbb{S}^1$ satisfies the assumptions of the Borsuk–Ulam Theorem. Hence, by the Borsuk–Ulam Theorem, $\deg_2(g) = 1$. But $g_{|\partial\mathbb{B}^2}$ can be extended to a continuous map on all of $\mathbb{B}^2$. Thus, by the Boundary Theorem for degrees, $\deg_2(g) = 0$. Hence the assumption that f exists, leads to a contradiction. Thus f cannot exist.

Solution (**Exercise 12.9**) Assume that there is no such point p. Then we can consider the map $g : \mathbb{S}^2 \to \mathbb{S}^1$ given by

$$g(p) = \frac{f(p) - f(-p)}{|f(p) - f(-p)|}.$$

By the assumption, g is smooth and satisfies $g(-p) = -g(p)$ for all $p \in \mathbb{S}^2$. This contradicts the result of the previous exercise.

Solution (Exercise 12.10) Let $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^n$ be open subsets. After possibly translating we can assume that V contains the origin. Moreover, since V is open, there is an $\varepsilon > 0$ such that $\mathbb{B}^n_\varepsilon(0) \subset V$, where $\mathbb{B}^n_\varepsilon(0)$ denotes the closed ball of radius ε centred at the origin. We can then consider the subset $\mathbb{S}^2_\varepsilon \subset \mathbb{B}^3_\varepsilon(0) \subset \mathbb{B}^n_\varepsilon(0) \subset V$. Assume now there was a homeomorphism $\varphi\colon V \to U$. Then $\varphi_{|\mathbb{S}^2_\varepsilon}$ is still a homeomorphism. Since scaling defines a homeomorphism $\mathbb{S}^2 \to \mathbb{S}^2_\varepsilon$, composition with $\varphi_{|\mathbb{S}^2_\varepsilon}$ would yield a continuous map $\mathbb{S}^2 \to \mathbb{R}^2$ with $f(p) \neq f(-p)$ for all $p \in \mathbb{S}^2$. This contradicts the assumed continuous version of the previous exercise.

A.13 Thom Transversality

Solution (**Exercise 13.1**) Let X and Y be submanifolds of $\mathbb{R}^N$. As in Section 13.4.2, we define a

$$F: X \times \mathbb{R}^N \to \mathbb{R}^N, (x, a) \mapsto x + a.$$

The derivative of F is given by

$$dF_{(x,a)}: T_x(X) \times \mathbb{R}^N \to \mathbb{R}^N, (v, w) \mapsto v + w.$$

Thus $dF_{(x,a)}$ is surjective at every point (x, a). Hence F is a submersion, and therefore transversal to every submanifold of $\mathbb{R}^N$. In particular, it is transversal to both boundaryless submanifolds $\mathrm{Int}(Y)$ and ∂Y.

By the Transversality Theorem 13.25, the map

$$t_a: X \to \mathbb{R}^N, \ x \mapsto x + a$$

is transversal to each of $\mathrm{Int}(Y)$ and ∂Y for almost every $a \in \mathbb{R}^N$. Hence it is transversal to both $\mathrm{Int}(Y)$ and ∂Y for almost every $a \in \mathbb{R}^N$. The derivative of the translation t_a is just

$$d(t_a)_x: T_x(X) \to \mathbb{R}^N, \ v \mapsto v.$$

Moreover, the tangent spaces of $X + a$ and X are equal, since any local parametrization ϕ of X defines a local parametrization $\phi + a$ of $X + a$. Since the derivatives of ϕ and $\phi + a$ are equal, we have $T_x(X) = T_{x+a}(X + a)$.

Hence the transversality $t_a \pitchfork Y$ implies

$$\mathbb{R}^N = \mathrm{Im}\,(d(t_a)_x) + T_{t_a(x)}(Y) = T_x(X) + T_{x+a}(Y) = T_{x+a}(X + a) + T_{x+a}(Y). \quad \text{(A.6)}$$

If $y = x + a \in Y$, then (A.6) means that $X + a$ and Y meet transversally in $y = x + a$. If $x + a \notin Y$, then $x + a \notin (X + a) \cap Y$, and $X + a$ and Y meet transversally in $x + a$ automatically.

Solution (**Exercise 13.2**) **(a)** Let X be a compact submanifold of $\mathbb{R}^n$, and let $w \in \mathbb{R}^n$. Since X is compact, the continuous function

$$X \to \mathbb{R}, \ x \mapsto |w - x|^2$$

has a minimum. Let $x \in X$ be a point, where this function has its minimum (there may be many such x's, we just pick one). Hence x is a point of X which is closest to w.

Now let $c: (-a, a) \to X$ be any smooth curve on Y with $c(0) = x$. The smooth function

$$f: (-a, a) \to \mathbb{R}, \ t \mapsto |w - c(t)|^2$$

then has a minimum at $t = 0$. Thus its derivative df_0 at 0 vanishes. Writing $f(t) = |w - c(t)|^2 = (w - c(t)) \cdot (w - c(t))$ using the scalar product, we see that df_0 is given by

$$df_0 = 2(w - c(0)) \cdot (-dc_0)$$

where we consider dc_0 as a vector in $\mathbb{R}^n$ (it really is a matrix with one row and n columns). In particular, we get $w - x = w - c(0)$ is orthogonal to dc_0 in $\mathbb{R}^n$.

Since every tangent vector in $T_x(X)$ is the velocity vector dc_0 for some smooth curve c on X with $c(0) = x$, this shows $w - x \in N_x(X)$ by definition of $N_x(X)$ as the orthogonal complement of $T_x(X)$ in $\mathbb{R}^n$.

(b) Let $N \subset N(X)$ be the open neighborhood of X (or rather of $X \times \{0\}$) in $N(X)$ which is mapped diffeomorphically onto $X^\varepsilon \subset \mathbb{R}^n$ by h. Given $w \in X^\varepsilon$, there is unique element $n \in N$ with $h(n) = w$. Since elements in $N(X)$ are pairs (x, v) with $v \in N_x(X)$, there is a unique $x \in X$ and $v \in N_x(X)$ such that $n = (x, v)$ and $\sigma(n) = x$. Since $h(x, v) = x + v$ by definition, and $h(n) = w$ by the choice of n, we must have $v = w - x \in N_x(X)$. Hence the pair (x, v) is uniquely determined by $w \in X^\varepsilon$.

Since we have the commutative diagram

$$\begin{tikzcd} N(X) \arrow[r, "h"] \arrow[dr, "\sigma"'] & X^\varepsilon \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

we know $\pi(w) = \sigma(n) = x$.

By the previous point, we know that any $x_0 \in X$ with minimal distance to w, must satisfy $w - x_0 \in N_{x_0}(X)$. We just learned that $\sigma(n) = x \in X$ is the unique element in X with this property. Hence $\pi(w)$ is the unique point of X closest to w.

Solution (Exercise 13.3) Let X be a submanifold of $\mathbb{R}^N$. Let V be a k-dimensional vector subspace of $\mathbb{R}^N$. Every such V has a basis consisting of a k-tuple of linearly independent k-tuples of vectors in $\mathbb{R}^N$. In particular, every V is the span of such a k-tuple in $\mathbb{R}^N$.

So let $S \subset (\mathbb{R}^N)^k$ be the set consisting of all linearly independent k-tuples of vectors in $\mathbb{R}^N$. For a k-tuple of vectors $[v] := v_1, \dots, v_k$ in $\mathbb{R}^N$, let $A_{[v]}$ be the $N \times k$-matrix with the v_i's as column vectors. Then the k-tuple $v_1, \dots, v_k$ is linearly independent if and only if he $k \times k$-matrix $A^t_{[v]} A_{[v]}$ is invertible, i.e. $\det(A^t_{[v]} A_{[v]}) \neq 0$. Hence S is the inverse image of the open subset $\mathbb{R} \setminus \{0\}$ under the continuous map

$$\mathbb{R}^{Nk} \to \mathbb{R},\ [v] \mapsto \det(A^t_{[v]} A_{[v]}).$$

Thus S is an open subset in $\mathbb{R}^{Nk}$.

We define the map $\varphi \colon \mathbb{R}^k \times S \to \mathbb{R}^N$ by

$$([t], [v]) := ((t_1, \dots, t_k), v_1, \dots, v_k) \mapsto t_1 v_1 + \cdots + t_k v_k.$$

Since S is open in $\mathbb{R}^{Nk}$, the tangent space to S at any $[v]$ is just $\mathbb{R}^{Nk}$. Moreover, φ is linear in each coordinate. Thus the derivative of φ at any point $([t], [v])$ is just φ. Since φ is surjective, $d\varphi_{([t],[v])}$ is surjective. Thus φ is a submersion.

Hence, the Transversality Theorem 13.25 implies that, for almost every $s = [v]$ in S, the map

$$\varphi_{[v]} \colon \mathbb{R}^k \to \mathbb{R}^N,\ (t_1, \dots, t_k) \mapsto t_1 v_1 + \cdots + t_k v_k$$

is transversal to every submanifold in $\mathbb{R}^N$. In particular, $\varphi_{[v]} \pitchfork X$ for almost every $s = [v]$. This means

$$\mathbb{R}^N = \operatorname{Im}(d\varphi_{[v]}) + T_x(X) = \operatorname{Im}(\varphi_{[v]}) + T_x(X)$$

for every $x \in X$. But the $\operatorname{Im}(\varphi_{[v]})$ is by definition of φ just the span of the k-tuple $[v] = \{v_1, \dots, v_k\}$ in $\mathbb{R}^N$.

By our opening remark, every k-dimensional vector subspace in $\mathbb{R}^N$ is the span of some $[v] \in S$. Thus we have shown that for almost every $V := \operatorname{span}([v]) = \operatorname{Im}(\varphi_{[v]})$ in $\mathbb{R}^N$ we have

$$V + T_x(X) = \mathbb{R}^N$$

for every $x \in X$. Thus $V \pitchfork X$.

Solution (Exercise 13.4) **(a)** Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with $n > 1$, and let $K \subset \mathbb{R}^n$ be compact and $\varepsilon > 0$. If $df_x \neq 0$ for all $x \in \mathbb{R}^n$, then we can take $g = f$.

Now assume there is an $x \in X$ such that $df_x = 0$. We would like to replace f with a suitable smooth function $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the two conditions

(a) $dg_x \neq 0$ for all $x \in X$, and

(b) $|f(x) - g(x)| < \varepsilon$ for all $x \in K$.

The idea for the solution is to replace f with $f + A$ for a suitable matrix $A \in M(n)$. For any given $A \in M(n) \setminus \{0\}$, the set of norms $\{|Ax| \in \mathbb{R} : x \in K\}$ has a maximum $\mu_A > 0$.

Hence **if** $\mu_A < \varepsilon$, then $|Ax| < \varepsilon$ for all $x \in K$, and we can define $g : \mathbb{R}^n \to \mathbb{R}^n$ by $g(x) = f(x) + Ax$. This map is smooth and satisfies condition (b).

Since $\frac{\varepsilon}{2\mu_A} A$ is a linear map, the derivative of g at x is $dg_x = df_x + \frac{\varepsilon}{2\mu_A} A$.

In order to prove the assertion, it remains to show that we can find an $A \in M(n)$ such that $df_x + A \neq 0$ and $\mu_A < \varepsilon$.

To do this, we define the map

$$F : \mathbb{R}^n \times M(n) \to M(n), \ (x, A) \mapsto df_x + A.$$

The derivative $dF_{(x,A)}$ of F at a point (x, A) is the sum of the derivative of df_x and the identity map on $M(n)$. In particular, $dF_{(x,A)} : \mathbb{R}^n \times M(n) \to M(n)$ is always surjective. Hence F is a submersion, and thus transversal to every submanifold of $M(n)$.

By the Transversality Theorem 13.25, this implies that, for almost all $A \in M(n)$, the map

$$F_A : \mathbb{R}^n \to M(n), \ x \mapsto F(x, A)$$

is transversal to the submanifold $\{0\}$ of $M(n)$.

But, **for $n > 1$**, $\dim \mathbb{R}^n = n$ is **strictly less** than $n^2 = \dim M(n)$.

Thus, since $\{0\}$ is a zero-dimensional submanifold of $M(n)$, F_A is transversal to $\{0\}$ if and only if the intersection $\mathrm{Im}\,(F_A) \cap \{0\}$ is empty, i.e., $F_A(x) \neq 0$ for all $x \in X$.

The subset of matrices in $M(n)$ with $\max\{|Ax| : x \in K\} < \epsilon$ is open in $M(n)$. This implies that the intersection of its complement with any subset of measure zero in $M(n)$ has measure zero. Thus, by the Transversality Theorem 13.25, we can choose an $A \in M(n)$ with $F_A(x) = df_x + A \neq 0$ for all $x \in X$ **and** $\max\{|Ax| : x \in K\} < \varepsilon$.

(b) For $n = 1$, we construct a counter-example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, let $K = [-2, 2] \subset \mathbb{R}$, and let $\varepsilon = 1$. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with

$$|f(x) - g(x)| < 1 \text{ for all } x \in K.$$

In particular, this implies

$$3 < g(-2) < 5, \; 3 < g(2) < 5, \; \text{and} \; -1 < g(0) < 1.$$

This shows

$$\frac{g(0) - g(-2)}{0 - (-2)} < 0 \text{ and } \frac{g(2) - g(0)}{2 - 0} > 0.$$

By the Mean Value Theorem, there are real numbers $c \in (-2, 0)$ and $e \in (0, 2)$ such that

$$g'(c) = \frac{g(0) - g(-2)}{0 - (-2)} < 0 \text{ and } g'(e) = \frac{g(2) - g(0)}{2 - 0} > 0.$$

Since g is smooth, g' is differentiable. Hence we can apply the Intermediate Value Theorem to g' and get a number $e \in (c, e)$ with $g'(d) = 0$. Hence we cannot find g with both $g'(x) \neq 0$ for all x and $|f - g| < \varepsilon$ on K.

A.14 Intersection Theory modulo 2

Solution (**Exercise 14.1**) We showed in a previous exercise that

$$f \pitchfork g^{-1}(W) \text{ if and only if } (g\circ f) \pitchfork W.$$

Moreover, since codimensions are preserved under taking preimages, we have

$$\dim X + \dim W = \dim Z \text{ if and only if } \dim X + \dim g^{-1}(W) = \dim Y.$$

Thus,

$$I_2(f, g^{-1}(W)) \text{ is defined if and only if } I_2(g\circ f, W) \text{ is defined.}$$

Now it remains to observe that the finite numbers satisfy

$$I_2(f, g^{-1}(W)) = \# f^{-1}(g^{-1}(W)) = \#(g\circ f)^{-1}(W) = I_2(g\circ f, W).$$

Solution (**Exercise 14.2**) **(a)** Let $Z \subset Y$ be any closed submanifold with $\dim X + \dim Z = \dim Y$. For $\dim X \geq 1$, this implies $\dim Y > 0$ and $\dim Z = \dim Y - \dim X < \dim Y$. In particular, Z is not all of Y and there exist points in Y which are not in Z.

