PROPOSALS FOR CONVERSION OF THE ENERGY IN OCEAN WAVES

by

K. Budal and J. Falnes

Institutt for eksperimentalfysikk
Fysikkseksjonen
Norges tekniske høgskole
Universitetet i Trondheim
Norway

1974

Internal Report, March 1974.
Reissued September 1974, March 1976,
June 1978.

PROPOSALS FOR CONVERSION OF THE ENERGY IN OCEAN WAVES

by

K. Budal and J. Falnes
Institutt for eksperimentalfysikk
Norges tekniske høgskole
Trondheim, Norway

ABSTRACT

Considerable amounts of energy are contained in the ocean waves. Available data indicate an average power transport of roughly 60 kW per meter of crest length for the North Atlantic. A general idea for conversion of ocean wave energy into useful forms is presented. A particular system consisting of a semisubmerged heaving tank is analysed in some detail. A numerical example shows that a cylindrical tank with diameter 16 m and depth 10 m, placed in waves comparable to those in the North Atlantic, may absorb an energy of roughly 10^7 kWh per year from the waves. It is believed that most of this energy can be extracted in terms of useful electric energy. Ideas for realisation of wave power stations based upon heaving tanks, oscillating water columns, and submerged systems are presented. Some of the proposed systems may be useful for protection of harbours against wave destruction (breakwater function).

CONTENTS Page

1. INTRODUCTION

The main aim of this report is to point to the potential energy sources of ocean waves, and present ideas for converting this energy into useful form.

Ocean waves contain a considerable amount of energy. In a regular (sinusoidal) progressive wave on deep water the power transmitted per unit length of the wave crest is

$$K = \frac{1}{4\pi} \rho g^2 T \overline{\eta^2(t)} \tag{1}$$

(Cf. Appendix A, eq. (A.15)). Here $\rho = 1.0 \cdot 10^3$ kg/m^3 is the density of water, $g = 9.8$ m/s^2 is the acceleration of gravity, T is the period of the wave and $\overline{\eta^2(t)}$ is the mean squared value of the surface elevation. For a sinusoidal wave $\overline{\eta^2(t)} = \frac{1}{2} \eta_0{}^2$, where η_0 is the wave amplitude. Consider as an example a regular wave with $\eta_0 = 1$ m and $T = 10$ s. Then $K \approx 40$ kW/m. In large ocean areas the energy transport in the waves averaged over all wave heights and all seasons are of this order of magnitude. This is illustrated in Fig. 1 which presents data from the North Atlantic. The left-hand vertical scale gives $\overline{\eta^2(t)}$ and the horizontal scale gives the number of hours of the year in which $\overline{\eta^2(t)}$ exceeds the indicated value. When calculating $\overline{\eta^2(t)}$ we have used the relation[3]

$$\overline{\eta^2(t)} = \frac{H_S{}^2}{16} \quad , \tag{2}$$

where H_S is a frequently observed quantity called the significant wave height. On the right-hand vertical scale of Fig. 1 is plotted the value of K as given by eq.(1) when T = 10 s. Actually this formula applies for regular waves only. However, when we put T equal to the average wave period of the sea, which is approximately 10 s for the North Atlantic, the formula gives roughly correct values for the transmitted wave power. It is seen that K exceeds 40 kW/m for more than half of the year in the North Atlantic. The average value is approximately 60 kW/m.

When the waves encounter the coast, the wave energy is partly transformed into thermal energy and partly reflected back to the sea. In principle the energy input on the shores could be transformed into electric energy. Assume that the average wave power input on the Norwegian West Coast is the same as in the North Atlantic. Then, if all this wave power were converted without loss into electric energy, a coast length of less than 150 km would produce an energy amount equal to the present electric power consumption in Norway ($7 \cdot 10^{10}$ kWh in 1972). This illustrates that the ocean waves represent a considerable energy resource. Although it is not sufficient for the whole global energy demand it seems worthwhile to study means to extract the energy content in ocean waves. This note presents some ideas to that end.

Although the emphasis in this note is placed in energy production, another interesting perspective should be mentioned. In harbour engineering a main objective is often to destroy wave energy. This is accomplished by a breakwater. The breakwater partly reflects and partly dissipates the wave power. Now, consider a system that efficiently converts the wave power into other forms, for instance electric power. This system would obviously calm the incoming waves and thus have the effect as a breakwater. In addition it might produce useful energy.

Section 2 of this note presents the general idea for absorbtion of wave power. In order to illustrate the idea a particular system is analysed in some detail in subsection 2.1. The analysis establishes on theoretical grounds the general features that must be fulfilled in order that the system shall absorb wave power efficiently. Theoretical estimates of the maximum power that can be absorbed in both swells and wind- generated waves are also given. Subsection 2.1.3 contains several proposals for practical realisation of the idea. Somewhat different systems are presented in subsections 2.2 to 2.4. These proposals are merely ideas. Obviously, a technological and economic study of these or similar systems is necessary before deciding on a real construction.

2. RESONANT ABSORBTION OF WAVE ENERGY

It is well-known that electromagnetic wave energy of harmonic waves is most efficiently absorbed by a resonator, that is, a circuit or a cavity that is tuned to the frequency of the wave. This principle of resonance absorbtion also applies in acoustics where an oscillating system may interact with a harmonic sound wave[4].

Although ocean waves are not harmonic, their frequency spectrum falls within a relatively narrow frequency band. It is particularly narrow for swells. Thus, it is possible to apply the principle of resonance absorbtion also on ocean wave energy. In the following several possible methods for realisation of this idea are discussed. They all consist of a resonator or an oscillating system that interacts with the ocean waves. This system is mechanically or electrically damped in such a way that useful mechanical energy or electric energy, respectively, is extracted from the system.

A resonator is characterised by a resonant frequency

$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{T_0} \tag{3}$$

and a quality factor

$$Q = \frac{\omega_0 W}{P'} \tag{4}$$

which is large in comparison with unity. Here W is the stored energy in the resonator and P' the average power delivered by the resonator. If the resonator is oscillating at its resonant frequency, the stored energy is completely transformed from potential energy to kinetic energy and back, twice every oscillation period T_0. During the time T_0 a fraction $2\pi/Q$ of this energy is delivered from the energy store and converted to other forms of energy. In the resonators considered here this energy is converted partly into wave energy and partly into electric energy. Inevitably, some of the energy is also dissipated. It is an aim of design to make this lost energy a relatively small fraction of the converted energy.

The incoming wave produces an oscillating force on the resonator. This force excites the resonator to oscillate with the frequency of the wave. The oscillation amplitude has an optimum value if the resonant frequency equals the frequency of the wave, that is, if

$$T_0 = T \qquad (5)$$

In general, the oscillating velocity has the same direction as the exciting force part of the oscillation period and opposite direction the rest of the period. Then the energy input from the wave to the resonator is sometimes positive and sometimes negative. As will be shown in the next subsection, however, the velocity is in phase with the exciting force if $T_0 = T$. Then the energy absorbtion, that is, the energy input from the incoming wave to the resonator, is always non-negative.

Eq.(5), $T_0 = T$, is but one of the requirements that must be satisfied in order to achieve resonance absorbtion. Another requirement concerns the partition of the absorbed energy into useful energy and energy given to the out-going radial waves, which are necessarily generated through the resonator motion. It turns out that the useful energy output has an optimum value if it is made equal to the energy delivered to the out-going wave. However, due to the finite bandwidth of the frequency spectrum of wind-generated waves the useful energy output has an optimum value if it is made larger than the energy delivered to the out-going radial wave. This is shown in subsection 2.1.2.

The term "useful energy" is used above because this is the energy that can be potentially converted into electric energy. However, from the harbour engineering viewpoint, the main aim of the resonator is to destroy the energy of the waves. It seems that conventional floating breakwaters are designed not to resonate with the waves[5]. Contrary to this we state that wave absorbtion is best realised by a resonant structure. It is a technical and economic matter whether the absorbed energy is delivered to an electric power transmission network or whether it is dissipated in the sea, for instance through an electrical resistance.

2.1 Heaving tank

One possible way of absorbing energy from ocean waves is by placing a semisubmerged tank in the sea, as illustrated in Fig. 2. The tank is performing heaving motions in the sea in response to the waves. The kinetic and potential energy of the tank is obviously absorbed from the energy reservoir of the waves. Assume now that a certain fraction of the absorbed mechanical energy of the tank is converted into electrical energy for each oscillation period. This can, for instance, be achieved by placing on the tank a propeller which drives an electric generator. Later on we shall discuss some alternative and, presumably, better methods. In any case, the net effect is that wave energy is converted into electric energy.

In the following we shall analyse this particular system with respect to maximising the electric power generation. The analysis will demonstrate that only a system with parameters closely matched to the wave parameters is an efficient power absorber. This applies for any system that deals with wave energy.

2.1.1 Regular waves

In this subsection we shall analyse the case of regular (sinusoidal) waves. Ocean waves are actually never regular. However, low-amplitude swells may come very close to sinusoidal waves, in which case the results of this subsection are applicable. In the next subsection we shall extend the analysis to include the more general case of waves with a continuous wave spectrum.

If the heaving tank of Fig.2 has a depth and a width small in comparison with the wavelength, Newton's law gives

$$m\ddot{\zeta} = (\eta-\zeta)A\rho g - R\dot{\zeta} \quad , \tag{6}$$

where ζ is the displacement of the tank, η is the elevation of the sea around the tank, m is the mass of the tank and A its cross sectional area at the water line. The last term in eq.(6) represents a damping force, assumed proportional to the heaving velocity $\dot{\zeta}$. The proportionality constant R is the mechanical resistance of the system, and it is primarily established by the electric generator.

The movement of the tank generates waves. The elevation $\eta(t)$ of the sea surface is therefore the sum of two terms:

$$\eta(t) = \eta_i(t) + \eta_r(t) \tag{7}$$

where η_i is the elevation due to the incoming wave and η_r is the elevation due to the wave generated by the movement of the tank itself. In the case that the wave is regular we have

$$\eta_i = \eta_0 e^{i\omega t} \tag{8}$$

where η_0 is the amplitude of the incident wave. For convenience, we take η_0 to be real. We assume that the relationship between the movement of the tank and the generated wave may be expressed by means of a linear operator Z_r. Then we write:

$$A\rho g\eta_r = -Z_r\dot{\zeta} = -(R_r+iX_r)\dot{\zeta} \tag{9}$$

Here $Z_r = R_r + iX_r$ is the so-called radiation impedance. The analytical expressions for R_r and X_r as well as explanation of their physical significance will be given later. Introducing eqs. (7)-(9) into eq.(6) gives

$$m\ddot{\zeta} + (R+R_r+iX_r)\dot{\zeta} + A\rho g\zeta = A\rho g\eta_0 e^{i\omega t} \tag{10}$$

It should be noted that the analysis is linear, which means that it holds for small-amplitude oscillations only. All of the considered systems are to some extent non-linear when the amplitude is as large as in a real ocean wave. However, the analysis is valid as a first order approximation. Also the analysis is based on the assumption of deep water. This is a fairly good approximation when the depth is at least a third of a wavelength. Practically this means a depth of roughly 50 m. In addition, the differential equation (10) for the tank movement is based upon the presupposition that the depth and the width of the tank is much smaller than the wavelength. *) Furthermore, it is assumed that the drag forces are negligible in comparison to the other forces working on the system. These assumptions put restrictions on the system that will be discussed later.

The stationary solution of eq.(10) is

$$\zeta = \zeta_0 e^{i(\omega t - \frac{\pi}{2} - \phi)} , \tag{11}$$

where the oscillation amplitude

$$\zeta_0 = \frac{1}{\omega\sqrt{1 + \left(\frac{\omega(m+X_r/\omega) - A\rho g/\omega}{R + R_r}\right)^2}} \cdot \frac{A\rho g \eta_0}{R + R_r} \tag{12}$$

and the phase angle ϕ is given by

$$tg\phi = \frac{\omega(m+X_r/\omega) - A\rho g/\omega}{R + R_r} . \tag{13}$$

The velocity of the tank is

$$u = \dot{\zeta} = \omega\zeta_0 e^{i(\omega t-\phi)} = u_0 e^{i(\omega t-\phi)} \tag{14}$$

---

*)

For a heaving body which is slender (i.e. the width is small in comparison to the wavelength) a generalisation of eq.(10) to an arbitrarily deep tank is[6]

$$m\ddot{\zeta} + (R+R_r+iX_r)\dot{\zeta} + A(0)\rho g\zeta = A_{eff}\rho g\eta_0 e^{i\omega t} \tag{10a}$$

where

$$A_{eff} = \int_{-H}^{0} \frac{dA(z)}{dz} e^{kz} dz$$

A(z) is the cross section of the tank at the depth z (z negative below the waterline). A(0) is the cross section in the water line and H is the depth of the tank. The diffraction of the incoming wave on the heaving slender body is of insignificant importance.

where the velocity amplitude is

$$u_0 = \omega\zeta_0 = \frac{1}{\sqrt{1 + \left(\frac{\omega(m+X_r/\omega) - A\rho g/\omega}{R + R_r}\right)^2}} \frac{A\rho g\eta_0}{R + R_r} \tag{15}$$

From eq.(10) it it seen that the incoming wave excites the tank with a force

$$F = A\rho g\eta_0 e^{i\omega t} \tag{16}$$

The average power that F deliveres to the system is

$$P' = \overline{\mathrm{Re}\{F\} \cdot \mathrm{Re}\{u\}} \tag{17}$$

By means of eq.(10) we easily find

$$P' = \frac{1}{2}Ru_0{}^2 + \frac{1}{2}R_r u_0{}^2 \tag{18}$$

The first term in eq.(18)

$$P = \frac{1}{2}Ru_0{}^2 \tag{19}$$

is the average power absorbed from the waves due to the mechanical resistance R of the system. If R is established by the electric generator alone and if the generator has negligible losses, then P is the electric power produced by the tank.

The second term in eq.(18)

$$P_r = \frac{1}{2}R_r u_0{}^2 \tag{20}$$

is the wave power radiated by the tank due to its oscillating movement. Cf. Appendix A, eqs.(A.31) and (A.33). This means that part of the total power P' is radiated back to the sea. This explains the significance of R_r (the radiation resistance) in eq.(9). The physical meaning of the radiation reactance X_r in the same equation is seen from eq. (12). Here we observe that the efficient mass of the system is

$$m_{eff} = m + \frac{X_r}{\omega} \quad , \tag{21}$$

i.e. X_r/ω represents an additional mass supplied by the fluid in which the tank is moving.

Analytical expressions for R_r and X_r are not easily obtained in the general case. Results for a semisubmerged sphere are known [7]. See Appendix A.3.3., eq.(A.69). According to the statements (A.64) and (A.65) these results also apply for a cylindrical tank with a hemispherical bottom, provided the depth and radius of the tank are small in comparison with the wavelength. Hence, for the tank of Fig.2 we have

$$R_r \approx \frac{2\pi}{3} \rho a^3 \omega \varepsilon \tag{22}$$

$$X_r \approx \frac{2\pi}{3} \rho a^3 \omega \mu \tag{23}$$

Here a is the radius of the tank (and the hemisphere). Further, ε and μ are dimensionless functions of ka as shown graphically in Fig. 3. The angular wave number of the wave is

$$k = \frac{2\pi}{\lambda} = \frac{\omega^2}{g} \tag{24}$$

Cf. eq.(A.11).

Introducing eq.(15) into eq.(19) we obtain the following expression for the potentially useful power production of the tank.

$$P = f \frac{R(A\rho g \eta_0)^2}{2(R + R_r)^2} \quad , \tag{25}$$

where

$$f = \frac{1}{1 + \left(\frac{\omega(m+X_r/\omega) - A\rho g/\omega}{R + R_r}\right)^2} \tag{26}$$

The function f has a peak value f = 1 for the angular frequency ω_0 which makes

$$\omega_0(m+\frac{X_r}{\omega_0}) - A\rho g/\omega_0 = 0 \qquad (27)$$

The frequency $\nu_0 = \omega_0/2\pi$ is the resonant frequency of the system. When ν_0 is tuned to the frequency ν of the waves (resonance),

$$P = \frac{R(A\rho g\eta_0)^2}{2(R + R_r)^2}. \qquad (28)$$

It is seen from eqs.(13) and (27) that the phase angle $\phi = 0$ at resonance. Furthermore eqs. (14) and (16) show that the tank velocity and the exciting force are in phase in this case. This means that the force feeds energy into the system during the whole oscillation cycle. This explains why P is a maximum at resonance.

It should be noted that in order to satisfy the resonance condition (27) the mass of the shallow tank in Fig.2 is too small when $T_0 = \nu_0^{-1} \sim 10$ s. The mass can be increased by connecting another, deeply submerged, body to the tank by a slender structure. If this additional body is placed at least one third of a wavelength below the surface its direct effect on the wave interaction is so small that it is of little significance. It is moving in essentially calm water. Therefore a propeller or turbine used to transform the translation oscillation into rotation oscillation may preferably be connected to this additional, deeply submerged, body. Later we shall discuss other methods to obtain effectively a large enough mass to satisfy the resonance condition (27).

