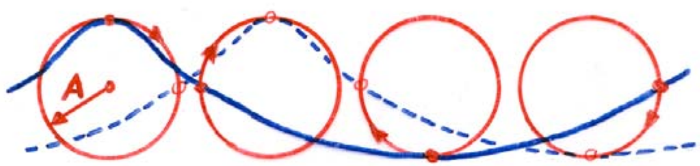
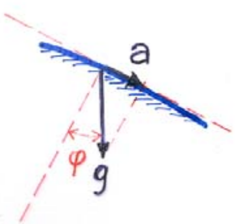
Overhead transparencies, handwritten by Johannes Falnes.

TYNGDEBØLGJE PÅ DJUPT VATN
GRAVITY WAVE ON DEEP WATER





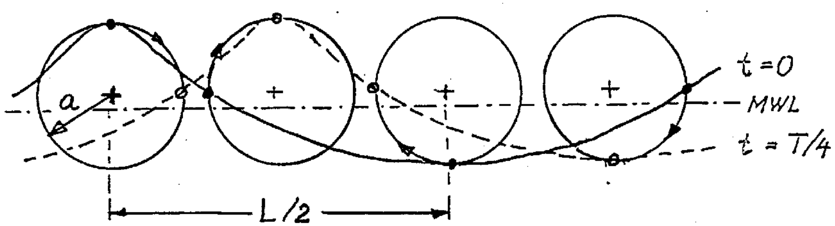
$$a = g \sin\varphi \approx g \tan\varphi = -g\,\frac{\partial\eta}{\partial x}$$

$$\eta = A \cos(\omega t - kx)$$

$\frac{\partial\eta}{\partial x}$ har amplitude kA

a har amplitude gkA

CIRCULAR ORBITS for water particles in a wave.



For a regular wave progressing on deep water the water particles move approximately in circular orbits.

A particular waterparticle moves once around its circle in a time T (the "period").

The crests of the wave project farther above the mean water level (MWL) than the troughs sink under it.

The troughs are wider than the crests.

The difference is negligible for small waves

$$a \ll L/2\pi = 1/k.$$

Then the water surface has a sinusoidal shape.

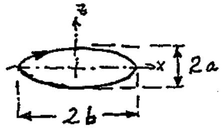
"Small waves" = "linear waves" (linear theory applicable)

$\lambda = L$ is the "wavelength"

$k = 2\pi/L$ is the "angular repetency" (or "wavenumber")

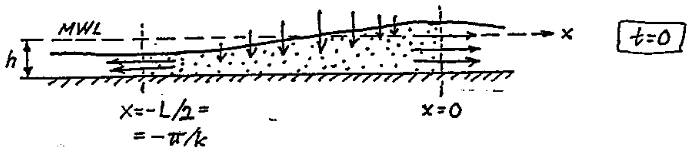
On <u>shallow water</u>: L5

- elliptical orbits



Excursion of water particle $\eta = s_z = a\cos(\omega t - kx)$
$s_x = b\sin(\omega t - kx)$

Velocity of water particle $\dot{s}_z = v_z = -\omega a \sin(\omega t - kx)$
$\dot{s}_x = v_x = \omega b\cos(\omega t - kx)$



Water flow outwards: $(v_x)_{max}\, 2h = \omega b 2h$ (shallow water)

Waterflow downwards:

$$-\int_{-L/2}^{0} v_z\big|_{t=0}\, dx = \omega a \int_{-L/2}^{0} \sin(-kx)\, dx = \omega a \left[\frac{\cos(kx)}{k}\right]_{-L/2}^{0}$$

$$= \frac{\omega a}{k}(1-(-1)) = \frac{2\omega a}{k} = \frac{\omega a L}{\pi}$$

Balance of flow: $\omega b 2h = \omega a L/\pi$ $\boxed{\frac{a}{b} = kh = \frac{2\pi h}{L}}$

On very shallow water the maximum values of the water particle excursion and the water velocity have a ratio

$$a/b = \omega a/\omega b = kh = 2\pi h/L$$

between the vertical and horizontal components.
("shallow water" means $kh \ll 1$)