We can assume that Y is connected, since f being homotopic to a constant map implies that the image of f lies in only one connected component of Y. Hence any part of Z in a different connected component satisfies $f^{-1}(Z) = \emptyset$.

Assume that f is homotopic to the constant map $g_0 : X \to \{y_0\} \subset Y$. If $y_0 \in Z$, then g and Z do not meet transversally, since $\dim Z < \dim Y$ and $\dim\{y_0\} = 0$. But since Y is connected and a smooth manifold, it is path-connected. Hence there is a path γ from y_0 to a point y_1 with $y_1 \notin Z$. Then g_0 is homotopic to the constant map $g_1 : X \to \{y_1\} \subset Y$ by composing g_0 with the path γ. Since being smoothly homotopic is a transitive relation, we have $f \simeq g_1$.

Since $y_1 \notin Z$, we have $g_1 \pitchfork Z$. This means $g_1^{-1}(Z) = \emptyset$, i.e.,

$$I_2(g_1, Z) = \# g_1^{-1}(Z) = 0.$$

By the homotopy invariance of $I_2(-, Z)$ we have $I_2(f, Z) = I_2(g_1, Z)$, and hence $I_2(f, Z) = 0$.

(b) Assume that $X = \{x\}$ is a one-point space. In particular, we have $\dim X = 0$. Thus any closed submanifold $Z \subset Y$ with $\dim X + \dim Z = \dim Y$ is of dimension $\dim Z = \dim Y$. This implies that every such Z intersects every other submanifold of Y transversally. In particular, $\{x\} \pitchfork Z$, i.e., $f \pitchfork Z$ for every map $f : X = \{x\} \to Y$. Hence, by definition of intersection numbers, we can calculate $I_2(f, Z)$ by assuming $f(x) = y \in Z$ and get

$$I_2(f, Z) = \# f^{-1}(Z) = \#\{y\} = 1 \neq 0.$$

(c) Since $\mathbb{S}^1$ is compact, the intersection number $I_2(\mathrm{id}_{\mathbb{S}^1}, \{x\})$ is defined for every point $x \in \mathbb{S}^1$ where we consider intersections in $\mathbb{S}^1$, i.e., $Y = X = \mathbb{S}^1$. Since the

$\mathrm{id}_{\mathbb{S}^1}^{-1}(x) = \{x\}$, we have $I_2(\mathrm{id}_{\mathbb{S}^1}, \{x\}) = 1 \neq 0$. But we just learned that if $\mathrm{id}_{\mathbb{S}^1}$ was homotopic to a constant map, then $I_2(\mathrm{Id}_{\mathbb{S}^1}, \{x\})$ had to be zero, since $\dim \mathbb{S}^1 = 1$. Thus there is at least one map $\mathbb{S}^1 \to \mathbb{S}^1$ which is not homotopic to a constant map. In fact, up to homotopy there is one homotopy class of maps $\mathbb{S}^1 \to \mathbb{S}^1$ for every integer $n \in \mathbb{Z}$ represented by taking nth powers in $\mathbb{C}$: $z \mapsto z^n$. We will learn more about that later.

Solution (Exercise 14.3) **(a)** Let Y be contractible and $\dim Y > 0$. Let $f: X \to Y$ be a smooth map with X compact and $Z \subset Y$ closed, and $\dim X + \dim Z = \dim Y$. Since Y is contractible, the identity map id_Y is homotopic to a constant map $Y \to \{y\} \subset Y$. Thus f is homotopic to the constant map $X \to \{y\} \subset Y$. Since $\dim Y > 0$, we have $\{y\} \neq Y$ and there are points in Y different from y. Hence, since $\dim X \geq 1$, the previous exercise implies $I_2(f, Z) = 0$.

Finally, we recall that Euclidean space $\mathbb{R}^k$ is contractible for all k.

(b) If X is compact, then the intersection number $I_2(\mathrm{id}_X, \{x\})$ is defined for every point $x \in X$ where we consider intersections in X (with $Y = X$ and $Z = \{x\}$). For the dimensions are complementary and id_X meets every submanifold transversally. This intersection number satisfies $I_2(\mathrm{id}_X, \{x\}) = \#\{x\} = 1$, since x has exactly one preimage under id_X.

If X is contractible and $\dim X \geq 1$, we can apply the previous point. Or we note that we have $\mathrm{id}_X \sim g$ where $g: X \to \{y\}$ is some constant map. Since $\dim X \geq 1$, we have $g \pitchfork \{x\}$ for $x \in X$ if and only if $x \neq y$. Hence $I_2(g, \{x\}) = \#g^{-1}(x) = 0$. However, since $\mathrm{id}_X \simeq g$, we have $I_2(\mathrm{id}_X, \{x\}) = I_2(g, \{x\})$. This is impossible.

Hence, if $\dim X \geq 1$, X cannot be both compact and contractible.

If $\dim X = 0$, then the only way X can be contractible is that it consists of just one point. This is a compact space.

Solution (Exercise 14.4) **(a)** Let $f: X \to \mathbb{S}^k$ be a smooth map with X compact and $0 < \dim X < k$. Let Z be a closed submanifold $Z \subset \mathbb{S}^k$ of dimension complementary to X, i.e., $\dim X + \dim Z = \dim \mathbb{S}^k$. The image of the derivative df_x at any point has at most dimension $\dim X = \dim T_x(X)$ in $T_{f(x)}(\mathbb{S}^k)$. Hence, since $\dim X < k$, df_x cannot be surjective. Hence the only points in $\mathbb{S}^k$ which are regular values of f are the points which are not in the image of f. By Sard's Theorem, however, we know that regular values exist. Hence there must be a point $p \in \mathbb{S}^k$ with $p \notin f(X) \cap Z$.

Now stereographic projection provides a diffeomorphism $\phi: \mathbb{S}^k \setminus \{p\} \to \mathbb{R}^k$. Since p is not in the image of f, the composition $\phi \circ f: X \to \mathbb{S}^k \setminus \{p\} \to \mathbb{R}^k$ is defined. Similarly, since p is not in Z, $\phi_{|Z}$ is a diffeomorphism. Hence we can consider $\phi(Z)$ as a submanifold of $\mathbb{R}^k$ of dimension $\dim Z$ which is diffeomorphic to Z. This implies

$$I_2(f, Z) = I_2(\phi \circ f, \phi(Z))$$

where the latter is the intersection number mod 2 in $\mathbb{R}^k$. But since $\mathbb{R}^k$ is contractible, the previous exercise implies $I_2(\phi \circ f, \phi(Z)) = 0$. Thus $I_2(f, Z) = 0$.

(b) Considering $\mathbb{S}^1 \subset \mathbb{C}$, we can define tow submanifolds X and Z as $X = \mathbb{S}^1 \times \{1\}$ and $Z = \{1\} \times \mathbb{S}^1$ in $\mathbb{S}^1 \times \mathbb{S}^1$. They intersect in the single point q given by $q := \{1\} \times \{1\} \in S^1 \times S^1$. The tangent space to $\mathbb{S}^1 \times \mathbb{S}^1$ at q is

$$T_q(\mathbb{S}^1 \times \mathbb{S}^1) = T_1(\mathbb{S}^1) \times T_1(\mathbb{S}^1) = T_1(X) \times T_1(Z).$$

Hence X and Z meet transversally in $\mathbb{S}^1 \times \mathbb{S}^1$. Thus their intersection number in $\mathbb{S}^1 \times \mathbb{S}^1$ is $I_2(X, Z) = 1$.

Now assume there was a diffeomorphism $\varphi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$. The map $\iota : \mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1,\ z \mapsto (z, 1)$ maps $\mathbb{S}^1$ diffeomorphically into X. Composition with φ gives us a map $\varphi \circ \iota : \mathbb{S}^1 \to \mathbb{S}^2$. Since φ is a diffeomorphism, $\varphi(Z)$ is the diffeomorphic image of Z, and we would have

$$I_2(X, Z) = I_2(\iota, Z) = I_2(\varphi \circ \iota, \varphi(Z))$$

where the latter is the intersection number in $\mathbb{S}^2$.

But by the previous point, $I_2(\varphi \circ \iota, \varphi(Z)) = 0$ since $\dim \mathbb{S}^1 = 1 < 2$.

Solution (Exercise 14.5) **(a)** Let $I : X \times [0,1] \to Y$ be a smooth homotopy with $I(x, 0) = i_0(x)$ being the embedding of X in Y and $i_1(X) = Z \subset Y$. We define a new map

$$F : X \times [0,1] \to Y \times [0,1],\ (x, t) \mapsto (I(x, t), t).$$

Since I is smooth, F is smooth.

Now we define $W := F(X \times [0,1]) \subset Y \times [0,1]$ to be the image of $X \times [0,1]$ under F. We claim that W is a compact smooth manifold with boundary. First, since X is compact, $X \times [0,1]$ is a compact, and hence its continuous image $F(X \times [0,1])$ in $Y \times [0,1]$ is compact.

Since $I(-, t)$ is an embedding for every $t \in [0,1]$, and the identity map on the second component is obviously an embedding as well, F is also an embedding. Thus the image of F is a smooth manifold.

The boundary of W is then given by

$$\begin{aligned}\partial W &= W \cap \partial(Y \times [0,1]) \\ &= F(X \times \{0\}) \cap Y \times \{0\} \cup F(X \times \{1\}) \cap Y \times \{1\} \\ &= X \times \{0\} \cup Z \times \{1\}.\end{aligned}$$

Hence W is a cobordism between X and Z.

(b) Let X and Z be cobordant in Y, and let C be a compact submanifold C in Y with dimension complementary to X and Z. Let f denote the restriction to W of the projection map $Y \times [0,1] \to Y$. Then $\partial f = f_{|\partial W}$ is a smooth map which can be extended to a map $f : W \to Y$. Hence, by the Boundary Theorem, $I_2(\partial f, V) = 0$ for any closed submanifold V of Y of dimension $\dim V = \dim Y - \dim \partial W$. Since $\dim \partial W = \dim X = \dim Z$, we have, in particular,

$$I_2(\partial f, C) = 0.$$

By the Transversality Homotopy Theorem 13.27, we can assume $f \pitchfork C$ and $\partial f \pitchfork C$. In particular, we can assume $X \pitchfork C$ and $Z \pitchfork C$. Hence $I_2(X, C) = \#(X \cap C)$ and $I_2(Z, C) = \#(Z \cap C)$. By definition of f, $(\partial f)^{-1}(C)$ is given by

$$\begin{aligned}(\partial f)^{-1}(C) &= \partial W \cap (C \times [0,1]) \\ &= (X \times \{0\}) \cap (C \times [0,1]) \cup (Z \times \{1\}) \cap (C \times [0,1]) \\ &= ((X \cap C) \times \{0\}) \cup ((Z \cap C) \times \{1\}).\end{aligned}$$

Thus

$$\begin{aligned}0 &= I_2(\partial f, C) \\ &= \#(\partial f)^{-1}(C) \\ &= \#((X \cap C) \times \{0\}) + \#((Z \cap C) \times \{1\}) \\ &= \#(X \cap C) + \#(Z \cap C) \\ &= I_2(X, C) + I_2(Z, C).\end{aligned}$$

Since we are working modulo 2, this implies

$$I_2(C, X) = I_2(C, Z).$$

Solution (Exercise 14.6) By Theorem 13.14, there is an open neighborhood $Z^\varepsilon \subset X$ of Z and a diffeomorphism $Z^\varepsilon \cong N^\varepsilon(Z, X)$ to an open subset $N^\varepsilon(Z, X) \subset N(Z, X)$. Now we use the assumption on Z. By Theorem 13.15, there is a diffeomorphism $Z \times \mathbb{R}^k \to N(Z, X)$. Hence we may identify $N(Z, X) = Z \times \mathbb{R}^k$, and $N^\varepsilon(Z, X) = Z \times (-\varepsilon, \varepsilon)$. However, for any $v \neq 0$ in $\mathbb{R}^k$, the subspaces $Z = Z \times \{0\}$ and the deformation $Z \times \{v\}$ do not intersect in $Z \times \mathbb{R}^k$. Thus, the existence of the diffeomorphism $Z^\varepsilon \cong Z \times (-\varepsilon, \varepsilon)$ implies that there is a deformation Z' of Z in X such that $Z \cap Z' = \emptyset$. Hence we have $I_2(Z, Z) = 0$.

Solution (Exercise 14.7) **(a)** We need to find a deformation Z' of Z which intersects Z transversally and then count the intersection points. For example, let us look at the map

$$f : [-1,1] \to [-1,1] \times (-1,1),\ t \mapsto \left(t, \frac{1}{2}\sin\left(t \cdot \frac{\pi}{2}\right)\right).$$

Let Z' be the image of f in X. There is a smooth homotopy

$$f_s : [-1,1] \times [0,1] \to X,\ (t,s) \mapsto \left(t, s \cdot \frac{1}{2}\sin\left(t \cdot \frac{\pi}{2}\right)\right)$$

with f_0 being the embedding of Z into X, i.e., the map

$$f_0 : [-1,1] \to X,\ t \mapsto (t, 0),$$

and f_1 being the embedding of Z' into X. (Note that, by abuse of notation, we have not distinguished between a point in $[-1,1] \times (-1,1)$ and its equivalence class in X.) Moreover, there is only one point where Z and Z' meet, namely the image of

$f(0)$ in X, i.e., the point with coordinates $p = (0, 0)$. Finally, the intersection of Z and Z' is transverse. We check this locally for the induced maps on the codomains of local charts: the tangent space to X at p is the x-axis in $\mathbb{R}^2$ and the tangent space to Z' at p is the line spanned by the vector $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \in \mathbb{R}^2$. Hence $T_p(Z)$ and $T_p(Z')$ span a two-dimensional subspace in the two-dimensional vector space $T_p(X)$, i.e., they span all of $T_p(X)$.

Thus we can conclude:

$$I_2(Z, Z) = I_2(Z, Z') = \#(Z \cap Z') = 1.$$

(b) By Exercise 14.6, the existence of a submersion $g : X \to \mathbb{R}$ such that $Z = g^{-1}(0)$ would imply $I_2(Z, Z) = 0$. This contradicts the previous point. Hence such a g cannot exist.

Solution (**Exercise 14.8**) **(a)** We have seen this map in Section 10.1 when we introduced the idea of intersection theory. There we showed that f is smooth by checking it for the induced map on local charts. It is proper, since its domain $\mathbb{S}^1$ is compact. It is injective, since $\cos(t_1/2) = \pm\cos(t_2/2)$ implies $t_2 = t_1 + 2\pi$, and similarly for sin.