Also the mechanical resistance R may be optimised. P has its maximum value

$$P_m = \frac{(A\rho g\eta_0)^2}{8R_r} \qquad (29)$$

when $R = R_r$ (resonance absorbtion). By means of eq.(22) we obtain

$$P_m = \frac{3\pi\rho g^2}{16} \cdot \frac{a\eta_0{}^2}{\omega\varepsilon} \qquad (30)$$

Fig.4 shows P_m as a function of the radius a of the tank when the wave amplitude $\eta_0 = 1$ m and the wave period $T_0 = 10$ s. In the same figure is also shown the displacement amplitude at resonance $(\omega=\omega_0)$

$$\zeta_0 = \frac{1}{\omega_0} \cdot \frac{A\rho g\eta_0}{R + R_r} \tag{31}$$

as given by eq.(12), when $R = R_r$. It is noticed that ζ_0 becomes very large for small values of the radius a. This is in fact prohibitive for two reasons. In the first place it was a presupposition for the theory that the depth ℓ of the tank (see Fig.2) is small in comparison to the wavelength of the wave. Since obviously $\zeta_0 \leq \ell$ this means that ζ_0 must be kept within limits, say $\zeta_0 \leq 5$ m for a wave period of $T_0 = 10$ s. Even more important is that drag forces become significant when ζ_0 is large. This is realised as follows: The drag force on a cylinder with a hemispherical bottom is

$$F_D = C_D \cdot \frac{1}{2} \rho u^2 \cdot A \quad , \tag{32}$$

where C_D is a dimensionless coefficient considerably less than 1, u is the velocity of the cylinder and A its cross section. The requirement is (see comments to eq.(10)) that

$$F_D << (R + R_r) u_0 \tag{33}$$

This condition is fulfilled if

$$\frac{1}{2} \rho u_0{}^2 \cdot A \leq (R + R_r) u_0 \tag{34}$$

or, since $u_0 = \omega_0 \zeta_0$ at resonance,

$$\frac{1}{2} \rho \omega_0 \zeta_0 A \leq R + R_r \tag{35}$$

If this inequality is fulfilled eq.(31) is correct, which together with eq.(35) gives the condition

$$\zeta_0 \leq \frac{\sqrt{2g\eta_0}}{\omega_0} \tag{36}$$

Taking $\eta_0 = 1$ m and the wave period T = 10 s, gives $\zeta_0 \leq 7$ m. The conclusion is that if we take the depth ℓ of the tank of the order of 5 m, i.e. $\zeta_0 \leq 5$ m, eq. (30) is correct for most common wave heights. Eq. (30) has obviously the limitation that the oscillation amplitude ζ_0 shall not exceed the depth ℓ of the tank.

This means according to eq.(31) that

$$\frac{1}{\omega_0}\frac{A\rho g\eta_0}{R + R_r} \leq \ell \qquad (37)$$

or

$$R \geq \frac{A\rho g\eta_0}{\omega_0\ell} - R_r \qquad (38)$$

The conslusion is that we take $R = R_r$ for such wave amplitudes η_0 that

$$2R_r > \frac{A\rho g\eta_0}{\omega_0\ell} \qquad (39)$$

Then eq.(30) applies. If eq. (39) is not fulfilled we take

$$R = \frac{A\rho g\eta_0}{\omega_0\ell} - R_r \qquad (40)$$

Then we must use eq. (28) to calculate P. In Fig.5 we have plotted P as a function of $\overline{\eta^2(t)} = \frac{1}{2}\eta_0^2$ for the case $\ell = 5$ m, a = 8 m and T = 10 s. This curve represents the maximum power that the tank can absorb from the regular wave.

2.1.2 Wind-generated waves

Ocean waves are not harmonic. Frequency analysis of real waves shows that they have a continuous frequency spectrum. Fortunately, the frequency spectrum is highly peaked, so that most of the energy in the wave is concentrated in a relatively narrow frequency band. The so-called power spectrum $E(\omega/\omega_m)$ of a wave system with peak frequency $\omega = \omega_m$ is related to the mean squared value of the surface elevation $\overline{\eta^2(t)}$ as follows

$$\overline{\eta^2(t)} = \int_0^\infty E(\omega/\omega_m)\cdot d\omega\,. \qquad (41)$$

$E(\omega/\omega_m)$ represents the contribution to $\overline{\eta^2(t)}$ per unit frequency interval. It can be shown that the absorbed power by the tank in this general case is

$$P = R\overline{u^2} = (A\rho g)^2 \int_0^\infty \frac{R}{(R + R_r)^2}\cdot f\, E(\omega/\omega_m)\,d\omega \quad , \qquad (42)$$

where f is given by eq.(26). A representative analytic expression for $E(\omega/\omega_m)$ is the so-called Jonswap spectrum[8,9]

$$E(\omega/\omega_m) = \alpha g^2 \omega_m^{-5} (\frac{\omega}{\omega_m})^{-5} \exp[-\frac{5}{4}(\frac{\omega}{\omega_m})^{-4}] \cdot \gamma^s \quad , \qquad (43)$$

where

$$s = \exp [- \frac{(\omega/\omega_m - 1)^2}{2\sigma^2}]$$

$$\alpha = 0.008$$

$$\omega_m = \text{peak angular frequency}$$

$$\sigma = \begin{cases} \sigma_a = 0{,}07 \text{ for } \omega \leq \omega_m \\ \sigma_b = 0{,}09 \text{ for } \omega > \omega_m \end{cases}$$

$$\gamma = \text{peakedness parameter}$$

The γ-value reflects the various wind conditions. The case $\gamma = 1$ leads to the Pierson-Moskowitz spectrum. However, the average value found for γ in a comprehensive observation project in the North Sea off the Island of Sylt was $\gamma = 3.3$.[8,9] Observations in the Lopp Sea off the coast of Northern Norway fitted eq.(43) very well with γ between 1 and 3[9]. In this note we have therefore based our numerical estimates on the average Jonswap spectrum (γ=3.3). However, final results for $\gamma = 1$ will also be given.

Eq.(43) shows that $E(\omega/\omega_m)$ is peaked around the angular frequency ω_m. Fig.6 shows E as a function of ω/ω_m when $\gamma = 3.3$. It is seen that the half width

$$\Delta\omega_E \approx 0.2 \cdot \omega_m . \qquad (44)$$

The function f in eq.(42) has a peak value f = 1 for $\omega = \omega_0$ (se eqs.(26) and (27)). The width of the peak is roughly

$$\Delta\omega_f = 2\kappa\omega_0 \tag{45}$$

where

$$\kappa = \frac{R + R_r}{2A\rho g}\omega_0 \tag{46}$$

If the frequency dependence of the factor $R/(R + R_r)^2$ in eq.(42) is neglected it is realised that f operates as a filter function for the power spectrum $E(\omega/\omega_m)$. Only those frequencies of $E(\omega/\omega_m)$ that falls within the peak region of the filter function f give a significant contribution to the absorbed power P. It is therefore obvious that P is closely maximised when the peaks of $E(\omega/\omega_m)$ and $f(\omega)$ coincide. This means that the resonant frequency $\nu_0 = 2\pi/\omega_0$ of the tank must be tuned to the peak frequency $\nu_m = 2\pi/\omega_m$ of the power spectrum of the waves, i.e. the tank must operate at resonance.

The wider the peak of f is the more of the spectrum of $E(\omega/\omega_m)$ can pass through the filter. However, a wider peak means a large value of $(R + R_r)$ according to eqs.(45) and (46). Since P from eq.(42) is inversely proportional to $(R + R_r)^2$ this means that R has an optimum value.

By means of eq.(41) we may write eq.(42):

$$P = \frac{R(A\rho g)^2}{[R + R_r(\omega_0)]^2}\, q \cdot \overline{\eta^2(t)}, \tag{47}$$

where

$$q = \frac{\int_0^\infty \left(\frac{R + R_r(\omega_0)}{R + R_r(\omega)}\right)^2 f(\omega)\, E(\omega/\omega_m)\, d\omega}{\int_0^\infty E(\omega/\omega_m)\, d\omega} \tag{48}$$

is a dimensionless quantity less than 1. The form of P as given by eq.(47) is particulary convenient since $\overline{\eta^2(t)}$ is a quantity that generally is easily obtained from published wave observations (see Fig.1)

If we compare eq.(47) with eq.(28) we see that $\frac{1}{2}\eta_0{}^2$ is replaced by $q\overline{\eta^2(t)}$. Since for a sinusoidal wave $\overline{\eta^2(t)} = \frac{1}{2}\eta_0{}^2$, we see that P is reduced by the factor q when the wave has a finite spectral width. The factor q is a function of R, ω_0, ω_m and a. When $\omega_0 = \omega_m$, q has its maximum value. It can be shown that this maximum value approximately can be expressed as a function of κ only

$$q \approx \frac{\int_0^\infty \frac{E(x)}{1 + (\frac{x^2-1}{2x\kappa})^2} dx}{\int_0^\infty E(x) \cdot dx} \qquad (49)$$

where

$$x = \omega/\omega_m \qquad (50)$$

The approximation consists in neglecting the frequency dependence of R_r and X_r/ω, i.e. we put $R_r(\omega) \approx R_r(\omega_0)$ and $X_r(\omega)/\omega \approx X_r(\omega_0)/\omega_0$. In Fig.7 q is plotted as a function of κ. We see that if $\kappa = 0.1$ the actually absorbed power P is about fifty percent of the corresponding power if the incident wave were purely sinusoidal. In this case eqs. (44) and (45) show that the wave spectrum and the filter function f have equal half widths. For larger values of κ Fig.7 shows that q approaches unity. This was to be expected since then almost all of the wave power spectrum falls within the peak region of the filter function f.

Since R is a function of κ, a and ω_0 according to eq.(46), the absorbed power P, as given by eq.(47), can be expressed as a function of a, ω_0 (which is equal to ω_m) and κ. In fig. 8 we have plotted $K = P/\overline{\eta^2(t)}$ as a function of κ for various values of a when $T_0 = 2\pi/\omega_0 = 10$ s. It is seen that K can be optimised by a proper choice of κ (or R since $R = R(\kappa)$). Fig.9 shows the maximum value of K as a function of the tank radius a for two different periods T_0. It is seen that the absorbed power is not very sensitive to the period T_m of the incident wave. This is explained as follows: It is seen from eq.(22) that R_r is increased when ω is increased. This means that the factor $R/(R + R_r)^2$ in eq.(47) is reduced. Further, eq.(46) shows that also κ increases. According to eq.(49) this means a larger q value in eq.(47). The net result is that K changes very little.

For the same reason as discussed previously in connection with eq.(30) and Fig.4, the curves in Fig.8 are valid only when the oscillation amplitude of the tank is kept within limit. This means that the relation

$$P = K \cdot \overline{\eta^2(t)} \qquad (51)$$

is valid up to a certain value of $\overline{\eta^2(t)}$ only. We want to estimate the relatively small contribution to the annual production which is due to the short times when eq.(51) is invalid. Let us as a rough approximation put $q\overline{\eta^2(t)}$ equal to $\frac{1}{2}\eta_0{}^2$ where η_0 is the amplitude of a harmonic wave, i.e. we put $\eta_0 = \sqrt{2q\overline{\eta^2(t)}}$. Then if eq.(39) is fulfilled we calculate P from eq.(51). If contrary, eq.(28) together with eq.(40) is used. In the latter case q changes as $\overline{\eta^2(t)}$ changes according to eqs.(46) and (49), and a graphical method must be used. Fig.10 shows the result for the case a = 8 m, T_0 = 10 s and ℓ = 5 m. The curve gives the power absorbed by the tank as a function of the variance $\overline{\eta^2(t)}$ of the wave elevation when the tank parameters are optimised, i.e. when the resonant frequency ν_0 of the tank is equal to the peak frequency ν_m of the power spectrum of the waves and when the mechanical resistance R of the system is properly chosen (resonant absorbtion).

By means of Fig.10 and Fig.1 we can now estimate the energy absorbed by the tank if it is placed in the North Atlantic. The result is presented as a duration curve in Fig.11. The area under this curve is the total wave energy absorbed by the tank per year. The energy amounts to $1.3 \cdot 10^7$ kWh. Fig.11 also shows the duration curve for a tank placed outside Halten, Norway during a 6 months winter season. The total energy absorbtion during this season is $6 \cdot 10^6$ kWh.

These numerical results are based upon the assumption that the Jonswap power spectrum is representative for real waves[9]. However, if the Pierson - Moskowitz spectrum is used (which corresponds to $\gamma = 1$ in eq.(43)) the results for the North Atlantic is $0.8 \cdot 10^7$ kWh. This is considerably less than the result $1.3 \cdot 10^7$ kWh obtained by the more sharply peaked Jonswap spectrum.

2.1.3 Tunable resonator

It should be clear from the previous analysis that power absorbtion from waves by a heaving tank is critically dependent upon a matching of tank parameters to the slowly changing wave parameters. It is particularly important to tune the resonant frequency ν_0 of the tank to the peak frequency ν_m of the power spectrum of the waves. Also the mechanical resistance R must be properly adjusted. This can be accomplished by changing the magnetic field in the electric generator and presents presumably no great problem. In the following, therefore, we will concentrate on the parameter ν_0.

Consider a tank performing free oscillations in the sea as illustrated in Fig.2. According to eq.(27) the angular frequency of the system is

$$\omega_0 = \sqrt{\frac{A\rho g}{m}} \tag{52}$$

when X_r is neglected. The mass m of the tank is equal to the mass of the displaced water when the tank is in its equilibrium position. Hence

$$m \approx A\rho\ell \tag{53}$$

Thus

$$\omega_0 = \frac{2\pi}{T_0} \approx \sqrt{\frac{g}{\ell}} \tag{54}$$

The necessary depth ℓ of the tank in order to obtain a period T_0 is

$$\ell \approx g\left(\frac{T_0}{2\pi}\right)^2 \tag{55}$$

The average wave period in large ocean areas is $T \approx 8 - 10$ s. However, the peak period T_m of the power spectrum is slighly larger than this. Assume that $T_m = T_0 = 10$ s. Then according to eq.(55) $\ell \approx 25$ m. But this is a dissapointingly large depth for two reasons: In the first place, the accelerating force from the waves decreases exponentially with the depth of the tank (see footnote on page 7). This means that the power absorbtion by a tank of 25 m depth is considerably less than that of a shallow tank. Furthermore the previous analysis has shown that the amplitude of the tank oscillations should not exceed 5 m (see eq.(30) and comments). A tank with 25 m depth would therefore have a considerable dead volume. In addition there would be the problem of tuning the resonant frequency of the tank.

A solution which presumably circumvents these problems is illustrated in Fig.12. The tank A is kept in a semisubmerged position by means of a wire S. The wire passes around the circumference of the axis of two flywheels F placed at the bottom of the sea, and is tightly stretched by means of the buoyancy of an empty tank B. When the tank A is forced to perform heaving motions in the sea due to incoming waves, the flyweeels F and the tank B become part of the oscillating system because of the force transmission established by the wire S.

Assume now that the axes of the flywheels have a radius r_1, and that the masses of the flywheels are concentrated at a radius r_2. Then it can be shown that the differential equation for the oscillating system is

$$(m_F(\frac{r_2}{r_1})^2 + m_A + m_B)\ddot{\zeta} + R\dot{\zeta} = (\eta-\zeta)A\rho g, \qquad (55)$$

where m_F is the total mass of the flywheels, m_A is the mass of the tank A and m_B the mass of the tank B. If we compare eq.(55) with eq.(6) we see that the system operates as if it had a mass

$$m_{eff} = m_F(\frac{r_2}{r_1})^2 + m_A + m_B \qquad (56)$$

The period of the tank oscillation is then approximately

$$T_0 = 2\pi\sqrt{\frac{m_{eff}}{A\rho g}} \qquad (57)$$

It is seen that m_{eff} can be raised considerably above the actual mass of the system by having $r_2 > r_1$. This means that we can have a shallow tank A and yet have a sufficiently large T_0. In addition this system probably permits a tuning of T_0, for instance by adding mass to the flywheels by means of a clutch system or by a gear box. The tuning facility should be able to shift the resonant period T_0 by a factor which is at most somewhat less than two.

The mechanical resistance R in eq.(55) is mainly established by a rotating electric generator. The generator (not shown in Fig. 12) is mechanically connected to the flywheels.

The essential point with the system illustrated in Fig.12 is that the kinetic energy in the resonator (see eq.(4) and comments) is mainly stored in the fast moving masses of the flywheels and not in the slowly moving mass of the tank A. In fact the tanks A and B may have negligible masses, i.e. the efficient mass of the system is practically all supplied by the flywheels. This means that the tanks could simply be two air-filled baloons contained in a strong wire netting. This would probably be a cheep solution.

It should be noted that there are many alternatives to the idea sketched in Fig.12. Figs.13-15 show some of these.

Still another possibility is to have force transmission from the tank to land, for instance by wires or hydraulic means. This has the advantage that the necessary mass of the oscillating system can be supplied on land (for instance by a flywheel or a water column) and that the power station can be placed on land. The force can in this case also be used to pump water into the water reservoir of a conventional hydroelectric power station.

2.2 Water column

Fig.16 shows an alternative resonator system. Here the ocean waves excite the motion of a water column in a U-shaped tube. The water column has a natural frequency given by the length of the tube. When the resonant frequency of the tube is tuned to the peak frequency in the power spectrum of the waves the water column will have its maximum oscillation amplitude. The water column may be damped, for instance, by placing a water turbin in the oscillating column or a gas turbine at the entrance B of the tube. The net result is then that wave energy is converted into electric energy. The tuning of the system may be achieved by letting a fraction of the kinetic energy of the system be stored in the rotating turbine (flywheel). The tuning is then established by changing the moment of inertia of the turbine.

Another way of tuning the resonator is by varying the length of the water column, which is proportional to the square of the resonant period. The effective length is adjustable by means of plane lid(s) as indicated for a construction shown on Fig.17. Energy is extracted from the resonator by means of a turbine which is connected to an electric generator (Fig.17). The water speed at the turbine is of the order of 1 m/s. It should be noted that the kinetic energy of the water leaving the turbine is not lost from the system. The efficiency requirement on the turbine design is that energy lost in friction and turbulence should be minimised. This turbine changes its sense of rotation twice each oscillation period.

