Chapter K (pages 68-103) is an extension the paper Falnes, J. "Wave-power absorption by an array of attenuators oscillating with unconstrained amplitudes", *Applied Ocean Research*, 6(1), pp 16-22, 1984. Figures K.5 (p.87) and K.10 (p.97) were first published in Falnes, J. & Budal, K "Wave power conversion by point absorbers", *Norwegian Maritime Research*, 6(4), pp 2-11, 1978.

# Chapter K. PARALLEL ROWS OF OSCILLATING BODIES AND OWC-S

## K.1. INTRODUCTION

We consider an array of groups of bodies and OWC-s as indicated in fig. K1. The array may be infinitely long, extending along the y axis. In each group there are $N_k$ OWC-s and $N_i$ oscillating bodies. In the general case, with six degrees of freedom for each body, the total number of oscillators in each group is

$$N = N_k + 6\ N_i$$

Note that a 2-dimensional problem (with no variation in the y direction) is a special case of this situation.

If each group has an extension, in the x direction, comparable to or larger than a wavelength, and if the group is used in order

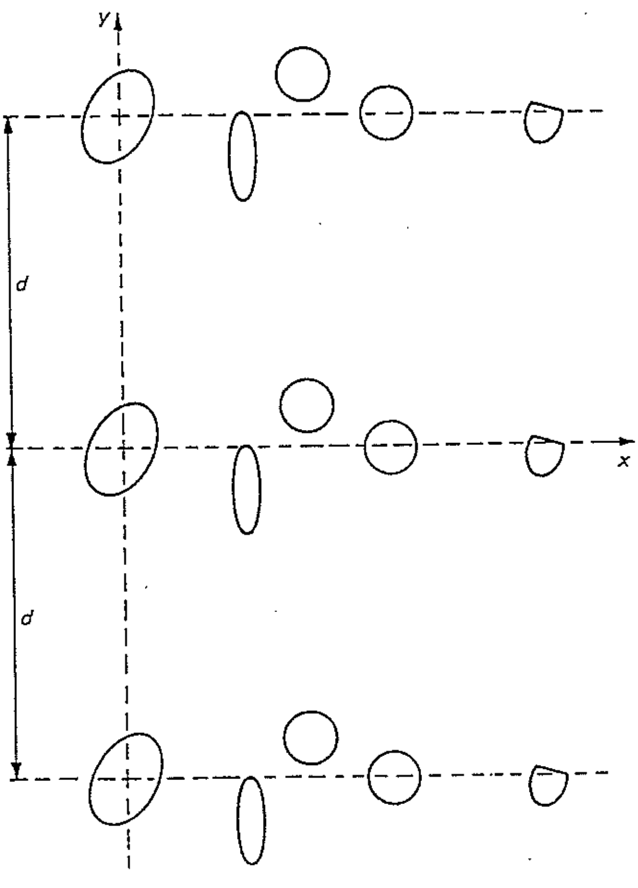


*Fig. K1. Array of equidistant equal groups of bodies and/or OWC-s distributed in the horizontal xy plane. In the figure each group (or "attenuator") consists of five bodies and/or OWC-s. Three groups of the infinite array are shown. Group no. m is placed in a localised region in the vicinity of the plane y = md (m=0,±1, ±2, ±3,...)*

to attenuate the wave (for instance, by extraction of wave energy), the group may be termed an "attenuator".

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With an incident wave

$$\eta_0 = A e^{-ik(x\cos\beta + y\sin\beta)}$$

the excitation volume fluxes and forces of the oscillators of group no. 0 are given by the $N_k$ dimensional vector

$$\underset{\sim}{Q} = \underset{\sim}{Q}(\beta) = -i\omega\rho\mathscr{I}(\phi_0, \underset{\sim}{\varphi}_p) = -\frac{\rho g D}{k}\, \underset{\sim}{h}_p(\beta \pm \pi) A \qquad (7.178)$$

and the $6N_i$ dimensional vector

$$\underset{\sim}{F} = \underset{\sim}{F}(\beta) = i\omega\rho\mathscr{I}(\phi_0, \underset{\sim}{\varphi}_u) = \frac{\rho g D}{k}\, \underset{\sim}{h}_u(\beta \pm \pi) A \qquad (7.177)$$