(b) We consider $Z = \mathrm{Im}(f)$ as a submanifold of $\mathbb{R}\mathrm{P}^2$. We now need to show we can deform Z such that it intersects itself in an odd number of points. Thinking of Z as being the image of the horizontal equator on $\mathbb{S}^2$ under the quotient map $\mathbb{S}^2 \to \mathbb{R}\mathrm{P}^2$ we could try the image of a vertical equator. So let $g : \mathbb{S}^1 \to \mathbb{R}\mathrm{P}^2$ be the map defined by sending $t \in [0, 2\pi]$ to $[0 : \sin(t/2) : \cos(t/2)]$. Just as for f, we can check that g is a smooth embedding. Properness and injectivity follow as for f. For smoothness we observe look at the effect of g on the codomains of local charts: We look at the diagram

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "g"] & \mathbb{R}\mathrm{P}^2 \arrow[d, "\psi_j"] \\ W_i \arrow[u, "\phi_i"] \arrow[r, dotted, "g"'] & \mathbb{R}^2 \end{tikzcd}$$

with maps ψ_j defined on the subsets $V_j = \{[x] : x_j \neq 0\}$ of $\mathbb{R}\mathrm{P}^2$ given by $\psi_2([x_1 : x_2 : x_3]) = \left(\frac{x_1}{x_2}, \frac{x_3}{x_2}\right)$ and $\psi_3([x_1 : x_2 : x_3]) = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$. Then we have:

- for the parametrizations $\phi_1 : W_1 = (0, \pi) \to \mathbb{S}^1$ and $\phi_2 : W_2 = (\pi, 2\pi) \to \mathbb{S}^1$ of $\mathbb{S}^1$ and $\psi_2 : V_2 = \{[x] : x_2 \neq 0\} \to \mathbb{R}^2$ of $\mathbb{R}\mathrm{P}^2$, g induces the map
$$t \mapsto \left(0, \frac{\cos(t/2)}{\sin(t/2)}\right).$$
- for the local parametrizations $\phi_3 : W_3 = (\pi/2, 3\pi/2) \to \mathbb{S}^1$ and $\phi_4 : W_4 = (3\pi/2, 5\pi/2) \to \mathbb{S}^1$ of $\mathbb{S}^1$ and $\psi_3 : V_3 = \{[x] : x_3 \neq 0\} \to \mathbb{R}^2$ of $\mathbb{R}\mathrm{P}^2$, g induces the map
$$t \mapsto \left(0, \frac{\sin(t/2)}{\cos(t/2)}\right).$$

All of these maps are smooth and hence g is smooth.

We write Z' for the image of g in $\mathbb{R}\mathrm{P}^2$. We have

$$f(t) = g(t) \iff \cos(t/2) = 0, \text{ i.e., if and only if } t = \pi.$$

Hence Z and Z' have exactly one intersection point $p = [0 : 1 : 0]$ in $\mathbb{R}\mathrm{P}^2$. We need to check that this intersection is transverse in p. We can do this locally using charts on $\mathbb{R}\mathrm{P}^2$ and check that df_π and dg_π together span a two-dimensional space.

We can also check transversality by looking at the actual tangent space of $T_p(\mathbb{R}\mathrm{P}^2)$. So let L denote the line in $\mathbb{R}^3$ through the origin corresponding to p, i.e., $L = \mathrm{span}(v_p)$ where v_p denotes the vector $v_p = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. In Remark 9.21 we pointed out that the tangent space of $\mathbb{R}\mathrm{P}^2 = \mathrm{Gr}_1(\mathbb{R}^3)$ at $p = L$ is given by

$$T_p(\mathbb{R}\mathrm{P}^2) = \mathrm{Hom}_{\mathbb{R}}(L, L^\perp).$$

To add some visual understanding note that L is the y-axis and $L^\perp$ is the xz-plane in $\mathbb{R}^3$. In particular, $T_p(\mathbb{R}\mathrm{P}^2)$ is a two-dimensional vector space. Hence, in order to show transversality, it suffices to show that the one-dimensional subspaces $T_p(Z)$ and $T_p(Z')$ are spanned by two linearly independent vectors.

A linear map $L \to L^\perp$ is determined by where it sends the vector $v_p = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ which spans L. By abusing notation a bit, we can think of

$$f_{|(\pi/2,3\pi/2)} \colon (\pi/2, 3\pi/2) \to \mathbb{R}\mathrm{P}^2 \text{ and } g_{|(\pi/2,3\pi/2)} \colon (\pi/2, 3\pi/2) \to \mathbb{R}\mathrm{P}^2$$

as local parametrizations of $\mathbb{R}\mathrm{P}^2$ around p. Then the image of df_π is the one-dimensional subspace in $\mathrm{Hom}_{\mathbb{R}}(L, L^\perp)$ spanned by linear maps which send L to the x-axis. And the image of dg_π is the one-dimensional subspace in $\mathrm{Hom}_{\mathbb{R}}(L, L^\perp)$ spanned by linear maps which send L to the z-axis. These two subspaces together span all of $T_p(\mathbb{R}\mathrm{P}^2)$. Hence $Z \pitchfork Z'$.

Finally, we need to check that f and g are homotopic. Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be a smooth bump function such that

$$\varphi(t) = \begin{cases} 0 & x \le 1/4 \\ 1 & x \ge 3/4. \end{cases}$$

We define a smooth homotopy by

$$\begin{aligned} F \colon\ & [0, 2\pi] \times [0, 1] \to \mathbb{R}\mathrm{P}^2, \\ & (t, s) \mapsto [\sqrt{1 - \varphi(s)}\cos(t/2) : \sin(t/2) : \sqrt{\varphi(s)}\cos(t/2)] \end{aligned}$$

such that $F(t, 0) = f(t)$ and $F(t, 1) = g(t)$ for all t. Summarising we have proven

$$I_2(Z, Z) = 1.$$

(c) Recall that any constant map c with value not in Z is transverse to Z and has mod 2-intersection number $I_2(c, Z) = 0$. Since $\mathbb{RP}^2$ is path-connected, all constant maps with value in $\mathbb{RP}^2$ are homotopic to each other. Hence f is not homotopic to a constant map.

(d) We pick a point on $\mathbb{RP}^2$, say $Q = [0 : 0 : 1]$, and let $q : \mathbb{S}^1 \to \mathbb{RP}^2$ be the constant map with value Q. Let $\varphi : \mathbb{R} \to \mathbb{R}$ again be a smooth bump function such that

$$\varphi(t) = \begin{cases} 0 & x \leq 1/4 \\ 1 & x \geq 3/4. \end{cases}$$

We define a smooth homotopy by

$$\begin{aligned} H : [0, 2\pi] \times [0, 1] &\to \mathbb{RP}^2, \\ (t, s) &\mapsto \left[\sqrt{1 - \varphi(s)}\cos(t) : \sqrt{1 - \varphi(s)}\sin(t) : \sqrt{\varphi(s)}\right] \end{aligned}$$

such that $H(t, 0) = 2f(t)$ and $H(t, 1) = q(t)$ for all t.

Finally, note that the map

$$\begin{aligned} \tilde{H} : [0, 2\pi] \times [0, 1] &\to \mathbb{RP}^2, \\ (t, s) &\mapsto \left[\sqrt{1 - \varphi(s)}\cos(t/2) : \sqrt{1 - \varphi(s)}\sin(t/2) : \sqrt{\varphi(s)}\right] \end{aligned}$$

does not work for f, i.e., it does not provide a homotopy between f and the constant map c. One problem is that it does not induce a map on $\mathbb{S}^1 \times [0, 1]$, since

$$\tilde{H}(0, s) = \left[\sqrt{1 - \varphi(s)} : 0 : \sqrt{\varphi(s)}\right]$$

whereas

$$\tilde{H}(2\pi, s) = \left[-\sqrt{1 - \varphi(s)} : 0 : \sqrt{\varphi(s)}\right].$$

There is a minus sign occurring in the first coordinate because

$$\cos(2\pi/2) = \cos(\pi) = -\cos(0).$$

Hence

$$\tilde{H}(0, s) \neq \tilde{H}(2\pi, s) \text{ in } \mathbb{RP}^2 \text{ when } 0 < \varphi(s) < 1.$$

This indicates that we cannot continuously deform the loop corresponding to f to a constant map without cutting it open at some point. And the above calculation of the intersection number for f shows that it is impossible to remedy this defect.

A.15 Orientation

Solution (**Exercise 15.1**) **(a)** Replacing v_i by a multiple cv_i corresponds to multiplying β with the matrix which equals the identity matrix except at the ith position on the diagonal where 1 is replaced with c. The determinant of this matrix is equal to c. Hence $(v_1, \dots, v_k)$ and $(v_1, \dots, cv_i, \dots, v_k)$ are in the same equivalence class if and only if $c > 0$. If $c < 0$, they have opposite orientations.

(b) Interchanging the places of v_i and v_j for $i \neq j$ corresponds to multiplying β with the matrix which equals the identity matrix with the ith and jth rows switched. We know from Linear Algebra that the determinant of this matrix is -1.

(c) Subtracting from one v_i a linear combination of the others corresponds to multiplying β with a matrix that we obtain from the identity matrix by subtracting the corresponding linear combination of rows from the ith row. We know from Linear Algebra that this operation does not change the determinant of the matrix. Hence the determinant of the change-of-basis-matrix is still $+1$.

(d) Suppose that V is the direct sum of V_1 and V_2. Let $(v_1, \dots, v_k)$ be an ordered positively oriented basis of V_1 and $(w_1, \dots, w_m)$ an ordered positively oriented basis of V_2. Then $(v_1, \dots, v_k, w_1, \dots, w_m)$ is an ordered positively oriented basis of $V_1 \oplus V_2$, and $(w_1, \dots, w_m, v_1, \dots, v_k)$ is an ordered positively oriented basis of $V_2 \oplus V_1$. Switching from the given positive basis of $V_1 \oplus V_2$ to the positive basis of $V_2 \oplus V_1$ corresponds to transposing exactly $(\dim V_1)(\dim V_2)$ many elements in the basis. Hence the determinant of the change-of-basis-matrix is $(-1)^{(\dim V_1)(\dim V_2)}$.

Solution (**Exercise 15.2**) Let $(e_1, \dots, e_k)$ be the ordered basis of $\mathbb{R}^k$ which defines the standard orientation of $\mathbb{R}^k$. The orientation of $\mathbb{H}^k$ is given by the standard orientation of $\mathbb{R}^k$ restricted to the subspace $\mathbb{H}^k \subset \mathbb{R}^k$. The boundary orientation of $\partial\mathbb{H}^k$ is given by requiring that, at any point $x \in \partial\mathbb{H}^k$, the outward pointing unit normal vector $n_x = -e_k$ fits into a positively oriented basis for $\mathbb{R}^k$

$$(n_x, e_1, \dots, e_{k-1}) = (-e_k, e_1, \dots, e_{k-1}).$$

But the matrix which sends $(e_1, \dots, e_k)$ to $(-e_k, e_1, \dots, e_{k-1})$ is given by

$$A = \begin{pmatrix} 0 & 1 & & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & & 1 \\ -1 & 0 & \dots & & 0 \end{pmatrix}.$$

The matrix A can be transormed into the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}$$

by interchanging two columns exactly $k-1$ times. Hence $\det(D) = (-1)^{k-1}\det(A)$. But $\det(D) = -1$. Thus $\det(A) = 1 > 0$ if and only if $(-1)^k = 1$, i.e., if k is even.

Solution (**Exercise 15.3**) **(a)** At $x = (a, b, c) \in \mathbb{S}^2$, the tangent space $T_x(\mathbb{S}^2)$ is the two-dimensional vector subspace of $\mathbb{R}^3$ which is orthogonal to x. Since $(a, b, c) \neq (0, 0, 0)$, let us assume that, say, $b \neq 0$. A basis for $T_x(\mathbb{S}^2)$ is given by, for example, $v = (-b, a, 0)$ and $v = (0, c, -b)$. The outward pointing normal vector is given by $n_x = (a, b, c)$. The boundary orientation of $\mathbb{S}^2$, is the orientation of $T_x(\mathbb{S}^2)$ determined by the basis (n_x, v, w). This basis is positively oriented in $T_x(\mathbb{R}^3) = \mathbb{R}^3$ if and only if the matrix $A = \begin{pmatrix} a & -b & 0 \\ b & a & c \\ c & 0 & -b \end{pmatrix}$ has positive determinant, since this is the matrix that transforms the standard basis of $\mathbb{R}^3$ into the basis (n_x, v, w). The determinant of A is

$$\det(A) = -a^2b - b^3 - bc^2 = -b(a^2 + b^2 + c^2) = -b.$$

Thus, if $b < 0$, (n_x, v, w) is a positively oriented basis of $T_x(\mathbb{S}^2)$. If $b > 0$, we take the basis (n_x, w, v). And if $b = 0$, we start over with either a or c replacing b.

(b) The boundary orientation of $\mathbb{S}^k$ is, at any point $x \in \mathbb{S}^k$, given on $T_x(\mathbb{S}^k)$ by chosing the ordered basis $(n_x, v_1, \dots, v_k)$ to be positively oriented where n_x is the outward pointing unit normal vector in $T_x(\mathbb{R}^{k+1}) = \mathbb{R}^{k+1}$ and $(v_1, \dots, v_k)$ is an ordered basis of $T_x(\mathbb{S}^k)$. But since $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ is of codimension one, n_x spans the orthogonal complement $N_x(\mathbb{S}^k, \mathbb{R}^{k+1})$ of $T_x(\mathbb{S}^k)$ in $\mathbb{R}^{k+1}$. Hence the orientation of $T_x(\mathbb{S}^k)$ induced by the direct sum

$$N_x(\mathbb{S}^k, \mathbb{R}^{k+1}) \oplus T_x(\mathbb{S}^k) = T_x(\mathbb{R}^{k+1}) = \mathbb{R}^{k+1}$$

equals the orientation of $\mathbb{S}^k$ as the preimage under g.

Solution (**Exercise 15.4**) Assume that $df_{x_0} : T_{x_0}(X) \to T_{f(x_0)}(Y)$ preserves orientation at some point $x_0 \in X$. Since f is a diffeomorphism, df_x is an isomorphism for all $x \in X$. Hence $\det(df_x) \neq 0$ for all $x \in X$. In particular, the two disjoint open subsets $U := \{x \in X : \det(df_x) > 0\}$ and $V := \{x \in X : \det(df_x) < 0\}$ cover X. By assumption $x_0 \in U$, and hence U is nonempty. Since X is connected, this implies $U = X$.