A system where the turbine rotates continuously in the same direction is indicated in Fig.18. For this purpose a buffer is used. Water is let into a water reservoir by opening a water valve when the water column is at its highest position. By proper controlling of this valve, the correct value of the mechanical resistance R is chosen. The water level in the buffer may be about 5 m above the sea level.*) The turbine is driven by dropping the water back to the sea. An alternative is to apply the increased pressure, when the water column is at the top, to drive a pump which pumps water into a higher water reservoir or pressurised air into an air tank. From the buffer a turbine is driven by water or air, respectively.

If a long cliffy shore as indicated in Fig.18 can be found, a battery of column resonators may be constructed. They may use a common buffer, which drives a large-scale power station.

The tuning may be realised by the lid-positioning method of Fig.17. An alternative method is to vary the length of the water column by adjusting the amount of air in the air cushion in the closed chamber above the upper end of the water column as shown in Fig.18. Of course, this method of tuning may also be combined with the energy extracting method of Fig.17.

---

*) The height of the water level in the buffer will be considerably more than 5 m if the water column tube is made considerably narrower in the upper part.

2.3 "Balloon" resonator

The resonator in Fig.19 drives air between two air reservoirs. Electric energy is extracted from the resonator by means of an air turbine placed in the connecting tube between the two systems. Since air has a low density the kinetic energy in the resonator must primarily be stored in the turbine which must also function as a flywheel.

2.3.1 Coupled "balloon" resonator

The resonator shown in Fig.20 is a variant of the system described above. Here both reservoirs are placed below the water surface, half a wave length apart. Then the exciting force will be twice as large as with one submerged tank only.

This system consists of two reservoirs They are coupled by means of the air column between the chambers A and C. Perhaps it is possible to design a system with a broader resonance curve than a single resonator, such that most of the wind-generated wave spectrum is contained within the resonator filter curve of the coupled system.

2.4 Controlled movement of wave interacting systems

Consider the system sketched in Fig.12. The movement of the tank A produces an out-going wave. This wave is superimposed on the incoming wave such that a resultant wave of reduced amplitude continues in the direction of the original incoming wave. As a consequence, wave energy is absorbed by the oscillator. Obviously, the power absorbtion from the waves would be optimum if the tank moved in such a fashion as to generate waves of the same time variation as the incoming waves. Unfortunately such a tank movement can only be achieved when the incoming waves are harmonic. This fact applies for any linear oscillating system and is the reason why wind-generated waves are less efficiently absorbed than harmonic ones (swells) by the oscillator. Quantitatively this is expressed by the spectral factor q in eq.(47). However, if the movement of the wave interacting system is controlled by an additional force to move it in this preferable fashion, then the efficiency (q-value) of the energy absorber can be increased considerably. Assume for instance that the flywheels in Fig.12 are replaced by a combined motor and generator that controls the movement of the tank A. This device is run in such a way that the incoming wave is reduced maximally. Then, obviously, the motor consumes less energy than the generator delivers with the net effect of the tank being an efficient absorber of wave energy.

This system needs a separate theoretical analysis before quantitative results can be given.

3. CONCLUSION

This analysis has shown that ocean waves contain considerable energy amounts and that this energy, in principle, can be converted into useful (electric) energy. It has been demonstrated that the parameters of the system that interacts with the waves must be continously tuned to the slowly varying parameters of the waves in order to absorb wave power efficiently (resonant absorbtion). Several interacting systems that meet these requirements are proposed. One of the systems (heaving tank) is analysed in detail on a theoretical basis. Estimates of the maximum energy amount that the system can produce are given. These estimates apply rougly for the other proposed systems also.

The proposed systems have not been given a technological and economic analysis. No doubt, there are much work left before a conclusion about the feasibility of the proposed systems or similar systems can be made. However, in view of the large energy amounts contained in the ocean waves and keeping in mind the shortage and raising prices of energy it seems reasonable that such a technological and economic study should be given high priority. In particular, a project of this kind would be natural for Norway due to its long and stormy coast. It is hoped that the present theoretical report can result in further research and development on the subject.

Appendix A. PLANE AND CIRCULAR SURFACE WAVES ON AN IDEAL INCOMPRESSIBLE FLUID

Here, we shall, for convenience, present a mathematical description of harmonic surface waves with plane or circular geometry (straight and ring-shaped wave front, respectively). The energy content and transport associated with surface waves is also considered. Further, the generation of radial (circular) waves from a floating object in heaving oscillations is studied.

It is assumed that the wave amplitudes are so small that a linearised theory applies. The variable quantities vary harmonically with time as $e^{i\omega t}$. For convenience, this factor will be suppressed in many of the following expressions. We let the plane $z = 0$ coincide with the undisturbed free surface of the fluid, with the z axis pointing upwards. Only the real part of the complex variable quantities has physical significance.

The basic equations for the waves can be found in books.[10] For convenience, they are resumed here as follows. Only deep water formulas are given. They give a very good approximation if the depth is larger than the wavelength. They give a rough approximation also for depths as small as one third to one half of a wavelength.

The fluid velocity $\vec{v}$ can be expressed by the velocity potential ϕ

$$\vec{v} = \nabla\phi \qquad \text{(A.1)}$$

which satisfies Laplace's equation

$$\nabla^2\phi = 0 \qquad \text{(A.2)}$$

and the following boundary conditions:

(1) On the solid boundaries the normal component of the velocity $\vec{u}$ of the surface of the solid must equal the normal component of the velocity

$$\frac{\partial\phi}{\partial n} = u_n \qquad \text{(A.3)}$$

(2) On the free surface, where the pressure equals the atmospheric pressure, we have

$$-\omega^2 \phi + g\frac{\partial \phi}{\partial z} = 0 \quad , \qquad (z = 0) \qquad \text{(A.4)}$$

where g is the acceleration of gravity.

The hydrodynamic pressure is given by

$$p = -i\omega\rho\phi \qquad \text{(A.5)}$$

where ρ is the mass density of the fluid and the surface elevation is

$$\eta = -\frac{i\omega}{g} \phi|_{z=0} \qquad \text{(A.6)}$$

A.1 Energy and power of plane waves

For a plane wave which propagates in the x direction (A.2) reads

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{(A.7)}$$

since there is no variation along the y axis. By the method of separation of variables the following particular solution of (A.7) is obtained

$$\phi = i\frac{g}{\omega}\eta_0 e^{-ikx} e^{kz} \qquad \text{(A.8)}$$

Here $i(g/\omega)\eta_0$ is a constant. The angular wave number k is assumed to be positive. The minus sign in e^{-ikx} means that the solution represent a wave which propagates in the positive x direciton. Since ϕ and all its derivatives are negligible when $-kz >> 1$, (A.8) satisfies fairly well the boundary condition (A.3) at the bottom, if the depth of the fluid is at least one wavelength. According to (A.6) the surface elevation is

$$\eta = \eta_0 e^{-ikx} \qquad \text{(A.9)}$$

The solution (A.8) must satisfy the boundary condition (A.4) at the free surface. This means that the angular wave number k and the angular frequency ω are related by

$$\omega^2 = gk \tag{A.10}$$

The wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi g}{\omega^2} = \frac{gT^2}{2\pi} \tag{A.11}$$

where $T = 2\pi/\omega$ is the period of the harmonic wave. The phase velocity of the wave is

$$v_p = \frac{\omega}{k} = \frac{\lambda}{T} = \frac{gT}{2\pi} \tag{A.12}$$

and the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{g}{2\omega} = \frac{gT}{4\pi} = \frac{1}{2}v_p \tag{A.13}$$

An energy amount of $p(x,z,t)v_x(x,z,t)$ passes through a unit area of the yz plane per unit time. The time average of this amount is

$$I = \overline{p(x,z,t)v_x(x,z,t)} = \frac{1}{2}\mathrm{Re}\{p(x,z)v_x^*(x,z)\}$$

Using (A.1), (A.5) and (A.8) this becomes

$$I = \frac{1}{2}\mathrm{Re}\{-i\omega\rho\phi\,\frac{\partial\phi^*}{\partial x}\} = \frac{1}{2}\omega k\rho|\phi|^2 = \frac{k\rho g^2}{2\omega}|\eta_0|^2 e^{2kz} \tag{A.14}$$

Integration over the interval $-\infty<z<0$ yields the average power per unit length (in the y direciton)

$$K = \int_{-\infty}^{0} I\, dz = \frac{\rho g^2}{4\omega}|\eta_0|^2 = \frac{\rho g^2 T}{8\pi}|\eta_0|^2 \tag{A.15}$$

It can be shown that

$$K = v_g E \tag{A.16}$$

where v_g is given by (A.13) and

$$E = \frac{\rho g}{2}|\eta_0|^2 \tag{A.17}$$

is the average stored energy by the wave (per unit surface area). In time average half of this energy is stored as potential energy and the rest as kinetic energy.

A.2 Radial waves

If a floating object, rotationally symmetric about a vertical axis, is oscillating harmonically up and down, out-going ring-shaped or "radial" waves are generated.

When circular symmetry around a vertical axis at r = 0 is assumed, (A.2) gives

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{(A.18)}$$

Using the method of separation of variables yields the particular solution

$$\phi = e^{kz} Z_0(kr)$$

Where Z_0 is a linear combination of the Bessel functions J_0 and N_0 of zero order. Still k is given by (A.10) if the free-surface boundary condition (A.4) is to be satisfied. In order to satisfy the requirement that the waves are out-going (the radiation condition) the solution must be of the form[11]

$$\phi = Be^{kz} H_0^{(2)}(kr) \qquad \text{(A.19)}$$

where B is an integration constant.

The Hankel function $H_0^{(2)}(kr)$ is given by setting n = 0 in

$$H_n^{(2)}(x) = J_n(x) - iN_n(x) \qquad \text{(A.20)}$$

We shall need the asymptotic approximation[13]

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\pi/4)} \qquad \text{(A.21)}$$

which is valid for $x = kr >> 1$. Further we need the following expressions valid for $x \to 0$,

$$H_0^{(2)}(x) = i\frac{2}{\pi} \ln \frac{2}{\gamma x} + 1 - i\frac{x^2}{2\pi} \ln \frac{2}{\gamma x} + \mathcal{O}\{x^2\} \qquad (A.22)$$

$$H_1^{(2)}(x) = i\frac{2}{\pi x} + i\frac{x}{\pi} \ln \frac{2}{\gamma x} + \mathcal{O}\{x\} \qquad (A.23)$$

where $\gamma = 1.781...$

According to (A.1) the velocity has a radial component

$$v_r = \frac{\partial \phi}{\partial r} = Be^{kz} \frac{d}{dr} H_0^{(2)}(kr) = -kBe^{kz} H_0^{(2)}(kr) \qquad (A.24)$$

The surface elevation of the ring-shaped wave is obtained from (A.6) and (A.19)

$$\eta = -\frac{i\omega}{g} B\, H_0^{(2)}(kr) . \qquad (A.25)$$

We consider a simple case where the constant B can easily be determined. Let the wave source consist of a vertical, infinitely long, circular tube, whose outer surface, of radius a, has a small-amplitude, radially directed velocity

$$u_r = u_0 e^{kz} e^{i\omega t} \quad , \qquad (A.26)$$

where k is given by (A.10) and u_0 is a constant which represents the velocity amplitude at z = 0. In this case the solution (A.19) is valid for all r > a. We insert (A.24) and (A.26) (suppressing the factor $e^{i\omega t}$) into the boundary condition (A.3) and obtain

$$B = - \frac{u_0}{kH_1^{(2)}(ka)} \qquad (A.27)$$

It should be noted that (A.19) is only a particular solution of (A.18). When radial waves are generated by a circularly symmetric wave source, the potential can be written as

$$\phi = \phi_\ell + Be^{kz} H_0^{(2)}(kr) \qquad (A.28)$$

where ϕ_ℓ is another (usually more complicated) particular solution of (A.18). The term ϕ_ℓ represents the local potential near the source since its amplitude decreases as $1/r$ as $r \to \infty$.[6,11] The last term in (A.28), which has an amplitude going as $\sqrt{1/r}$ when $r \to \infty$, is the dominating term for distances r a few wavelengths away from the source. However, the term ϕ_ℓ is important in the interaction between the oscillation of the source and the wave in the fluid. In general, the term ϕ_ℓ plays a very dominant role when the boundary condition (A.3) at the wave generator is to be taken explicitly into consideration.

When $r \to \infty$ we can neglect ϕ_ℓ in comparison with the last term in (A.28) and use the asymptotic expression (A.21) for $H_0^{(2)}(kr)$. Thus, (A.28) gives for $r \to \infty$

$$\phi \to Be^{kz} \sqrt{\frac{2}{\pi kr}}\, e^{-i(kr - \pi/4)} \tag{A.29}$$

This expression is a good approximation if $|kr| >> 1$. For the surface elevation (A.6) gives

$$\eta \to B\frac{\omega}{g} \sqrt{\frac{2}{\pi kr}}\, e^{-i\pi/4}\, e^{-ikr} \tag{A.30}$$

Since $\sqrt{1/r}$ varies relatively very little over one wavelength when r is very large, the radial wave can be considered as a plane wave when $r \to \infty$. Hence, the energy transport formulas (A.14) and (A.15) are applicable when

$$\eta_0 \text{ is replaced by } B\frac{\omega}{g} \sqrt{\frac{2}{\pi kr}}\, e^{-i\pi/4}$$

Thus the average power radiated in the out-going radial waves is

$$P_r = K\cdot 2\pi r = \frac{\rho g^2}{4\omega}\left|\frac{\omega}{g}B\right|^2 \frac{2}{\pi kr} 2\pi r = \frac{\rho\omega}{k}|B|^2 \tag{A.31}$$

A.3 Radial waves from an oscillating heaving source.

The excitation of waves from oscillating heaving bodies which have a vertical axis of rotational symmetry has been analysed before. Havelock has considered a cylinder of finite length[12] and a sphere.[7] Newman has considered a more arbitrary rotationally symmetric body whose maximum radius is small in comparison with the wavelength.[6] Results of these analyses will be given after the next subsection, which considers a simpler case where (A.19) and

(A.25) with (A.27) can be used directly.

Let the vertical position of the vertically oscillating body of revolution be denoted by

$$\zeta e^{i\omega t} \tag{A.32}$$

Suppressing the factor $e^{i\omega t}$ the corresponding velocity is $i\omega\zeta$. The radiation resistance R_r of the wave generator can be defined by

$$P_r = \frac{1}{2}R_r|i\omega\zeta|^2 \tag{A.33}$$

where P_r is the average power which is delivered from the generator to the waves. Cf. eq.(20) of section 2.1. Since power dissipation is neglected in the present analysis (ideal fluid), this power is the same as the average power at an arbitrary radius r. Hence, using (A.31) we find the following general expression for the radiation resistance

$$R_r = \frac{2\rho\omega}{k}\left|\frac{B}{i\omega\zeta}\right|^2 \tag{A.34}$$

The relation between B and ζ is studied for different forms of the body of revolution in the following subsections.

A.3.1 Exponentially varying cross section

We consider a body of revolution whose radius R varies with z as

$$R = R(z) = a - b(1-e^{kz}) \tag{A.35}$$

(Fig.21). We shall assume that

$$b << a. \tag{A.36}$$

Dropping the factor $e^{i\omega t}$, the velocity (positive in the z-direction) is $i\omega\zeta$. The normal component at the surface of the body is (Fig.22) $u_n = -i\omega\zeta\sin\alpha$ and the fluid displacement velocity has a radial component

$$u_r = u_n\cos\alpha = -i\omega\zeta\sin\alpha\cos\alpha$$

Because of (A.36) α is small. Hence,

$$u_r \approx -i\omega\zeta\alpha \approx -i\omega\zeta\frac{dR}{dz} = -i\omega\zeta kbe^{kz},$$

and this can be considered to be the radial component of the fluid velocity v at r = a. This corresponds to setting

$$u_0 = -i\omega kb\zeta \tag{A.37}$$

in (A.26). Thus (A.27) becomes

$$B = \frac{i\omega kb\zeta}{kH_1^{(2)}(ka)} \tag{A.38}$$

Expressions for ϕ are obtained by inserting (A.38) into (A.19). Using (A.34) we find

$$R_r = \frac{2\rho\omega b^2}{k|H_1^{(2)}(ka)|^2} \tag{A.39}$$

This result may also be obtained by considering the reaction force from the waves on the oscillating body. According to (A.5), (A.19) and (A.38) the dynamic pressure on the surface of the body is

$$p\big|_{r=a} = \frac{\omega^2\rho b\zeta H_0^{(2)}(ka)}{H_1^{(2)}(ka)}\, e^{kz} \tag{A.40}$$

The vertically directed reaction force is

$$F_r = \int_{-\infty}^{0} dz 2\pi a\frac{dR}{dz}\, p\big|_{r=a} = \int_{-\infty}^{0} dz 2\pi abke^{kz}\, \frac{\omega^2\rho b\zeta H_0^{(2)}(ka)}{H_1^{(2)}(ka)}\, e^{kz} =$$

$$= \pi ab^2\, \frac{\omega^2\rho\zeta H_0^{(2)}(ka)}{H_1^{(2)}(ka)} \tag{A.41}$$

The time average of the power delivered from the body to the fluid is

$$P_r = -\frac{1}{2}\text{Re}\{F_r(i\omega\zeta)^*\} = \frac{1}{2}\text{Re}\{Z_r i\omega\zeta(i\omega\zeta)^*\} = \frac{1}{2}\text{Re}Z_r|i\omega\zeta|^2 \quad (A.42)$$

where we have introduced the radiation impedance

$$Z_r = R_r + iX_r = \frac{-F_r}{i\omega\zeta} \quad (A.43)$$

Since $\text{Re}Z_r = R_r$, and since there is no power loss, this definition is in agreement with (A.33). The radiation reactance X_r may be written as

$$X_r = \omega m_r \quad (A.44)$$

where m_r is the so-called "added mass". Using (A.41) and (A.43) we get

$$Z_r = R_r + iX_r = i\omega\rho\pi ab^2\frac{H_0^{(2)}(ka)}{H_1^{(2)}(ka)} \quad (A.45)$$

Using (A.20) and a Wronskian relationship[13] (A.45) gives

$$R_r = \omega\rho\pi ab^2\frac{J_1(ka)N_0(ka) - J_0(ka)N_1(ka)}{\{J_1(ka)\}^2 + \{N_1(ka)\}^2} =$$

$$\frac{2\omega\rho b^2}{k\{J_1(ka)\}^2 + k\{N_1(ka)\}^2} \quad (A.46)$$

This agrees with (A.39). Further, (A.20) gives

$$X_r = \omega\rho\pi ab^2\frac{J_0(ka)J_1(ka) + N_0(ka)N_1(ka)}{\{J_1(ka)\}^2 + \{N_1(ka)\}^2} \quad (A.47)$$

From these formulas the curves in Fig. 23 have been computed. Using the asymptotic expansion for $H_1^{(2)}(ka)$ in (A.39) it is easily shown that $R_r \to \omega\rho\pi ab^2$ as $ka \to \infty$. This corresponds to the dashed horisontal line in Fig. 23. For $ka \ll 1$, (A.22), (A.23) and (A.45) give

$$R_r \approx \frac{1}{2}\omega k\rho(\pi ab)^2 \tag{A.48}$$

$$X_r \approx \omega k\rho(\pi ab)^2\frac{1}{\pi}\ln\frac{2}{\gamma ka} \tag{A.49}$$

where terms of $\mathcal{O}\{k^2a^2\ln(ka)\}$ compared to 1 are neglected. We see from Fig.23 that these approximations are fairly good for $ka < 0.3$. Further we see that the radiation reactance $X_r = \omega m_r$ is smaller than the radiation resistance R_r if $ka > 0.3$. It should be noted that if the floating body of Fig.21 is not infinitely deep, but cut such that it has a flat bottom at a depth $z = -H$, an additional added mass, excluded above, must be included. If $H>>1/k$ this bottom end does not contribute much to the radiated wave power and, hence, to the radiation resistance. Cf.(A.33), (A.34), (A.57) and (A.58). Including the effect of the bottom end would, in contrast to Fig.23, probably make the total radiation reactance larger that the radiation resistance. This is actually true for the floating hemisphere. Cf.(A.69) and Fig.3.

