

OPTIMUM OPERATION
OF
WAVE POWER CONVERTER
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ABSTRACT

An improved version of the resonance-tuned point absorber¹⁻² of ocean wave power is discussed. This absorber utilises a machine to control its movement in an optimum way relative to the time evolution of the incident wave. As a result, the absorber becomes theoretically as efficient in non-harmonic waves as in harmonic waves. An expression for the maximum obtainable absorbed energy is derived both for small waves and for large waves. In the latter case the physical limitations of the absorber make restraints on the movement. The general relationship between the incident wave and the power absorbed by the system is studied. With small harmonic waves it is derived that the optimum absorbed power equals the power transmitted in a wave crest of width $\lambda/2\pi$, where λ is the wavelength. Further, on the basis of wave data⁹ from the North Atlantic (59°N, 19°W) numerical values are obtained for the optimum energy absorbed in an average year by systems of various physical dimensions. As an example, a heaving, cylindrical tank of diameter 16 m and maximum oscillation amplitude 3 m absorbs 8 GWh/year. Finally, a discussion is given of the problem of controlling the vertical movement of the tank in the optimum way.

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1. INTRODUCTION

In previous papers¹⁻² we have pointed to the possibility of converting wave power into useful form by means of resonant absorbers. Resonant absorbers work most efficiently in harmonic or nearly harmonic waves. If the power in the wave is shared between several frequencies, the absorber efficiency is reduced, particularly so if the frequency spectrum of the wave is wide. The reason being, that the resonator can only be tuned to one frequency at a time.

The ideal absorber is one which, so to speak, is tuned to all frequencies simultaneously. Such a system can in fact be achieved as has been pointed out in a previous report³. The present paper is a theoretical analysis of this type of absorber.

The general idea can be explained as follows: Consider a wave power converter as illustrated in Fig. 1. The system consists of a partly submerged tank that can be forced to make arbitrary heaving motions in the sea. The movement of the tank, of course, produces an outgoing, ring-shaped wave which is superimposed on the incoming wind-generated wave. Obviously, if the ring-shaped wave interferes destructively with the incoming wave, power is removed from the sea. This power is then absorbed by the system. Hence, the point is to force the system to generate a wave of such an amplitude and phase relative to the incoming wave that an optimum destructive interference results.

It is quite obvious that the generated wave must have roughly the same time variation but of opposite phase as the incoming wave. A linear, resonant absorber can never generate a wave which has the same time variation as the wave which excites its motion, except in one case, namely when the incoming wave is harmonic. This is the reason for the shortcoming of a resonant absorber. On the other hand, the system shown in Fig. 1 can be given an arbitrary motion and can, therefore, generate an arbitrary wave. Consequently it is more efficient than the tuned resonator.

In paragraph 2 a general expression for the power absorbed by a body with arbitrary heaving motions in an arbitrary sea is derived. Analytical expressions for the optimum power and the corresponding optimum motion of the body are presented for small and large waves. By "small" we infer that the finite height of the absorber does not introduce any restraints. Some numerical results are also given.

In paragraph 3, the means by which the optimum motion can be established, are discussed.

2. OPTIMUM POWER ABSORPTION BY CONTROLLED MOVEMENT OF A HEAVING TANK.

The tank shown in Fig.2 is moving under the influence of the force $F_w(t)$ due to the surface elevation around the tank and the restoring force $F_s(t)$ due to the movement of the tank. In addition, it is presumed that the tank is subject to a force $F(t)$ which can give the tank an arbitrary vertical movement. In Fig.1 the force $F(t)$ is established by the combined hydraulic pump and motor. The value of $F(t)$ is positive when it is pointing upwards and negative when it is pointing downwards. The force $F(t)$ may feed energy into the system or may extract energy from the system.

The equation of motion for the tank, when friction forces are neglected, is

$$m \ddot{\zeta}(t) = F_w(t) - F_s(t) + F(t) \quad (1)$$

Here m is the mass of the tank and $\zeta(t)$ is the displacement of the tank. The restoring force is

$$F_s(t) = A g \rho \zeta(t) \quad (2)$$

where A is the cross sectional area of the tank at the water line, ρ is the density of the water and g is the acceleration of gravity.

The time average value of the power absorbed by means of the hydraulic machine is

$$P = - \overline{F(t) \cdot \dot{\zeta}(t)} = - \frac{1}{\tau} \int_0^{\tau} F(t) \cdot \dot{\zeta}(t) \cdot dt \quad (3)$$

Here the averaging time τ must be much larger than the characteristic period of the wave, which is typically 5 to 15 seconds.

Combining Eqs. (1), (2) and (3) we obtain

$$P = - m \overline{\ddot{\zeta}(t) \cdot \dot{\zeta}(t)} - A\rho g \overline{\zeta(t) \cdot \dot{\zeta}(t)} + \overline{F_w(t) \cdot \dot{\zeta}(t)} \quad (4)$$

Here, the first two right-hand terms represent average rate of change of kinetic energy and potential energy, respectively. Both of these terms vanish, as is shown in the following:

$$\begin{aligned} \overline{\ddot{\zeta}(t) \cdot \dot{\zeta}(t)} &= \frac{1}{\tau} \int_0^{\tau} \ddot{\zeta}(t) \cdot \dot{\zeta}(t) \cdot dt = \frac{1}{\tau} \frac{1}{2} \int_0^{\tau} \frac{d}{dt} [\dot{\zeta}(t)]^2 dt \\ &= \frac{1}{2} \frac{1}{\tau} [\dot{\zeta}(t)]^2 \Big|_0^{\tau} \approx 0 \end{aligned} \quad (5)$$

The expression is approximately zero because τ can be made arbitrarily large whereas $\zeta(t)$ is an oscillating function of limited amplitude. Similarly, we find

$$\overline{\zeta(t) \cdot \dot{\zeta}(t)} = \frac{1}{2} \frac{1}{T} [\zeta(t)]^2 \Big|_0^T \approx 0 \quad (6)$$

The net power absorbed by the tank can thus be expressed by the simple equation

$$P = \overline{F_w(t) \cdot \dot{\zeta}(t)} \quad (7)$$

The force $F_w(t)$ due to the total wave at the location of the tank can be separated into two terms

$$F_w(t) = F_e(t) + F_r(t) \quad (8)$$

The term $F_e(t)$ is the excitation force established by the incident wave $\eta_i(t)$. The second term $F_r(t)$ is the reaction force on the tank due to the wave $\eta_r(t)$ generated by its own movement. Consequently, Eq. (7) may be written

$$P = \overline{F_e(t) \cdot \dot{\zeta}(t)} + \overline{F_r(t) \cdot \dot{\zeta}(t)} \quad (9)$$

The physical interpretation of the two terms is as follows:

$$P_e = \overline{F_e(t) \cdot \dot{\zeta}(t)} \quad (10)$$

is the total absorbed power from the incident wave, whereas

$$P_r = - \overline{F_r(t) \cdot \dot{\zeta}(t)} \quad (11)$$

is the power radiated back to sea by the outgoing, ring-shaped wave generated by the movement of the tank.

It should be noted that P_e may equally well be written:

$$P_e = - \overline{\hat{F}_e(t) \cdot \zeta(t)} \quad (12)$$

This is easily shown by partial integration.

When the radius and the depth of the tank are much smaller than the wavelength of the incident wave, an approximate expression for the excitation force is

$$F_e(t) = A \rho g \eta_i(t) \quad (13)$$

This formula was used in previous works¹⁻³. In the general case of an arbitrary wave working on an arbitrary body, the heaving excitation force $F_e(t)$ can be found as follows: The function $F_e(t)$ is decomposed into harmonic components

$$F_e(t) = \sum_n F_{en}(t) = \sum_n F_n \sin(\omega_n t + \alpha_n) \quad (14)$$

where α_n is a phase angle. The amplitudes of the harmonic components can, within the limit of linear theory, be written⁴

$$F_n = \left| \omega \rho \iint dS \left(\phi_i \frac{\partial \phi_r}{\partial n} - \phi_r \frac{\partial \phi_i}{\partial n} / u \right) \right| \quad (15)$$

Here ϕ_i and ϕ_r are frequency dependent, complex amplitudes of the velocity potential of the incident and the radiated

wave, respectively, and u is the complex amplitude of the velocity of the heaving body. The integral (15) is taken over any closed surface which encloses the body, but only that part of the surface which is in the water. The outwards pointing, normal component of the gradient is denoted by $\partial/\partial n$ in Eq. (15). Notice that Eq. (15) includes the effect of diffraction of the incident wave on the body, although the diffracted wave is not explicitly expressed in the equation.