respectively.(*) Note that, because waves generated by group no. 0 are diffracted from the other groups, the Kochin functions $\underset{\sim}{h}(\theta)$ and the corresponding far-field coefficients

$$\underset{\sim}{a}(\theta) = \begin{bmatrix} \underset{\sim}{a}_p(\theta) \\ \underset{\sim}{a}_u(\theta) \end{bmatrix} = (2\pi)^{-1/2} e^{-i\pi/4}\, \underset{\sim}{h}(\theta) = (2\pi)^{-1/2} e^{-i\pi/4} \begin{bmatrix} \underset{\sim}{h}_p(\theta) \\ \underset{\sim}{h}_u(\theta) \end{bmatrix}$$

are, in general, different from those of the case with only one single group.

For an infinite row of groups, the excitation parameters for group no. m are given by the vector

$$\underset{\sim}{Q}_m = \underset{\sim}{Q}\, e^{-ikmd\sin\beta}$$

and

$$\underset{\sim}{F}_m = \underset{\sim}{F}\, e^{-ikmd\sin\beta}$$

With neglection of end effects, the same relations are valid as an approximation even for a finite row of many groups.

Assuming that mechanical load impedances (or load control devices) are identical in all groups, the oscillator states in group no. m is given by

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(*) according to extended Haskind relation (7.187).

$$\begin{bmatrix} \underset{\sim}{p}_m \\ \underset{\sim}{u}_m \end{bmatrix} = \begin{bmatrix} \underset{\sim}{p} \\ \underset{\sim}{u} \end{bmatrix} e^{-ikmd \sin\beta} \qquad [L24]$$

where $\underset{\sim}{p} = \underset{\sim}{p}_o$ gives the oscillating air pressure in the $N_k$ OWC-s of group no. 0, and $\underset{\sim}{u} = \underset{\sim}{u}_o$ gives the oscillating velocity for each mode of its $N_i$ oscillating bodies.

In terms of the $N = N_k + 6N_i$ dimensional oscillator state vector $\tilde{\pi} = (\tilde{p}\ \tilde{u})$ for group no. 0 we have for group no. m

$$\underset{\sim}{\pi}_m = \underset{\sim}{\pi}\, e^{-ikmd \sin\beta} \qquad [L24a]$$

For the N dimensional excitation vector $\tilde{\kappa} = (\tilde{Q}\ \tilde{F})$ we have similarly

$$\underset{\sim}{\kappa}_m = \underset{\sim}{\kappa}\, e^{-ikmd \sin\beta} \qquad [L21a]$$

The contribution from group no. 0 to the radiated waves is

$$\tilde{\varphi}_p(r,\theta,z)\underset{\sim}{p} + \tilde{\varphi}_u(r,\theta,z)\underset{\sim}{u} = \tilde{\varphi}(r,\theta,z)\underset{\sim}{\pi}$$

where the corresponding contribution from group no. m (see fig. K2) may be written

