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ADDED-MASS MATRIX AND ENERGY STORED IN THE "NEAR FIELD"

by

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The so-called addes mass, associated with a body which oscillates sinusoidally in water, is usually a positive quantity. However, in some cases negative values occur, as discussed in a recent paper by McIver and Evans (Ref. 1). Negative added mass occurs when the oscillating body produces a water motion where the associated potential energy is larger than the associated kinetic energy.

This follows from the formula

$$E_k - E_p = \frac{1}{4} \sum_i \sum_j m_{ij} \; u_i u_j^* \tag{1}$$

where E_k is the kinetic energy and E_p the potential energy of the oscillating water. Further m_{ij} is an element of the added mass matrix and u_i is the complex velocity amplitude of "oscillator" no. i. Since each body may oscillate in six different modes, the system contains up to 6N "oscillators" if there are N separate bodies. The summation indices i , j in eq. (1) run over all "oscillators".

The original derivation of eq. (1) is included in the Appendix.

The occurrence of negative added mass and its relation to eq. (1) have been discussed by McIver and Evans (Ref. 1)

We shall here use eq. (1) to show that the added mass is associatied with the "near field" (a term which is traditionally used in antenna theory). When oscillating bodies generate a wave it is, at large distance, i.e. in the "far field", a circular wave in the three-dimensional case, and a plane wave in the two-dimensional case. Apart from exceptional cases the simple far-field wave can not match the boundary conditions on the wetted surface of the bodies. The near-field part of the wave is defined as the difference between the exact wave and the far-field wave.

In the three-dimensional case the exact wave (assuming linear theory) is given by the following complex amplitude of the velocity potential:

$$\phi = \sum_i \varphi_i(r,\theta,z)\ u_i \tag{2}$$

where r , θ , z are cylindrical coordinates, the z axis pointing upwards. (The free surface is at $z = 0$.) The complex coefficient φ_i satisfies the Laplace equation, the radiation condition at infinity $(r\to\infty)$ and the usual boundary conditions at the free surface, at fixed surfaces and at oscillating surfaces. (See e.g. Ref. 1 or Ref. 2.)

Assuming that all bodies are located within a limited region and that the water depth h is constant at large distance, the far-field wave is given by the asymptotic expression for φ_i

$$\varphi_{i,FF} = b_i(\theta)e(kz)\ (kr)^{-1/2}e^{-ikr} \tag{3}$$

Here k is the angular repetency, which is related to the angular frequency ω and to the acceleration of gravity g through the dispersion relation

$$\omega^2 = gk\ \tanh(kh) \tag{4}$$

Further

$$e(kz) = \cosh(kz+kh)/\cosh(kh) \tag{5}$$

and the far-field coefficient $b_i(\theta)$ depends on the geometry and on the frequency. The resulting far-field wave is

$$\phi_{FF} = B(\theta)e(kz)\ (kr)^{-1/2}e^{-ikr} \tag{6}$$

where

$$B(\theta) = \sum_i b_i(\theta)\ u_i \tag{7}$$

The exact velocity potential is given by

$$\phi(r,\theta,z) = \phi = \phi_{NF} + \phi_{FF} \tag{8}$$

where ϕ_{NF} is the near-field part of the solution. Note that $r\phi_{NF}$ remains finite,

$$r|\phi_{NF}| < \infty \quad \text{as} \quad r \to \infty \tag{9}$$

(see Ref. 3 pp. 475 - 478).

The time-average value E_p of the potential energy per unit area, associated with the surface elevation, of complex amplitude

$$\eta = \eta(r,\theta) = -\frac{i\omega}{g}\ \phi(r,\theta,0), \tag{10}$$

is

$$E_p = \frac{\rho g}{4}\ |\eta|^2 \tag{11}$$

At large distance $(r \to \infty)$, where the near-field part is negligible, we have

$$E_{p,FF} = \frac{\rho g}{4}\ |\eta_{FF}|^2 = \frac{\rho\omega^2}{4gkr}\ |B(\theta)|^2 \tag{12}$$

The time-average value E_p of the kinetic energy per unit area is

$$E_k = \int_{-h}^{0} \frac{\rho}{4} \left(\left|\frac{\partial\phi}{\partial r}\right|^2 + \left|\frac{1}{r}\ \frac{\partial\phi}{\partial\theta}\right|^2 + \left|\frac{\partial\phi}{\partial z}\right|^2 \right) dz \tag{13}$$

Carrying out differentiation and integration for the dominating term we find after some manipulation,

$$E_k = \frac{\rho\omega^2}{4gkr} \left(|B(\theta)|^2 + O\{(kr)^{-1/2}\} \right) \tag{14}$$

In the final step of the derivation eq. (4) was used.