For mathematical convenience we shall assume that the angular frequencies ω_n of the spectrum of the wind-generated wave are multiples of some fundamental frequency. This corresponds to the assumption that the wave reproduces itself after a certain time, for instance, 20 minutes. Then, the elevation due to the incoming wave can be written as

$$\eta_i(t) = \sum_n \eta_{in}(t) = \sum_n \eta_n \sin(\omega_n t + \phi_n) \quad (16)$$

at the location of the oscillating body. By means of Eq. (15) it can then be shown (see Appendix A) that

$$F_n = \left\{ \frac{2\rho g^3 R(\omega_n)}{\omega_n^3} \right\}^{\frac{1}{2}} \cdot \eta_n \quad (17)$$

Here $R(\omega)$ is the radiation resistance of the body.* The displacement function $\zeta(t)$ of the oscillating body may, of course, be chosen quite independently of the incident wave $\eta_i(t)$. However, any reasonable choice of $\zeta(t)$ has obviously the same period as $\eta_i(t)$. Hence, we can write

$$\zeta(t) = \sum_n \zeta_n(t) = \sum_n s_n \sin(\omega_n t + \beta_n) \quad (18)$$

* An expression for $R(\omega)$ will be given later.

The reaction force $F_r(t)$ is related to the movement of the body through the relation

$$F_r(t) = - \sum_n [R(\omega_n) \cdot \dot{\zeta}_n(t) + m_r(\omega_n) \cdot \ddot{\zeta}_n(t)] \quad (19)$$

Here $m_r(\omega)$ is the so-called added mass.

Introducing Eqs. (14), (18) and (19) into Eq. (9) gives

$$P = \sum_n [\overline{F_{en}(t) \cdot \dot{\zeta}_n(t)} - \overline{R(\omega_n) \cdot \dot{\zeta}_n^2(t)}] \quad (20)$$

Notice that the last term in Eq. (19) does not contribute to P because of Eq.(5)

The present problem is to find that particular displacement function $\zeta(t)$ which makes the absorbed power P , as given by Eq.(20) a maximum.

2.1. Small Incident Waves

Since the functions $F_{en}(t)$ and $\dot{\zeta}_n(t)$ in Eq.(20) are independent it is easily seen that maximum power is absorbed when

$$\dot{\zeta}_n(t) = \frac{F_{en}(t)}{2R(\omega_n)} \quad (21)$$

Consequently the optimum velocity is

$$\dot{\zeta}_s(t) = \sum_n \frac{F_{en}(t)}{2R(\omega_n)} \quad (22)$$

and the optimum absorbed power is found to be

$$P_s = \sum_n \frac{\overline{F_{en}^2(t)}}{4R(\omega_n)} \quad (23)$$

According to Eqs. (14), (16) and (17) we then obtain

$$P_s = \frac{1}{2} \rho g^3 \sum_n \frac{1}{\omega_n^3} \cdot \overline{\eta_{in}^2(t)} \quad (24)$$

If the incident wave is harmonic, i.e.

$$\eta_i(t) = \eta_0 \sin \omega t \quad (25)$$

Eq. (24) reads

$$P_s = \frac{\rho g^3 \eta_0^2}{4\omega^3}$$

It is noticed that this can be written

$$P_s = \frac{\lambda}{2\pi} \cdot K \quad (27)$$

where $\lambda = 2\pi g/\omega^2$ is the wavelength and

$$K = \frac{\rho g^2 \eta_0^2}{4 \omega} \quad (28)$$

is the power per unit crest length of the incident wave. Eq. (27) then shows that the tank absorbs an amount of power equal to that passing a crest length of $\lambda/2\pi$. This quantity can be defined as the absorption length d_a of the tank,¹

$$d_a = \frac{P_s}{K} = \frac{\lambda}{2\pi} \quad (29)$$

It is a remarkable fact that the absorption length is a function of λ only. In fact, this result is valid for any circularly symmetric optimized heaving body on deep water. The explanation is that d_a has a purely geometrical origin and gives the optimum limit to which extent a circularly symmetric outgoing wave may interfere destructively with a plane incident wave.

Looking at Eq. (24) it is realized from the discussion above, that the controlled system absorbs every single component of the incident wave optimally. This is in contrast to the resonance-tuned system which only absorbs optimally the frequency to which it is tuned. The more the frequency deviates from the resonance frequency, the less efficient is the absorption. If the frequency spectrum of the incident wave is wide, the difference in power absorption, by the two systems, may be substantial.

We shall now extend the analysis to the general case when the incident wave $\eta_1(t)$ is non-periodic. We define the power spectrum $E(\omega)$ of the incident wave by the relation

$$E(\omega) = \frac{\Delta\omega}{\sum \overline{\eta_{in}^2(t)}} / \Delta\omega \quad (30)$$

Here the sum includes all the terms $\overline{\eta_{in}^2(t)}$ within the frequency interval between ω and $\omega + \Delta\omega$. Transforming the series (24) into an integral yields for the optimum power

$$P_s = \frac{1}{2} \rho g^3 \int_0^{\infty} \frac{1}{\omega^3} E(\omega) \cdot d\omega \quad (31)$$

We now define an average angular frequency ω_s by the equation

$$\int_0^{\infty} \frac{E(\omega) d\omega}{\omega^3} = \frac{1}{\omega_s^3} \int_0^{\infty} E(\omega) d\omega \quad (32)$$

Observing from Eq. (30) that

$$\int_0^{\infty} E(\omega) \cdot d\omega = \overline{\eta_i^2(t)} \quad (33)$$

we then finally obtain

$$P_s = \frac{\rho g^3}{2\omega_s^3} \cdot \overline{\eta_i^2(t)} \quad (34)$$

This is the same amount of power that would be absorbed in a harmonic wave with angular frequency ω_s and variance $\overline{\eta_i^2(t)}$.

A condition for the above results - Eqs. (21) to (34) - to be valid, is that the displacement $\zeta_s(t)$ of the tank

does not exceed $\pm \ell$, where ℓ is the depth of the tank (see Fig.2). The displacement function $\zeta_s(t)$ derived from Eq. (22) is, of course, irregular since all the $F_{en}(t)$ are directly related to the incident wave $\eta_i(t)$.

If $\eta_i(t)$ is of such small amplitude that the mean square displacement

$$\overline{\zeta_s^2(t)} = \frac{1}{2} \rho g^3 \sum_n \frac{\overline{\eta_{in}^2(t)}}{\omega_n^5 \cdot R(\omega_n)} \quad (35)$$

or

$$\overline{\zeta_s^2(t)} = \frac{1}{2} \rho g^3 \int_0^{\infty} \frac{E(\omega) \cdot d\omega}{\omega^5 R(\omega)} \quad (36)$$

is substantially less than $\frac{1}{2}\ell^2$, the equations above are essentially correct. In the next paragraph we shall discuss the case when this criterium cannot be fulfilled.

It is seen from Eq. (24) or (34) that P_s is independent of the dimensions of the tank. However, this does not mean that the shape and size of the tank is unimportant. The radiation resistance $R(\omega)$ is a function that increases with the radius a of the tank (when a is small $R \propto a^4$). If a is reduced, Eq. (35) or (36) then shows that the average displacement of the tank is increased. This means that the height of the tank must be increased in order to absorb the same amount of power. The conclusion is that for a given incident wave, the volume of the tank must be above a certain limit in order to absorb the optimum power P_s .

