MECHANICAL OSCILLATOR

AND ITS APPLICATION FOR

ABSORPTION OF WAVE ENERGY.

Lecture notes, handwritten by

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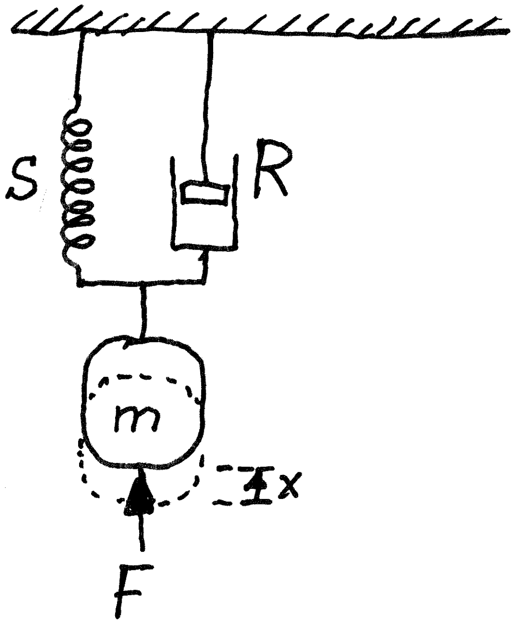
Mechanical oscillator



Mass m displaced a distance x from equilibrium position

Forces:

F = external applied force

$F_s = -Sx$ spring force

$F_R = -R\dot{x}$ damping force

For mathematical simplicity the damping force is assumed proportional to the velocity $\dot{x} = \frac{dx}{dt}$

R = "mechanical resistance"

The spring force is assumed proportional to the displacement x from equilibrium.

Newton: $m\ddot{x} = F + F_S + F_R = F - Sx - R\dot{x}$

$$m\ddot{x} + R\dot{x} + Sx = F \qquad (2.2)$$

(External force balanced against 1) inertial force 2) damping force and 3) spring force)

Energy = Force × Displacement
Power = Force × Velocity

Power applied to the mechanical system

$$P = F\dot{x} = m\ddot{x}\dot{x} + R\dot{x}^2 + Sx\dot{x} =$$
$$= m\dot{x}\frac{d\dot{x}}{dt} + R\dot{x}^2 + S\frac{dx}{dt}x =$$
$$= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) + R\dot{x}^2 + \frac{d}{dt}\left(\frac{1}{2}Sx^2\right)$$
$$= R\dot{x}^2 + \frac{d}{dt}\left(W_k + W_p\right)$$

$$W_k = \frac{1}{2}m\dot{x}^2 = \text{kinetic energy} \qquad (2.73)$$
$$W_p = \frac{1}{2}Sx^2 = \text{potential energy (of the spring)} \qquad (2.74)$$

Assume periodic oscillation. After one period $T = 2\pi/\omega$ the stored energy $W_k + W_p$ is the same (as one period earlier)

The average applied power $\overline{P}$ equals the average damping power $\overline{P_R}$

$$\overline{P} = \overline{F\dot{x}} = \overline{R\dot{x}^2} = \overline{P_R}$$

Assume the applied force F is given $\underline{F = F_0 \cos(\omega t)}$

What is $\dot{x}$?

Try

$$x = x_0 \sin(\omega t - \varphi) \qquad (2.11)$$

$$\dot{x} = \omega x_0 \cos(\omega t - \varphi) = u_0 \cos(\omega t - \varphi)$$

$$\ddot{x} = -\omega^2 x_0 \sin(\omega t - \varphi)$$

$$F = R\dot{x} + Sx + m\ddot{x}$$

$$F_0 \cos(\omega t) = R u_0 \cos(\omega t - \varphi) + \left(\frac{S}{\omega} - \omega m\right) u_0 \sin(\omega t - \varphi) =$$

$$= R u_0 \{\cos\varphi \cos\omega t + \sin\varphi \sin\omega t\}$$

$$+ \left(\frac{S}{\omega} - \omega m\right) u_0 \{-\sin\varphi \cos\omega t + \cos\varphi \sin\omega t\} =$$