From the above results we have

$$E_k - E_p = \mathcal{O}\{(kr)^{-3/2}\} \quad \text{as} \quad kr \to \infty \tag{15}$$

and

$$\mathcal{E}_k - \mathcal{E}_p = \int_0^{2\pi} d\theta \int_0^{\infty} r\, dr (E_k - E_p) \tag{16}$$

Note that the integral exists for $r \to \infty$ because of the result (15). However, both of $\mathcal{E}_k$ and $\mathcal{E}_k$ are infinite except when no far-field wave is generated, that is, when $B(\theta) \equiv 0$.

From the above it is obvious that it is the near-field part which contributes to the integral (16). In the far-field region there is no difference between the kinetic energy and the potential energy.

While the above derivation was made for the three-dimensional case, similar results can be obtained for the two-dimensional case. The far-field wave is then given by an expression as

$$\phi_{FF} = A^{\pm}\, e(kz) e^{\mp ikx} \tag{17}$$

On the basis of the formula (1) we now attribute the added mass matrix to the difference between the kinetic energy and the potential energy of the water in the near field. According to the definition of the added mass, the right-hand side of formula (1) is expressed formally as a kinetic energy. This interpretation is, however, not physically valid, unless the potential energy is zero.

Using the equations (4), (25) and (27) in ref. 2, we have the following expression for the added mass:

$$m_{ij} = \rho \iint_{S_\infty} \varphi_i \frac{\partial \varphi_j^*}{\partial r}\, dS - \rho \iint_S \varphi_i \frac{\partial \varphi_j^*}{\partial n}\, dS \tag{18}$$

where the first integral is taken over a vertical cylinder S_∞ (r = constant) in the far-field region, while the second integral is taken over the totality S of the wetted surfaces of the oscillating bodies. Note that the first integral is independent of the radius r of the cylinder S_∞ provided r is so large that the near-field parts of φ_i and φ_j are negligible.

This result (18) also demonstrates that the added mass is attributed to the near field. It follows that the added mass vanishes in a special case when there is no near-field, that is if the far-field wave, as defined by eq. (6) or eq.(17), matches the boundary condition at the wetted surface S. (Note that, in order to obtain this result for the three-dimensional case, we have to define the far-field solution (6) by the asymptotic form of the Hankel function $H_n^{(2)}(kr)$ and not by the Hankel function itself.)

However, it may happen that the added mass is zero in spite of a non-vanishing near field. Several cases are seen e.g. in Ref. 1.

A simple interpretation of the added mass would be to consider it as the mass of a lump of water which is pushed back and forth together with the oscillating body. However, even in a case where the added mass is zero water is pushed by the oscillating body. And it is evidently not correct to interpret a negative added mass as a "subtracted mass" of water. Hence this interpretation is not physically correct. Negative added mass means that there is more potential energy than kinetic energy in the near field.

The correct interpretation of added mass is to attribute it to reactive power or to stored energy in the near field. This is analogous to the energy stored in a resonator.

References

1. P. McIver and D.V. Evans: "The occurrence of negative added mass in free-surface problems involving submerged oscillating bodies". Report no. AM-83-03, School of Mathematics, University of Bristol, UK (1983). Published in Engineering Mathematics

2. J. Falnes: "Radiation impedance matrix and optimum power absorption for interacting oscillators in surface waves". Applied Ocean Research, vol 2, pp. 75-80 (1980).

3. J.V. Wehausen and E.V. Laitone: "Surface Waves". Handbuch der Physik, Bd. IX, pp. 446-778 (1960).

Appendix

I derived the relation (1) while lecturing the graduate course "Energy from ocean waves" (NTH dr.ing.-fag nr. 71090 "Energi frå havbølgjer") and it was presented in a lecture 15th October 1981.

After having found the relation for an example I found it was more simple to derive it for the general case. The following two pages is a copy of my original derivation. This is my "unpublished note" which is mentioned in Ref. 1. The next four pages are copied from my notes to the lecture 15th October 1981. In the notes some equations are labeled (Jn) where n is the same as equation number in Ref. 2.

3/10-81

Addert-masse-matrisa og lagra energi i nærfeltet.