We can define the angular frequency ω_L by the equation

$$\int_0^{\infty} \frac{E(\omega) \cdot d\omega}{\omega^5 R(\omega)} = \frac{1}{\omega_L^2 R(\omega_L)} \int_0^{\infty} \frac{E(\omega) \cdot d\omega}{\omega^3} \quad (37)$$

or, taking into account Eqs.(32) and (33)

$$\int_0^{\infty} \frac{E(\omega) d\omega}{\omega^5 R(\omega)} = \frac{1}{\omega_L^2 \cdot \omega_S^3 R(\omega_L)} \overline{\eta_i^2(t)} \quad (38)$$

We can then write

$$\overline{\zeta_s^2(t)} = \frac{1}{2} \frac{\rho g^3}{R(\omega_L) \cdot \omega_L^2 \cdot \omega_S^3} \overline{\eta_i^2(t)} \quad (39)$$

The condition for Eq. (34) to be an essentially correct expression for the optimum absorbed power is that

$$\overline{\zeta_s^2(t)} \ll \frac{1}{2} \ell^2 \quad (40)$$

or, expressed in terms of the incident wave

$$\overline{\eta_i^2(t)} \ll \frac{R(\omega_L) \omega_L^2 \cdot \omega_S^3 \ell^2}{\rho g^3} \quad (41)$$

Notice that for harmonic incident waves Eq.(34) gives exactly the optimum power as long as

$$\overline{\zeta_s^2(t)} \leq \frac{1}{2} \ell^2 \quad (42)$$

In the case of irregular incident waves the true optimum absorbed power will deviate from P_s when $\overline{\zeta_s^2(t)}$ approaches $\frac{1}{2}\ell^2$. The reason is that $\zeta_s(t)$ at times will overshoot $\pm\ell$, whereas the true optimum displacement function $\zeta_{opt}(t)$ is bound by the restriction

$$|\zeta_{\text{opt}}(t)| \leq \ell \quad (43)$$

However, as long as Eq. (42) is fulfilled the deviation between $\zeta_{\text{opt}}(t)$ and $\zeta_s(t)$ is so modest that Eq.(34) for the optimum power may be used with fair accuracy for the whole range.

2.2. Large Incident Waves

Firstly, harmonic incident waves will be considered. In this case the true displacement function $\zeta_{\text{opt}}(t)$ is significantly different from $\zeta_s(t)$ when

$$\overline{\zeta_s^2(t)} > \frac{1}{2}\ell^2 \quad (44)$$

In order to analyse this case we have to go back to the general expression (20). According to Eqs. (14) and (18)

$$F_e(t) = F_{e1}(t) = F_1 \sin(\omega t + \alpha) \quad (45)$$

and

$$\zeta(t) = \sum_{n=1}^{\infty} s_n \sin(n\omega t + \beta_n) \quad (46)$$

It is readily found that the maximum absorbed power is

$$P = \frac{1}{2} (F_1 \omega s_1 - \sum_{n=1}^{\infty} R(n\omega) n^2 \omega^2 s_n^2) \quad (47)$$

It is observed that only the first harmonic component

$$\zeta_1(t) = -s_1 \cos(\omega t + \alpha) \quad (48)$$

in $\zeta(t)$ contributes to the first term in P . If we choose $\zeta(t) = \zeta_1(t)$, then according to Eq.(43) the largest value s_1 can have is l . On the other hand, s_1 may be larger than l if higher harmonics are included in $\zeta(t)$. This is illustrated in Fig.3. Consequently, the first term (P_e) in Eq.(47) can be raised by appropriately adding higher harmonics to $\zeta(t)$. However, the radiated power P_r (the sum in Eq.(47)) is also raised. As a result, the amplitudes (s_n) of the higher harmonics at optimum power absorption are generally small. Thus, by introducing higher harmonics in the displacement function $\zeta(t)$, the additional absorbed power is small.

In the extreme case, when the incident wave is so large (F_1 very large) that $P_e \gg P_r$ even if higher harmonics are included in P_r , we may, according to Eq.(12) write

$$P \approx P_e = - \overline{\dot{F}_e(t) \cdot \zeta(t)} \quad (49)$$

Taking into account Eq.(43) the maximum of this expression is evidently obtained when $\zeta(t)$ is a square-wave function with $\zeta(t) = -l$ when $\dot{F}_e(t)$ is positive and $\zeta(t) = l$ when $F(t)$ is negative. The maximum absorbed power is

$$P_1 \approx \frac{2}{\pi} F_1 \omega l \quad (50)$$

On the other hand, taking $\zeta(t)$ to be a sinusoidal function the maximum power is

$$P \approx \frac{1}{2} F_1 \omega \ell \quad (51)$$

The ratio is

$$\frac{P_1}{P} = \frac{4}{\pi} = 1.27 \quad (52)$$

Hence, by adding higher harmonics to the displacement function $\zeta(t)$ the absorbed power is increased by an amount which is certainly less than 27 per cent.

The square wave function leading to Eq. (50) is, of course, unrealistic since it implies an extremely large (infinite) velocity and acceleration of the oscillating body. Looking at Fig.1 it is seen that the flow of oil through the combined hydraulic motor and pump is proportional to the velocity of the tank. A large velocity, therefore, requires a hydraulic machine with large capacity. The high cost of such a machine would, however, not be justifiable due to the small power increase.

From these considerations it is evident that with large incident sinusoidal waves, the optimum displacement function ζ_{opt} is given approximately by

$$\zeta_{opt} \approx \zeta_L(t) = - \ell \cos(\omega t + \alpha) \quad (53)$$

Accordingly we find from Eq.(47) for the optimum absorbed power

$$P_{opt} \approx P_L = \frac{1}{2} (F_1 \omega \ell - R(\omega) \omega^2 \ell^2) \quad (54)$$

It is seen from Eqs. (53) and (22) that

$$\zeta_L(t) = C \zeta_S(t) \quad (55)$$

where C is a constant given by

$$C = \left\{ \frac{\frac{1}{2} \ell^2}{\zeta_S^2(t)} \right\}^{\frac{1}{2}} \quad (56)$$

and where ζ_S is the optimum displacement function in the case no maximum limitation is set on the displacement, velocity and acceleration of the oscillator. It follows that the displacement function ζ_L we have chosen is proportional to ζ_S but with reduced amplitude such as to fulfil the inequality (43).

Now, turning to the case of large irregular incident waves, we argue that Eqs. (55) and (56) may be generalised. We know that the true optimum displacement function ζ_{opt} is more square shaped than $\zeta_L(t)$ (see Fig.3) and the more so the larger the waves. Moreover, unlike ζ_L (notice that $\overline{\zeta_L^2(t)} = \frac{1}{2} \ell^2$) ζ_{opt} does not overshoot $\pm \ell$. However, following the line of reasoning as above and at the end of § 2.1. it is evident that ζ_{opt} deviates only modestly from $\zeta_L(t)$. Hence, by means of Eqs. (55), (22) and (20), the optimum absorbed power is found to be

$$P_{opt} \approx P_L = (2C - C^2) P_S, \quad (57)$$

where P_S is given by Eq. (24) and

$$C = \ell \left\{ \rho g^3 \sum_{n=1}^{\infty} \frac{\overline{\eta_{in}^2(t)}}{\omega_n^5 R(\omega_n)} \right\}^{-\frac{1}{2}} \quad (58)$$

When the wave spectrum is continuous P_S is given by Eq.(34) and

$$C = \ell \left\{ \rho g^3 \int_0^{\infty} \frac{E(\omega) \cdot d\omega}{\omega^5 R(\omega)} \right\}^{-\frac{1}{2}} = \left\{ \frac{R(\omega_L) \cdot \omega_S^3}{\rho g^3 \overline{\eta_i^2(t)}} \right\}^{\frac{1}{2}} \omega_L \ell \quad (59)$$

It may be convenient to express P_L in an alternative manner:

$$P_L = \frac{1}{2} (F_L \omega_L \ell - R(\omega_L) \omega_L^2 \ell^2) \quad (60)$$

where

$$F_L = \left\{ 4 \rho g^3 R(\omega_L) \overline{\eta_i^2(t)} / \omega_S^3 \right\}^{\frac{1}{2}} \quad (61)$$

Eq. (60) has the same form as Eq. (54) and applies when $C \leq 1$. It is seen that F_L increases linearly with the root mean squared value of the surface elevation. In comparison, for small waves, the optimum power is proportional to $\overline{\eta^2(t)}$, as is evident from Eq. (34).