$$\tilde{\varphi}_m(r_m,\theta_m,z)\underset{\sim}{\pi}_m$$

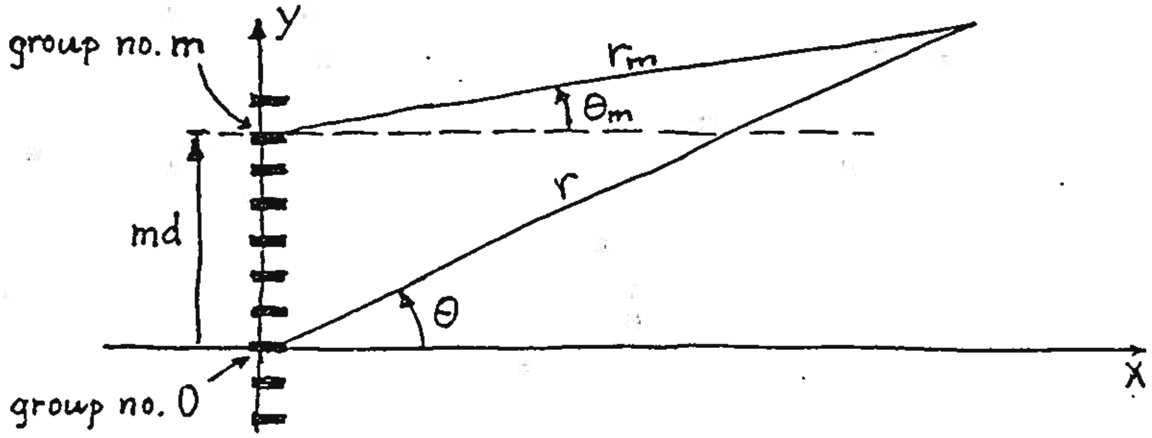


*Fig. K2. Polar coordinates $(r,\theta)$ and $(r_m,\theta_m)$ referred to the global origin and to the origin of group no. m, respectively.*

For an infinite row of groups (or with neglected end effects for a finite row) the coefficient function vector $\tilde{\varphi}_m = (\tilde{\varphi}_{p,m}\ \tilde{\varphi}_{u,m})$ is independent of $m$. Thus the contribution from group no. m is

$$\tilde{\varphi}(r_m,\theta_m,z)\underset{\sim}{\pi}_m = \tilde{\varphi}(r_m,\theta_m,z)\underset{\sim}{\pi}\ e^{-ikmd\ \sin\beta}$$

The radiation coefficient vectors are at large distance ($kr_m >> 1$) given asymptotically by

$$\underset{\sim}{\varphi}(r_m,\theta_m,z) \sim \underset{\sim}{a}(\theta_m)\ e(kz)(kr_m)^{-½}\ \exp\{-ikr_m\}$$

$$\sim \underset{\sim}{a}(\theta)\ e(kz)(kr)^{-½}\ e^{-ik(r-md\ \sin\theta)}$$

Note that for large distance we use the approximations $\theta_m \approx \theta$ and $(kr_m)^{-½} \simeq (kr)^{-½}$ in the modulus (amplitude), and the better approximation

$$r_m \simeq r - md\ \sin\theta$$

in the argument (phase). Similar approximations are commonly used in studies of optical interference problems.

Note that the far-field coefficient vector $\underset{\sim}{a}(\theta_m)$ is referred to the origin (0, md) of group no. m. Alternatively we may introduce the far-field coefficient vector $\underset{\sim}{a}_{(m)}(\theta)$ referred to the common origin (0,0)

$$\underset{\sim}{a}_{(m)}(\theta) = \underset{\sim}{a}(\theta)\ e^{ikmd\ \sin\theta}$$

$$\begin{bmatrix} \underset{\sim}{a}_{p(m)} \\ \underset{\sim}{a}_{u(m)} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{a}_{p} \\ \underset{\sim}{a}_{u} \end{bmatrix} e^{ikmd\ \sin\theta}$$

Then we have asymptotically

$$\underset{\sim}{\varphi}(r_m,\theta_m,z) \sim \underset{\sim}{a}_{(m)}(\theta)\ e(kz)(kr)^{-½}\ e^{-ikr}$$

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K.2. RADIATION DAMPING MATRICES

Within group no. 0 we have the $N\times N$ radiation damping matrix

$$\underset{\sim}{A}_{00} = \frac{\omega\rho D}{2k}\int_0^{2\pi} \underset{\sim}{a}_{(0)}(\theta)\tilde{a}^*_{(0)}(\theta)d\theta = \frac{\omega\rho D}{2k}\int_0^{2\pi} \underset{\sim}{a}(\theta)\tilde{a}^*(\theta)d\theta$$

which represent radiation coupling between the oscillators of group no. 0. In general, we may introduce an $N\times N$ radiation damping matrix

$$\underset{\sim}{A}_{mm'} = \frac{\omega\rho D}{2k}\int_0^{2\pi} \underset{\sim}{a}_{(m)}(\theta)\tilde{a}^*_{(m')}(\theta)d\theta =$$

$$= \frac{\omega\rho D}{2k}\int_0^{2\pi} \underset{\sim}{a}(\theta)\tilde{a}^*(\theta)\ e^{ik(m-m')d\ \sin\theta}\ d\theta$$

which represent radiation coupling between the oscillators of group no. m and those of group no. m'. For a case with no oscillating body $(N_i = 0,\ N_k \neq 0)$ this reduces to the radiation conduction matrix

$$\underset{\sim}{G}_{mm'} = \frac{\omega\rho D}{2k}\int_0^{2\pi} \underset{\sim}{a}_p(\theta)\tilde{a}^*_p(\theta)\ e^{ik(m-m')d\ \sin\theta}\ d\theta$$