Solution (**Exercise 15.5**) Let X and Z be transversal submanifolds in Y and assume X, Z and Y are oriented. Let $i : X \hookrightarrow Y$ be the inclusion of X into Y. The intersection $X \cap Z$ equals the preimage $i^{-1}(Z)$. By Section 15.5, the preimage orientation on $S := i^{-1}(Z)$ is induced, at any $y \in X \cap Z$, by the direct sum

$$N_y(S, X) \oplus T_y(S) = T_y(X),$$

where $N_y(S, X)$ is the orthogonal complement of $T_y(S)$ in $T_y(X)$. The orientation on

$N_y(S, X)$ is induced by the direct sum

$$di_y(N_y(S, X)) \oplus T_y(Z) = T_y(Y),$$

and the fact that $d(i_y)_{|N_y(S,X)}$ is an isomorphism onto its image. Since all these vector spaces are subspaces in $T_y(Y)$, and are oriented as subspaces of $T_y(Y)$, we can identify $N_y(S, X)$ with its image under di_y in $T_y(Y)$ and can rewrite this equation as

$$N_y(S, X) \oplus T_y(Z) = T_y(Y).$$

Now let $N_y(S, Z)$ be the orthogonal complement of $T_y(S)$ in $T_y(Z)$. Then the orientation of $T_y(S)$ is determined by the direct sum

$$N_y(S, X) \oplus N_y(S, Z) \oplus T_y(S) = T_y(Y).$$

Now if we start with the inclusion $j\colon\ Z \hookrightarrow Y$ of Z in Y instead, we get that the orientation of S considered as the preimage $j^{-1}(X)$ in Z, is determined by the direct sum

$$N_y(S, Z) \oplus N_y(S, X) \oplus T_y(S) = T_y(Y).$$

We learned in the first exercise that the signs of the orientations of $N_y(S, X) \oplus N_y(S, Z)$ and $N_y(S, Z) \oplus N_y(S, X)$ differ by $(-1)^{(\dim N_y(S,X))(\dim N_y(S,Z))}$. Now it remains to remark that, by definition of the normal spaces as orthogonal complements, we have

$$\begin{aligned}\dim N_y(S, X) &= \operatorname{codim} X \cap Z \text{ in } X = \operatorname{codim} Z \text{ in } Y, \text{ and}\\ \dim N_y(S, Z) &= \operatorname{codim} X \cap Z \text{ in } Z = \operatorname{codim} X \text{ in } Y.\end{aligned}$$

Solution (Exercise 15.6) First, we assume that Z is orientable. At any point $z \in Z$, let $(v_1, \dots, v_n)$ be an oriented basis of T_zZ. Since T_zZ is a vector subspace of codimension one in T_zX we can choose a single vector $n_z \in (T_zZ)^\perp = N_z(Z, X) \subset T_zX$ such that $(n_z, v_1, \dots, v_n)$ is an oriented basis of T_zX. Since both X and Z are oriented we can make this choice of n_z smoothly for every $z \in Z$. Since n_z has to be non-trivial in order to be part of a basis we can define a diffeomorphism

$$\varphi\colon\ Z \times \mathbb{R} \to N(Z, X),\ (z, t) \mapsto (z, t \cdot n_z).$$

Hence $N(Z, X)$ is a trivial line bundle over Z.

Now we assume that $N(Z, X)$ is trivial and let $\varphi\colon\ Z \times \mathbb{R} \to N(Z, X)$ be a trivializing diffeomorphism. Let n_z denote the vector in $(T_zZ)^\perp = N_z(Z, X) \subset T_zX$ such that $\varphi(z, 1) = (z, n_z)$. We then define a basis $(v_1, \dots, v_n)$ of T_zZ to be positively oriented if $(n_z, v_1, \dots, v_n)$ is a positively oriented basis of T_zX. Since φ is a diffeomorphism this defines a smooth choice of an orientation for all $z \in Z$. Hence Z is orientable.

Solution (Exercise 15.7) **(a)** Any basis of $V \times V$ consists of the product $(\alpha \times 0, 0 \times \beta)$

where α and β are ordered bases of V. The sign of this basis satisfies

$$\operatorname{sign}(\alpha \times 0, 0 \times \beta) = \operatorname{sign}(\alpha) \cdot \operatorname{sign}(\beta).$$

Switching the orientation of V changes both signs, $\operatorname{sign}(\alpha)$ and $\operatorname{sign}(\beta)$. Changing both signs simultaneously results in multiplying with $(-1)^2 = 1$. Hence the sign of the basis of $V \times V$ is independent of the choice of orientation for V.

(b) Let X be an orientable manifold. The orientation of $X \times X$ is given by a smooth choice of orientation of each tangent space

$$T_{(x,y)}(X \times X) = T_x(X) \times T_y(X).$$

Changing the orientation of X means changing the orientation of both $T_x(X)$ and $T_y(X)$. As in the previous point, this means multiplying the sign of any ordered basis of $T_{(x,y)}(X \times X)$ by $+1$. Hence the product orientation on $X \times X$ is the same for all choices of orientation on X.

(c) Let X be a smooth manifold which is not orientable. Any Euclidean space $\mathbb{R}^m$ is oriented as a manifold by the choice of the standard orientation of the tangent space $T_z(\mathbb{R}^m) = \mathbb{R}^m$ for any $z \in \mathbb{R}^m$. For any points $x \in X$ and $z \in \mathbb{R}^m$, the tangent space $T_{(x,z)}(X \times \mathbb{R}^m)$ is just $T_x(X) \times \mathbb{R}^m$. If there was a smooth choice for an orientation of $X \times \mathbb{R}^m$, then each tangent space $T_x(X)$ of X would inherit a smooth choice of orientation from the product $T_x(X) \times \mathbb{R}^m$. This contradicts the non-orientability of X.

Now let Y by any smooth manifold. If $X \times Y$ was orientable, then also $X \times U$ for an open subspace $U \subset Y$ which is diffeomorphic to some $\mathbb{R}^m$. But then $X \times \mathbb{R}^m$ would also inherit an orientation which is not possible. Applied to $Y = X$, we see that $X \times X$ is not orientable.

(d) We can cover X by local parametrizations $\phi\colon U \to X$. The union of the images of the maps $\phi \times \phi\colon U \times U \to X \times X$ is then an open subspace V of $X \times X$ which includes Δ. We orient each individual $\phi(U)$ by requiring the diffeomorphism $\phi\colon U \to \phi(U)$ to be orientation preserving. This induces an orientation on $(\phi \times \phi)(U \times U) = \phi(U) \times \phi(U)$. As we argued before, changing the orientation on $\phi(U)$ does not change the orientation on the product $\phi(U) \times \phi(U)$, since we multiply the signs of all tangent spaces by $+1$. Hence there is a well-defined orientation on $\phi(U) \times \phi(U)$ which is independent on the local parametrizations chosen. Thus V which is an open neighborhood of Δ in $X \times X$ is orientable.

However, this does not mean that Δ is always orientable. For, the tangent space to Δ at any point (x, x) is the diagonal of $T_x(X) \times T_x(X)$. This diagonal is isomorphic to $T_x(X)$. Hence changing the orientation of $T_x(X)$ does change the orientation of the diagonal in $T_x(X) \times T_x(X)$. Thus if we had a smooth choice of orientations for all diagonals in $T_x(X) \times T_x(X)$, then we had a smooth choice of orientations for all $T_x(X)$. In other words, Δ is orientable if and only if X is orientable.

Solution (**Exercise 15.8**) To determine the fiber over b, we write $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$. Then we get

$$\begin{aligned} \pi(z_0, z_1) = (0,1,0) &\Rightarrow 2z_0\bar{z}_1 = i \text{ and } |z_0|^2 = |z_1|^2 = \frac{1}{2} \\ &\Rightarrow y_0 = x_1,\ y_1 = -x_0 \text{ and } x_0^2 + x_1^2 = \frac{1}{2}. \end{aligned}$$

Thus the fiber over b has the form

$$\begin{aligned} \pi^{-1}(b) &= \{(z_0, z_1) \in \mathbb{S}^3 : \bar{z}_1 = \frac{i}{2z_0}\} \\ &= \{(x_0, y_0, x_1, y_1) \in \mathbb{S}^3 : y_0 = x_1, y_1 = -x_0\}. \end{aligned}$$

Let $q = (x_0, x_1, x_1, -x_0) \in \pi^{-1}(b)$ be a point in the fiber over b. Since not both x_0 and x_1 can be zero, we assume that $x_0 \neq 0$. The tangent space $T_q\mathbb{S}^3$ is the vector space

$$\begin{aligned} T_q\mathbb{S}^3 &= \{\mathbf{u} \in \mathbb{R}^4 : \mathbf{u} \perp q\} \\ &= \operatorname{span}\left\{ q_1^\perp = \begin{pmatrix} -x_1 \\ x_0 \\ 0 \\ 0 \end{pmatrix}, q_2^\perp = \begin{pmatrix} -x_1 \\ 0 \\ x_0 \\ 0 \end{pmatrix}, q_3^\perp = \begin{pmatrix} x_0 \\ 0 \\ 0 \\ x_0 \end{pmatrix} \right\}. \end{aligned}$$

The orientation of $T_q\mathbb{S}^3$ as a boundary of the unit ball is such that the **outward** pointing vector q together with the basis vectors of $T_q\mathbb{S}^3$ form a positively oriented basis of $\mathbb{R}^4$. The matrix expressing the basis $(q, q_1^\perp, q_2^\perp, q_3^\perp)$ in the standard basis of $\mathbb{R}^4$ is

$$\begin{pmatrix} x_0 & -x_1 & -x_1 & x_0 \\ x_1 & x_0 & 0 & 0 \\ x_1 & 0 & x_0 & 0 \\ -x_0 & 0 & 0 & x_0 \end{pmatrix}.$$

The determinant of this matrix is

$$2x_0^4 + 2x_0^2x_1^2 = 2x_0^2(x_0^2 + x_1^2) = x_0^2 > 0.$$

In particular, it is positive and the basis $(q_1^\perp, q_2^\perp, q_3^\perp)$ is a positively oriented basis of $T_q\mathbb{S}^3$.

The tangent space $T_q\pi^{-1}(b)$ equals the kernel of $d\tilde{\pi}_q$. In a previous exercise, we computed this map as represented by the matrix

$$d\tilde{\pi}_q = 2 \cdot \begin{pmatrix} x_1 & -x_0 & x_0 & x_1 \\ x_0 & x_1 & x_1 & -x_0 \\ x_0 & x_1 & -x_1 & x_0 \end{pmatrix}.$$

The kernel of this map is the span of the vector $q_0^\perp = \begin{pmatrix} -x_1 \\ x_0 \\ x_0 \\ x_1 \end{pmatrix}$. The normal space $N_q(\pi^{-1}(b); \mathbb{S}^3) \subset T_q\mathbb{S}^3$ of vectors which are orthogonal to $T_q\pi^{-1}(b)$ is the span of $(q_1^\perp - q_2^\perp, q_3^\perp)$. The map $d\tilde{\pi}_q$ sends $q_1^\perp - q_2^\perp$ and $q_3^\perp$ to, respectively,

$$d\tilde{\pi}_q(q_1^\perp - q_2^\perp) = 2\begin{pmatrix} -2x_0^2 \\ 0 \\ -2x_0x_1 \end{pmatrix}, \ d\tilde{\pi}_q(q_3^\perp) = 2\begin{pmatrix} -2x_0x_1 \\ 0 \\ 2x_0^2 \end{pmatrix}.$$

These two vectors form a basis $(d\tilde{\pi}_q(q_1^\perp - q_2^\perp), d\tilde{\pi}_q(q_3^\perp))$ of $T_b\mathbb{S}^2$. We need to check the orientation of this basis.

The tangent space $T_b\mathbb{S}^2$ has a basis $(\mathbf{e}_1^3, \mathbf{e}_3^3)$ as a subspace in $\mathbb{R}^3$. This basis is negatively oriented, since, together with the outward pointing vector $b = \mathbf{e}_2^3$, the basis $(\mathbf{e}_2^3, \mathbf{e}_1^3, \mathbf{e}_3^3)$ is a negatively oriented basis of $\mathbb{R}^3$. For we need to make one permutation to get the standard basis which leads to multiplying the sign with -1. The matrix B which expresses $(d\tilde{\pi}_q(q_1^\perp - q_2^\perp), d\tilde{\pi}_q(q_3^\perp))$ in terms of the basis $(\mathbf{e}_1^3, \mathbf{e}_3^3)$ is given by

$$B = 4 \cdot \begin{pmatrix} -x_0^2 & -x_0x_1 \\ -x_0x_1 & x_0^2 \end{pmatrix}.$$

We see that $\det B = 16(-x_0^4 - x_0^2x_1^2) = -16x_0^2(x_0^2 + x_1^2) = -8x_0^2 < 0$ is negative. Hence the basis $(d\tilde{\pi}_q(q_1^\perp - q_2^\perp), d\tilde{\pi}_q(q_3^\perp))$ is a negatively oriented basis of $T_b\mathbb{S}^2$. This defines an orientation on the normal space $N_q(\pi^{-1}(b); \mathbb{S}^3)$ by declaring the orientation of the basis $(q_1^\perp - q_2^\perp, q_3^\perp)$ to be negative.

Finally, the orientation of $T_q\pi^{-1}(b)$ is such that the direct sum

$$N_q(\pi^{-1}(b); \mathbb{S}^3) \oplus T_q\pi^{-1}(b) = T_q\mathbb{S}^3$$

induces the given orientation on $T_q\mathbb{S}^3$. We check this by looking at the basis $(q_1^\perp - q_2^\perp, q_3^\perp, q_0^\perp)$ of $N_q(\pi^{-1}(b); \mathbb{S}^3) \oplus T_q\pi^{-1}(b)$. The transition matrix from the basis $(q_1^\perp, q_2^\perp, q_3^\perp)$ to the basis $(q_1^\perp - q_2^\perp, q_3^\perp, q_0^\perp)$ is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & x_1/x_0 \end{pmatrix}.$$

The determinant of this matrix is -2. In particular, it is negative. Since we checked that the basis $(q_1^\perp, q_2^\perp, q_3^\perp)$ is a negatively oriented basis, we see that the orientation of the basis $(q_1^\perp - q_2^\perp, q_3^\perp, q_0^\perp)$ of $T_q\mathbb{S}^3$ is positive. Since the sign of $(q_1^\perp - q_2^\perp, q_3^\perp)$ is positive as a basis of $N_q(\pi^{-1}(a); \mathbb{S}^3)$, we need that the basis $q_0^\perp$ also has positive sign. Hence the vector $q_0^\perp$ provides a positively oriented basis of $T_q\pi^{-1}(b)$.