(128)

$$m_{ij} = -\frac{\rho}{2}\iint_S \frac{\partial}{\partial n}(\varphi_i^* \varphi_j)\, dS = -\frac{\rho}{2}\iint_S \frac{\partial}{\partial n}(\varphi_i \varphi_j^*)\, dS$$

Resiprositetssatsen

$$\hat{\phi} = \sum_i \varphi_i \hat{u}_i$$

$$\hat{\eta} = -\frac{i\omega}{g}\sum_i \varphi_i|_{z=0}\, \hat{u}_i \qquad \hat{\vec{v}} = \nabla\hat{\phi} = \sum_i \nabla\varphi_i\, \hat{u}_i$$

Potensiell energi pr. flate: $E_p = \frac{\rho g}{4}|\hat{\eta}|^2 = \frac{\rho g}{4}\frac{\omega^2}{g^2}\sum_{ij}(\varphi_i \varphi_j^*)_{z=0}\, \hat{u}_i \hat{u}_j^*$

Kinetisk energi pr flate

$$E_k = \frac{\rho}{2}\,\frac{1}{2}\int_{-h(x,y)}^{0} \hat{\vec{v}}\cdot\hat{\vec{v}}^*\, dz = \frac{\rho}{4}\sum_{ij}\hat{u}_i \hat{u}_j^* \int_{-h(x,y)}^{0} (\nabla\varphi_i)\cdot(\nabla\varphi_j^*)\, dz$$

Potensiell energi $\mathcal{E}_p = \iint_{S_\ell} E\, dS \quad \frac{\rho\omega^2}{4g}\sum_i\sum_j \hat{u}_i \hat{u}_j^* \iint_{S_\ell} \varphi_i \varphi_j^*\, dS$

S_ℓ fri overflate (grenseflate mot luft)

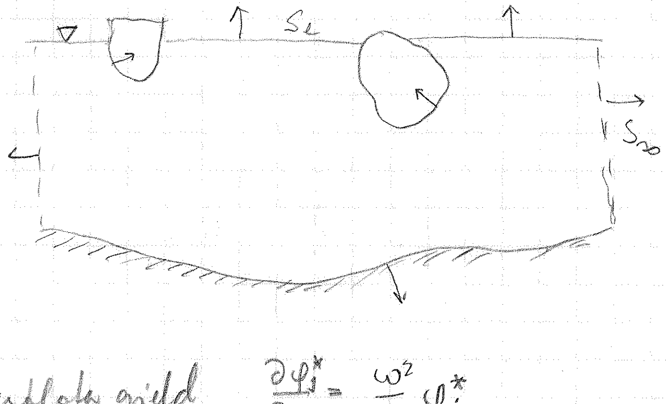
Kinetisk energi $\mathcal{E}_k = \iint_{S_\ell} E_k\, dS = \frac{\rho}{4}\sum_i\sum_j \hat{u}_i \hat{u}_j \iiint_\tau (\nabla\varphi_i)\cdot(\nabla\varphi_j^*)\, d\tau$

$$\underbrace{\nabla\cdot(\varphi_i \nabla\varphi_j^*)} = (\nabla\varphi_i)\cdot(\nabla\varphi_j^*) + \varphi_i \underbrace{\nabla^2\varphi_j^*}_{=0}$$

Generalisert Gauss' setning

$$\mathcal{E}_k = \frac{\rho}{4}\sum_i\sum_j \hat{u}_i \hat{u}_j \oiint \vec{n}\,\varphi_i\cdot\nabla\varphi_j^*\, dS$$

($\vec{n}$ retta ut av væskevolumet)



På sjøbotnen er integranden null

$\frac{\partial\varphi_j^*}{\partial n} = 0$ På den fri overflata gjeld $\frac{\partial\varphi_j^*}{\partial z} = \frac{\omega^2}{g}\varphi_j^*$

3/10-81

Elles har me (J27): $R_{ij} = i\omega\rho \iint_{S_\infty} \varphi_j \frac{\partial \varphi_i^*}{\partial n} dS$ med innoverwendt normalvektor

$$R_{ji} = R_{ij}^* = -i\omega\rho \iint_{S_\infty} \varphi_i \frac{\partial \varphi_j^*}{\partial n} dS$$ med utoverwendt normalvektor

På dei svingande kroppane vil me snu normalvektorane til å peika ~~ut~~ frå kropp og inn i væske ($\vec{n} \to -\vec{n}$ på S)