A.3.2 Slender body with arbitrary cross section

Having analysed the wave excitation from the rotationally symmetric oscillating body of Fig.21 where R(z) is given by (A.35), we give now some results where R(z) is a more arbitrary function, but under the assumption that $|kR|<<1$. The depth of the floating body is H. The the velocity potential is[6]

$$\phi = \frac{i\omega\zeta}{4\pi}\int_{-H}^{0}\frac{dA(z_1)}{dz_1}g(r,z,z_1)dz_1 \tag{A.50}$$

where

$$A(z) = \pi\{R(z)\}^2 \tag{A.51}$$

and

$$g(r,z,z_1) = \{r^2+(z-z_1)^2\}^{-\frac{1}{2}} + P\int_0^\infty \frac{m+k}{m-k}e^{(z+z_1)}J_0(mr)\,dm$$

$$- 2\pi ike^{k(z+z_1)}J_0(kr), \tag{A.52}$$

P denoting principle value of the integral. By deforming the path of integration one finds[11]

$$g(r,z,z_1) = -i2\pi k e^{k(z+z_1)} H_0^{(2)}(kr) + g_\ell(r,z,z_1) \tag{A.53}$$

where

$$g_\ell(r,z,z_1) = \{r^2+(z-z_1)^2\}^{-\frac{1}{2}} + \{r^2+(z+z_1)^2\}^{-\frac{1}{2}}$$

$$- \frac{4k}{\pi} \int_0^\infty \{k \cos m(z+z_1) - m \sin m(z+z_1)\} \frac{K_0(mr)}{m^2+k^2} dm \tag{A.54}$$

Here K_0 is the modified Bessel function of zero order and second kind. When mr is positive, $K_0(mr)$ decreases monotonically with mr, in fact, for $mr>>1$, $K_0(mr) \approx e^{-mr}\sqrt{\pi/2mr}$. But $K_0(mr)$ has a logaritmic singularity at $mr = 0$. It is easy to show that all of the three terms in (A.54) goes as $1/r$ when $r \to \infty$. Thus, on the right-hand side of (A.53) the first term dominates over the second when r is sufficiently large.

The potential is given by (A.28)

$$\phi = B\, e^{kz} H_0^{(2)}(kr) + \phi_\ell \tag{A.55}$$

where now

$$\phi_\ell = \frac{i\omega\zeta}{4\pi} \int_{-H}^{0} \frac{dA(z_1)}{dz_1} g_\ell(r,z,z_1) dz_1 \tag{A.56}$$

$$B = \frac{\omega k \zeta}{2} A_{eff} \tag{A.57}$$

$$A_{eff} = \int_{-H}^{0} \frac{dA(z)}{dz} e^{kz} dz \tag{A.58}$$

Thus, a given volum displacement ($-\zeta dA$) gives the smaller contribution to the amplitude of the distant wave the deeper the displacement takes place.

By inserting B from (A.57) into (A.34) we find the radiation resistance

$$R_r = \frac{\rho\omega k}{2} A^2_{eff} \tag{A.59}$$

For the body shown in Fig.21 (A.35), (A.36), (A.51) and (A.58) give

$$A_{eff} = 2\pi abk \int_{-\infty}^{0} e^{2kz} dz = \pi ab \tag{A.60}$$

We see that (A.48) agrees with (A.59).

Finally, we consider the reaction force on the oscillating body. When $r \to 0$ the leading term in $g(r,z,z_1)$ is the first right-hand term in (A.52). Inserting this term into (A.50) the following expression for the velocity potential at the body, $r = R(z)$ is obtained[6]

$$\phi|_{r=R} = i\omega\zeta \; R\frac{dR}{dz} \ln\frac{1}{R} + \mathcal{O}\{R^2\} \tag{A.61}$$

The reaction force F_r is

$$F_r = \int_{-H}^{0} dz \; 2\pi R\frac{dR}{dz} p|_{r=R} \tag{A.62}$$

Inserting from (A.5) and (A.61) gives

$$\frac{-F_r}{i\omega\zeta} = i\omega\rho 2\pi \int_{-H}^{0} (R\frac{dR}{dr})^2 \ln\frac{1}{R} \, dz + \cdots$$

Since the leading term is imaginary, it represents the radiation reactance X_r or added mass $m_r = X_r/\omega$. Thus, for a slender body ($R \to 0$) the leading term of the radiation reactance is

$$X_r \sim \omega\rho 2\pi \int_{-H}^{0} \left(R\frac{dR}{dz}\right)^2 \ln\frac{1}{R} dz \qquad (A.63)$$

For the body of Fig.21 (A.35), (A.36) and (A.63) give

$$X_r \sim \omega\rho 2\pi a^2 \ln\frac{1}{a} \int_{-\infty}^{0} (bke^{kz})^2 dz = \omega k\rho\pi a^2 b^2 \ln\frac{1}{a}$$

This agrees with (A.49) when $a \to 0$.

Now consider a rotationally symmetric body which is cylindrical down to a depth ℓ and which has a given shape further down. Thus $R = R(z)$ varies from 0 to a when z varies from $z=-H$ to $z=-\ell$. In the region $0>z>\ell$, $R(z) = a = \text{constant}$ and $dR/dz = 0$. For this case (A.63) shows that (to the present approximation) the added mass

$$m_r = X_r/\omega \text{ is independent of } \ell \qquad (A.64)$$

However (A.58) and (A.59) show that the radiation resistance

$$R_r \text{ is proportional with } e^{-2k\ell} \qquad (A.65)$$

(to the present approximation). The result (A.64) is reasonable: The added mass associated with the heaving motion is given only by the shape and magnitude of the bottom part of the body and independent of the upper cylindrical part. The result (A.65) was to be expected for the following reason. The factor e^{kz} in (A.58) means that if a given oscillating volum displacement is lowered from the surface down to a depth ℓ, the wave amplitude far away is reduced by a factor $e^{-k\ell}$. Thus the power and, hence, the radiation resistance are reduced by a factor $e^{-2k\ell}$. Cf.(A.33).

A.3.3 Heaving hemisphere

We consider a semi-submerged floating sphere of radius a. Then

$$R(z) = \sqrt{a^2 - z^2} \qquad , \ -a<z<0$$

If a is small compared to the wavelength the preceding results are applicable. From (A.58) we find

$$A_{eff} = \int_{-a}^{0} \pi \frac{d(a^2-z^2)}{dz} e^{kz} dz =$$

$$= \frac{2\pi}{k^2}\{1 - (1+ka)e^{-ka}\} = \pi a^2(1+\mathcal{O}\{ka\}) \qquad \text{(A.66)}$$

From (A.59) we then find the radiation resistance

$$R_r = \frac{\omega k \rho}{2}\pi^2 a^4 (1+\mathcal{O}\{ka\})$$

$$= \omega \frac{2\pi a^3 \rho}{3} \varepsilon \qquad \text{(A.67)}$$

where

$$\varepsilon = \frac{3\pi}{4} ka + \mathcal{O}\{k^2 a^2\} \qquad \text{(A.68)}$$

This first order term for small a is shown as the dashed line in Fig.3.

For larger values of the radius a an analysis has been carried out by Havelock.[7] The radiation impedance can be written as

$$Z_r = \omega\rho \frac{2\pi a^3}{3} (\varepsilon + i\mu) \qquad \text{(A.69)}$$

where ε and μ are given by the numerically computed, fully drawn, curves of Fig.3. The parameter μ is a dimensionless measure of the added mass referred to the mass of the displaced fluid. It is seen from Fig.3. that the first order theory for small a - cf. (A.66) - gives large deviation from the correct values when ka > 0.1. Comparison with Fig.23 indicates that the first order approximation has a smaller region of validity for the floating hemisphere than for the body of Fig.21.

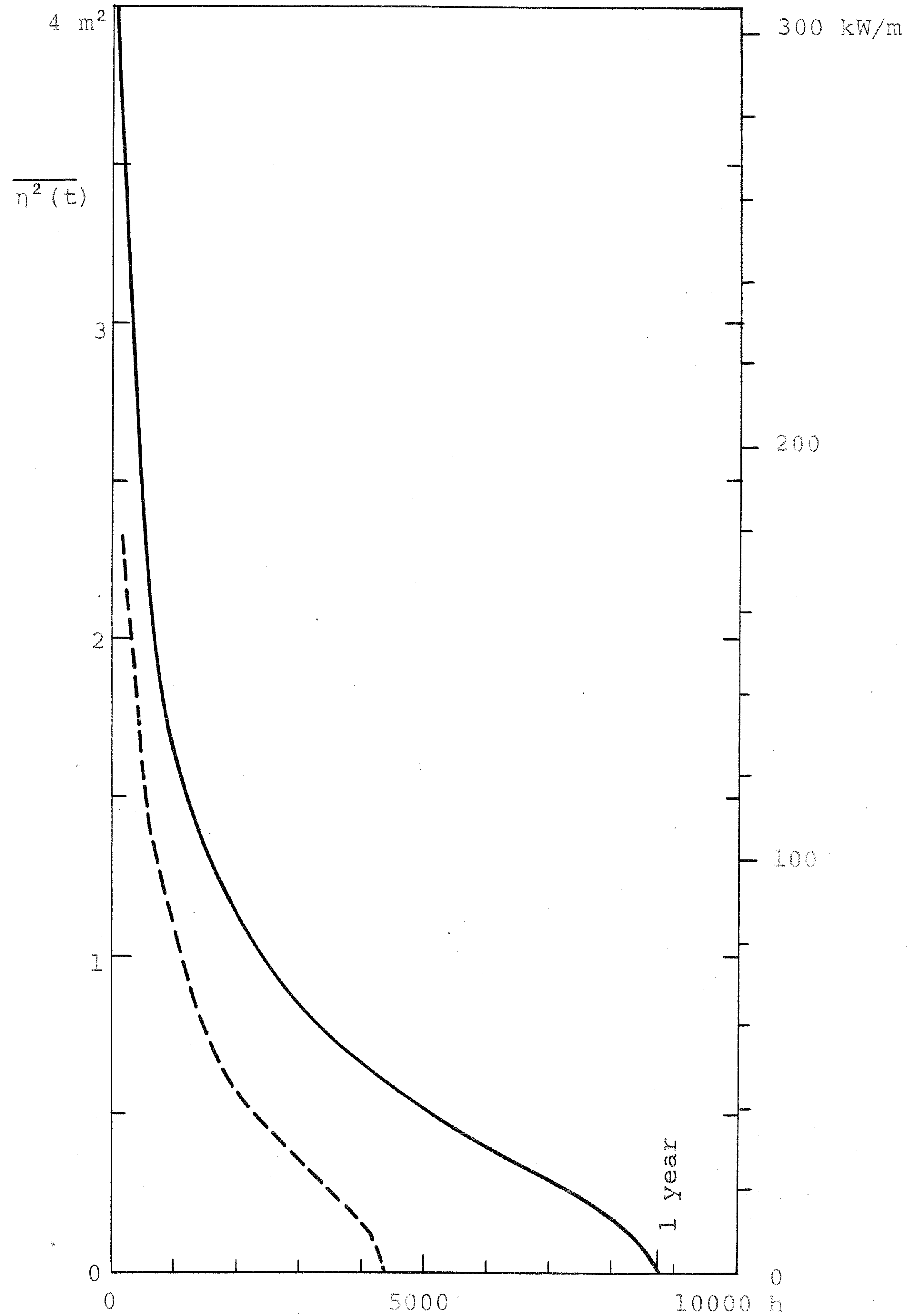


Fig.1

Duration curve for $\overline{\eta^2(t)}$ - the mean value of the square of the surface elevation. The curve gives the number of hours of the year in which $\overline{\eta^2(t)}$ exceeds the indicated value. The fully drawn curve is a 10-year average based on visual observation from 9 differently positioned weather ships in the North Atlantic[1,*]. The dashed curve is based on measurements over 6 winter months 1972/73 outside Halten, Norway.[2] If the period of the waves is taken as T = 10 s, the power transmitted per unit length of the wave crest is as given by the right-hand vertical scale.

---

*) We have put $\overline{\eta^2(t)} = H_S^2/16$, where H_S is the significant wave height. H_S is related to the observed average wave height H_V in ref. 1 by the relation $H_S = 1.68 \cdot H_V^{0.75}$, where H_s and H_v are given in metres.

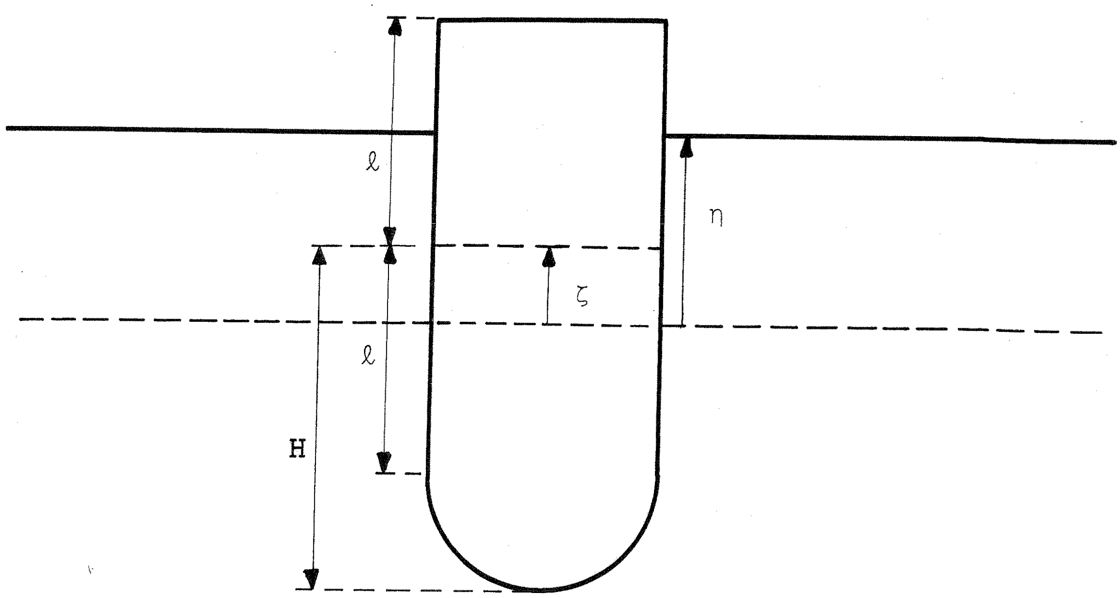


Fig.2

A tank performing heaving motions in the sea. The undisturbed sea level is indicated by the dashed line, η is the surface elevation of the sea and ζ is the displacement of the water line of the tank. The diameter and the depth of the tank are small in comparison to the wavelength of the waves.