A.16 The Brouwer Degree

A.16.1 Brouwer degree

Solution (Exercise 16.4) **(a)** For $1 \leq i \leq k+1$, let r_i be the reflection map on the ith coordinate:

$$r_i : \mathbb{S}^k \to \mathbb{S}^k, (x_1, \ldots, x_{k+1}) \mapsto (x_1, \ldots, -x_i, \ldots, x_{k+1}).$$

The antipodal map is equal to the composition of reflections $r_1 \circ r_2 \circ \cdots \circ r_{k+1}$. Each reflection r_i is a diffeomorphism which reverses the orientation on $\mathbb{S}^k$. The composition of two such reflections $r_i \circ r_{i+1}$, however, is then a diffeomorphism which preserves the orientation on $\mathbb{S}^k$. Hence, if $k = 2n$ is even, $a = r_1 \circ r_2 \circ \cdots \circ r_{2n+1}$ has degree -1. Whereas if $k = 2n-1$ is odd, $a = r_1 \circ r_2 \circ \cdots \circ r_{2n}$ has degree $+1$. In other words, $\deg(a) = (-1)^{k+1}$.

(b) As pointed out, we know that a is homotopic to the identity if k is odd. By the previous point, $\deg(a) = -1$ if k is even. Since deg is homotopy invariant, $\deg(a) = -1 \neq 1 = \deg(1\mathrm{id})$ implies that the antipodal map is not homotopic to the identity if k is even.

Solution (Exercise 16.5) Recall that we have proven the existence part for k odd before:

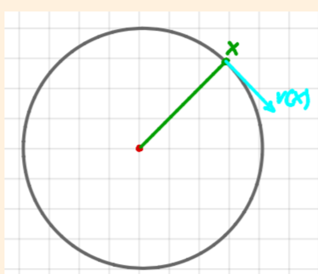
If k is odd, then $k+1$ is even and we can define the map

$$s : \mathbb{S}^k \to \mathbb{R}^{k+1}, (x_1, \ldots, x_{k+1}) \mapsto (-x_2, x_1, -x_3, x_4, \ldots, -x_{k+1}, x_k).$$

This map can be extended to a linear map $\mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ and therefore s is smooth. For each $x \in \mathbb{S}^k$, $s(x)$ is nonzero and satisfies $x \perp s(x)$. Thus $s(x)$ is a tangent vector at x, i.e. $s(x) \in T_x(\mathbb{S}^k) \setminus \{0\}$. Hence

$$\sigma : \mathbb{S}^k \to T(\mathbb{S}^k),\ \sigma(x) := (x, s(x))$$

is the desired non-vanishing vector field on $\mathbb{S}^k$.



Recall that we also have shown that if $\mathbb{S}^k$ has a vector field which has no zeros, then its antipodal map $x \mapsto -x$ is homotopic to the identity:

Given a vector field $\sigma : \mathbb{S}^k \to T(\mathbb{S}^k)$ which has no zeros. Let $\sigma(x) = (x, v(x))$. Since $v(x) \neq 0$ for every $x \in \mathbb{S}^k$, we can define a new vector field by

$$x \mapsto w(x) = \frac{v(x)}{|v(x)|}.$$

By replacing s with this new non-vanishing vector field, we can assume $|v(x)| = 1$. Hence we can assume $v(x) \in \mathbb{S}^k$ and $v(x) \cdot x = 0$ for every $x \in \mathbb{S}^k$.

Now we define the map

$$F : \mathbb{S}^k \times [0,1] \to \mathbb{S}^k, \ (x,t) \mapsto \cos(\pi t)x + \sin(\pi t)v(x).$$

We check that $F(x,t)$ is in fact an element in $\mathbb{S}^k$ for every $x \in \mathbb{S}^k$:

$$\begin{aligned} F(x,t) \cdot F(x,t) &= (\cos(\pi t)x + \sin(\pi t)s(x)) \cdot (\cos(\pi t)x + \sin(\pi t)v(x)) \\ &= \cos^2(\pi t)(x \cdot x) + 2\cos(\pi t)\sin(\pi t)(x \cdot v(x)) + \sin^2(\pi t)(s(x) \cdot v(x)) \\ &= \cos^2(\pi t) + \sin^2(\pi t) \\ &= 1 \end{aligned}$$

where we use $x \cdot x = 1 = v(x) \cdot v(x)$ and $x \cdot v(x) = 0$. Thus $F(x,t)$ is a vector of norm 1 for every x and every t. Moreover, F is a smooth map with $F(x,0) = x$ and $F(x,1) = -x$, i.e. F is a smooth homotopy from the identity to the antipodal map on $\mathbb{S}^k$. Hence, by homotopy invariance of deg, if $\mathbb{S}^k$ has a vector field which has no zeros, then $\deg(a) = \deg(\mathrm{id}) = 1$. By the previous exercise, since $\deg(a) = (-1)^{k+1}$ and hence k must be even.

Note that we would not have been able to make this conclusion with the mod 2-degree. For in $\mathbb{Z}/2$, we cannot distinguish between 1 and -1.

Solution (**Exercise 16.6**) We begin as in the solution to Exercise 12.4. We use the homotopy from $p_0(z) = z^m$ to $p_1(z) = p(z)$ defined by

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

We then observe that, if W is a closed ball around the origin in $\mathbb{C}$ with sufficiently large radius, none of the p_t has a zero on ∂W. Hence the homotopy

$$\frac{p_t}{|p_t|} : \partial W \to \mathbb{S}^1$$

is defined for all $t \in [0,1]$. Thus

$$\deg\left(\frac{p}{|p|}\right) = \deg\left(\frac{p_0}{|p_0|}\right).$$

Since $p_0(z) = z^m$, the degree of $p_0/|p_0|$ is the same as $\deg(z^m) = m$, that is

$$\deg\left(\frac{p}{|p|}\right) = m.$$

Thus, if $m > 0$, $p/|p|$ does not extend to all of W, since otherwise its degree had to be zero. Hence p **must have a zero** inside W.[a]

[a] Note that the final argument was not available for $\deg_2$ if m is even.

Solution **(Exercise 16.7)** Let us try to set up the argument we used in Exercise 16.6: Let

$$p(x) = x^m + a_1 x^{m-1} + \cdots + a_m$$

be a monic real polynomial. Define a homotopy from $p_0(x) = x^m$ to $p_1(x) = p(x)$ by

$$p_t(x) = tp(x) + (1-t)x^m = x^m + t(a_1 x^{m-1} + \cdots + a_m).$$

If $W = [-a, a]$ is a large enough closed interval in $\mathbb{R}$ containing the origin, none of the p_t has a zero on $\partial W = \{-a, a\}$. Hence the homotopy

$$\frac{p_t}{|p_t|} : \partial W = \{-a, a\} \to \{-1, +1\}$$

is defined for all $t \in [0, 1]$. Thus

$$\deg\left(\frac{p}{|p|}\right) = \deg\left(\frac{p_0}{|p_0|}\right).$$

However, for each $t \in [0, 1]$, the map $\frac{p_t}{|p_t|}(x)$ is constant with value either -1 or $+1$. A constant map has degree 0, since all points not in the image of the map are regular values and have empty fibers. Hence we get $\deg\left(\frac{p}{|p|}\right) = 0$. But this does not imply that we cannot extend $\frac{p}{|p|}$ to all of W. In particular, we do not get a contradiction to the assumption that p does not have a zero in W.

Solution **(Exercise 16.8)** **(a)** First, g is in fact a map $\mathbb{S}^1 \to \partial\mathbb{D}_0$, since any point z with $|z| = 1$ is sent to point with

$$|g(z) - z_0| = z_0 + rz - z_0 = |rz| = r \text{ since } |z| = 1.$$

Moreover, g is smooth and has an inverse given by

$$g^{-1} : \partial\mathbb{D}_0 \to \mathbb{S}^1,\ z \mapsto \frac{z - z_0}{|z - z_0|}.$$

Note that g^{-1} is also smooth, since both taking norms and dividing by $|z - z_0|$ are smooth operations in $\mathbb{C} \setminus \{z_0\}$.

Hence we know that g either preserves or reverses orientations. To check that g preserves orientations, it suffices to note that the derivative of g at any point, for example dg_1, is given by multiplication with $r > 0$.

(b) Let $y \in \mathbb{S}^1$. Since g is a diffeomorphism, y is a regular value for p if and only if it is a regular value for $p \circ g$. Moreover, g defines a bijection between the finite sets $\{z \in \partial D_0 | p(z) = y\}$ and $\{z \in \mathbb{S}^1 | p(g(z)) = y\}$. Since g preserves orientations, the orientation numbers at each point in these sets agree. Hence the degrees of the above maps are the same.

(c) Since $p(z) \neq 0$ for all $z \in \mathbb{D}_0 \setminus \{z_0\}$ by our choice of $\mathbb{D}_0$, we have $|q(z_0 + trz)| \neq 0$ for all $|z| = 1$. Hence we have a well-defined smooth map

$$H : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^1,\ (z, t) \mapsto \frac{z^m q(z_0 + trz)}{|q(z_0 + trz)|}.$$

Write $h_t(z) = H(z,t)$. For all $z \in \mathbb{S}^1$, we have

$$h_0(z) = \frac{z^m q(z_0)}{q(z_0)|} = \frac{q(z_0)}{q(z_0)|} \cdot z^m$$

and

$$\begin{aligned} h_1(z) = \frac{z^m q(z_0 + rz)}{|q(z_0 + rz)|} &= \frac{(rz)^m q(z_0 + rz)}{r^m |q(z_0 + rz)|} \\ &= \frac{(z_0 + rz - z_0)^m q(z_0 + rz)}{|z_0 + rz - z_0|^m |q(z_0 + rz)|} \\ &= \frac{p(g(z))}{|p(g(z))|}. \end{aligned}$$

Hence h_t is the desired homotopy.

(d) We know from the calculation in Example 16.11 that the map $\mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^m$, has degree m. Since multiplying with a constant c does not change the degree, we also have $\deg(z \mapsto c \cdot z^m) = \deg(h_0) = m$. Since degrees are homotopy invariant, the previous point shows $\deg(h_1) = \deg(h_0)$. As we observed before, $\deg(p/|p|) = \deg(h_1)$. Hence we can conclude $\deg(p/|p|) = m$.

Solution (Exercise 16.9) Let z be a regular value for g. If z is not in the image of g, then $\deg(g)$ and $\deg(g \circ f)$ both vanish and the claim is true. So let us assume that z is in the image of g. We showed in a previous exercise that z then is a regular value for $g \circ f$ if and only if every $y \in g^{-1}(z)$ is a regular value for f. Hence we can use z to compute $\deg(g \circ f)$ if and only if we can use $y \in g^{-1}(z)$ to compute $\deg(f)$. If z is not in the image of $g \circ f$, then y is not in the image of f. In this case both $\deg(g \circ f)$ and $\deg(f)$ vanish and the claim is true.

So assume there are points $x \in X$ with $g(f(x)) = z$. Given such an x, the chain rule gives us

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Since x and $f(x)$ are regular points by assumption, all three derivatives in this equation are isomorphisms. Recall that we equip x with orientation number $+1$ if df_x preserves orientation and with orientation number -1 if df_x reverses orientation. Moreover, whether df_x preserves or reverses orientation is determined by the sign of its determinant. Similarly, for $f(x)$ with respect to $dg_{f(x)}$ and x with respect to $d(g \circ f)_x$. By the chain rule and since the determinant is a multiplicative function, we get the formula for signs:

$$\operatorname{sign}(\det(d(g \circ f)_x)) = \operatorname{sign}(\det(dg_{f(x)})) \cdot \operatorname{sign}(\det(df_x)).$$

Since the degree is the sum of the orientation numbers at all points in the fiber, we get

$$\begin{aligned}
\deg(g\circ f) &= \sum_{x\in(g\circ f)^{-1}(z)} \operatorname{sign}\left(\det(d(g\circ f)_x)\right) \\
&= \sum_{x\in(g\circ f)^{-1}(z)} \left[\operatorname{sign}\left(\det(dg_{f(x)})\right)\cdot \operatorname{sign}\left(\det(df_x)\right)\right] \\
&= \left(\sum_{y\in g^{-1}(z)} \operatorname{sign}\left(\det(dg_y)\right)\right)\cdot\left(\sum_{x\in f^{-1}(y)} \operatorname{sign}\left(\det(df_x)\right)\right) \\
&= \deg(g)\cdot\deg(f).
\end{aligned}$$

Note that we used here that we can compute $\deg f$ using any regular value $y \in Y$ for f with $g(y) = z$.

Solution (**Exercise 16.10**) Let $a\colon \mathbb{S}^k \to \mathbb{S}^k$ denote the antipodal map. We assume that f had no fixed point. Then x and $a(f(x))$ are never antipodal points and we get $|x-(a\circ f)(x)| < 2$ for all $x \in \mathbb{S}^k$. By Exercise 8.6, this implies that $\mathrm{id}_{\mathbb{S}^k}$ and $a\circ f$ are homotopic. By Exercise 16.9, we have $\deg(a\circ f) = \deg(a)\cdot\deg(f)$ and by Exercise 16.4 we know $\deg(a) = (-1)^{k+1}$. Thus we obtain

$$1 = \deg(\mathrm{id}) = (-1)^{k+1}\deg(f),$$

which implies $\deg(f) = (-1)^{k+1}$. Thus, if f does not have fixed point, then $\deg(f) = (-1)^{k+1}$. This proves the claim.

Solution (**Exercise 16.11**) At every point $[x] \in \mathbb{R}\mathrm{P}^k$, the fiber under q consists of two antipodal points $q^{-1}([x]) = \{x, -x\} \subset \mathbb{S}^k$. Let $a\colon \mathbb{S}^k \to \mathbb{S}^k$ be the antipodal map. We have $q(x) = q(-x)$ for all $x \in \mathbb{S}^k$, i.e., we have $q = q\circ a$. This implies $dq_{-x}\circ da_x = dq_x$ for all $x \in \mathbb{S}^k$. In particular, this implies $\det(dq_{-x})\cdot\det(da_x) = \det(dq_x)$. If k is odd, then a preserves orientation, i.e, $\det(da_x) > 0$. Thus, $\det(dq_{-x})$ and $\det(dq_x)$ have the same sign. This implies that x and $-x$ contribute to the degree with the same orientation number. Hence $\deg(q)$ is either $+2$ or -2. This also shows that, if $\det(dq_x) > 0$, i.e., if q preserves orientations, then $\deg(q) = +2$.