It is noted that $(2C-C^2)$ has its maximum value for $C=1$ in which case Eq.(57) is identical to Eq.(34). If we, therefore, define a quantity γ by

$$\left. \begin{aligned} \gamma &= 1 && \text{when } C > 1 \\ \gamma &= (2C-C^2) && \text{when } C < 1 \end{aligned} \right\} \quad (62)$$

where C is given by Eq. (59) we may write the expression for the absorbed power in the form

$$P = \gamma \frac{1}{2} \frac{\rho g^3}{\omega_s^3} \overline{\eta^2(t)} \quad (63)$$

This equation is quite general and applies to large and small, sinusoidal as well as irregular incident waves.

Formula (63) gives the exact expression for the optimum power for small waves. For large waves the formula (63) slightly underestimates the optimally obtainable power.

2.3. Numerical Results

Presented below are the numerical results pertaining to the power absorbed by a semi-submerged cylindrical tank, with hemispherical bottom, placed in irregular waves.

For the power spectrum of the incident waves, the so-called JONSWAP spectrum has been used.^{5,*} This may be written in the form:

$$E(\omega) = E(\omega_m x) = K \cdot x^{-5} \exp\left[-\frac{5}{4} x^{-4}\right] \cdot \nu S \quad (64)$$

where

ν = peakedness parameter

$S = \exp[-(x-1)^2/2\sigma^2]$

$x = \omega/\omega_m$

ω_m = peak angular frequency

$\sigma = \begin{cases} 0.07 & \text{for } x < 1 \\ 0.09 & \text{for } x > 1 \end{cases}$

* For general review of wave spectra see ref. 6.

The constant K is defined by the condition

$$\int E(\omega) \cdot d\omega = \overline{\eta^2(t)} = \frac{H_S^2}{16} \quad (65)$$

where H_S is the significant wave height. H_S is a quantity frequently presented in wave data.

Another characteristic wave parameter is the so-called zero-upcrossing period T_Z . The angular frequency $\omega_Z = 2\pi/T_Z$ is related to ω_m by the formula⁷

$$\omega_Z^2 \cdot \int_0^{\infty} E(\omega) \cdot d\omega = \int_0^{\infty} \omega^2 E(\omega) d\omega \quad (66)$$

or

$$\omega_m = q \cdot \omega_Z \quad (67)$$

where

$$q = \left\{ \frac{\int_0^{\infty} E(x) \cdot dx}{\int_0^{\infty} x^2 E(x) dx} \right\}^{1/2} \quad (68)$$

The factor q can be found from Eq.(64) when v is specified. The quantity P_S in Eq.(31) can be expressed in terms of H_S and ω_Z as follows

$$P_S = \frac{1}{2} \rho g^3 \int_0^{\omega_0} \frac{E(\omega) \cdot d\omega}{\omega^3} = \frac{1}{32} \rho g^3 \frac{H_S^2}{(\omega q \omega_Z)^3} \quad (69)$$

where

$$w = \left\{ \int_0^{\infty} E(x) \cdot dx / \int_0^{x_0} \frac{E(x) dx}{x^3} \right\}^{1/3} \quad (70)$$

The limit of integration ω_0 or x_0 will be commented on below. Notice that $w q \cdot \omega_z = \omega_s$ (see Eq. (32)).

The radiation resistance $R(\omega)$ of the tank is ^{8, 3}.

$$R(\omega) \approx \frac{2\pi}{3} \rho a^3 \omega \varepsilon e^{-2\ell\omega^2/g} \quad (71)$$

Here a is the radius of the tank (and of the hemisphere). Further ε is a dimensionless function of $ka = a\omega^2/g$ as shown in Fig. 4. Knowing $R(\omega)$ and $E(\omega)$, the constant C entering Eq. (63) can be calculated as follows

$$C = \ell / \left\{ \rho g^3 \int_0^{\omega_0} \frac{E(\omega) \cdot d\omega}{\omega^5 R(\omega)} \right\}^{1/2} = s \cdot \frac{\ell \cdot (q\omega_z)^5}{H_S} \quad (72)$$

where

$$s = 4(\rho g^3)^{-1/2} \cdot \int_0^{\infty} E(x) \cdot dx / \int_0^{x_0} \frac{E(x) \cdot dx}{x^5 R(xq\omega_z)} \quad (73)$$

The limit of integration x_0 in Eqs. (70) and (73) is chosen so that the absorbed power (Eq. (63)) is a maximum. The reason that s and therefore, also P becomes small when x_0 is very large is because $R(\omega)$ decreases rapidly with ω due to the exponential factor $\exp(-2\ell\omega^2/g)$ in Eq. (71). Physically this means that the tank must oscillate heavily in order to generate a high frequency wave, i.e. to absorb the power from the high frequency components of the incident wave. Due to the

limited oscillation amplitude $\pm l$ of the tank, the absorption of wave power at the most energetic part (peak region) of the wave spectrum then becomes correspondingly small. The obvious remedy is to allow the displacement function to contain only the most energetic frequency components of the incident wave $\eta_i(t)$, say the interval from $\omega = 0$ to $\omega = 2\omega_m$ or $x = 0$ to $x = x_0 = 2$. Since the absorbed power P is a slowly varying function of x_0 the value of x_0 is not very critical. Of course, there may also be technical reasons for avoiding rapid movements of the tank.

On the basis of the above formulae the power absorbed by an optimally moved tank in irregular waves is calculated. Input data are radius a and depth l of the tank and the wave parameters v , T_z and H_s . Knowing the distribution of T_z and H_s throughout the year, the average energy absorption per year by the tank can be calculated. Results with wave data from the weather ship station India (59°N , 19°W) are shown in Fig. 5 for two different values of the spectral peakedness parameter v . It is noticed that results are not very sensitive to the shape of the wave spectrum.

3. OPTIMUM OPERATION OF THE TANK

In the previous paragraph it was stated that the optimum velocity function of the tank is

$$\dot{\zeta}_{\text{opt}}(t) = \dot{\zeta}_s(t) = \sum \frac{F_{\text{en}}(t)}{n \cdot 2R(\omega_n)} \quad (74)$$

in small waves and that

$$\zeta_{\text{opt}}(t) \approx \zeta_L(t) = C \zeta_s(t) \quad (75)$$

in large waves, where C is a proper constant. If the power spectrum of the incident wave is sufficiently narrow, we may put $R(\omega) \approx R(\omega_m) = R$. Then Eq.(74) shows that

$$\dot{\zeta}_s(t) \approx \frac{1}{2R} F_e(t) , \quad (76)$$

i.e. the function $\dot{\zeta}_s(t)$ is approximately proportional to the excitation force $F_e(t)$ due to the incident wave. This is a reasonable result, as can be seen from the general expression (9) for the absorbed power

$$P = \overline{F_e(t) \cdot \dot{\zeta}(t)} + \overline{F_r(t) \cdot \dot{\zeta}(t)}$$

which reduces to

$$P \approx \overline{F_e(t) \cdot \dot{\zeta}(t)} - R \overline{\dot{\zeta}^2(t)} , \quad (77)$$

when $R \approx$ constant. It is noticed that the first, dominating term is always positive when $\dot{\zeta}(t) \propto F_e(t)$, i.e. the waves are always feeding power into the system. Hence, it is very important that $\dot{\zeta}(t)$ has the same sign as $F_e(t)$. Combining Eqs. (75) and (76) we state that

$$\dot{\zeta}_{opt}(t) \approx \dot{\zeta}_L(t) = b F_e(t) \quad (78)$$

where b is a constant to be optimised with regard to the useful power output in every particular state of the sea.

The constant b is slowly changing as the state of the sea changes, say from one hour to the other. In large waves $\zeta_L(t)$ may exceed the maximum displacement $\pm l$ of the tank. Then $\zeta_{opt}(t)$ must be modified with respect to $\zeta_L(t)$ as indicated in Fig. 6.

The constant b is equal to $C/2R$ according to the equations above. It should certainly never exceed $1/2R$. However, in large waves where C is substantially less than l , it is probable that optimisation with regard to power output gives $b > C/2R$. The reason is, as pointed out in §§ 2.1. and 2.2, that the true displacement function is somewhat more square-like than the $\zeta_L(t)$ function. It should be mentioned, however, that the efficiency of the hydraulic and electric machinery may influence the optimum value of b significantly. This is particularly so if the efficiency of the machinery is low. The reason is that the machinery will have a larger average load than the average useful power delivered by the system. The ratio between power load and useful power is a function of the ratio between the amplitude of the tank and the amplitude of the wave. Of course, if the machinery has a low efficiency, then optimum power output requires that this ratio is relatively small, i.e. b is small.