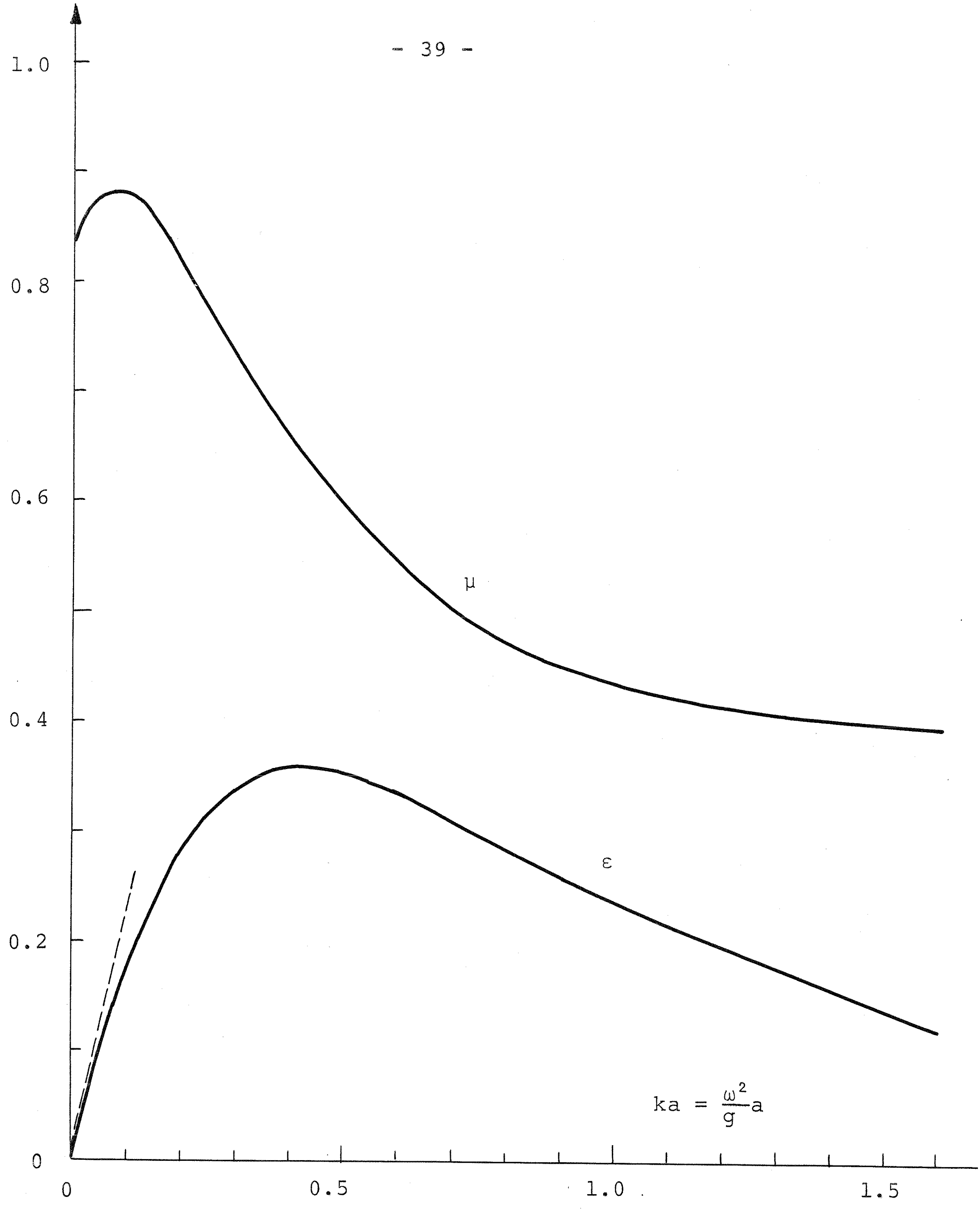


Fig.3. The dimensionless functions ε and μ (taken from Ref.7) give the radiation resistance R_r and the radiation reactance X_r for a cylindrical tank with a semispherical bottom in accordance with eqs. (22) and (23). The dashed line corresponds to the first order term in eq.(A.68).

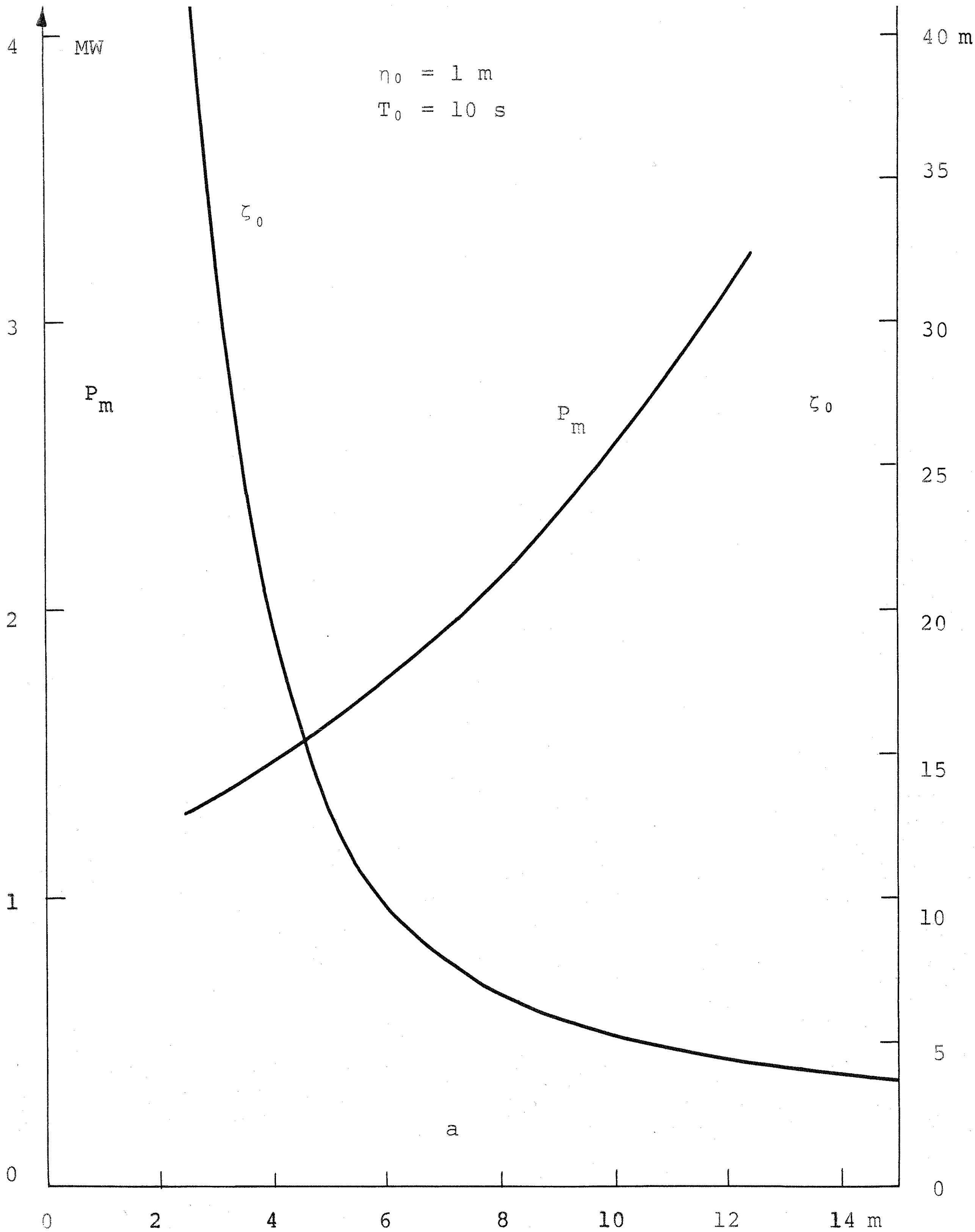


Fig.4 The maximum absorbed power P_m and the displacement amplitude ζ_0 of a tank as a function of the tank radius a, when the wave amplitude and period are η_0 = 1 m and T_0 = 10 s, respectively.

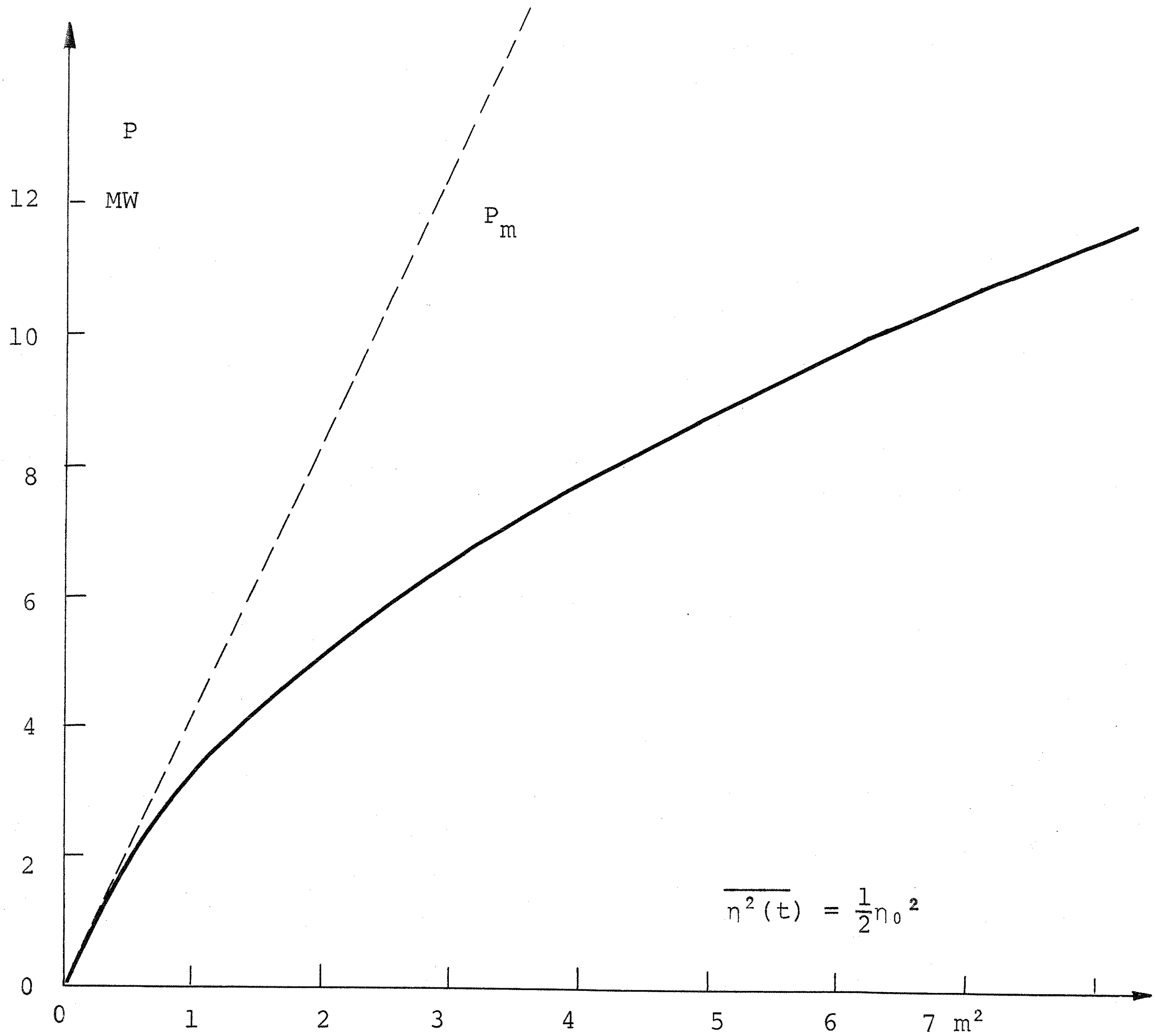


Fig.5

The curve gives the maximum power that a heaving tank with radius $a = 8$ m and height $2\ell = 10$ m can absorb from a sinusoidal wave with period $T = 10$ s. The dashed curve represents the function P_m as given by eq.(30).

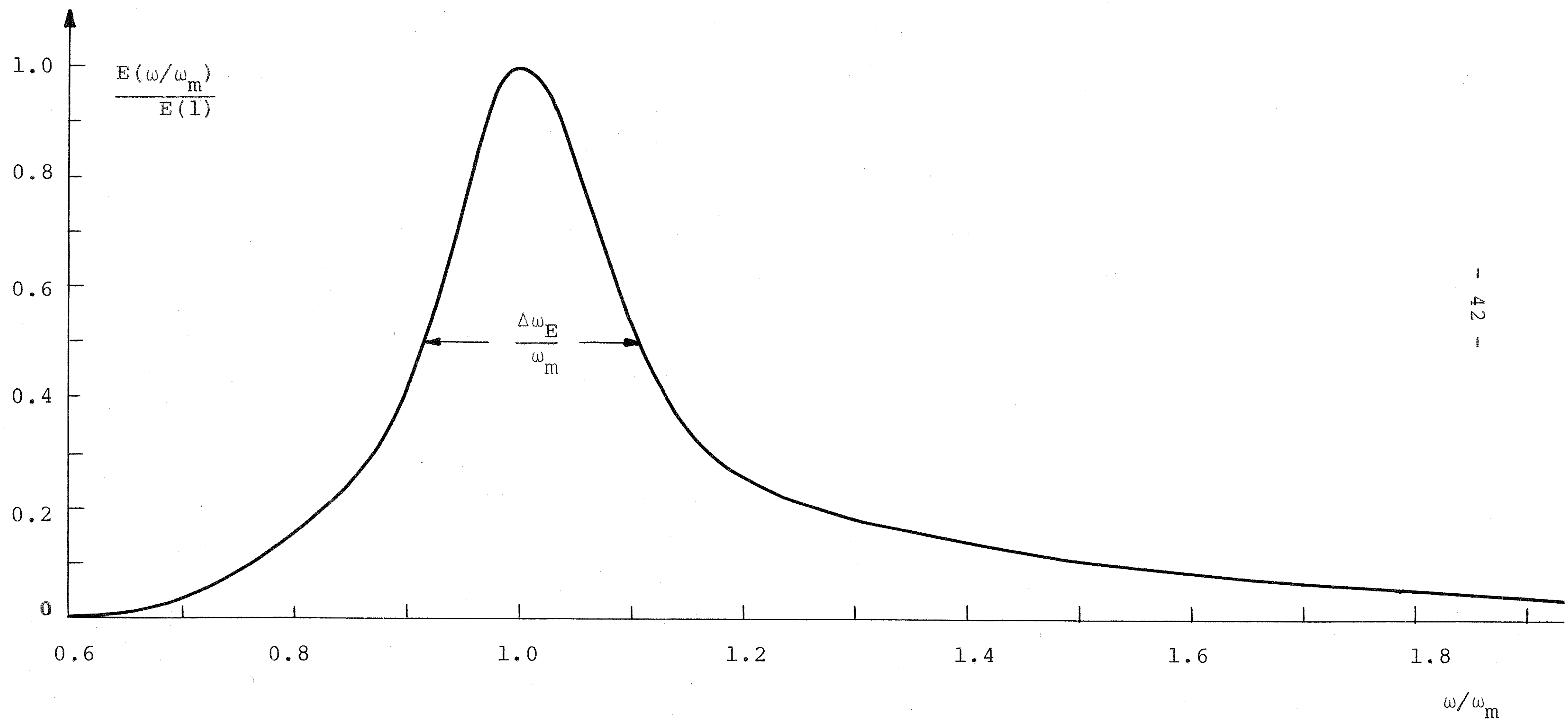


Fig.6 The Jonswap-power spectrum of wind-generated waves[8,9] for the case $\gamma = 3.3$ (see eq.(43)).

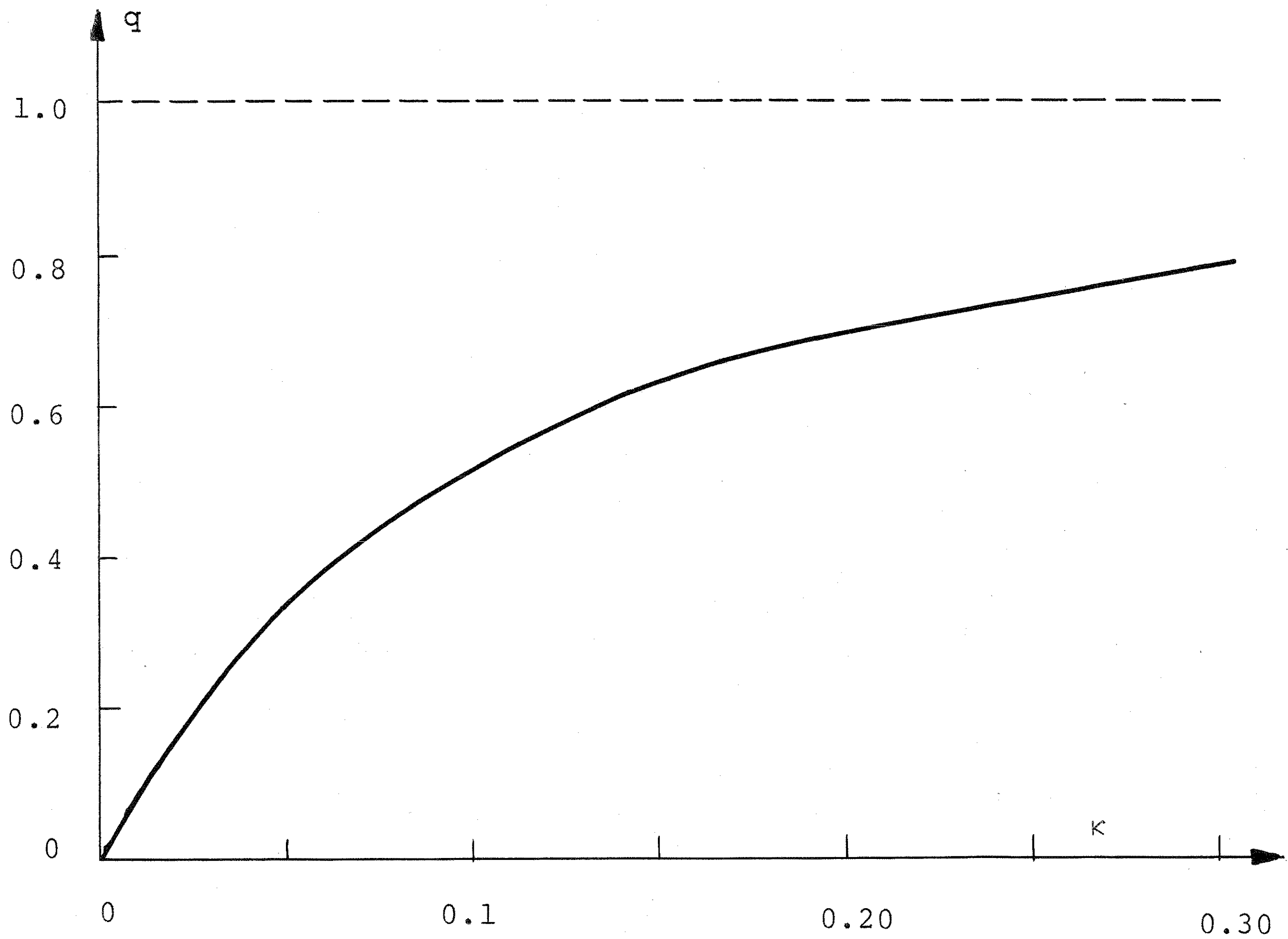


Fig. 7 The spectral factor q as given by eq.(49) plotted as a function of κ when γ = 3.3 in the Jonswap spectrum.