Solution (**Exercise 16.12**) **(a)** If $f\colon \mathbb{S}^k \to \mathbb{S}^k$ sends no pair of antipodal points to antipodal points, then the straight line in $\mathbb{R}^{k+1}$ between $f(x)$ and $f(-x)$ does not go through the origin. Thus the map

$$H\colon \mathbb{S}^k\times[0,1]\to\mathbb{S}^k,\ x\mapsto \frac{(1-t)f(x)+tf(-x)}{|(1-t)f(x)+tf(-x)|}$$

is well-defined and smooth for every x and t. Hence H defines a homotopy between f and $f\circ a$. For $t = 1/2$, we have $H(x,1/2) = \frac{1}{2}(f(x)+f(-x)) = g(x)$ for all $x \in \mathbb{S}^k$. Thus f and $f\circ a$ are both smoothly homotopic to the map g as well. The invariance of the degree under homotopy implies $\deg(f) = \deg(f\circ a) = \deg(g)$.

(b) The previous point gives us, using Exercise 16.9,

$$\deg(g) = \deg(f \circ a) = \deg(f) \cdot \deg(a) = \deg(f) = \deg(g).$$

Since we assume that k is even, we have $\deg(a) = -1$ by Exercise 16.4. Thus we get $\deg(g) = -\deg(g)$ which implies $\deg(g) = 0$.

(c) Since k is odd, the degree of the quotient map $q : \mathbb{S}^k \to \mathbb{R}\mathrm{P}^k$ is defined. Since g satisfies $g(x) = g(-x)$ and q is a quotient map, g can be written as a composition

$$\mathbb{S}^k \xrightarrow{q} \mathbb{R}\mathrm{P}^k \xrightarrow{[g]} \mathbb{S}^k.$$

By Exercise 16.11, we have $\deg(q) = 2$. Thus, by Exercise 16.9, we get

$$\deg(g) = \deg(q) \cdot \deg([g]) = 2 \cdot \deg([g]).$$

This shows that $\deg(g)$ is even.

(d) The previous points show that the assumption that $f : \mathbb{S}^k \to \mathbb{S}^k$ sends no pair of antipodal points to antipodal points implies that $\deg(f) = \deg(g)$ must be even. Since $\deg(f)$ is odd, there must be at least one pair of antipodal points $x_0, -x_0$ such that $f(-x_0) = -f(x_0)$. This proves the claim.

A.17 Linking Number and the Hopf Invariant

A.17.1 Linking number and the Hopf invariant

Solution (Exercise 16.1) **(a)** Let $z \in \mathbb{S}^k$ be a regular value for λ and let $(x, y) \in X \times Y$ with $\lambda(x, y) = z$. Consider the derivative

$$d\lambda_{(x,y)} \colon T_x(X) \times T_y(Y) = T_{(x,y)}(X \times Y) \to T_z(\mathbb{S}^k).$$

Then the degree of λ is defined as the sum

$$\deg(\lambda) = \sum_{(x,y)\in\lambda^{-1}(z)} \operatorname{sign}(d\lambda_{(x,y)}).$$

As we learned in Section 15.2.1, switching the order of the factors in the domain corresponds to making $\dim X \cdot \dim Y$ many flips of basis vectors. Each flip requires to multiply orientation numbers with a factor (-1). Hence the map $s \colon X \times Y \to Y \times X$ induces multiplication by $(-1)^{mn}$ on the degree. In addition, we change the value of λ by sending $x - y$ to $y - x = -(x - y)$. Hence we compose with the antipodal map a on $\mathbb{S}^k$. This map has degree $(-1)^{k+1}$ as we learned in a previous exercise. Still using that we showed in an exercise that deg sends composition of maps to products, this implies in total

$$\begin{aligned} L(Y, X) &= \deg(a \circ \lambda \circ s) = (-1)^{k+1} \cdot \deg(\lambda) \cdot (-1)^{mn} \\ &= (-1)^{(m+1)(n+1)} \deg(\lambda) = (-1)^{(m+1)(n+1)} L(X, Y) \end{aligned}$$

where we use $k = m + n$.

(b) Since Y does not have a boundary, the product $W \times Y$ is a smooth manifold with boundary $\partial(W \times Y) = \partial W \times Y = X \times Y$. Since W and Y are disjoint, λ extends to a smooth map on $W \times Y$. Then the Boundary Theorem for degrees implies that $\deg(\lambda) = 0$.

Solution (Exercise 16.2) **(a)** Since taking preimages preserves codimensions, the dimension of $f^{-1}(w)$ and $f^{-1}(z)$ for any regular values w and z for f is given by

$$\dim f^{-1}(w) = \dim f^{-1}(z) = \dim \mathbb{S}^{2n-1} - \dim \mathbb{S}^n = n - 1.$$

Hence, if n is odd, then $\dim f^{-1}(w) + 1 = \dim f^{-1}(z) = n$ are odd and hence $H(f) = L(f^{-1}(w), f^{-1}(z)) = -L(f^{-1}(z), f^{-1}(w)) = -H(f)$ by a previous exercise. Thus we must have $H(f) = 0$.

(b) By Sard's Theorem we can find a point $a \in \mathbb{S}^n$ which is a regular value for both g and $g \circ f$. The fiber $g^{-1}(a)$ consists of a finite number of points, say $a_1, \dots, a_r$ in $\mathbb{S}^n$. We can assume that these points are ordered such that the orientation numbers of g at $a_1, \dots, a_p$ are positive, while the orientation numbers of g at $a_{p+1}, \dots, a_r$ are negative. Then we have $\deg(g) = 2p - r$. The fiber $f^{-1}(g^{-1}(a))$ then consists of a disjoint union of the oriented submanifolds $f^{-1}(a_i) \subset \mathbb{S}^{2n-1}$:

$$f^{-1}(g^{-1}(a)) = f^{-1}(a_1) \sqcup \ldots \sqcup f^{-1}(a_p) \sqcup (-f^{-1}(a_{p+1})) \sqcup \ldots \sqcup (-f^{-1}(a_r)).$$

Similarly, we can choose b such that the fiber $f^{-1}(g^{-1}(b))$ consists of a disjoint union of the submanifolds $f^{-1}(b_j) \subset \mathbb{S}^{2n-1}$ for $j = 1, \ldots, s$ and such that the $f^{-1}(a_i)$ and $f^{-1}(b_j$ are mutually disjoint. Again we order these points are ordered such that the orientation numbers of g at $b_1, \ldots, b_s$ are positive, while the orientation numbers of g at $b_{q+1}, \ldots, b_s$ are negative. Then we have $\deg(g) = 2q - s$. The fiber $f^{-1}(g^{-1}(b))$ then consists of a disjoint union of the oriented submanifolds $f^{-1}(a_i) \subset \mathbb{S}^{2n-1}$:

$$f^{-1}(g^{-1}(b)) = f^{-1}(b_1) \sqcup \ldots \sqcup f^{-1}(b_q) \sqcup (-f^{-1}(b_{q+1}) \sqcup \ldots \sqcup (-f^{-1}(b_s).$$

Since orientations behave multiplicatively on products of manifolds, the product fiber $f^{-1}(g^{-1}(a)) \times f^{-1}(g^{-1}(b))$ is the disjoint union of the submanifolds in $\mathbb{S}^{2n-1}$

$$\begin{aligned} & f^{-1}(g^{-1}(a)) \times f^{-1}(g^{-1}(b)) \\ = & \sqcup_{i,j} (\pm f^{-1}(a_i)) \times (\pm f^{-1}(b_j)) \\ = & \sqcup_{i=1,j=1}^{p,q} (f^{-1}(a_i) \times f^{-1}(b_j)) \sqcup \sqcup_{i=p+1,j=1}^{r,q} (-(f^{-1}(a_i) \times f^{-1}(b_j))) \\ & \sqcup \sqcup_{i=1,j=q+1}^{p,s} (-(f^{-1}(a_i) \times f^{-1}(b_j))) \sqcup \sqcup_{i=p+1,j=q+1}^{r,s} (f^{-1}(a_i) \times f^{-1}(b_j)). \end{aligned}$$

Now we write λ for the map

$$(g \circ f)^{-1}(a) \times (g \circ f)^{-1}(b) \to \mathbb{S}^{2n-2},\ (x, y) \mapsto \frac{x - y}{|x - y|}$$

and we write λ_{ij} for the map

$$f^{-1}(a_i) \times f^{-1}(b_j) \to \mathbb{S}^{2n-2},\ (x, y) \mapsto \frac{x - y}{|x - y|}.$$

Note that $H(f) = \deg(\lambda_{ij})$ for every pair i, j, since $H(f)$ does not depend on the choice of regular values. Since the degree is additive on oriented connected components, the above decomposition implies

$$\begin{aligned} & H(g \circ f) = \deg(\lambda) \\ = & \sum_{i=1,j=1}^{p,q} \deg(\lambda_{ij}) - \sum_{i=p+1,j=1}^{r,q} \deg(\lambda_{ij}) - \sum_{i=1,j=q+1}^{p,s} \deg(\lambda_{ij}) + \sum_{i=p+1,j=q+1}^{r,s} \deg(\lambda_{ij}) \\ = & H(f) \cdot [pq + (p - \deg(g))(q - \deg(g)) - (q(p - \deg(g))) - (p(q - \deg(g)))] \\ = & H(f) \cdot (\deg(g))^2. \end{aligned}$$

Solution (Exercise 16.3) **(a)** For $(z_0, z_1) \in \mathbb{S}^3$, we need to check $\pi(z_0, z_1) \in \mathbb{S}^2$. To do this, we recall that $|z| = z\bar{z}$ and $\bar{\bar{z}} = z$ for any complex number z. Now we compute:

$$\begin{aligned}&(2z_0\bar z_1)\cdot(2\bar z_0 z_1)+\left(|z_0|^2-|z_1|^2\right)^2\\&=4|z_0|^2|z_1|^2+|z_0|^4-2|z_0|^2|z_1|^2+|z_1|^4\\&=\left(|z_0|^2+|z_1|^2\right)^2\\&=1\end{aligned}$$

where the final step uses that $(z_0,z_1)\in\mathbb{S}^3$.

(b) First we assume $\pi(z_0,z_1)=\pi(w_0,w_1)$: Then we get

$$\begin{aligned}&\left(2w_0\bar w_1,|w_0|^2-|w_1|^2\right)=\left(2z_0\bar z_1,|z_0|^2-|z_1|^2\right)\\\iff\ &w_0\bar w_1=z_0\bar z_1 \text{ and } |w_0|^2-|w_1|^2=|z_0|^2-|z_1|^2.\end{aligned}$$

Remembering that neither of the numbers can be zero, the left hand equation gives us $\frac{w_0}{z_0}=\frac{\bar z_1}{\bar w_1}$. Moreover, we have $|w_0|^2-|w_1|^2=|z_0|^2-|z_1|^2$ and $|w_0|^2+|w_1|^2=1=|z_0|^2+|z_1|^2$. Putting these together implies $z_0^2=w_0^2$ and $z_1^2=w_1^2$. This shows that the desired α with $\alpha\bar\alpha=1$, i.e., $\bar\alpha=\frac{1}{\alpha}$, exists.

Now we assume that $(w_0,w_1)=(\alpha z_0,\alpha z_1)$ with $|\alpha|^2=\alpha\bar\alpha=1$: Then we compute

$$\begin{aligned}\pi(w_0,w_1)=&\ \left(2w_0\bar w_1,|w_0|^2-|w_1|^2\right)\\=&\ \left(2\alpha z_0\bar\alpha\bar z_1,|\alpha|^2|z_0|^2-|\alpha|^2|z_1|^2\right)\\=&\ \left(2z_0\bar z_1,|z_0|^2-|z_1|^2\right)\\=&\ \pi(z_0,z_1).\end{aligned}$$

(c) Let $p\in\mathbb{S}^2$ be a fixed point and fix a point (z_0,z_1) in $\mathbb{S}^3$ with $\pi(z_0,z_1)=p$. By the previous point, we have that the points in $\pi^{-1}(p)$ is parametrized by the complex number α with $|\alpha|^2=1$. The latter condition means $\alpha\in\mathbb{S}^1\subset\mathbb{C}$. Hence we get a bijective map

$$\mathbb{S}^1\to\pi^{-1}(p),\ \alpha\mapsto(\alpha z_0,\alpha z_1).$$

Since this map just consists of multiplication with nonzero complex numbers, we can conclude that it is a diffeomorphism (where we consider$\mathbb{S}^1$ and $\pi^{-1}(p)$ as subsets in real Euclidean space).

(d) First we look at the map $\tilde\pi:\ \mathbb{R}^4\cong\mathbb{C}^2\to\mathbb{C}\times\mathbb{R}\cong\mathbb{R}^3$ using the same formula as for π, i.e., $\pi=\tilde\pi_{|\mathbb{S}^3}$, and compute its derivative at a point $q=(z_0,z_1)$. To do this we write $(z_0,z_1)=(x_0,y_0,x_1,y_1)$ for real coordinates x_0,y_0,x_1,y_1. First we get

$$\begin{aligned}\tilde\pi(z_0,z_1)&=\left(2z_0\bar z_1,|z_0|^2-|z_1|^2\right)\\&=\left(2(x_0x_1+y_0y_1)+i2(-x_0y_1+y_0x_1),(x_0^2+y_0^2)-(x_1^2+y_1^2)\right).\end{aligned}$$

Then we can compute

$$d\tilde\pi_q=2\cdot\begin{pmatrix}x_1&y_1&x_0&y_0\\-y_1&x_1&y_0&-x_0\\x_0&y_0&-x_1&-y_1\end{pmatrix}.$$

Note that if $q \in \mathbb{S}^3$, the restriction of $d\tilde{\pi}_q$ to $T_q\mathbb{S}^3$ has image contained in $T_{\pi(q)}\mathbb{S}^2$.

Recall: Since it has been a while that we looked at tangent spaces, let us see why this claim is true. So let $g_4 : \mathbb{R}^4 \to \mathbb{R}$ and $g_3 : \mathbb{R}^3 \to \mathbb{R}$ be the usual smooth maps such that $\mathbb{S}^3 = g_4^{-1}(1)$ and $\mathbb{S}^2 = g_3^{-1}(1)$ respectively. Then we have $T_q\mathbb{S}^3 = \text{Ker}\,(d(g_4)_q) \subset T_q\mathbb{R}^4 = \mathbb{R}^4$ and $T_p\mathbb{S}^2 = \text{Ker}\,(d(g_3)_p) \subset T_p\mathbb{R}^3 = \mathbb{R}^3$. We know that $\tilde{\pi}(q) \in \mathbb{S}^2$ if $q \in \mathbb{S}^3$. Actually, our calculation above shows that $\tilde{\pi}(q) \in \mathbb{S}^2$ if and only if $q \in \mathbb{S}^3$. This implies $\mathbb{S}^3 = \tilde{\pi}^{-1}(g_3^{-1}(1)) = (g_3 \circ \tilde{\pi})^{-1}(1)$. In particular, $g_3(\tilde{\pi}(q)) = 1$ is constant on $\mathbb{S}^3$. Hence, for every $q \in \mathbb{S}^3$, the image of the restriction $(d\tilde{\pi}_q)_{|T_q\mathbb{S}^3}$ is contained in the kernel of $d(g_3)_{\tilde{\pi}(q)}$ which is $T_{\tilde{\pi}(q)}\mathbb{S}^2$.