On <u>deep water</u> ($kh \gg 1$)

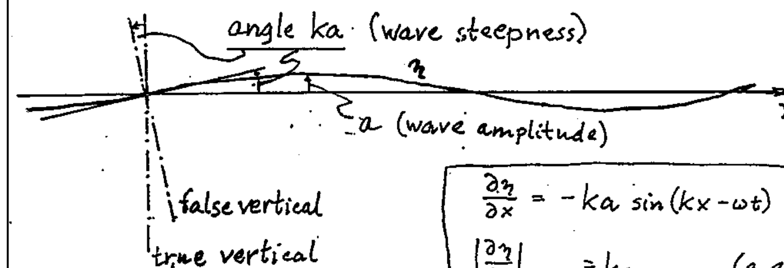
- circular orbits $b = a$
$(v_x)_{max} = (v_z)_{max} = \omega a$

<u>The phase velocity $c = \omega/k$ for water waves:</u>

Assume regular wave of small amplitude, that is of sinusoidal shape.
$\eta = a\cos(kx - \omega t)$

The maximum tilt (the "steepness") of the wave is

$$ka = 2\pi a/L$$



$$\boxed{\begin{array}{l} \dfrac{\partial\eta}{\partial x} = -ka\sin(kx-\omega t) \\ \left|\dfrac{\partial\eta}{\partial x}\right|_{max} = ka \qquad \text{(q.e.d.)} \end{array}}$$

By "small" amplitude a we mean $ka \ll 1$

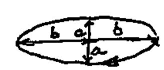
Then $\sin(ka) \approx \tan(ka) \approx ka$

This is also angle between false vertical and true vertical

$$\frac{\text{horizontal acceleration}}{\text{gravity acceleration } (g)} = \text{tangent of angle of tilt}$$

Maximum horizontal acceleration: $(\dot{v}_x)_{max} = g\tan(ka) \approx gka$

From orbit motion: $(\dot{v}_x)_{max} = (\ddot{s}_x)_{max} = \omega^2 b$



$$gka = \omega^2 b \qquad \boxed{\omega^2 = gk\frac{a}{b}}$$

Phase velocity: $c = \frac{L}{T} = \frac{\omega}{k} = \frac{g}{\omega}(a/b) = \sqrt{\frac{g}{k}(a/b)}$

The wavelength: $L = 2\pi/k = \frac{g}{2\pi}T^2(a/b)$

The period: $T = 2\pi/\omega = \left\{2\pi\frac{L}{g}(b/a)\right\}^{1/2}$

For "deep" water (that is $kh \gg 1$):

we have $b/a = 1$ (circular orbit)

Then $c = \frac{g}{\omega} = \left\{\frac{g}{k}\right\}^{1/2} = \frac{g}{2\pi}T = \left\{\frac{g}{2\pi}L\right\}^{1/2}$

$$L = cT = \frac{g}{2\pi}T^2$$

$\frac{g}{2\pi} = \frac{9.81\ m/s^2}{2\pi} = 1.56\ m/s^2$

Example: $T = 10\,s$: $c = 15.6\ m/s$ $L = 156\ m$

($= 56.2\ km/h$)

The deep-water approximation is usually acceptable if $h > L/3$ ($kh > 2$)

For shallow water (that is $kh \ll 1$):

we found $a/b = kh = 2\pi h/L$

$$\therefore\ \omega^2 = gk(a/b) = gkkh = ghk^2$$

$$c = \frac{L}{T} = \frac{\omega}{k} = \sqrt{gh}$$

$$L = cT = T\sqrt{gh}$$

Example: $h = 6\ m$ $c = \sqrt{9.81 \cdot 6.0} = 7.7\ m/s$

($= 27.6\ km/h$)

If $T = 20\,s$ $L = 153\ m$

The shallow-water approximation is usually acceptable if $h < L/20$ ($kh < 0.3$)

Superposition of two waves of the same amplitude but slightly different frequencies:

$\eta_1 = a\cos(\omega_1 t - k_1 x)$ $\eta_2 = a\cos(\omega_2 t - k_2 x)$

$\eta(x,t) = \eta_1(x,t) + \eta_2(x,t)$

Recalling the trigonometric identity

$$\cos\alpha_1 + \cos\alpha_2 = 2\cos\frac{\alpha_1 - \alpha_2}{2}\cos\frac{\alpha_1 + \alpha_2}{2}$$

$$\eta = 2a\cos\left(\frac{\omega_2 - \omega_1}{2}t - \frac{k_2 - k_1}{2}x\right)\cos\left(\frac{\omega_2 + \omega_1}{2}t - \frac{k_2 + k_1}{2}x\right)$$

Set $\omega_1 = \omega - \Delta\omega$ $\omega_2 = \omega + \Delta\omega$

$k_1 = k - \Delta k$ $k_2 = k + \Delta k$

$$\eta = 2a\underbrace{\cos(\Delta\omega\, t - \Delta k\, x)}\cos(\omega t - kx)$$

Varies slowly if $\Delta\omega \ll \omega$ (and hence $\Delta k \ll k$)

Resulting wave of "angular frequency" ω and a slowly varying "amplitude"

$$2a\cos(\Delta\omega\, t - \Delta k\, x)$$

This "amplitude" propagates with a speed

$$c_g = \frac{\Delta\omega}{\Delta k}$$

which (if $\Delta\omega \to 0$) is termed the <u>group velocity</u>

We found $\omega^2 = gk\,(a/b)$ (the "dispersion" equation)

On deep water $(a/b = 1)$ $\omega^2 = gk \Rightarrow 2\omega\,d\omega = g\,dk$

Phase velocity $c = \frac{\omega}{k} = \frac{g}{\omega} = \sqrt{g/k}$

Group velocity $c_g = \frac{d\omega}{dk} = \frac{g}{2\omega} = \frac{1}{2}c$

Thus we have the important result that on deep water the group velocity is half of the phase velocity

On very shallow water $(a/b = kh)$ $\omega = k\sqrt{gh}$

$c_g = \frac{d\omega}{dk} = \frac{\omega}{k} = \sqrt{gh} = c$

The group velocity and phase velocity are equal and independent of the frequency so long as the water may be considered to be shallow $(\omega\sqrt{h/g} = kh << 1)$

Example:

Assume that the sea is calm. Then, suddenly a storm develops $l = 300$ km from land. How long time afterwards can we record swells of period $T = 14$ s at the shore? What if the period is $T = 10$ s? Assume that the water depth is more than 200 m.

Solution:

Deep-water formulas are applicable because $L = 1{,}56\,T^2 = 1{,}56 \cdot 14^2 = 306$ m, that is $h > 200\text{ m} > L/3 = 102$ m

The group velocity is

$c_g = \frac{1}{2}c = \frac{1}{2}\frac{L}{T} = \frac{1}{2}\,1{,}56 \times T = 0{,}78\,T$

$= 0.78 \times 14 = 10.9$ m/s

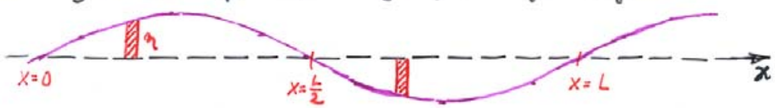
Time before swell record

$\Delta t = \frac{l}{c_g} = \frac{l}{0{,}78\,T} = \frac{300 \cdot 10^3}{0{,}78\,T} = \frac{3.8 \cdot 10^5}{T}$

$T = 14$ s: $\Delta t = \frac{3{,}8 \cdot 10^5}{14} = 27 \cdot 10^3$ s $= 7.6$ hours

$T = 10$ s $\Delta t = \frac{3{,}8 \cdot 10^5}{10} = 38 \cdot 10^3$ s $= 10.7$ hours