Difor har me:

$$\mathcal{E}_k = \frac{\rho}{4} \sum_i \sum_j \hat{u}_i \hat{u}_j^* \Big\{ \underbrace{-\iint_S \varphi_i \frac{\partial \varphi_j^*}{\partial n} dS}_{\text{Etter (J25)} \;=\; \frac{Z_{ij}}{i\omega\rho} = \frac{R_{ij}}{i\omega\rho} + \frac{i\omega m_{ij}}{i\omega\rho}} + \frac{\omega^2}{g} \iint_{S_f} \varphi_i \varphi_j^* dS - \frac{R_{ij}}{i\omega\rho} \Big\}$$

$$\mathcal{E}_k = \frac{1}{4} \sum_i \sum_j \hat{u}_i \hat{u}_j^* m_{ij} + \underbrace{\frac{\rho\omega^2}{4g} \sum_i \sum_j \hat{u}_i \hat{u}_j^* \iint_{S_f} \varphi_i \varphi_j^* dS}_{\mathcal{E}_p}$$

$$\mathcal{E}_k - \mathcal{E}_p = \sum_i \sum_j \frac{1}{4} m_{ij} \hat{u}_i \hat{u}_j^*$$

Addert-masse-matrise m_{ij}

Står ikkje i kompendiet

$$Z_{ij} = -i\omega\rho \iint_S \varphi_j \frac{\partial \varphi_i^*}{\partial n} dS$$

$$m_{ij} = \frac{X_{ij}}{\omega} = \frac{1}{\omega} \mathrm{Im}\, Z_{ij} = \frac{1}{2i\omega}\left(Z_{ij} - Z_{ij}^*\right) = \frac{1}{2i\omega}\left(Z_{ij} - Z_{ji}^*\right)$$

$$Z_{ij}^* = Z_{ji}^* = i\omega\rho \iint_S \varphi_i^* \frac{\partial \varphi_j}{\partial n} dS$$

$$m_{ij} = -\frac{\rho}{2} \iint_S \left(\varphi_j \frac{\partial \varphi_i^*}{\partial n} + \varphi_i^* \frac{\partial \varphi_j}{\partial n}\right) dS$$

$$\boxed{m_{ij} = -\frac{\rho}{2} \iint_S \frac{\partial}{\partial n}\left(\varphi_i^* \varphi_j\right) dS = -\frac{\rho}{2} \iint_S \frac{\partial}{\partial n}\left(\varphi_i \varphi_j^*\right) dS} \quad \text{(J28)}$$

Me kan assosiera den adderte massen med ein kinetisk energi som er lagra i nærfeltet.

Potensiell energi pr. flateeining – jfr. (B84) –

$$E_p = \frac{\rho g}{4} |\hat{\eta}|^2 = \frac{\rho g}{4} \frac{\omega^2}{g^2} \left|\hat{\Phi}|_{z=0}\right|^2 = \frac{\rho \omega^2}{4g} \left|\sum_i (\varphi_i)_{z=0} \hat{u}_i\right|^2$$

$$= \frac{\rho \omega^2}{4g} \sum_i \sum_j (\varphi_i \varphi_j^*)_{z=0} \hat{u}_i \hat{u}_j^*$$

Potensiell energi i heile bølgja

$$\mathcal{E}_p = \iint_{S_0} E_p \, dS = \frac{\rho\omega^2}{4g} \sum_i \sum_j \hat{u}_i \hat{u}_j^* \iint_{S_0} \varphi_i \varphi_j^* dS$$

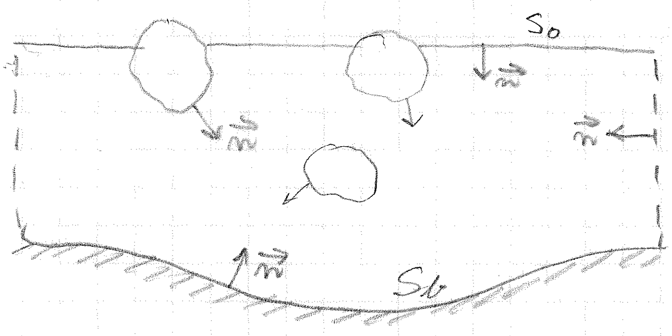
Integralet går over den fri overflata mot luft (planet $z=0$, med unntak av den svingande kroppens bryt overflata)

14/10-81 DJ26

Kinetisk energi:

$$\mathcal{E}_k = \iiint_\tau \frac{1}{2}\rho \overline{\vec{v}(t)\cdot\vec{v}(t)}\, d\tau = \frac{1}{4}\rho \iiint_\tau \hat{\vec{v}}\cdot\hat{\vec{v}}^*\, d\tau$$

der integralet går over vassvolumet τ avgrensa av den lukka flata



$S_0 \cup S \cup S_b \cup S_\infty$

$$\hat{\vec{v}} = \nabla\hat{\Phi} = \sum_i \hat{u}_i \nabla\varphi_i$$

$$\mathcal{E}_k = \frac{\rho}{4}\sum_i\sum_j \hat{u}_i\hat{u}_j^* \iiint_\tau \nabla\varphi_i\cdot\nabla\varphi_j^*\, d\tau$$