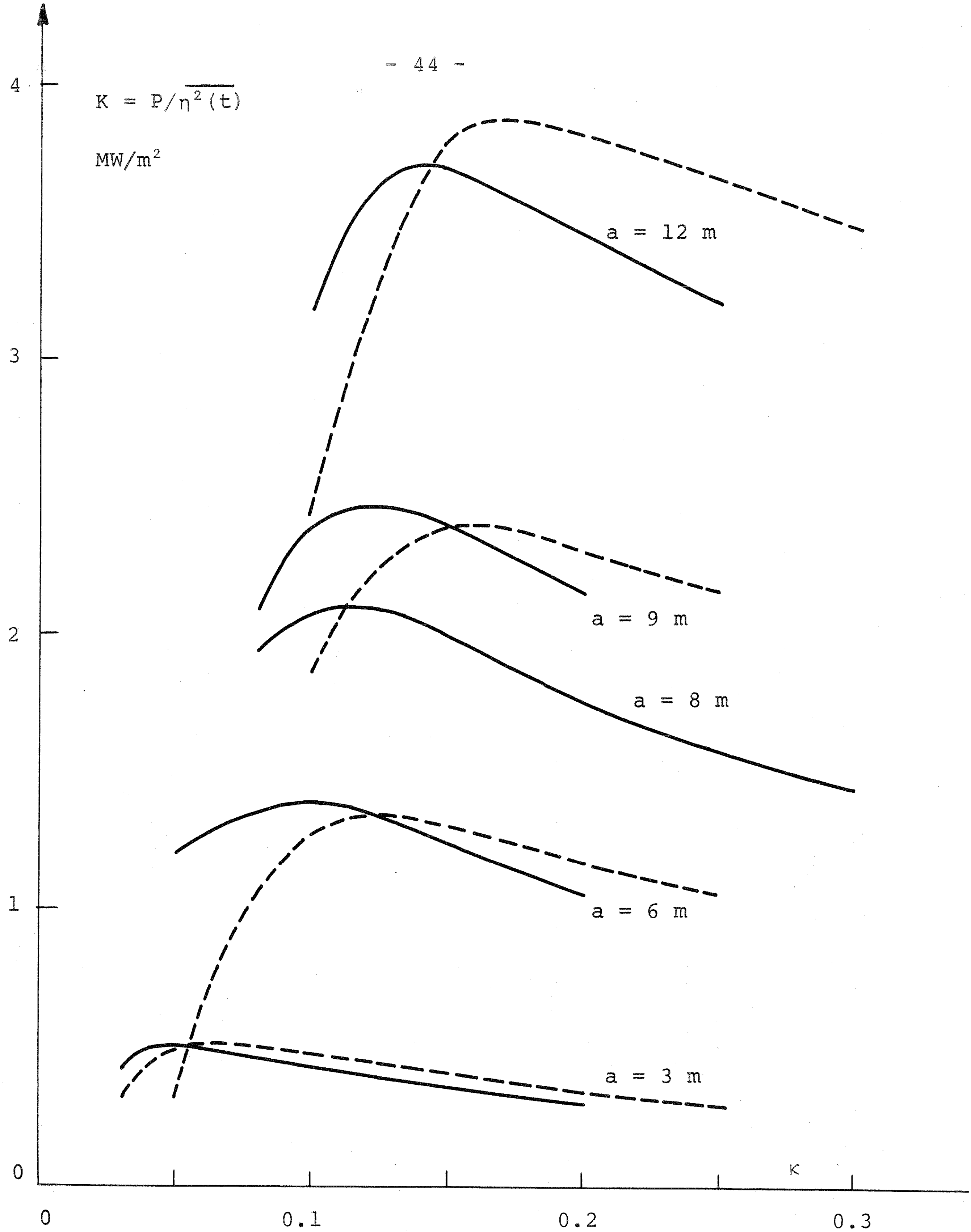


Fig.8 The ratio $P/\overline{\eta^2(t)}$ as given by eq.(47) plotted as a function of the parameter κ for various values of the tank radius a. The fully drawn curves are for the case $T_0 = T_m = 10$ s. The dashed curves are obtained when $T_0 = T_m = 8$ s.

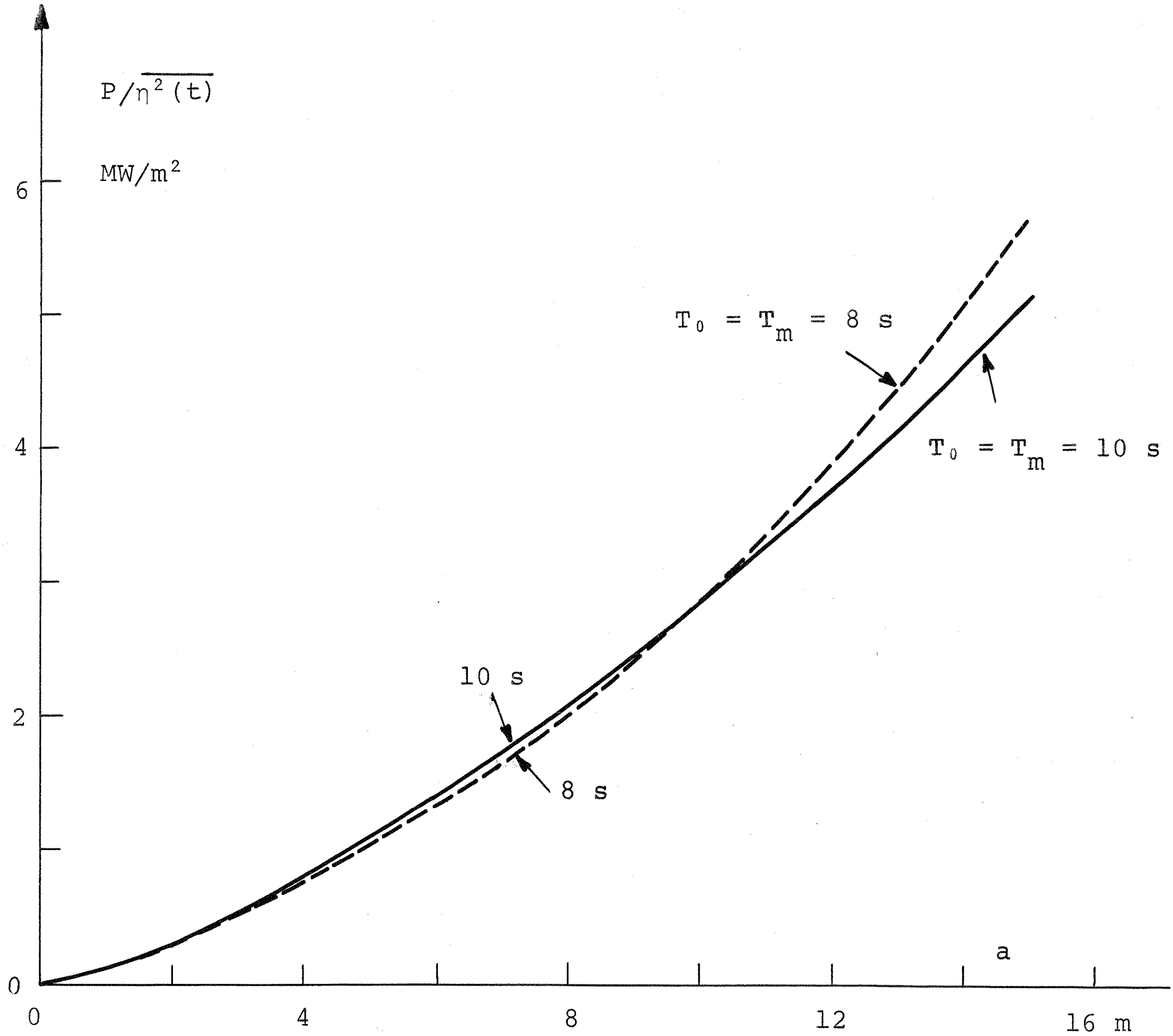


Fig.9 The maximum value of the ratio $P/\overline{\eta^2(t)}$ as given by eq.(47) as a function of the tank radius a for two different values of the period T_0.

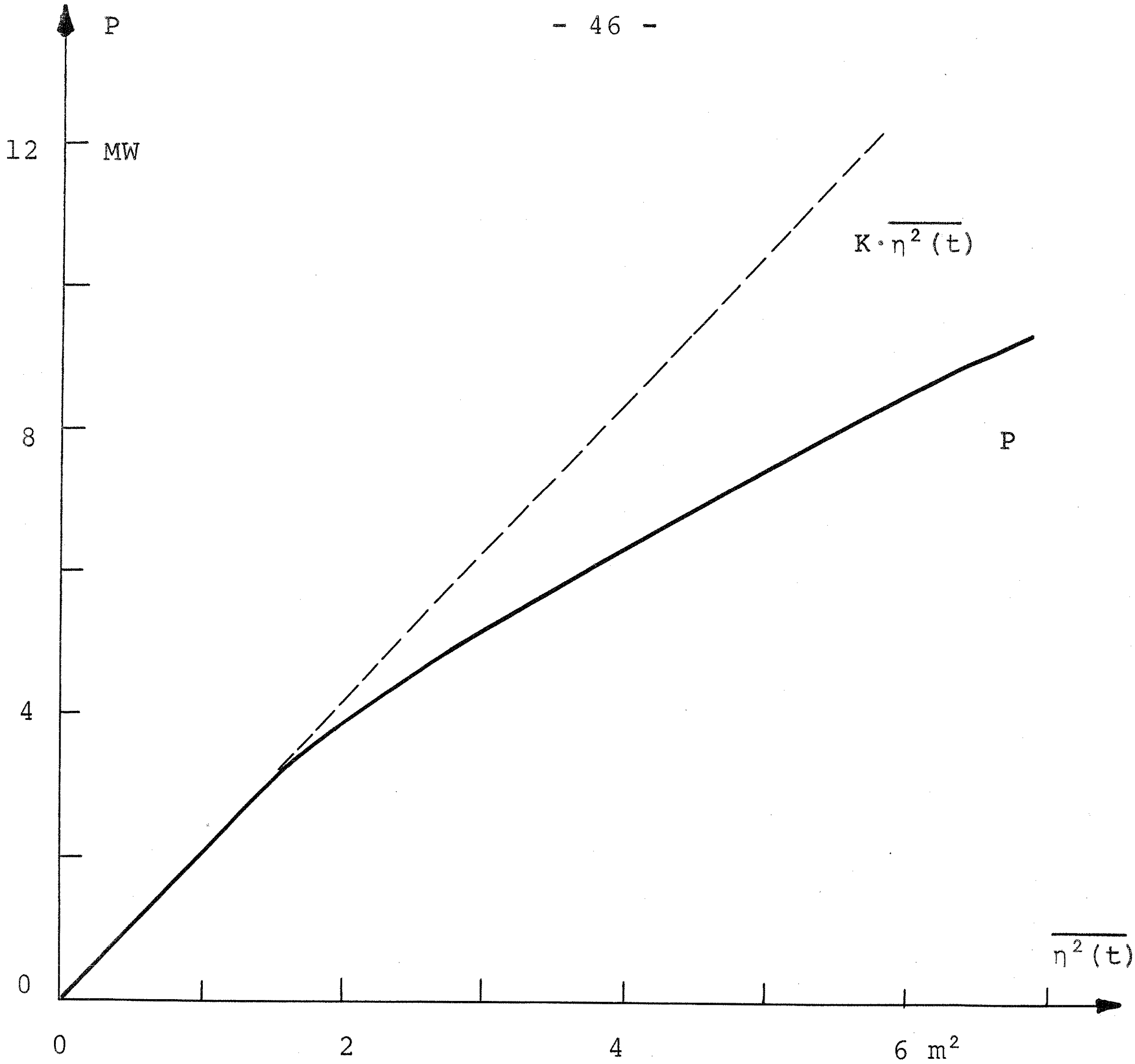


Fig. 10 The maximum power that a tank of radius a = 8 m and depth ℓ = 5 m can absorb, plotted as a function of $\overline{\eta^2(t)}$. $T_0 = T_m$ = 10 s.

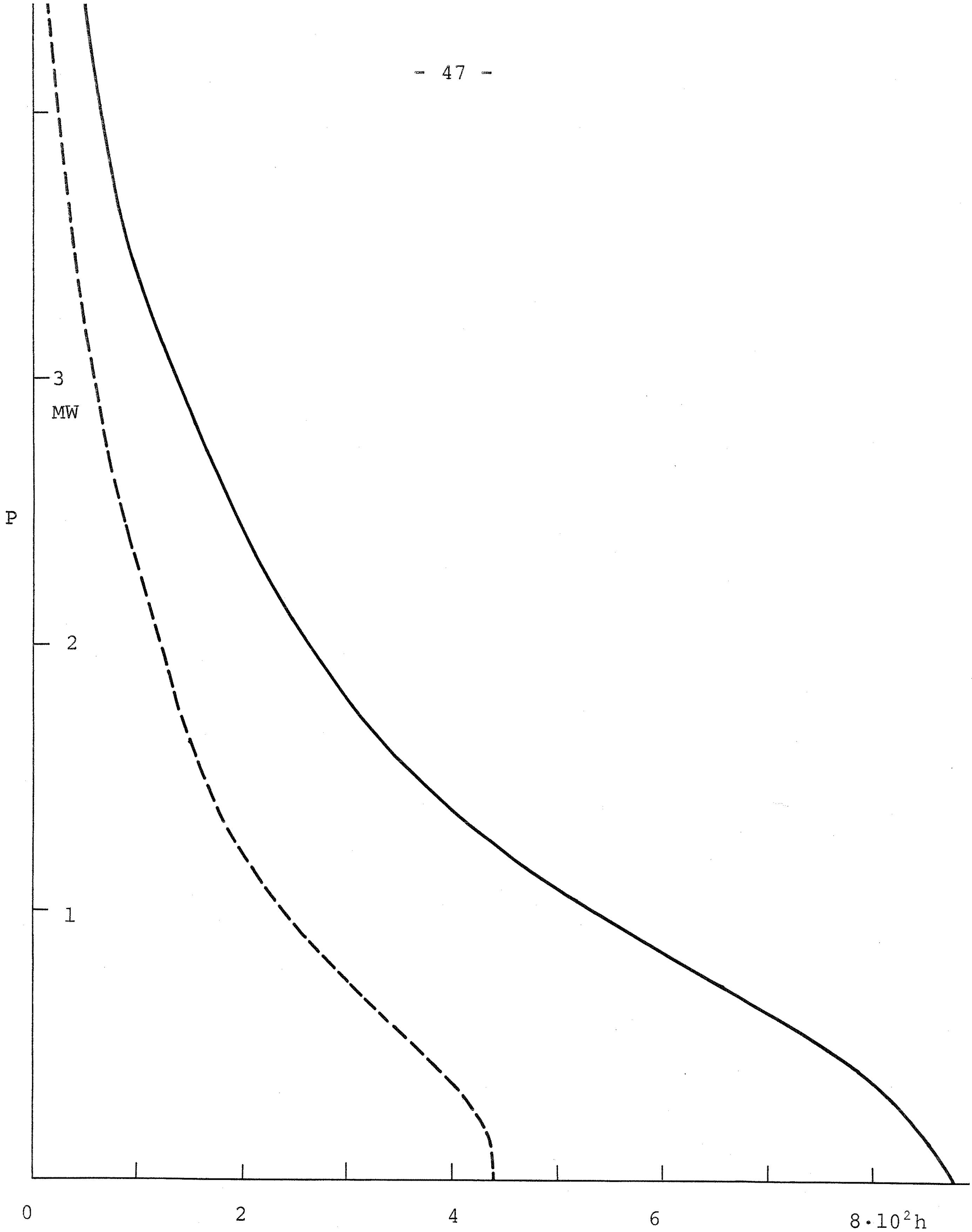


Fig.11 Duration curve for optimum power absorbtion by a tank with diameter equal to 16 m and height $2\ell = 10$ m placed in the North Atlantic. The curve gives the number of hours of the year in which P exceeds the indicated value. The dashed curve is the duration curve for a tank placed outside Halten, Norway during a 6 months winter season (see Fig.1).