In order to establish the displacement function $\zeta_{opt}(t)$ it is, according to Eq. (78), necessary to know the excitation force $F_e(t)$, due to the incident wave. According to Eq. (8)

$$F_e(t) = F_w(t) - F_r(t), \quad (79)$$

where $F_w(t)$ is the heaving force due to the total wave and $F_r(t)$ is the reaction force due to the generated wave. The force $F_w(t)$ can be measured and $F_r(t)$ can be computed when $R(\omega)$ and $m_r(\omega)$ are known.

We shall first discuss how $F_w(t)$ can be measured. If a pressure transducer is placed in a fixed position directly under the tank (for instance on the wire) then

$$F_w(t) \propto p(t) \quad (80)$$

where $p(t)$ is the measured pressure. The elevation of the surface of the water is not evenly distributed around the circumference of the tank except when the radius of the tank is very small in comparison to the wavelength. However, a pressure transducer placed on the axis of the tank will be roughly proportional to the average pressure on the bottom of the tank and is, therefore, approximately proportional to the total wave force on the system. It is important that the transducer is placed as close to the bottom of the tank as possible, i.e. at a depth slightly larger than $2\lambda + a$. The reason is the exponential decay of pressure with depth. However, this effect will influence only the high frequency (and less important) part of the frequency spectrum of $F_e(t)$.

Another possibility for measuring $F_w(t)$ is to place pressure transducers in the bottom of the tank. The pressure on these transducers is also influenced by the position of the tank. Therefore F_w must be calculated according to the formula

$$F_w(t) = A p(t) + A \rho g \zeta(t) \quad (81)$$

Here the instantaneous position $\zeta(t)$ of the tank must be measured together with the average pressure $p(t)$.

The force $F_r(t)$ may be found as follows: From Eqs. (18) and (19) it is observed that

$$F_r(t) \approx -R \cdot \dot{\zeta}(t) - m_r \cdot \ddot{\zeta}(t) \quad (82)$$

when R and m_r can be considered constants. Hence, knowing the constants R and m_r and measuring the displacement $\zeta(t)$, $F_r(t)$ can be determined. The constant $R \approx R(\omega_m)$ where $R(\omega)$ is given by Eq. (71). The added mass of the tank is given by

$$m_r(\omega) \approx \frac{2\pi}{3} \rho a^3 \mu \quad (83)$$

where μ is a dimensionless function of ω shown in Fig. 4. The constant m_r in Eq. (82) is approximately $m_r \approx m_r(\omega_m) = m_r(q\omega_z)$.

Having determined $F_w(t)$ and $F_r(t)$ according to the equations above, the excitation force $F_e(t)$ may then be calculated from Eq. (79). Eq. (78) then gives the displacement function $\zeta_L(t)$. The actual displacement function $\zeta_{opt}(t)$ is modified with regard to $\zeta_L(t)$ only when $|\zeta_L(t)| > \ell$, in which case $\zeta_{opt}(t)$ is smoothly levelled off to the values $+\ell$ or $-\ell$. Cf. Fig. 6.

APPENDIX A

Excitation force on heaving body due to waves

We assume that the partly submerged body has a vertical axis of circular symmetry and that the heaving motion is the only degree of freedom for the body. If the width of the body is much smaller than one wavelength it is a good approximation to neglect the contribution from the diffracted wave to the dynamic force which acts on the body. In order to judge the validity of this approximation we shall now derive an exact expression for the dynamic force. In this derivation we shall assume that the waves and the oscillation are given by sinusoidal functions of time, with an angular frequency ω . Then, it is convenient to represent the dynamic variables by their complex amplitudes. The velocity potential of the incoming plane wave, assumed to propagate in the the positive x direction, is represented by

$$\phi_i = i \frac{g}{\omega} \eta_0 e^{kz} e^{-ikx} , \quad (\text{A.1})$$

where η_0 is the complex amplitude of the surface elevation. It is assumed that $|\eta_0|$ is so small that non-linear terms are negligible. The z axis is vertical and collinear with the symmetry axis of the heaving body. On deep water the dispersion relationship is

$$k = \omega^2/g \quad (\text{A.2})$$

where g is the acceleration of gravity.

The heaving body generates an outgoing (radiated) wave which contributes a reaction force with complex amplitude

$$F_r = -Z_r u = -(R+i\omega m_r)u \quad (\text{A.3})$$

on the body. Here u represents the velocity of the heaving

body. Further, m_r is the hydrodynamic added mass, R is the radiation resistance and Z_r is the radiation impedance, which is a complex quantity depending on ω and on the shape and magnitude of the heaving body.

The total force acting on the body is

$$F = F_e + F_r, \quad (\text{A.4})$$

where

$$F_e = F_i + F_d \quad (\text{A.5})$$

Here F_i and F_d are contributions due to the incident wave and the diffracted wave, respectively.

The "excitation force" is given by the formula

$$F_e = i\omega\rho \iint dS \left(\phi_i \frac{\partial \phi_r}{\partial n} - \phi_r \frac{\partial \phi_i}{\partial n} \right) / u \quad (\text{A.6})$$

which is due to Haskind.⁴ Here ϕ_i represents the velocity potential of the incident wave in accordance with (A.1). Further, ϕ_r represents the velocity potential of an outgoing wave which is generated by the heaving body in otherwise calm water, when the velocity of the body is given by u . The integral is taken over any closed surface which encloses the body, but only that part of the surface which is in the water, $z \leq 0$. The unit normal is pointing outwards.

In this way it is possible to calculate F_e without having explicit knowledge on the diffracted wave.

Firstly, we shall integrate over the surface of the rotationally symmetric body, for which the radius at depth z is given by the function $a(z)$. Using the plane wave expression (A.1) for ϕ_i and the boundary condition that

$(\partial\phi_r/\partial n) = \dot{u}_n$, where u_n is the normal component of u , we find from (A.6)

$$F_e = -i\omega\rho \int_{-H}^0 dz \int_0^{2\pi} d\theta \frac{da}{dz} a(z) \left\{ 1 - \frac{k\phi_r}{u} \left(1 + i \frac{dz}{da} \cos \theta \right) \right\} \phi_i \quad (\text{A.7})$$

Here H is the height of the submerged part of the body.

Secondly, we take the integral in (A.6) over a cylindrical surface, which is far outside the body and which has an arbitrarily large radius r . Then we have

$$F_e = i\omega\rho \int_0^{2\pi} d\theta \int_{-\infty}^0 dz \left(\phi_l \frac{\partial\phi_r}{\partial r} - \phi_r \frac{\partial\phi_l}{\partial r} \right) r/u \quad (\text{A.8})$$

Of course, F_e is independent of r . Since the integration surface and the heaving body have a common axis of rotational symmetry, we have

$$\phi_r \rightarrow u C r^{-\frac{1}{2}} e^{kz} e^{-ikr} \quad \text{as } r \rightarrow \infty \quad (\text{A.9})$$

where C is a complex parameter depending on ω and on the shape and magnitude of the heaving body. If $r \gg \lambda = 2\pi/k$, the ring-shaped outgoing wave (A.9) is very similar to a plane wave when considered within a spaceous region having a dimension of a few wavelengths. Noting that $x = r \cos \theta$, insertion of (A.1) and (A.9) into (A.8) gives after integrating over z ,

$$F_e = \lim_{r \rightarrow \infty} \frac{i}{2} \rho g \eta_0 C r^{\frac{1}{2}} e^{-ikr} \int_0^{2\pi} (1 + \cos \theta) e^{-ikr \cos \theta} d\theta \quad (\text{A.10})$$

Using the identities

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \theta} d\theta \quad (\text{A.11})$$

and

$$J_1(x) = -J_0'(x) = \frac{-i}{2\pi} \int_0^{2\pi} \cos \theta e^{ix \cos \theta} d\theta \quad (\text{A.12})$$

we obtain

$$F_e = i\pi\rho g\eta_0 C \lim_{r \rightarrow \infty} e^{-ikr} \left\{ J_0(-kr) - iJ_1(-kr) \right\} \quad (\text{A.13})$$