With no OWC $(N_k = 0,\ N_i \neq 0)$ it reduces to the radiation resistance matrix

$$\underset{\sim}{R}_{mm'} = \frac{\omega\rho D}{2k}\int_0^{2\pi} \underset{\sim}{a}_u(\theta)\tilde{a}^*_u(\theta)\ e^{ik(m-m')d\ \sin\theta}\ d\theta \qquad [L23]$$

Note that $\underset{\sim}{A}_{mm'}$ is a function of $(m-m')kd$ .

## K.3. ABSORBED POWER

The absorbed power is (cf. 7.96)

$$P = P_e - P_r \qquad (7.96)\ \text{~~[P28]~~}$$

where $P_e$ is the excitation power and $P_r$ is the radiated power. For a single group of $N = N_k + 6N_i$ oscillators we have

$$P_e = \frac{1}{4}\left(\tilde{\kappa}\,\underset{\sim}{\pi}^* + \tilde{\kappa}^*\underset{\sim}{\pi}\right) = \frac{1}{4}\sum_{n=1}^{N}\left(\kappa_n\pi_n^* + \kappa_n^*\pi_n\right)$$

and

$$P_r = \frac{1}{2}\,\tilde{\pi}\,\underset{\sim}{\Lambda}\,\underset{\sim}{\pi}^* = \frac{1}{2}\sum_{n=1}^{N}\sum_{n'=1}^{N}\pi_n\,\Lambda_{nn'}\,\pi_{n'}^*$$

For an array of

$$M_o \equiv 2M + 1$$

groups (that is, $m = 0, \pm1, \pm2, \ldots \pm M$) we have

$$P_e = \sum_{m=-M}^{M}\left(\tilde{\kappa}_m\underset{\sim}{\pi}_m^* + \tilde{\kappa}_m^*\,\underset{\sim}{\pi}_m\right) = M_o\,\frac{1}{4}\left(\tilde{\kappa}\,\underset{\sim}{\pi}^* + \tilde{\kappa}^*\,\underset{\sim}{\pi}\right)$$

and

$$P_r = \sum_{m=-M}^{M}\sum_{m'=-M}^{M}\frac{1}{2}\,\tilde{\pi}_m\,\underset{\sim}{\Lambda}_{mm'}\,\underset{\sim}{\pi}_{m'}^* =$$

$$\sum_{m=-M}^{M}\sum_{m'=-M}^{M}\frac{1}{2}\,\tilde{\pi}\,\underset{\sim}{\Lambda}_{mm'}\,\underset{\sim}{\pi}^*e^{-ik(m-m')d\,\sin\beta} =$$

$$= M_o\,\frac{1}{2}\,\tilde{\pi}\underset{\sim}{\Lambda}_M(\beta)\underset{\sim}{\pi}^*$$

where

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$$\underset{\sim}{A}_M(\beta) = \frac{1}{M_o} \sum_{m=-M}^{M} \sum_{m'=-M}^{M} \underset{\sim}{A}_{mm'} \, e^{-ik(m-m')d \sin\beta} =$$

$$= \frac{1}{M_o} \sum_{m=-M}^{M} \sum_{m'=-M}^{M} \frac{\omega\rho D}{2k} \int_0^{2\pi} \underset{\sim}{a}(\theta)\tilde{\underset{\sim}{a}}^*(\theta) e^{ik(m-m')d(\sin\theta - \sin\beta)} \, d\theta =$$

$$= \frac{\omega\rho D}{2k} \int_0^{2\pi} \underset{\sim}{a}(\theta)\tilde{\underset{\sim}{a}}^*(\theta) \frac{1}{M_o} \sum_m \sum_{m'} e^{ik(m-m')d(\sin\theta - \sin\beta)} \, d\theta$$