$$= (R\cos\varphi + (\omega m - S/\omega)\sin\varphi)\, u_0 \cos(\omega t)$$

$$+ (R\sin\varphi - (\omega m - S/\omega)\cos\varphi)\, u_0 \sin(\omega t)$$

$$R\sin\varphi = (\omega m - S/\omega)\cos\varphi$$

$$\tan\varphi = \frac{\omega m - S/\omega}{R} \qquad \cos\varphi = \frac{R}{|Z|} \qquad \sin\varphi = \frac{\omega m - S/\omega}{|Z|}$$

$$|Z| = \{R^2 + (\omega m - S/\omega)^2\}^{1/2} \qquad (2.16)$$

$$F_0 = (R\cos\varphi + (\omega m - S/\omega)\sin\varphi)\, u_0$$

$$= (R^2/|Z| + (\omega m - S/\omega)^2/|Z|)\, u_0 = |Z| u_0$$

$$\dot{x} = u_0 \cos(\omega t - \varphi) \qquad \cos\varphi = \frac{R}{|Z|}$$

$$u_0 = F_0/|Z| = F_0\{R^2 + (\omega m - \tfrac{S}{\omega})^2\}^{-1/2} \qquad \sin\varphi = \frac{\omega m - S/\omega}{|Z|}$$

Resonance

Mechanical resistance R

Mechanical reactance $X = \omega m - S/\omega$

For $\omega = \sqrt{S/m} \equiv \omega_0 \quad X = 0$

$$|Z|_{min} = R \qquad u_{0,max} = \frac{F_0}{R} \qquad \varphi = 0$$

This is resonance.

If $\omega > \sqrt{S/m} \qquad X > 0 \qquad \frac{\pi}{2} > \varphi > 0$

If $\omega \to \infty \quad X \to +\infty \qquad \varphi \to \frac{\pi}{2} \qquad u_0 \to \frac{F_0}{\omega m} \to 0$

(Dynamics dominated by inertia)

If $\omega < \sqrt{S/m} \qquad X < 0 \qquad -\frac{\pi}{2} < \varphi < 0$

If $\omega \to 0 \quad X \to -\infty \qquad \varphi \to -\frac{\pi}{2} \qquad u_0 \to \frac{\omega F_0}{S} \to 0$

Dynamics dominated by the spring

$\frac{\omega}{\omega_0} = \omega\sqrt{\frac{m}{S}}$ normalised frequency [non-dimensional]

$\delta \equiv \frac{R}{2m}$ damping coefficient [dimension 1/s]

$\varkappa = \frac{\delta}{\omega_0} = \frac{R}{2m}\sqrt{\frac{m}{S}} = \frac{R}{2\sqrt{mS}}$ = damping factor [non-dimensional]