Now back to our task:

We want to show that every $q \in \mathbb{S}^3$ is a regular point. Hence we need to show that $d\tilde{\pi}_q$ restricted to $T_q\mathbb{S}^3$ is surjective onto $T_{\pi(q)}\mathbb{S}^2$. Since the tangent space $T_{\pi(q)}\mathbb{S}^2$ of $\mathbb{S}^2$ is two-dimensional, we need to check that the image of $d\tilde{\pi}_q$ restricted to Ker $(d(g_4)_q)$ spans a two-dimensional subspace. Since Ker $(d(g_4)_q)$ is of dimension 3, it suffices to show that $d\tilde{\pi}_q$ has rank 3, which implies that the kernel of $d\tilde{\pi}_q$ has dimension 1. Hence we need to show that $d\tilde{\pi}_q$ always has 3 linear independent columns.

We can show this for example by calculating the determinants of appropriate 3×3-minors. Ignoring the factor 2 in our formula for $d\tilde{\pi}_q$ we look at the minors A_j of the remaining matrix where we omit the jth column:

- The determinant of A_4 is $-x_1(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -x_1$.
- The determinant of A_3 is $-y_0(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -y_0$.
- The determinant of A_2 is $-x_0(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -x_0$.
- The determinant of A_1 is $-y_1(x_0^2 + y_0^2 + x_1^2 + y_1^2) = -y_1$.

For every point $q \in \mathbb{S}^3$, at least one of the coordinates x_0, y_0, x_1, y_1 is nonzero. Hence the matrix always has three linear independent columns and $d\tilde{\pi}_q$ has rank 3. This shows that each point in $\mathbb{S}^3$ is a regular point for π, and hence every point in $\mathbb{S}^2$ is a regular value for π.

(e)
- The fiber over a is

$$\pi^{-1}(a) = \{(z_0, 0) \in \mathbb{S}^3 \subset \mathbb{C}^2 \,:\, |z_0|^2 = 1\}.$$

Let $q = (x_0, y_0, 0, 0) \in \pi^{-1}(a)$ be a point in the fiber over a. The tangent space $T_q\mathbb{S}^3$ is the vector space

$$\begin{aligned} T_q\mathbb{S}^3 &= \{\mathbf{u} \in \mathbb{R}^4 \,:\, \mathbf{u} \perp q\} \\ &= \text{span}\left\{ q^\perp = \begin{pmatrix} -y_0 \\ x_0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_3^4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_4^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

The orientation of $T_q\mathbb{S}^3$ as a boundary of the unit ball is such that the **outward** pointing vector q together with the basis vectors of $T_q\mathbb{S}^3$ form a positively oriented basis of $\mathbb{R}^4$. The determinant of the matrix expressing the basis $(q, q^\perp, \mathbf{e}_3^4, \mathbf{e}_4^4)$ in the standard basis of $\mathbb{R}^4$ equals $x_0^2 + y_0^2 = 1 > 0$. In particular, it is positive and the basis $(q^\perp, \mathbf{e}_3^4, \mathbf{e}_4^4)$ is a positively oriented basis of $T_q\mathbb{S}^3$.

The tangent space $T_q\pi^{-1}(a)$ equals the kernel of $d\tilde{\pi}_q$. We computed this map as represented by the matrix

$$d\tilde{\pi}_q = 2 \cdot \begin{pmatrix} 0 & 0 & x_0 & y_0 \\ 0 & 0 & y_0 & -x_0 \\ x_0 & y_0 & 0 & 0 \end{pmatrix}.$$

The kernel of this map is the span of the vector $q^\perp$ that we have just seen above. The normal space $N_q(\pi^{-1}(a); \mathbb{S}^3) \subset T_q\mathbb{S}^3$ of vectors which are orthogonal to $T_q\pi^{-1}(a)$ is the span of $(\mathbf{e}_3^4, \mathbf{e}_4^4)$. The map $d\tilde{\pi}_q$ sends $\mathbf{e}_3$ and $\mathbf{e}_4$ to, respectively,

$$d\tilde{\pi}_q(\mathbf{e}_3^4) = 2\begin{pmatrix} x_0 \\ y_0 \\ 0 \\ 0 \end{pmatrix}, \; d\tilde{\pi}_q(\mathbf{e}_4^4) = 2\begin{pmatrix} y_0 \\ -x_0 \\ 0 \\ 0 \end{pmatrix}.$$

These two vectors form a basis $(d\tilde{\pi}_q(\mathbf{e}_3^3), d\tilde{\pi}_q(\mathbf{e}_4^4))$ of $T_a\mathbb{S}^2$. We need to check the orientation of this basis. The tangent space $T_a\mathbb{S}^2$ has a basis $(\mathbf{e}_1^3, \mathbf{e}_2^3)$ as a subspace in $\mathbb{R}^3$. This basis is positively oriented, since, together with the outward pointing vector $a = \mathbf{e}_3^3$, the basis $(\mathbf{e}_3^3, \mathbf{e}_1^3, \mathbf{e}_2^3)$ is a positively oriented basis of $\mathbb{R}^3$.[a] The matrix A which expresses $(d\tilde{\pi}_q(\mathbf{e}_3^3), d\tilde{\pi}_q(\mathbf{e}_4^4))$ in terms of the basis $(\mathbf{e}_1^3, \mathbf{e}_2^3)$ is given by

$$A = 2 \cdot \begin{pmatrix} x_0 & y_0 \\ y_0 & -x_0 \end{pmatrix}.$$

We see that $\det A = 4(-x_0^2 - y_0^2) = -4 < 0$ is negative. Hence the basis $(d\tilde{\pi}_q(\mathbf{e}_3^3), d\tilde{\pi}_q(\mathbf{e}_4^4))$ is a negatively oriented basis of $T_a\mathbb{S}^2$. This defines an orientation on the normal space $N_q(\pi^{-1}(a); \mathbb{S}^3)$ by declaring the orientation of the basis $(\mathbf{e}_3^4, \mathbf{e}_4^4)$ to be negative.

Finally, the orientation of $T_q\pi^{-1}(a)$ is such that the direct sum

$$N_q(\pi^{-1}(a); \mathbb{S}^3) \oplus T_q\pi^{-1}(a) = T_q\mathbb{S}^3$$

induces the given orientation on $T_q\mathbb{S}^3$. We check this by looking at the basis $(\mathbf{e}_3^4, \mathbf{e}_4^4, q^\perp)$ of $N_q(\pi^{-1}(a); \mathbb{S}^3) \oplus T_q\pi^{-1}(a)$. As a basis of $T_q\mathbb{S}^3$ this basis is positively oriented, since it arises by two permutations from the positively oriented basis $(q^\perp, \mathbf{e}_3^4, \mathbf{e}_4^4)$. Since the sign of $(\mathbf{e}_3^4, \mathbf{e}_4^4)$ is negative as a basis of $N_q(\pi^{-1}(a); \mathbb{S}^3)$, we need that $q^\perp$ also has negative sign. Hence the vector $q^\perp$ provides a negatively oriented basis of $T_q\pi^{-1}(a)$.

Comparing this orientation with the standard orientation of $\mathbb{S}^1 \subset \mathbb{C} \subset \mathbb{C}^2$, we see that $\pi^{-1}(a)$ has the opposite orientation.

- To determine the fiber over b, we write $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$. Then we get

$$\pi(z_0, z_1) = (0, 1, 0) \Rightarrow 2z_0\bar{z}_1 = i \text{ and } |z_0|^2 = |z_1|^2 = \frac{1}{2}$$
$$\Rightarrow y_0 = x_1,\ y_1 = -x_0 \text{ and } x_0^2 + x_1^2 = \frac{1}{2}.$$

Thus the fiber over b has the form

$$\begin{aligned}\pi^{-1}(b) &= \{(z_0, z_1) \in \mathbb{S}^3 \,:\, \bar{z}_1 = \frac{i}{2z_0}\} \\ &= \{(x_0, y_0, x_1, y_1) \in \mathbb{S}^3 \,:\, y_0 = x_1, y_1 = -x_0\}.\end{aligned}$$

Let $q = (x_0, x_1, x_1, -x_0) \in \pi^{-1}(b)$ be a point in the fiber over b. Since not both x_0 and x_1 can be zero, we assume that $x_0 \neq 0$. The tangent space $T_q\mathbb{S}^3$ is the vector space

$$\begin{aligned}T_q\mathbb{S}^3 &= \{\mathbf{u} \in \mathbb{R}^4 \,:\, \mathbf{u} \perp q\} \\ &= \operatorname{span}\left\{ q_1^\perp = \begin{pmatrix} -x_1 \\ x_0 \\ 0 \\ 0 \end{pmatrix}, q_2^\perp = \begin{pmatrix} -x_1 \\ 0 \\ x_0 \\ 0 \end{pmatrix}, q_3^\perp = \begin{pmatrix} x_0 \\ 0 \\ 0 \\ x_0 \end{pmatrix} \right\}.\end{aligned}$$

The orientation of $T_q\mathbb{S}^3$ as a boundary of the unit ball is such that the **outward** pointing vector q together with the basis vectors of $T_q\mathbb{S}^3$ form a positively oriented basis of $\mathbb{R}^4$. The matrix expressing the basis $(q, q_1^\perp, q_2^\perp, q_3^\perp)$ in the standard basis of $\mathbb{R}^4$ is

$$\begin{pmatrix} x_0 & -x_1 & -x_1 & x_0 \\ x_1 & x_0 & 0 & 0 \\ x_1 & 0 & x_0 & 0 \\ -x_0 & 0 & 0 & x_0 \end{pmatrix}.$$

The determinant of this matrix is

$$2x_0^4 + 2x_0^2x_1^2 = 2x_0^2(x_0^2 + x_1^2) = x_0^2 > 0.$$

In particular, it is positive and the basis $(q_1^\perp, q_2^\perp, q_3^\perp)$ is a positively oriented basis of $T_q\mathbb{S}^3$.

The tangent space $T_q\pi^{-1}(b)$ equals the kernel of $d\tilde{\pi}_q$. We computed this map as represented by the matrix

$$d\tilde{\pi}_q = 2 \cdot \begin{pmatrix} x_1 & -x_0 & x_0 & x_1 \\ x_0 & x_1 & x_1 & -x_0 \\ x_0 & x_1 & -x_1 & x_0 \end{pmatrix}.$$

The kernel of this map is the span of the vector $q_0^\perp = \begin{pmatrix} -x_1 \\ x_0 \\ x_0 \\ x_1 \end{pmatrix}$. The normal space $N_q(\pi^{-1}(b); \mathbb{S}^3) \subset T_q\mathbb{S}^3$ of vectors which are orthogonal to $T_q\pi^{-1}(b)$ is the span of $(q_1^\perp - q_2^\perp, q_3^\perp)$. The map $d\tilde{\pi}_q$ sends $q_1^\perp - q_2^\perp$ and $q_3^\perp$ to, respectively,

$$d\tilde{\pi}_q(q_1^\perp - q_2^\perp) = 2\begin{pmatrix} -2x_0^2 \\ 0 \\ -2x_0x_1 \end{pmatrix}, \; d\tilde{\pi}_q(q_3^\perp) = 2\begin{pmatrix} -2x_0x_1 \\ 0 \\ 2x_0^2 \end{pmatrix}.$$

These two vectors form a basis $(d\tilde{\pi}_q(q_1^\perp - q_2^\perp), d\tilde{\pi}_q(q_3^\perp))$ of $T_b\mathbb{S}^2$. We need to check the orientation of this basis. The tangent space $T_b\mathbb{S}^2$ has a basis $(\mathbf{e}_1^3, \mathbf{e}_3^3)$ as a subspace in $\mathbb{R}^3$. This basis is negatively oriented, since, together with the outward pointing vector $b = \mathbf{e}_2^3$, the basis $(\mathbf{e}_2^3, \mathbf{e}_1^3, \mathbf{e}_3^3)$ is a negatively oriented basis of $\mathbb{R}^3$. For we need to make one permutation to get the standard basis which leads to multiplying the sign with -1. The matrix B which expresses $(d\tilde{\pi}_q(q_1^\perp - q_2^\perp), d\tilde{\pi}_q(q_3^\perp))$ in terms of the basis $(\mathbf{e}_1^3, \mathbf{e}_3^3)$ is given by

$$B = 4 \cdot \begin{pmatrix} -x_0^2 & -x_0x_1 \\ -x_0x_1 & x_0^2 \end{pmatrix}.$$

We see that $\det B = 16(-x_0^4 - x_0^2x_1^2) = -16x_0^2(x_0^2 + x_1^2) = -8x_0^2 < 0$ is negative. Hence the basis $(d\tilde{\pi}_q(q_1^\perp - q_2^\perp), d\tilde{\pi}_q(q_3^\perp))$ is a negatively oriented basis of $T_b\mathbb{S}^2$. This defines an orientation on the normal space $N_q(\pi^{-1}(b); \mathbb{S}^3)$ by declaring the orientation of the basis $(q_1^\perp - q_2^\perp, q_3^\perp)$ to be negative.

Finally, the orientation of $T_q\pi^{-1}(b)$ is such that the direct sum

$$N_q(\pi^{-1}(b); \mathbb{S}^3) \oplus T_q\pi^{-1}(b) = T_q\mathbb{S}^3$$

induces the given orientation on $T_q\mathbb{S}^3$. We check this by looking at the basis $(q_1^\perp - q_2^\perp, q_3^\perp, q_0^\perp)$ of $N_q(\pi^{-1}(b); \mathbb{S}^3) \oplus T_q\pi^{-1}(b)$. The transition matrix from the basis $(q_1^\perp, q_2^\perp, q_3^\perp)$ to the basis $(q_1^\perp - q_2^\perp, q_3^\perp, q_0^\perp)$ is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & x_1/x_0 \end{pmatrix}.$$

The determinant of this matrix is -2. In particular, it is negative. Since we checked that the basis $(q_1^\perp, q_2^\perp, q_3^\perp)$ is a negatively oriented basis, we see that the orientation of the basis $(q_1^\perp - q_2^\perp, q_3^\perp, q_0^\perp)$ of $T_q\mathbb{S}^3$ is positive. Since the sign of $(q_1^\perp - q_2^\perp, q_3^\perp)$ is positive as a basis of $N_q(\pi^{-1}(a); \mathbb{S}^3)$, we need that the basis $q_0^\perp$ also has positive sign. Hence the vector $q_0^\perp$ provides a positively oriented basis of $T_q\pi^{-1}(b)$.