Noting that J_0 is an even function and J_1 an odd function and using the asymptotic expansion for the Bessel functions, we finally obtain

$$F_e = e^{i\pi/4} \sqrt{2\pi/k} \rho g C \eta_0 \quad (\text{A.14})$$

In this expression for F_e , C is still an unspecified complex constant. An expression for $|C|$ is easily obtained as follows. Consider the case when the heaving body generates waves on otherwise calm water. The time-average power delivered by the generator is

$$P_r = \frac{1}{2} \text{Re}(F_r u^*) = \frac{1}{2} R |u|^2, \quad (\text{A.15})$$

where (A.3) has been used. On the other hand, the radiated power crossing the vertical cylinder of large radius is

$$P_r = 2\pi r \int_{-\infty}^0 \frac{1}{2} \operatorname{Re}(p v_r^*) dz = \pi r \int_{-\infty}^0 \operatorname{Re}(-i\omega\rho\phi \frac{\partial\phi^*}{\partial r}) dz \quad (\text{A.16})$$

Inserting for ϕ_r from (A.9) and integrating over z , we find

$$P_r = \frac{\pi}{2} \omega\rho |Cu|^2 \quad (\text{A.17})$$

Conservation of energy requires that P_r is independent of r in the present model of an ideal fluid. Hence, from (A.15) and (A.17) we get

$$|C| = \left\{ \frac{R}{\pi\omega\rho} \right\}^{\frac{1}{2}} \quad (\text{A.18})$$

Insertion into (A.14) gives

$$|F_e| = \left\{ \frac{2\rho R}{k\omega} \right\}^{\frac{1}{2}} g |\eta_0| \quad (\text{A.19})$$

Here we have found the amplitude but not the phase of the excitation force.

Both the amplitude and the phase can be found when the radius of the heaving body is small in comparison with the wavelength, $\max a(z) \ll 1/k$. Then we can set $e^{-ikx} \approx 1$ when inserting for ϕ_i from (A.1) into (A.7). This gives

$$F_e \approx \rho g \eta_0 \int_{-H}^0 2\pi a(z) \frac{da}{dz} \left(1 - \frac{k\phi_r}{u}\right) e^{kz} dz \quad (\text{A.20})$$

We note that the function ϕ_r and its derivative are singular at $r = 0$. Therefore, when $ka \ll 1$, we have

$$F_e \approx \rho g A_{\text{eff}} \eta_0 \quad (\text{A.21})$$

where

$$A_{\text{eff}} = \int_{-H}^0 2\pi a(z) \frac{da}{dz} e^{kz} dz = \int_{-H}^0 e^{kz} \frac{d}{dz} (\pi a^2) dz. \quad (\text{A.22})$$

The same result is obtained by integrating over all surface elements the vertical component of the force due to the hydrodynamic pressure of the incoming wave. Hence, the diffraction has a negligible effect on the excitation force F_e when the radius of the body is much smaller than one wavelength.

Further, if the body is shallow, $kH \ll 1$, then

$$A_{\text{eff}} \approx \pi \{a(0)\}^2 = A \quad (\text{A.23})$$

where A is the cross section of the body at the water surface $z = 0$. In this case, the excitation force is

$$F_e \approx \rho g A \eta_0 \quad (\text{A.24})$$

An identical result is obtained by simple buoyancy considerations.

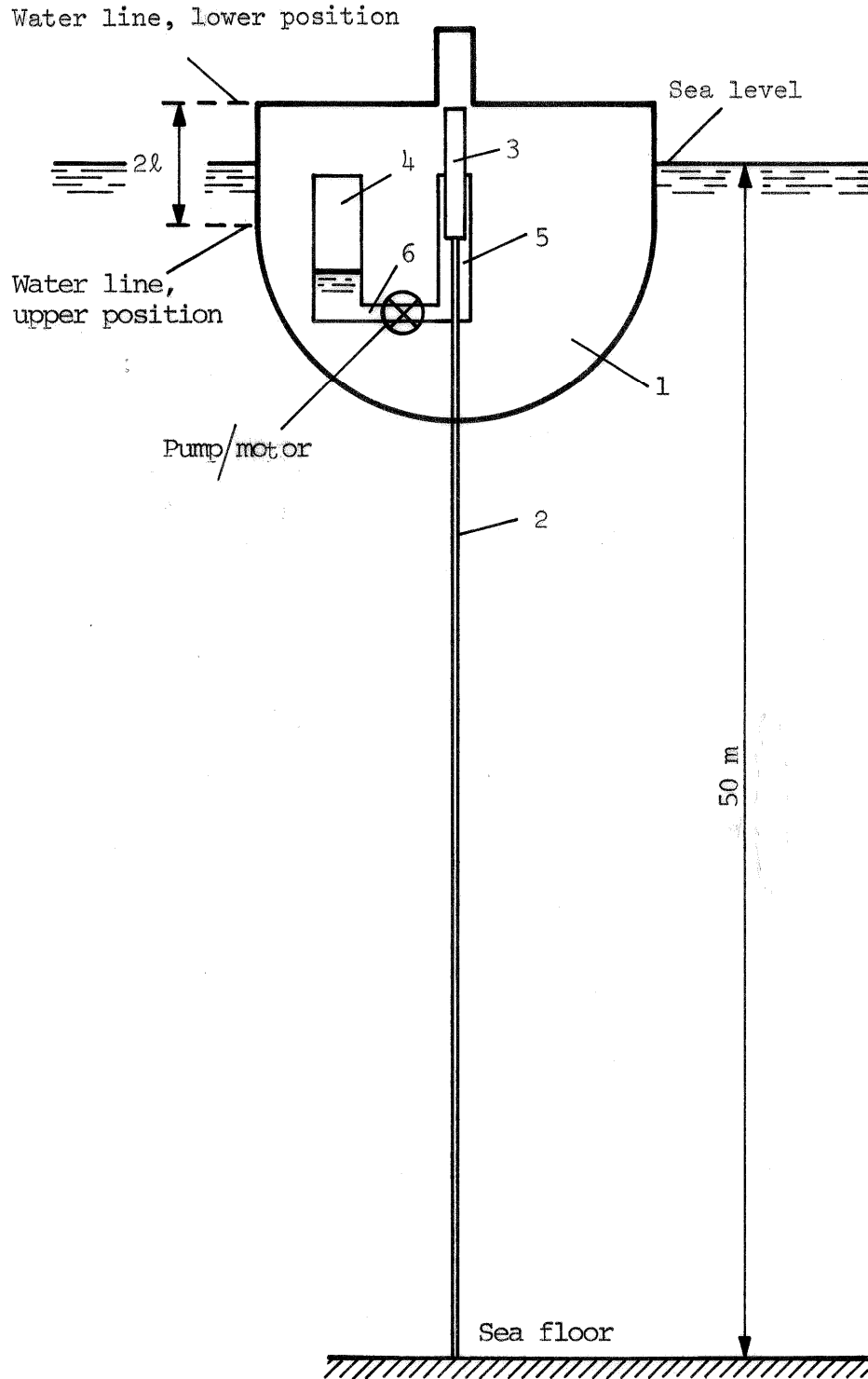


Fig. 1. Example of an efficient wave power absorber. The essentially weightless tank (1) is kept in a partly submerged position by means of the pull in the wire (2) connected to the sea floor. The pull in the wire is established by means of the hydraulic piston (3). The equilibrium pressure on the piston is obtained by means of the compressed gas in the tank (4). In the conduit (6) between the tank (4) and the cylinder (5), a combined hydraulic pump and motor is placed, which can regulate the pressure in the cylinder and henceforth control the vertical position of the tank.