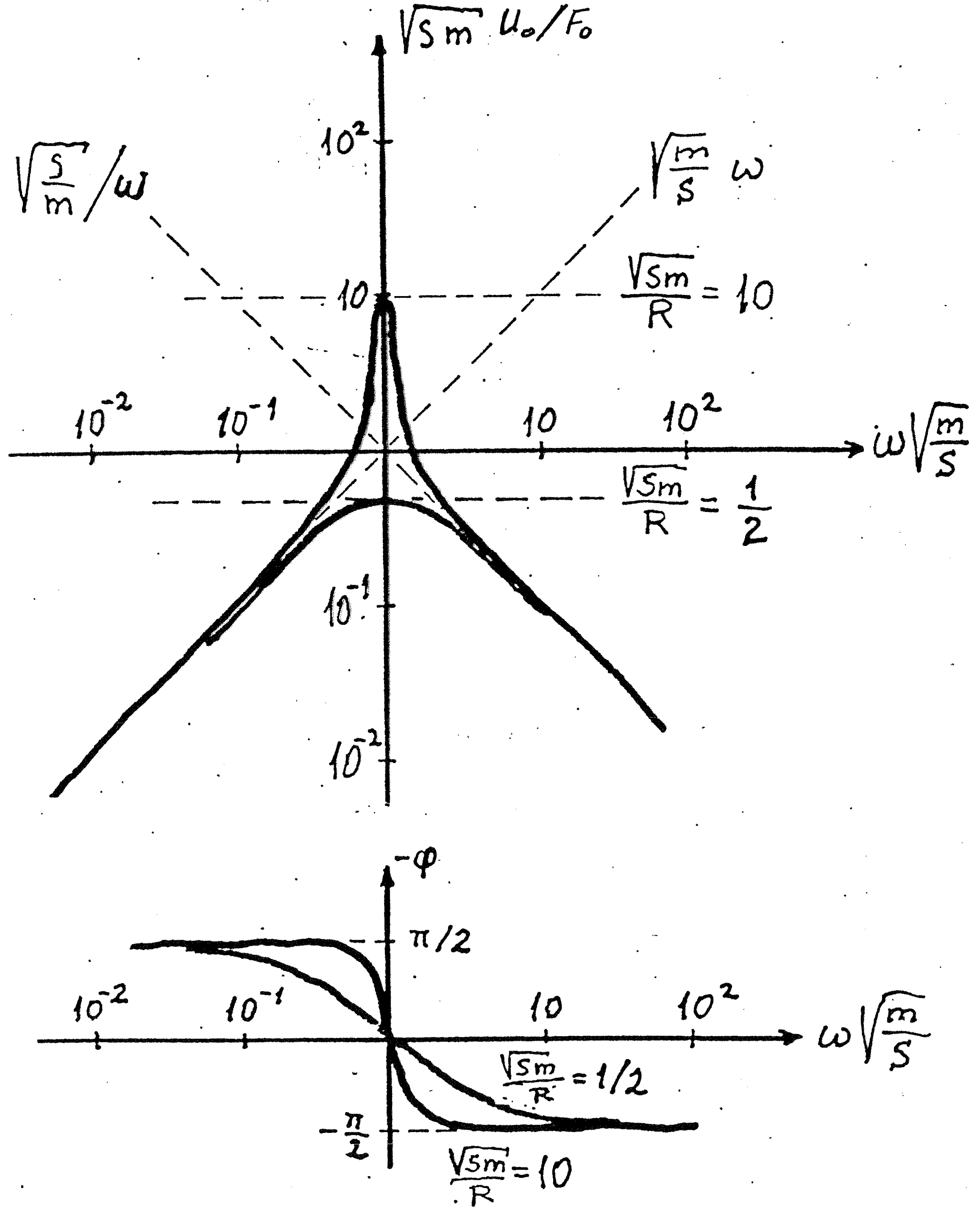


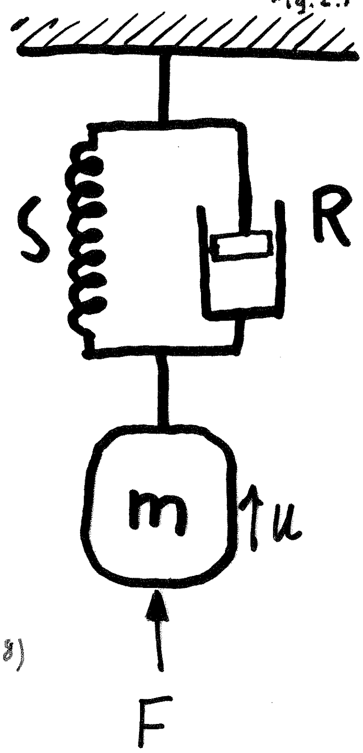
Fig. ~~A.9~~ 2.2. "Bode diagram". Frequency responce of relation between velocity u and applied force F in normalised units, for two different values of the damping coefficient.

a. Amplitude (modulus) response with both scales logaritmic.

b. Phase response with linear scale for the phase difference.

POWER AND ENERGY RELATIONS.

Fig. 2.1



Mechanical power (rate of work) delivered by supplying the external force $F(t)$:

$$P(t) = F(t)\,u(t) = F(t)\,\dot{x}(t)$$

$$F(t) = F_m(t) - F_R(t) - F_S(t)$$

$$= m\,a(t) + R\,u(t) + S\,x(t) \qquad (2.68)$$

$$P(t) = P_R(t) + P_k(t) + P_p(t)$$

$$P_R(t) = -F_R(t)\,u(t) = R\,u^2 \qquad (2.70)$$

$$P_k(t) = F_m(t)\,u(t) = m\,\dot{u}\,u = \frac{d}{dt}W_k(t)$$

$$W_k(t) = \frac{1}{2}m\,(u(t))^2$$ -kinetic energy.

$$P_p(t) = -F_S(t)\,u(t) = S\,x\,\dot{x} = \frac{d}{dt}W_p(t) \qquad (2.72)$$

$$W_p(t) = \frac{1}{2}S\,(x(t))^2$$ -potential energy.