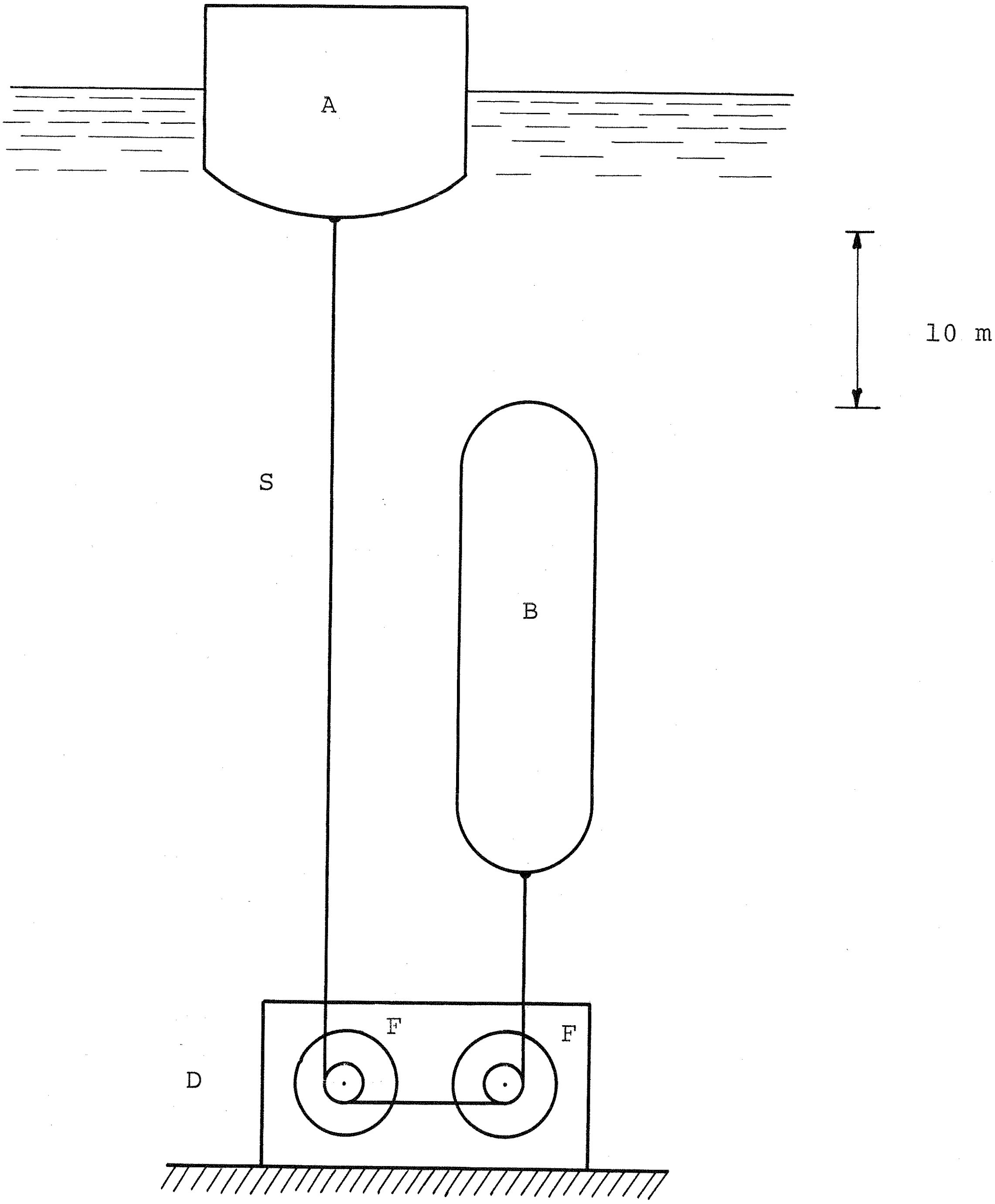


Fig.12 Heaving tank resonator. The tank A is semisubmerged by means of a wire S streched by the buoyancy of the tank B. The wire S drives the flywheels F placed in a housing D on the ocean bottom. There is no water access to the flywheel housing. An electric generator (not shown) is mechanically connected to the movement of the flywheels.

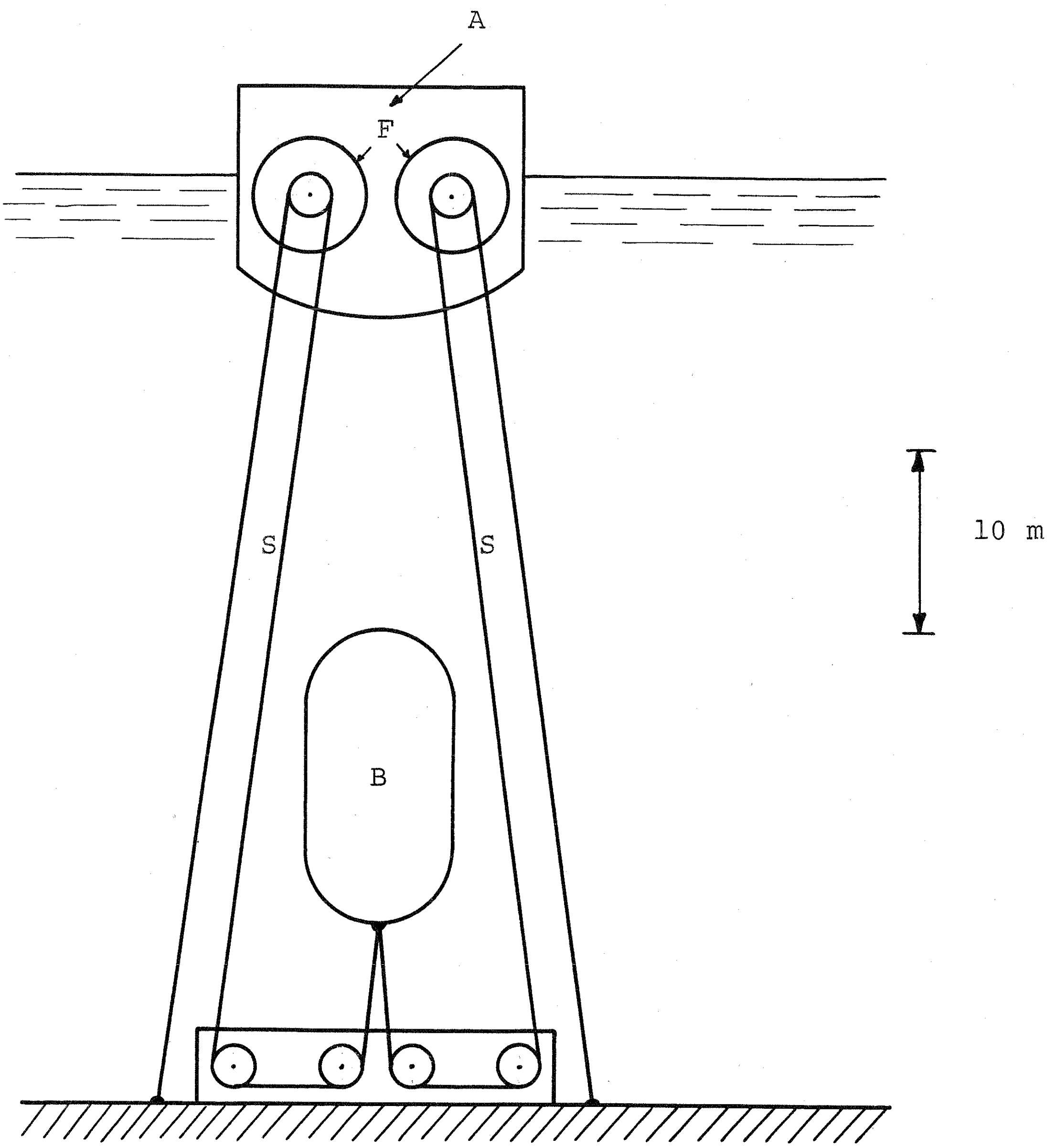


Fig. 13 Heaving tank resonator. The tank A is semisubmerged by means of wires S streched by the buoyancy of the tank B. The wires drive two flywheels F placed in the tank A. An electric generator is mechanically connected to the movement of the flywheels. This system can extract energy also from the rolling motion of the tank A.

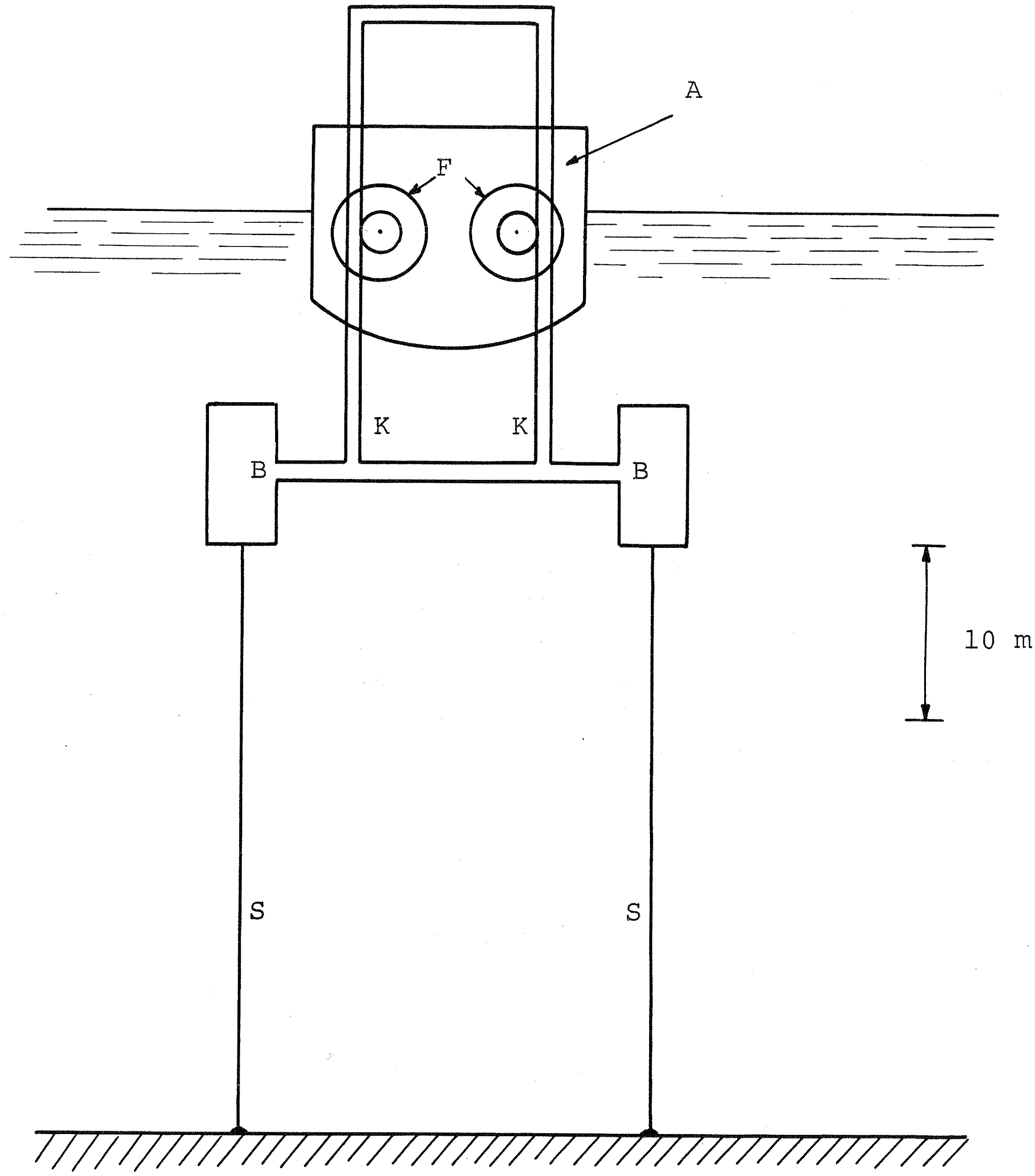


Fig.14 Heaving tank resonator. The tank A is semisubmerged by its own weight. Below A is placed a ring shaped, empty tank B held in position by means of the wires S. The tank B supports the tooth rails K that drive the flywheels F.

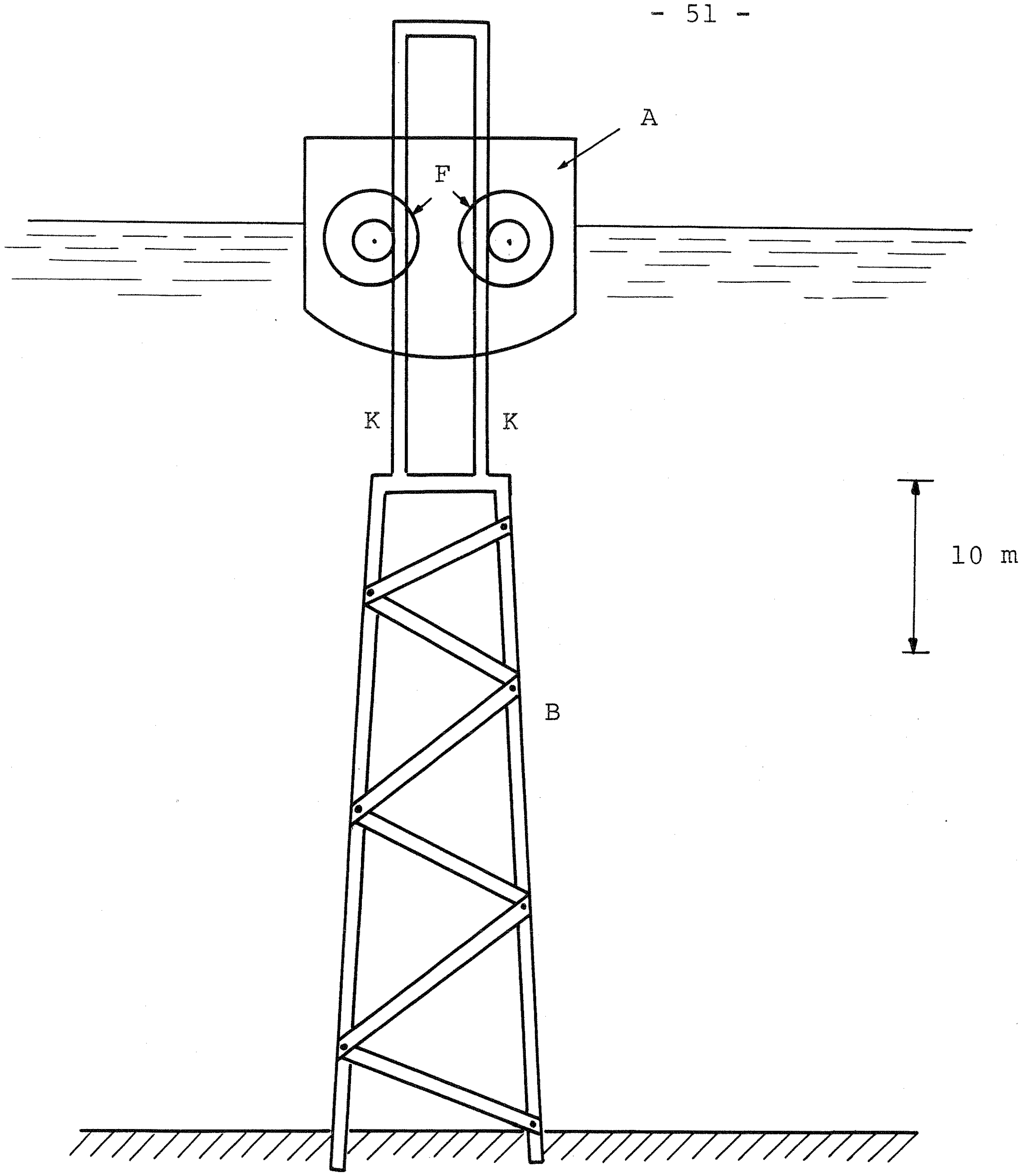


Fig.15 Heaving tank resonator. The tank A is semisubmerged by its own weight. Below A is placed a framework B whose leg piles are rigidly connected to the ocean bottom. The framework supports the tooth rails K that drive the flywheels F.

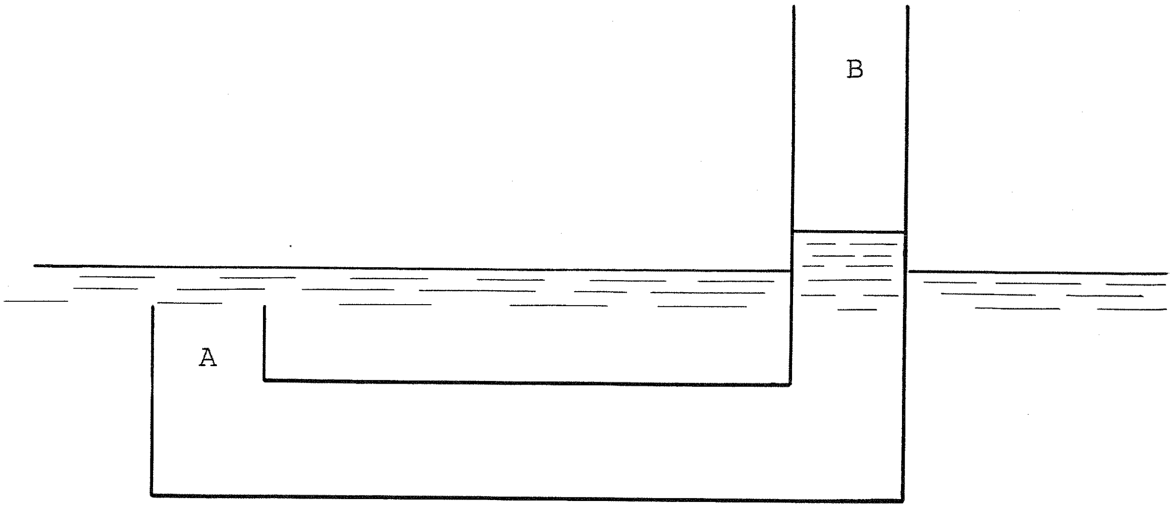


Fig.16 Water column resonator. The entrance A of a U-shaped tube is placed a few metres below the water surface. The other end B of the tube is placed above the water level. The incoming waves establishes a time-varying pressure at the entrance A of the tube. As a result the water column in the tube is put into oscillations.