(f) By definition of $H(\pi)$, we need to choose two distinct regular values a and b of π and calculate the linking number of $\pi^{-1}(a)$ and $\pi^{-1}(b)$. Since we showed that each value is regular, we can for example choose $a = (0, 0, 1)$ and $b = (0, 1, 0)$ on $\mathbb{S}^2 \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$.

To calculate the linking number of $\pi^{-1}(a)$ and $\pi^{-1}(b)$ we need to choose a point on $\mathbb{S}^3$ disjoint from these two subsets and stereographically project $\mathbb{S}^3$ from this point onto $\mathbb{R}^3$. By our choice of a and b, we get that the north pole $N = (0, 0, 0, 1)$

is neither on $\pi^{-1}(a)$ nor on $\pi^{-1}(b)$. Recall that the formula for the stereographic projection $\phi_N^{-1} : \mathbb{S}^3 \setminus \{N\} \to \mathbb{R}^3$ is, with the notation we use here, given by

$$(x_0, y_0, x_1, y_1) \mapsto \frac{1}{1-y_1}(x_0, y_0, x_1).$$

Hence we get

$$\begin{aligned} S_a &:= \phi_N^{-1}(\pi^{-1}(a)) \\ &= \{\mathbf{v} = (v_0, v_1, v_2) \in \mathbb{R}^3 \, : \, v_0^2 + v_1^2 = 1 \text{ and } v_2 = 0\} \end{aligned}$$

and

$$\begin{aligned} S_b &:= \phi_N^{-1}(\pi^{-1}(b)) \\ &= \left\{ \mathbf{w} = (w_0, w_1, w_2) \in \mathbb{R}^3 \, : \, w_1 = w_2 \text{ and } w_0^2 + w_1^2 = \frac{(1-w_0)^2}{2} \right\}. \end{aligned}$$

Now we can calculate $H(\pi)$ as $\deg(\lambda)$ with

$$\lambda : \, S_a \times S_b \to \mathbb{S}^2, (\mathbf{v}, \mathbf{w}) \mapsto \frac{\mathbf{v}-\mathbf{w}}{|\mathbf{v}-\mathbf{w}|}.$$

To compute the degree of λ we pick a convenient point of $\mathbb{S}^2$ and determine the fiber over this point. Then we check that we actually picked a regular value.

So let us look at $p = (1, 0, 0)$. The equation $\lambda(\mathbf{v}, \mathbf{w}) = p$ then implies

$$v_1 = w_1 = 0 \text{ and } v_0 - w_0 = |v_0 - w_0|.$$

The latter condition, implies that $v_0 - w_0$ is positive. This does not look very helpful at first glance, but we also know

$$1 = v_0^2 + v_1^2 = v_0^2, \text{i.e.}, v_0 = \pm 1,$$

and

$$w_0^2 = \frac{(1-w_0)^2}{2} \iff w_0 = \pm\sqrt{2} - 1.$$

Now we can check for the four possible permutations of the signs whether they yield $v_0 - w_0 \geq 0$ and get three points: one with $v_0 = 1$, $w_0 = \sqrt{2} - 1$, one with $v_0 = 1$, $w_0 = -\sqrt{2} - 1$, and one with $v_0 = -1$, $w_0 = -\sqrt{2} - 1$. Hence we get three points $(\mathbf{v}, \mathbf{w})$ in $S_a \times S_b$ with $\lambda(\mathbf{v}, \mathbf{w}) = p$.

It remains to check the derivatives of λ at these points. We have to show that the determinants at each point are nonzero and that the sum of the signs is $+1$. For then we get $\deg(\lambda) = +1$ as claimed.

Since S_a is the unit circle in the xy-plane, the tangent space of S_a at a point $\mathbf{v}$ is given by

$$T_{\mathbf{v}} S_a = \{\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{R}^3 \, : \, u_2 = 0 \text{ and } u_0 v_0 + u_1 v_1 = 0\}.$$

Similarly, S_b lies in the plane P in $\mathbb{R}^3$ of points $\mathbf{w} = (w_0, w_1, w_2)$ with $w_1 = w_2$. Then S_b is the fiber of the map

$$g_b : P \to \mathbb{R}, \ \mathbf{w} = (w_0, w_1, w_1) \mapsto w_0^2 + w_1^2 = \frac{(1-w_0)^2}{2}.$$

After a simple computation we get

$$S_b = g_b^{-1}(1) = \left\{ \mathbf{w} = (w_0, w_1, w_1) \in P \,:\, 2w_1^2 + w_0^2 + 2w_0 = 1 \right\}.$$

The derivative of g_b as a map from $P \to \mathbb{R}$ is given by the matrix (we could also consider it as a map $\mathbb{R}^3 \to \mathbb{R}^2$)

$$d(g_b)_{\mathbf{w}} = (2w_0 + 2, 2w_1).$$

Hence we get

$$T_{\mathbf{w}}S_b = \{\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{R}^3 \,:\, u_2 = u_1 \text{and } u_0(w_0+1) + u_1 w_1 = 0\}.$$

Now we calculate the derivative of λ. First we do this as a map

$$\tilde{\lambda} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \ (\mathbf{v}, \mathbf{w}) \mapsto \frac{\mathbf{v} - \mathbf{w}}{|\mathbf{v} - \mathbf{w}|}.$$

This will help us, since $g_3 \circ \tilde{\lambda}$ is constant on $S_a \times S_b$. Hence $d\tilde{\lambda}_{(\mathbf{v},\mathbf{w})}$ sends the subspace $T_{\mathbf{v}} \times T_{\mathbf{w}}S_b \in \mathbb{R}^3 \times \mathbb{R}^3$ to $T_p\mathbb{S}^2 \subset \mathbb{R}^3$.

We determine $d\tilde{\lambda}_{(\mathbf{v},\mathbf{w})}$ by computing its partial derivatives $\frac{\partial \tilde{\lambda}_i}{\partial v_j}(\mathbf{v}, \mathbf{w})$ and $\frac{\partial \tilde{\lambda}_i}{\partial w_j}(\mathbf{v}, \mathbf{w})$ with respect to the variables v_0, v_1, v_2 and w_0, w_1, w_2: For $i \neq j$, we have

$$\frac{\partial \tilde{\lambda}_i}{\partial v_j}(\mathbf{v}, \mathbf{w}) = \frac{(v_i - w_i)(v_j - w_j)}{|\mathbf{v} - \mathbf{w}|^3} = \frac{\partial \tilde{\lambda}_i}{\partial w_j}(\mathbf{v}, \mathbf{w}).$$

For $i = j$, we get

$$\frac{\partial \tilde{\lambda}_i}{\partial v_i}(\mathbf{v}, \mathbf{w}) = \frac{1}{|\mathbf{v} - \mathbf{w}|^3} \cdot \begin{cases} (v_1 - w_1)^2 + (v_2 - w_2)^2 & \text{if } i = 0 \\ (v_0 - w_0)^2 + (v_2 - w_2)^2 & \text{if } i = 1 \\ (v_0 - w_0)^2 + (v_1 - w_1)^2 & \text{if } i = 2 \end{cases}$$

and

$$\frac{\partial \tilde{\lambda}_i}{\partial w_i}(\mathbf{v}, \mathbf{w}) = -\frac{\partial \tilde{\lambda}_i}{\partial v_i}(\mathbf{v}, \mathbf{w}).$$

Now we evaluate these formulae at the points $(\mathbf{v}, \mathbf{w})$ with $\lambda(\mathbf{v}, \mathbf{w}) = p$. For each such point we found $v_1 = v_2 = w_1 = w_2 = 0$. Hence we get

$$\frac{\partial \tilde{\lambda}_i}{\partial v_j}(\mathbf{v}, \mathbf{w}) = 0 = \frac{\partial \tilde{\lambda}_i}{\partial w_j}(\mathbf{v}, \mathbf{w})$$

for $i \neq j$,

$$\frac{\partial \tilde{\lambda}_i}{\partial v_i}(\mathbf{v}, \mathbf{w}) = \frac{1}{|v_0 - w_0|} \cdot \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1, 2 \end{cases}$$

and

$$\frac{\partial \tilde{\lambda}_i}{\partial w_i}(\mathbf{v}, \mathbf{w}) = \frac{1}{|v_0 - w_0|} \cdot \begin{cases} 0 & \text{if } i = 0 \\ -1 & \text{if } i = 1, 2. \end{cases}$$

Now we are equipped to study the linear map $d\lambda_{(\mathbf{v},\mathbf{w})} : T_{\mathbf{v}}S_a \times T_{\mathbf{w}}S_b \to T_p\mathbb{S}^2$:

A basis of $T_p\mathbb{S}^2$ is given by the vectors $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Since the vector $\mathbf{p} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an outward pointing normal vector, this is a positively oriented basis of $T_p\mathbb{S}^2$.

At each of the points $(\mathbf{v}, \mathbf{w})$ we found with $\lambda(\mathbf{v}, \mathbf{w}) = \mathbf{p}$, the vector $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis of $T_{\mathbf{v}}S_a$ and the vector $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ provides a basis of $T_{\mathbf{w}}S_b$. The map $d\lambda_{(\mathbf{v},\mathbf{w})}$ sends $\mathbf{a}$ to $\frac{1}{|v_0 - w_0|} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{b}$ to $\frac{1}{|v_0 - w_0|} \cdot \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$. Hence we have

$$d\lambda_{(\mathbf{v},\mathbf{w})}(\mathbf{a}) = \frac{1}{|v_0 - w_0|} \cdot \mathbf{e}_2$$

and

$$d\lambda_{(\mathbf{v},\mathbf{w})}(\mathbf{b}) = -\frac{1}{|v_0 - w_0|} \cdot \mathbf{e}_2 - \frac{1}{|v_0 - w_0|} \cdot \mathbf{e}_3.$$

These two vectors form a basis of $T_p\mathbb{S}^2$ and we see that $(\mathbf{v}, \mathbf{w})$ is a regular point. Since this is true for all points in the fiber of $p \in \mathbb{S}^2$, we conclude that p actually is a regular value.

To check the orientation of the basis, we calculate the determinant of

$$d\lambda_{(\mathbf{v},\mathbf{w})} = \frac{1}{|v_0 - w_0|} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

which is also the transition matrix from $(\mathbf{e}_2, \mathbf{e}_3)$ to the new basis. The determinant of this matrix is negative. Hence the basis $(d\lambda_{(\mathbf{v},\mathbf{w})}(\mathbf{a}), d\lambda_{(\mathbf{v},\mathbf{w})}(\mathbf{b}))$ of $T_p\mathbb{S}^2$ is negatively oriented.

In order to understand the effect of $d\lambda_{(\mathbf{v},\mathbf{w})}$ on the orientations, we must determine whether $(\mathbf{a}, \mathbf{b})$ is a positively or negatively oriented basis of $T_{\mathbf{v}}S_a \times T_{\mathbf{w}}S_b$:

- $(\mathbf{v},\mathbf{w})$ with $v_0 = 1$, $w_0 = \sqrt{2}-1$ and $v_1 = v_2 = w_1 = w_2 = 0$: As we checked in a previous point, the orientation of S_a is opposite to the standard orientation of the circle. Hence the basis ($\mathbf{a}$) is a negatively oriented basis of $T_{\mathbf{v}}S_a$. The basis ($\mathbf{b}$), however, is a positively oriented basis of $T_{\mathbf{w}}S_b$.[b] For, at this $\mathbf{w}$, the corresponding vectors are $q = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$ and $q_0^\perp = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, and $\mathbf{b}$ is a positive multiple of $d(\phi_N^{-1})_q(q_0^\perp) = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. Since $q_0^\perp$ has sign $+1$ and ϕ_N preserves orientations, $\mathbf{b}$ also has sign $+1$. Hence $(\mathbf{a}\times 0, 0\times\mathbf{b})$ is a negatively oriented basis of $T_{\mathbf{v}}S_a \times T_{\mathbf{w}}S_b$. Thus $d\lambda_{(\mathbf{v},\mathbf{w})}$ sends a negatively oriented basis to a negatively oriented basis and therefore preserves orientations. Hence the orientation number at $(\mathbf{v},\mathbf{w})$ is $+1$.

- $(\mathbf{v},\mathbf{w})$ with $v_0 = 1$, $w_0 = -\sqrt{2}-1$ and $v_1 = v_2 = w_1 = w_2 = 0$: The vector $\mathbf{a}$ is a negatively oriented basis of $T_{\mathbf{v}}S_a$. The basis $\mathbf{b}$, however, is now also a negatively oriented basis of $T_{\mathbf{w}}S_b$. For, at this $\mathbf{w}$, the corresponding vector are $q = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ and $q_0^\perp = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$, and $\mathbf{b}$ is a negative multiple of $d(\phi_N^{-1})_q(q_0^\perp) = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$. Since $q_0^\perp$ has sign $+1$ and ϕ_N preserves orientations, $\mathbf{b}$ has sign -1. Hence $(\mathbf{a}\times 0, 0\times\mathbf{b})$ is a positively oriented basis of $T_{\mathbf{v}}S_a \times T_{\mathbf{w}}S_b$. Thus $d\lambda_{(\mathbf{v},\mathbf{w})}$ sends a positively oriented basis to a negatively oriented basis and therefore reverses orientations. Hence the orientation number at $(\mathbf{v},\mathbf{w})$ is -1.

- $(\mathbf{v},\mathbf{w})$ with $v_0 = -1$, $w_0 = -\sqrt{2}-1$ and $v_1 = v_2 = w_1 = w_2 = 0$: Now the vector $\mathbf{a}$ is a positively oriented basis of $T_{\mathbf{v}}S_a$ and $\mathbf{b}$ is a negatively oriented basis of $T_{\mathbf{w}}S_b$. Hence $(\mathbf{a}\times 0, 0\times\mathbf{b})$ is a negatively oriented basis of $T_{\mathbf{v}}S_a \times T_{\mathbf{w}}S_b$. Thus $d\lambda_{(\mathbf{v},\mathbf{w})}$ sends a negatively oriented basis to a negatively oriented basis and therefore preserves orientations. Hence the orientation number at $(\mathbf{v},\mathbf{w})$ is $+1$.

Thus in total we get that the sum of the orientation numbers is $+1-1+1=+1$. Hence we have shown $H(\pi) = \deg(\lambda) = 1$.

[a] We make two permutations which lead to multiplying with $(-1)^2 = +1$.

[b] Imagining S_b as a deformation of the unit circle lying in the plane $y = z$ in $\mathbb{R}^3$ and the vector $\mathbf{b}$ points upwards at a point with positive x-coordinate. We checked that the orientation of S_b is opposite to the standard orientation of the circle.

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Index