Energy stored in the oscillating system:

$$W(t) = W_k(t) + W_p(t) \qquad (2.75)$$

$$P_k(t) + P_p(t) = \frac{d}{dt}W(t) \qquad (2.76)$$

$$F(t) = F_0 \cos(\omega t)$$

$$u(t) = \dot{x}(t) = u_0 \cos(\omega t - \varphi)$$

$$u_0 = F_0 / |Z| \qquad \cos\varphi = R/|Z|$$

$$|Z| = \sqrt{R^2 + X^2} \qquad X = \omega m - S/\omega \qquad (2.16)$$

$$x(t) = x_0 \sin(\omega t - \varphi) \qquad x_0 = u_0/\omega$$

$$\cos\alpha \cos\beta \equiv \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$

(2.67) $$P(t) = F(t)\, u(t) = F_0 u_0 \cos(\omega t) \cos(\omega t - \varphi)$$
$$= \frac{1}{2} F_0 u_0 \left[\cos\varphi + \cos(2\omega t - \varphi)\right]$$

Delivered power: $$P \equiv \overline{P(t)} = \frac{1}{2} F_0 u_0 \cos\varphi =$$
$$= \frac{1}{2} F_0^2 R/|Z|^2 = \frac{1}{2} R u_0^2 \qquad (2.79)$$

(2.70) $$P_R(t) = R\,[u(t)]^2 = R u_0^2 \cos^2(\omega t - \varphi)$$
$$= \frac{1}{2} R u_0^2 \left[1 + \cos(2\omega t - 2\varphi)\right]$$

Consumed power: $$P_R \equiv \overline{P_R(t)} = \frac{1}{2} R u_0^2 \qquad (2.81)$$

Obs! : $P_R = P$

But $P_R(t) \neq P(t)$ if $\varphi \neq 0$

The difference $P(t) - P_R(t)$ must be power exchanged with the energy store.

Instantaneous

kinetic energy $$W_k(t) = \frac{m}{2}[u(t)]^2 = \frac{m}{2}u_0^2\cos^2(\omega t-\varphi)$$
$$= \frac{m}{4}u_0^2\left[1+\cos(2\omega t-2\varphi)\right]$$

and

potential energy $$W_p(t) = \frac{1}{2}S[x(t)]^2 = \frac{1}{2}S x_0^2\sin^2(\omega t-\varphi)$$
$$= \frac{1}{4}S x_0^2\left[1-\cos(2\omega t-2\varphi)\right]$$

Average kinetic and potential energies

$$W_k \equiv \overline{W_k(t)} = \frac{m}{4}u_0^2 \qquad W_p \equiv \overline{W_p(t)} = \frac{1}{4}S x_0^2 = \frac{S u_0^2}{4\omega^2}$$

Instantaneous total stored energy:

$$W(t) = W_k(t) + W_p(t) = (W_k + W_p) + (W_k - W_p)\cos(2\omega t - 2\varphi)$$

Average total stored energy:

$$W = \overline{W(t)} = W_k + W_p = \frac{1}{4}(mu_0^2 + Sx_0^2) = \frac{1}{4}mu_0^2\left(1+\frac{\omega_0^2}{\omega^2}\right) \qquad (2.89)$$

Amplitude of oscillating part of total stored energy:

$$W_k - W_p = \frac{1}{4\omega}(\omega m - S/\omega)u_0^2 = \frac{1}{4\omega}X u_0^2 \qquad (2.90)$$

Instantaneous reactive power (rate of change of stored energy):

$$P_k(t) + P_p(t) = \frac{d}{dt}W(t) = (W_k - W_p)\,2\omega\left[-\sin(2\omega t - 2\varphi)\right] =$$
$$= -\frac{1}{2}X u_0^2\sin(2\omega t - 2\varphi) \qquad (2.91)$$

Checking that $P_k(t) + P_p(t) = P(t) - P_R(t)$

$P(t) - P_R(t) =$

$= P - P_R + \frac{1}{2} F_0 u_0 \cos(2\omega t - \varphi) - \frac{1}{2} R u_0^2 \cos(2\omega t - 2\varphi)$