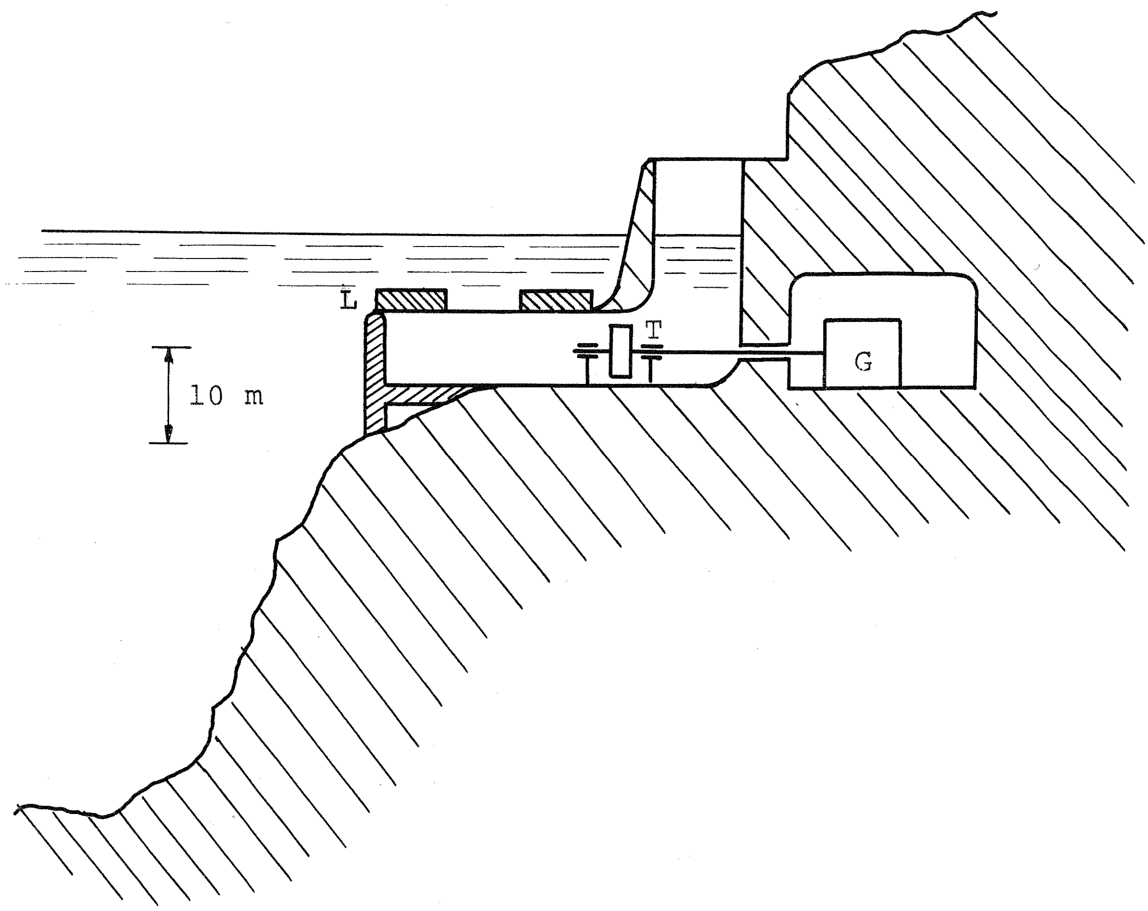


Fig. 17 Water column resonator of rectangular cross section, partly constructed as a tunnel and extended as a concrete bed. The length of the oscillating water column and, hence, the resonant frequency are adjusted by horisontally positioning the lids L along the bed. A fraction of the energy in the water oscillation is delivered to the electric generator G by means of the rotationally oscillating turbine T.

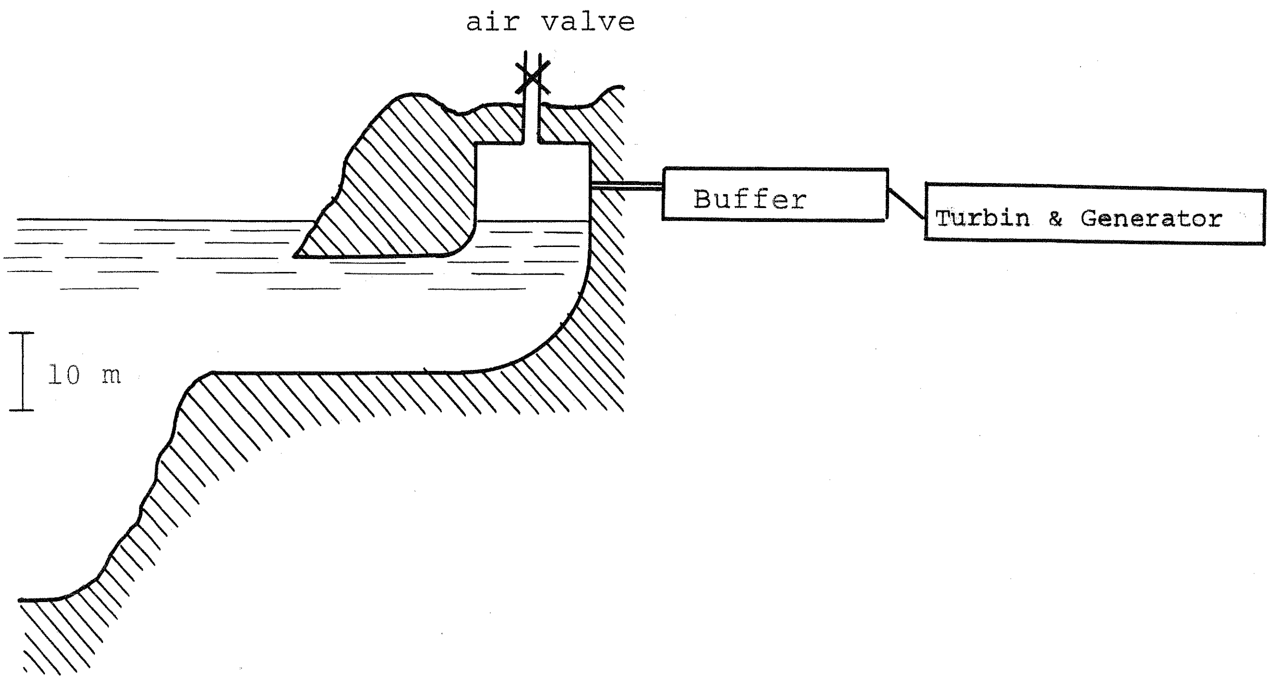


Fig.18 Water column resonator with enclosed air cushion at the upper end. The resonator is tuned by controlling the amount of air in the cushion. Power is taken from the resonator by pumping water or pressurised air into a buffer. A turbine is driven continuously by water or air, respectively, from this buffer. At a cliffy shore which is precipitous to a depth of roughly 40 to 50 m or more, the resonator chamber may be constructed as a tunnel in the cliff.

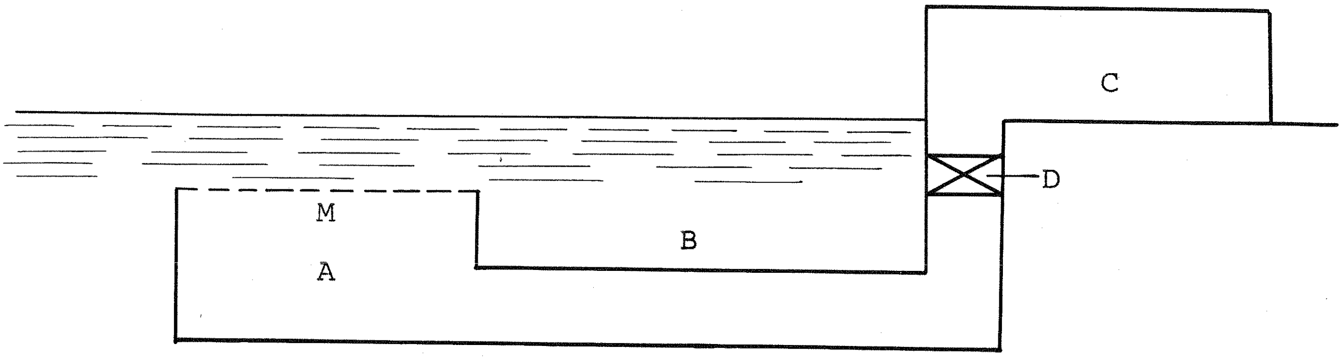


Fig.19 "Balloon" resonator. An air filled tank A placed below the water surface is connected to a tube B. The other entrance of the tube is connected to a large air reservoir C on land. The tank A has a flexible membrane M. When the water is still the membrane is kept plane by the air pressure in the system A-B-C. Ocean waves causes the membrane M to oscillate. Air is driven between A and C. The turbine D in the tube B generates electric energy.

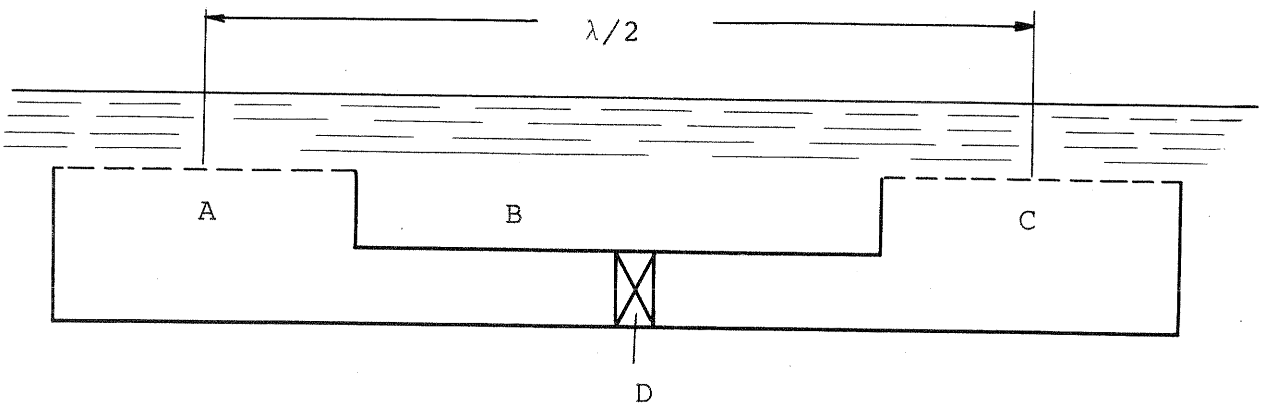
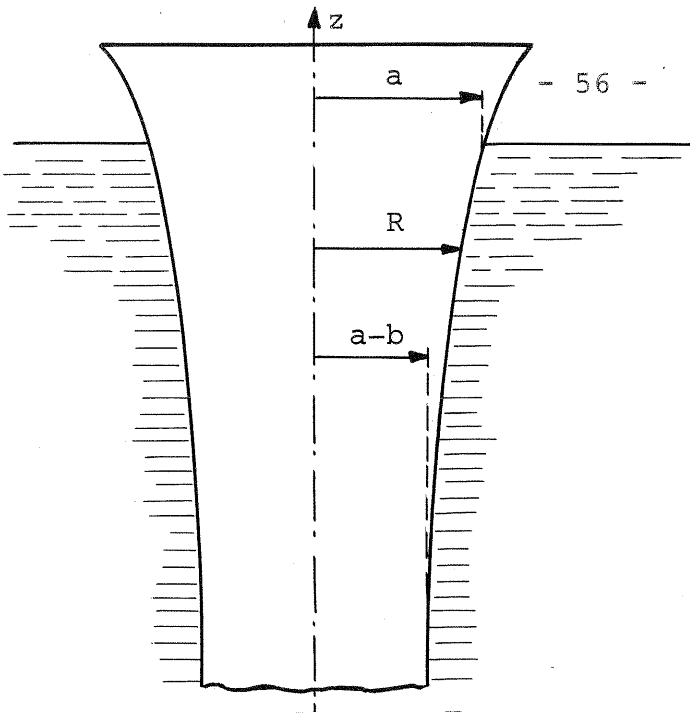


Fig.20 Two air filled tanks A and C with flexible membrams M are placed below the water surface. The tanks are placed half a wavelength apart and are connected by a tube B with an air turbine D.



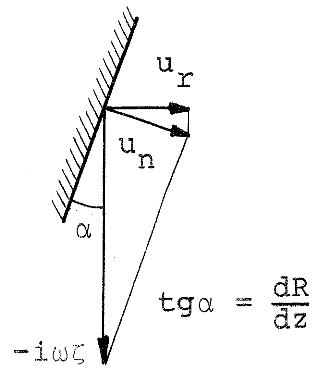


Fig.21 Rotationally symmetric floating body with radius $R = R(z) = a-b+be^{kz}$

Fig.22 Velocity components at the surface of the body

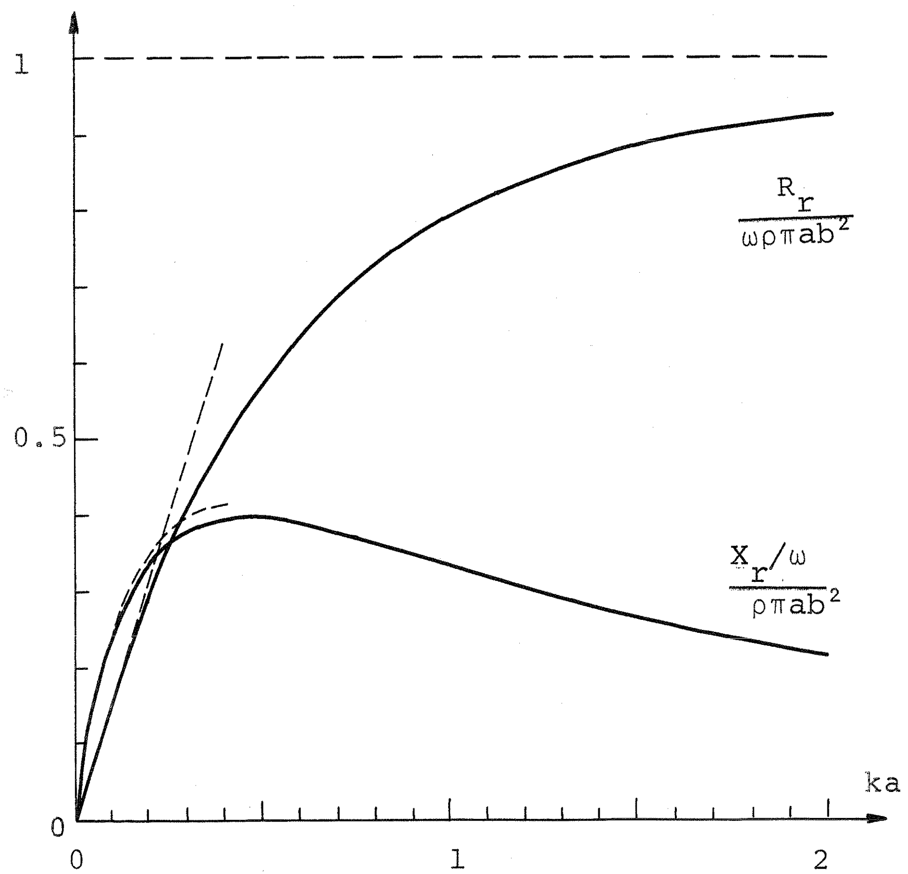


Fig.23 Radiation impedance $Z_r = R_r + X_r$ as a function of ka for body with exponentially varying cross section. Cf. eqs.(A.39) and (A.47). The horisontal dashed line gives the asymptotic value of R_r for larger values of ka. The two other dashed curves correspond to the approximations (A.48) and (A.49) for small ka.

REFERENCES

1. See: J. Flatseth and B. Pedersen, Distribution of wave height in different ocean areas around the world, Det Norske Veritas - Research Department, Report no. 70-7-S.

2. O.G. Houmb, private communication.

3. See for instance: J.N. Svašek, The Statistical Evaluation of Wave Conditions in a Deltaic Area, Symposium "Research on Wave Action", Proceedings, Vol.1. See also A. Paape, General report on Session 1, in the same volume. (Delft Hydraulics Laboratory, Delft, The Netherlands 1969).

4. See for instance: E. Meyer and E.G. Neumann, Physikaliske und Technische Akustik (Friedr. Vieweg & Sohn, Braunschweig, 1967), p. 180.

5. R.L. Wiegel, Oceanographic Engineering (Prentice - Hall, 1964), p. 139.

6. J.N. Newman, The motions of a spar buoy in regular waves, Report 1499, David Taylor Model Basin (May, 1963).

7. T. Havelock, Waves due to a floating sphere making periodic heaving oscillations, Proc. Roy.Soc. Vol. 231 A, pp. 1-7 (1955).

8. K. Hasselman et al., Measurements of Wind-Wave Growth and Swell Decay During the Joint North Sea Wave Project (Jonswap), Deutches Hydrographisches Zeitschrift, No. 12, 1973.

9. O.G. Houmb and H. Rye, Analysis of Wave Data from the Norwegian Continental Shelf, Second International Conference on Port and Ocean Engineering Under Arctic Conditions, University of Ice-land (1973).

10. J.V. Wehausen and E.V. Laitone, Surface Waves, Handbuch der Physik, Bd IX, pp. 446-778, (1960).

11. See for instance, Ref. 10, pp. 475-478.

12. T.H. Havelock, Forced surface-waves on water, Phil. Mag. S. 7. Vol. 8, pp. 569-576 (1929).

13. See e.g. Jahnke-Emde-Lösch, Tafeln höherer Funktionen (Teuber, Stuttgart, 1960) p. 155.