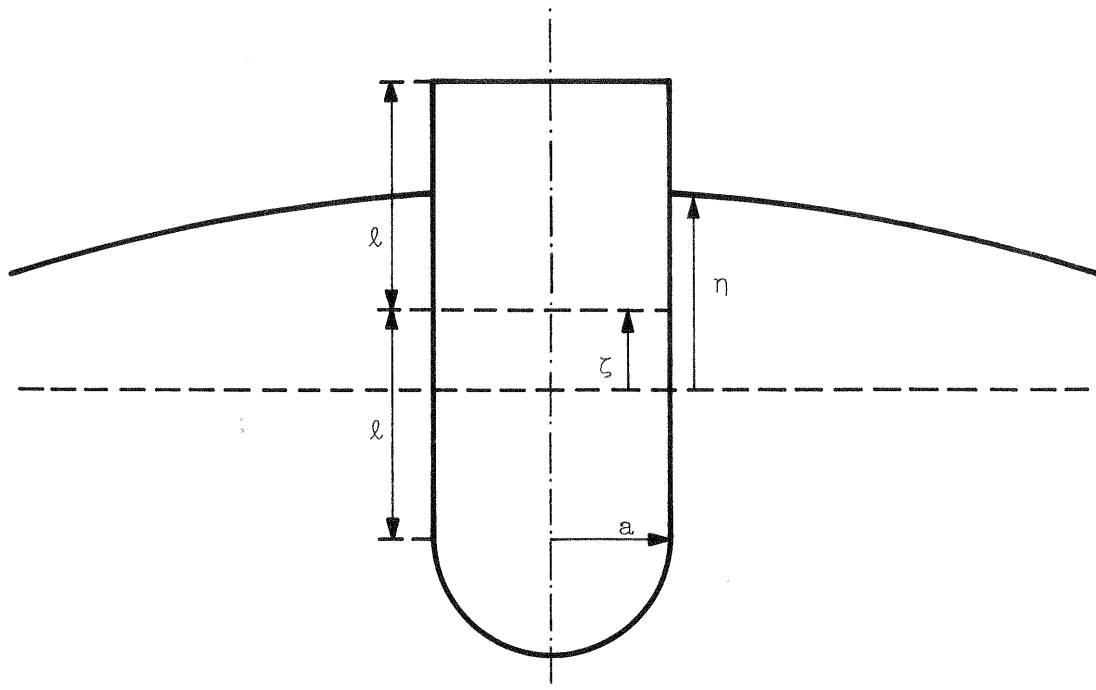


Fig. 2. A tank performing heaving motions in the sea. The undisturbed sea level is indicated by the dashed line, η is the surface elevation of the sea and ζ is the displacement of the water line of the tank.

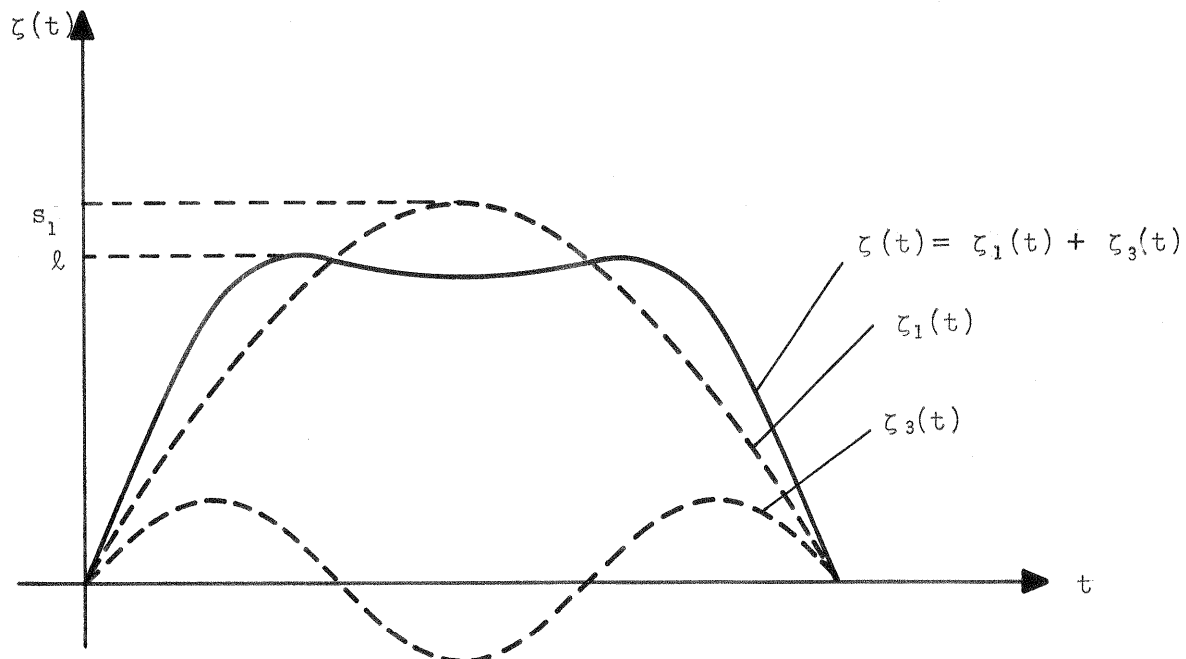


Fig. 3. Increasing power absorption by introducing higher harmonics in the displacement function $\zeta(t)$. With harmonic incident wave, the fundamental component $\zeta_1(t)$ can be increased beyond the maximum amplitude l by appropriately introducing the third harmonic $\zeta_3(t)$.

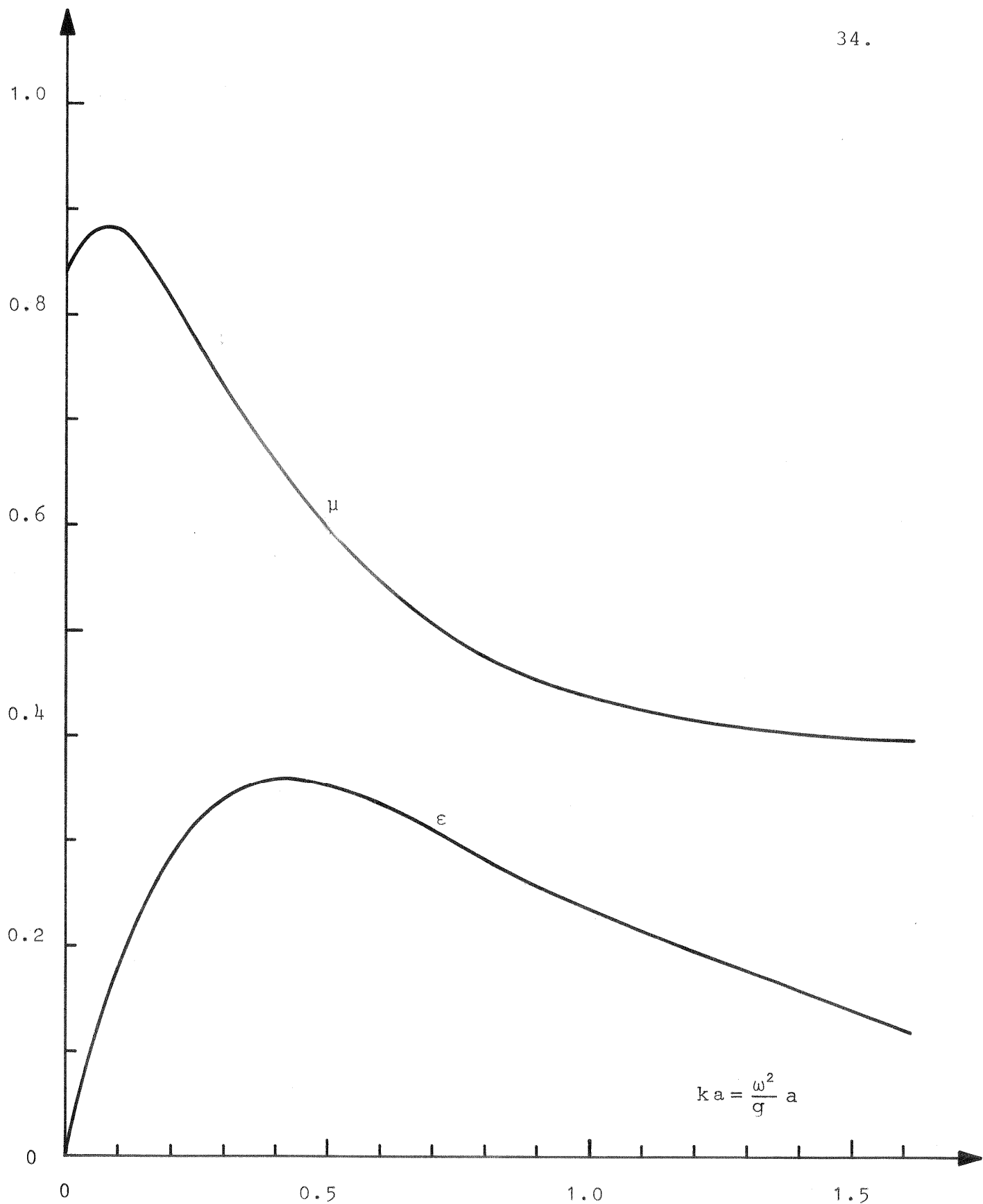


Fig. 4. The dimensionless functions ϵ and μ (taken from Ref. 8) give the radiation resistance R and the added mass m_r for a cylindrical tank with hemispherical bottom in accordance with Eqs. (71) and (83) respectively.