$= 0 + \frac{1}{2} F_0 u_0 \cos(\psi + \varphi) - \frac{1}{2} R u_0^2 \cos\psi \quad (\psi \equiv 2\omega t - 2\varphi)$

$P(t) - P_R(t) = \frac{1}{2} F_0 u_0 (\cos\psi \cos\varphi - \sin\psi \sin\varphi) - \frac{1}{2} R u_0^2 \cos\psi =$

$= \left(\frac{1}{2} F_0 u_0 \cos\varphi - \frac{1}{2} R u_0^2\right) \cos\psi - \frac{1}{2} F_0 u_0 \sin\varphi \sin\psi$

$= (P - P_R) \cos(2\omega t - 2\varphi) - \frac{1}{2} F_0 u_0 \sin\varphi \sin(2\omega t - 2\varphi)$

$F_0 = |Z| u_0 \qquad R = |Z| \cos\varphi \qquad X = |Z| \sin\varphi$

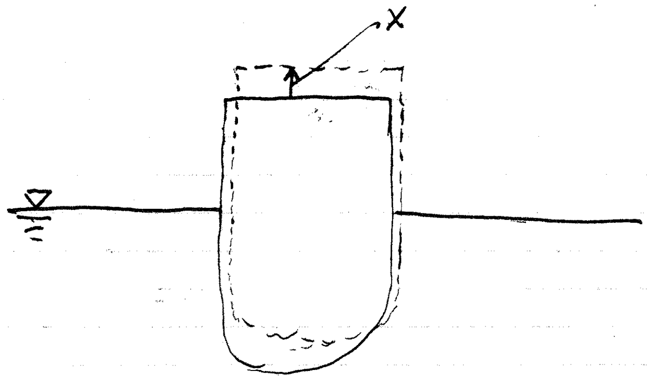
$P = \frac{1}{2} F_0 u_0 \cos\varphi = \frac{1}{2} R u_0^2 = P_R$

$P(t) - P_R(t) = -\frac{1}{2} F_0 u_0 \sin\varphi \sin(2\omega t - 2\varphi) =$

$= -\frac{1}{2} X u_0^2 \sin(2\omega t - 2\varphi) = P_k(t) + P_p(t)$

(Q.E.D.)

Stiffness S of buoyant floating body



Water plane area A_W. If body is lifted up a distance x from its equilibrium position, the upward buoyancy force is reduced by $\rho g A_w x$. Thus there will be a restoring force

$$F_s = -\rho g A_W x$$

acting in a direction to restore the equilibrium

(Note: position and force are chosen to be positive upwards)

$$F_s = -S x$$

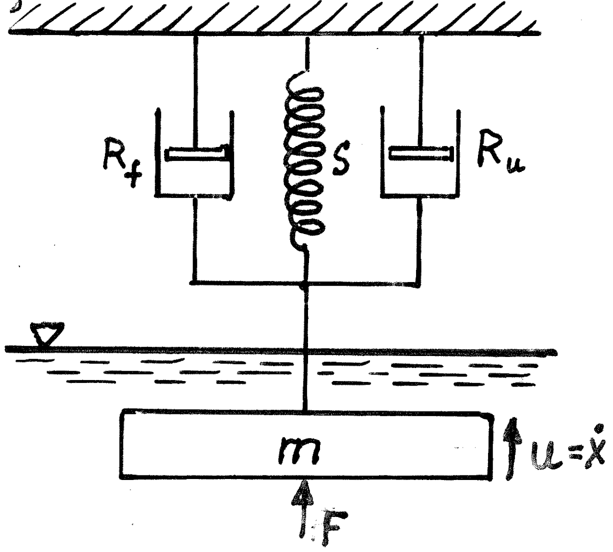
Hydrostatic stiffness (buoyancy stiffness) $S = \rho g A_W$

If the floating body is axisymmetric and of radius a (diameter $2a$) the "water plane area" is $A_W = \pi a^2$

$$S = \rho g \pi a^2 \qquad (5.3/4)$$

Mechanical oscillator interacting with waves.

Fig. 3.2



Mass m arranged to oscillate in water

It has an equilibrium position determined by a spring (e.g. hydrostatic "spring" for a floating body arranged to oscillate)

Mechanical resistances: R_f R_u R_r

$\overline{P_f} = R_f \overline{u^2}$ = power lost by friction (including viscous losses in the water, etc.)