Reglane for derivering av produkt gir

$$\nabla(\varphi_i\cdot\nabla\varphi_j^*) = (\nabla\varphi_i)\cdot(\nabla\varphi_j^*) + \varphi_i\nabla^2\varphi_j^*$$

Då φ_i og φ_j stettar Laplace-likninga, blir siste leddet null. Altså

$$\mathcal{E}_k = \frac{\rho}{4}\sum_i\sum_j \hat{u}_i\hat{u}_j^* \iiint_\tau \nabla(\varphi_i\cdot\nabla\varphi_j^*)\, d\tau$$

Ved hjelp av den generaliserte Gauss-setninga kan dette volumintegralet omformast til eit integral over den lukka flata $S_0 \cup S \cup S_b \cup S_\infty$

$$\mathcal{E}_k = \frac{\rho}{4}\sum_i\sum_j \hat{u}_i\hat{u}_j^* \oiint -\vec{n}\,\varphi_i\cdot\nabla\varphi_j^*\, dS$$

$$= -\frac{\rho}{4}\sum_i\sum_j \hat{u}_i\hat{u}_j^* \oiint \varphi_i\frac{\partial\varphi_j^*}{\partial n}\, dS$$

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På S_b er $\frac{\partial \varphi_j^*}{\partial n} = 0$

På S_0 er $-\frac{\partial \varphi_j^*}{\partial n} = \frac{\partial \varphi_j^*}{\partial z} = \frac{\omega^2}{g}\varphi_j^*$

Elles har me (J27)

$$R_{ij} = R_{ji} = -i\omega\rho \iint_{S_\infty} \varphi_i \frac{\partial \varphi_j^*}{\partial n} dS = i\omega\rho \iint_{S_\infty} \varphi_i \frac{\partial \varphi_j^*}{\partial n} dS$$

og (J25)

$$Z_{ij} = Z_{ji} = -i\omega\rho \iint_S \varphi_i \frac{\partial \varphi_j^*}{\partial n} dS$$

Difor har me

$$\mathcal{E}_k = \frac{\rho}{4}\sum_i\sum_j \hat{u}_i \hat{u}_j^* \left\{ \frac{\omega^2}{g} \iint_{S_0} \varphi_i \varphi_j^* dS + \frac{Z_{ij}}{i\omega\rho} - \frac{R_{ij}}{i\omega\rho} \right\}$$

$$\mathcal{E}_k = \mathcal{E}_p + \frac{\rho}{4}\sum_i\sum_j \hat{u}_i \hat{u}_j^* \frac{R_{ij} + i\omega m_{ij} - R_{ij}}{i\omega\rho}$$

$$\mathcal{E}_k - \mathcal{E}_p = \frac{1}{2}\sum_i\sum_j m_{ij} \frac{1}{2} \hat{u}_i \hat{u}_j^*$$

For ei plan bølgje gjeld (B85) $E = E_k + E_p = 2E_k = 2E_p = \frac{\rho g}{2}|\hat{\eta}|^2$
Det er tilsvarande for ei sirkulær bølgje i fjernfeltet. Men i nærfeltet er det skilnad på kinetisk energi og potensiell energi. Tilleggsleddet i kinetisk energi kan assosierast med den adderte massen
(Hugs $\overline{u^2(t)} = \frac{1}{2}\hat{u}\hat{u}^*$)

Jfr. og (J25)

$$Z_{ij} = Z_{ji} = -i\omega\rho \iint_S \varphi_i \frac{\partial \varphi_j^*}{\partial n} dS$$

og (J27)

$$R_{ij} = R_{ji} = -i\omega\rho \iint_{S_\infty} \varphi_i \frac{\partial \varphi_j^*}{\partial r} dS$$

Dette integralet over S_∞ kan takast overalt i fjernfeltet der skilnaden

$$\varphi_{i,j} - b_{i,j}(\theta) \frac{e(kz)}{\sqrt{kr}} e^{-ikr}$$

dvs. nærfeltet, er neglisjerbart. Me har og følgjande uttrykk for den adderte massen:

$$m = \frac{Z_{ij} - R_{ij}}{i\omega} = \rho \left\{ \iint_{S_\infty} \varphi_i \frac{\partial \varphi_j^*}{\partial r} dS - \iint_S \varphi_i \frac{\partial \varphi_j^*}{\partial n} dS \right\}$$