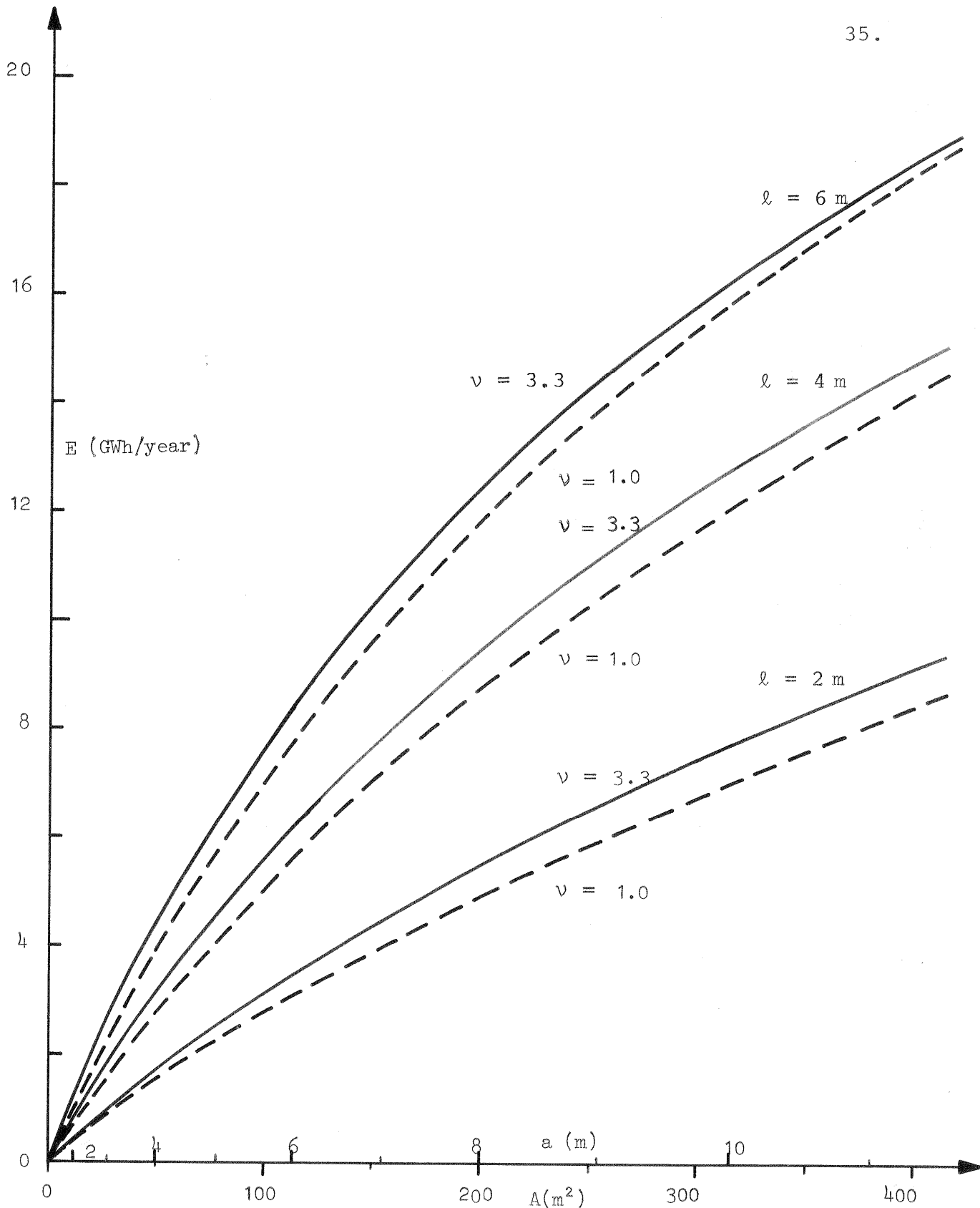


Fig. 5. Average annual absorption of energy by a cylindrical tank of radius a and depth ℓ (cf. Fig. 2.) from North Atlantic waves, based on observations of wave heights and wave periods at station India ($59^{\circ}\text{N}, 19^{\circ}\text{W}$). The absorbed energy is plotted versus the cross sectional area $A = \pi a^2$ for different values of the maximum oscillation amplitude ℓ . The peakedness parameter of the power spectrum is $\nu = 3.3$ (mean JONSWAP spectrum) for the fully drawn curve and $\nu = 1.0$ (Pierson-Moskowitz-Kitaigorodskii spectrum) for the dotted curve.

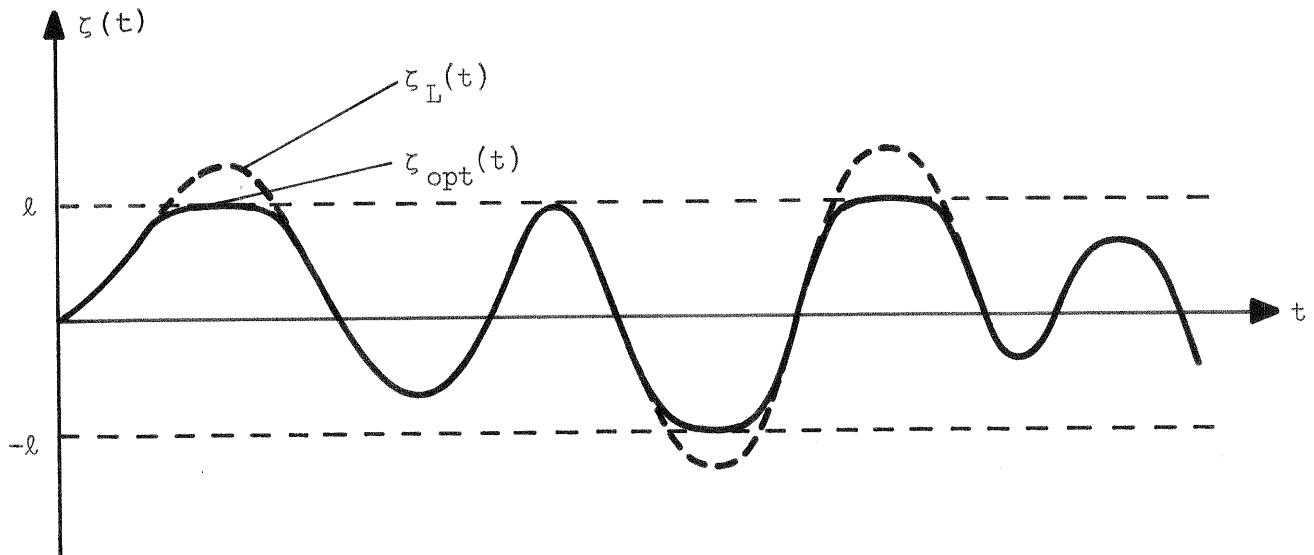


Fig. 6. The fully drawn curve shows the approximate optimum displacement function $\zeta_{\text{opt}}(t)$ for the tank in irregular incident waves.

The dotted curve $\zeta_L(t)$ is obtained from the relation $\zeta_L(t) = b F_e(t)$, where $F_e(t)$ is the excitation force on the tank due to the incident wave and where b is an optimised constant.

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