$\overline{P_u} = R_u \overline{u^2}$ = useful converted power

$\overline{P_r} = R_r \overline{u^2}$ = radiated power (if a body in water oscillates, a wave is generated which carries away wave energy.)

R_f = <u>loss resistance</u>
R_u = (useful) <u>load resistance</u> (e.g. a pump providing fluid to a turbine)
R_r = <u>radiation resistance</u>

Not only the mass m but also the surrounding water attain velocity (and acceleration). Hence an <u>added hydrodynamic mass</u> m_r is included in the dynamic system.

Dynamic equation

$$m\ddot{x} + m_r * \ddot{x} + R_r * \dot{x} + R_u \dot{x} + R_f \dot{x} + Sx = F_e$$

F_e is the <u>wave excitation force</u> (that is, the wave force which the body experiences if it is not moving)

[$$m\ddot{x} + R_u \dot{x} + R_f \dot{x} + Sx = F_e - m_r * \ddot{x} - R_r * \dot{x} \equiv F_w$$

F_w is the total wave force when the body oscillates It is due to the total wave including the radiated wave which is generated by the body's motion.]

For simplicity let us neglect that m_r and R_r depend on frequency. [Thus, if m_r and R_r are constants the convolution multiplication (*) is just ordinary multiplication.]

Let $F_e = F_0 \cos(\omega t)$

$F_0 = f_0 A$ $\quad$ A = amplitude of incident wave

f_0 = coefficient of proportionality ("excitation force coefficient")

If $\quad F_e = F_e(t) = F_0 \cos(\omega t)$

then $\quad \dot{x} = u = u_0 \cos(\omega t - \varphi)$

$$u_0 = \frac{F_0}{|Z|} \qquad |Z| = \left\{(R_u + R_f + R_r)^2 + X_{tot}^2\right\}^{1/2}$$

$$\cos\varphi = (R_u + R_f + R_r)/|Z| \qquad X_{tot} = \omega(m + m_r) - S/\omega$$

$$\sin\varphi = X_{tot}/|Z|$$

Useful power:

$$\overline{P}_u = R_u \overline{u^2} = R_u \frac{1}{2} u_0^2 =$$

$$= \frac{R_u F_0^2/2}{(R_u + R_f + R_r)^2 + (\omega m + \omega m_r - S/\omega)^2} \qquad (3.39a)$$

Note $\overline{P}_u \to 0$ when $\omega \to 0$ and when $\omega \to \infty$

Otherwise $\overline{P}_u > 0$

When is $\overline{P}_u$ a maximum?

If it can be arranged that the total reactance vanishes $X_{tot} = 0$

or $\omega^2(m + m_r) = S$ $\qquad \omega = \omega_0 = \{S/(m+m_r)\}^{1/2} \qquad (3.47)$

then we have resonance.

This is the optimum condition w.r.t. m (or alternatively w.r.t. S)

If we have resonance

$$(\overline{P}_u)_{res} = \frac{R_u F_0^2/2}{(R_u + R_f + R_r)^2}$$

Assume R_r and R_f are given, what is the optimum choice of R_u?

The answer is $\boxed{(R_u)_{opt} = R_f + R_r}$ $\qquad (3.44a)$

[Show this, Hint: Differentiate $(\overline{P}_u)_{res}$ w.r.t. R_u]

Thus $\boxed{(\overline{P}_u)_{max} = \frac{F_0^2}{8(R_r + R_f)}}$ $\qquad (3.45a)$

Alternatively: Set $R_f + R_r = R_i$

$$(\bar{P}_u)_{res} = \frac{R_u F_0^2/2}{(R_u + R_i)^2} =$$

$$= \frac{F_0^2}{8R_i} \cdot \frac{4R_u R_i}{(R_u+R_i)^2} =$$

$$= \frac{F_0^2}{8R_i}\left[1 - \frac{R_u^2 + 2R_u R_i + R_i^2 - 4R_u R_i}{(R_u+R_i)^2}\right] =$$

$$= \frac{F_0^2}{8R_i}\left[1 - \left(\frac{R_u - R_i}{R_u + R_i}\right)^2\right]$$