Potential energy for wave on sea surface (averaged over time)
Potensiell energi pga. bølgje på havflata (midla over tid)
Rectangle: Length = 1 wavelength, Width: 1 unit of Length (e.g. 1 m)
Rektangel 1 bølgjelengd langt og 1 lengdeining breitt



$$E_p L = \int_0^{L/2} \rho g \eta \frac{\eta}{2} 2\,dx = \rho g \int_0^{L/2} \eta^2 dx = \rho g \frac{L}{2}\frac{1}{2}|\eta|^2_{max}$$

Pot. energy per area unit of ocean surface
Potensiell energi pr. flateining av havflata:

$$E_p = \rho g \frac{1}{4} |\eta|^2_{max}$$

It can be shown that there is equally much kinetic energy
Det kan visast at der er like mykje kinetisk energi E_k

Stored energy per area unit ocean surface
Lagra energi pr. flateining havflate:

$$E = E_p + E_k = 2E_p = 2E_k = \frac{1}{2} \rho g |\eta|^2_{max}$$

$$[E] = J/m^2$$

Flow of energy per unit length of the wave front
Energistraum J pr. lengdeining av bølgjefronten:

$$J = c_g E = \frac{c_g}{2}\rho g|\eta|^2_{max}$$

c_g = gruppefarten = group velocity

$$[J] = \frac{m}{s}\frac{J}{m^2} = \frac{W}{m}$$

På djupt vatn: $c_g = c_f/2 = \frac{g}{2\omega} = \frac{gT}{4\pi}$

$$J = \frac{\rho g^2}{4\omega}|\eta|^2_{max} = \frac{\rho g^2 T}{8\pi}|\eta|^2_{max} = \frac{\rho g^2 T}{32\pi} H^2 = \left(976 \frac{W}{s\cdot m^3}\right) T H^2$$

$H = 2|\eta|_{max}$ = bølgjehøgda = wave height

[For $\eta = A\cos(\omega t - kx + \alpha) + B\cos(\omega t + kx + \beta)$:

$$J = \frac{\rho g^2}{4\omega}(A^2 - B^2) \qquad \langle E\rangle = \frac{\rho g}{2}(A^2+B^2)$$

For ståande bølgje er $B=A$ og dermed $J = 0$]
For standing wave: and hence

Djupt vatn — Deep water, Progressive wave
Progressiv bølgje $\eta = A\cos(\omega t - kx + \alpha)$

$$J = \frac{\rho g^2}{4\omega}A^2 = \frac{\rho g^2}{8\pi f}A^2$$

$$E = 2E_p = 2E_k = \frac{1}{2}\rho g A^2 = \rho g \overline{\eta^2}$$

Composed wave
Samansett bølgje $\eta = \sum_m A_m\cos(\omega_m t - k_m x + \alpha_m)$

$$J = \sum_m \frac{\rho g^2}{8\pi f_m}A_m^2 \qquad E = \frac{1}{2}\rho g\sum_m A_m^2$$

Meir generelt — More generally

$$\eta = \eta(x,y,t)$$

$$E = \rho g\overline{\eta^2(x,y,t)} \equiv \rho g\int_0^\infty S(f)\,df$$

$S(f)$ = „energispektret", „spektret" energy spectrum

Spectrally defined "significant wave height"
Spektralt definert „signifikant bølgjehøgd"

$$H_{m0} = 4\sqrt{m_0} \qquad m_0 \equiv \int_0^\infty S(f)\,df$$

$$J = \frac{\rho g^2}{4\pi}\int_0^\infty \frac{S(f)}{f}\,df = \frac{\rho g^2}{4\pi}m_{-1}$$

$$m_{-1} \equiv \int_0^\infty f^{-1}S(f)\,df$$

(Spectral moment of order
Spektralt moment av orden j: $m_j \equiv \int_0^\infty f^j S(f)\,df$)

„Energiperioden" $T_J = T_{-1,0} \equiv \frac{m_{-1}}{m_0}$ = "energy period"

$$J = \frac{\rho g^2}{4\pi}T_J m_0 = \frac{\rho g^2}{64\pi}T_J H_{m0}^2$$