$$(\bar{P}_u)_{res} \leq \frac{F_0^2}{8R_i} = (P_u)_{MAX}$$

Maximum obtained for $R_u = R_i (= R_f + R_r)$

Useful power

$$\overline{P_u} = R_u \overline{u^2} = R_u \tfrac{1}{2} u_0^2 \qquad u = \dot{x}$$

$$R_u \dot{x} = F_e - R_r \dot{x} - R_f \dot{x} - (m + m_r)\ddot{x} - Sx$$

$$R_u \dot{x}^2 = F_e \dot{x} - R_r \dot{x}^2 - R_f \dot{x}^2 \underbrace{- (m+m_r)\ddot{x}\dot{x} - Sx\dot{x}}$$

Time-average value

$$- \frac{m+m_r}{2} \frac{d}{dt} \dot{x}^2 - \frac{S}{2} \frac{d}{dt} x^2$$
$$= -\frac{d}{dt}\left(W_{kinetic} + W_{potential}\right)$$

$$\overline{P_u} = R_u \overline{\dot{x}^2} = \overline{F_e \dot{x}} - R_r \overline{\dot{x}^2} - R_f \overline{\dot{x}^2}$$

$$F_e \dot{x} = F_e u = F_0 \cos(\omega t)\, u_0 \cos(\omega t - \varphi)$$

$$\cos\alpha_1 \cos\alpha_2 = \tfrac{1}{2}\cos(\alpha_2 - \alpha_1) + \tfrac{1}{2}\cos(\alpha_1 + \alpha_2)$$

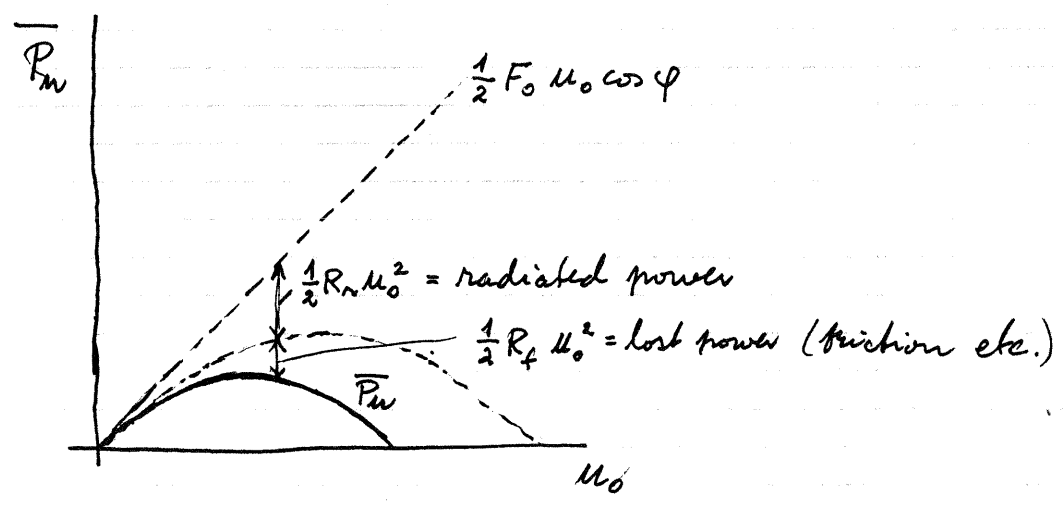
$$F_e \dot{x} = F_0 u_0 \left(\tfrac{1}{2}\cos\varphi + \tfrac{1}{2}\cos(2\omega t - \varphi)\right)$$

$$\overline{F_e \dot{x}} = \tfrac{1}{2} F_0 u_0 \cos\varphi$$

$$R_f \overline{\dot{x}^2} = \tfrac{1}{2} R_f u_0^2$$

$$\overline{P_u} = \tfrac{1}{2} F_0 u_0 \cos\varphi - \tfrac{1}{2} R_r u_0^2 - \tfrac{1}{2} R_f u_0^2$$

How does $\overline{P_u}$ depend on oscillation amplitude u_0 ?



F_0 is proportional to the wave amplitude. To make $\overline{P_u}$ large (for a given wave) ~~the~~ the phase angle φ should be small (resonance), since $(\cos\varphi)_{max} = 1$ for $\varphi = 0$. Resonance, or if not resonance use control system to obtain small φ.