This double sum is of the form

$$\sum_{m=-M}^{M} \sum_{m'=-M}^{M} e^{i\alpha(m-m')} = \left| \sum_{m=-M}^{M} e^{i\alpha m} \right|^2$$

Summing the geometrical series gives

$$\left| \sum_{m=-M}^{M} e^{i\alpha m} \right|^2 = \frac{\sin^2 (M_o \alpha/2)}{\sin^2(\alpha/2)}$$

This function is well-known in studies of optical gratings. Then we have

$$\underset{\sim}{A}_M(\beta) = \frac{\omega\rho D}{2k} \int_0^{2\pi} \underset{\sim}{a}(\theta)\tilde{\underset{\sim}{a}}^*(\theta) \frac{\sin^2\{M_o kd(\sin\theta - \sin\beta)/2\}}{M_o \sin^2\{kd(\sin\theta - \sin\beta)/2\}} \, d\theta \qquad [L29]$$

Thus the absorbed power per group is

$$\frac{P}{M_o} = \frac{1}{4} (\tilde{\kappa}\underset{\sim}{\pi}^* + \tilde{\kappa}^*\underset{\sim}{\pi}) - \frac{1}{2} \tilde{\pi} \, \underset{\sim}{A}_M(\beta)\underset{\sim}{\pi}^* \qquad [L28]$$

Note that we have neglected end effects of this finite array consisting of $M_o = 2M + 1$ equidistant groups.

Infinitely long array

For an infinite array ( $M \to \infty$) there are no end effects and the absorbed power per group is

$$P'd = \lim_{M\to\infty} \frac{P}{M_o} = P_e'd - P_r'd \qquad [L30]$$

where

$$P_e'd = \frac{1}{4}\left(\tilde{\kappa}\underset{\sim}{\pi}^* + \tilde{\kappa}^*\underset{\sim}{\pi}\right)$$

and

$$P_r'd = \frac{1}{2}\,\tilde{\pi}\,\underset{\sim}{A}'d\,\underset{\sim}{\pi}^*$$

Note that for a two-dimensional case $P'$ , $P_e'$ and $P_r'$ represent the absorbed, excitation and radiated power, respectively, per unit width (in the y direction)

We have here introduced the "array radiation damping matrix"

$$\underset{\sim}{A}'d = \lim_{M\to\infty} \underset{\sim}{A}_M(\beta) =$$

$$= \frac{\omega\rho D}{2k} \lim_{M\to\infty} \int_0^{2\pi} \underset{\sim}{a}(\theta)\tilde{a}^*(\theta)\, \frac{\sin^2\{M_o kd(\sin\theta-\sin\beta)/2\}}{M_o\ \sin^2\{kd(\sin\theta-\sin\beta)/2\}}\, d\theta \qquad [L31]$$

When $M_o \to \infty$ the integrand tends to zero except for the discrete values of the angle $\theta$ which satisfy

$$\sin\{kd(\sin\theta-\sin\beta)/2\} = 0 \qquad [L32]$$

In those directions wave radiated from the different groups interfere constructively.

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Radiation rays from the array

The solutions $\theta_\nu$ and $(\pi-\theta_\nu)$ of eq. [L32] are given by

$$\sin\theta_\nu = \sin\beta - \nu\pi\frac{2}{kd} = \sin\beta - \nu\frac{\lambda}{d} \qquad [L33]$$

where

$$\nu = \nu_{min}, \ldots, -1, 0, 1, 2, \ldots, \nu_{max}$$

Consider rays incident under an angle of incidence $\beta$ on two adjacent groups. See fig. K3. Between rays radiated from those groups in direction $\theta_\nu$ there is a ray path length difference

$$d\sin\beta - d\sin\theta_\nu = \nu\lambda$$

(Note that $\theta_\nu$ is negative in fig. K.3.) If $\nu$ is an integer, we have constructive interference. We choose

$$-\frac{\pi}{2} < \theta_\nu < \frac{\pi}{2}$$

that is, this solution corresponds to a ray in the 1st or 4th quadrant (i.e. in positive x-direction). The associated solution $\pi-\theta_\nu$ corresponds to a ray in the 2nd or 3rd quadrant (i.e. in negative x-direction).

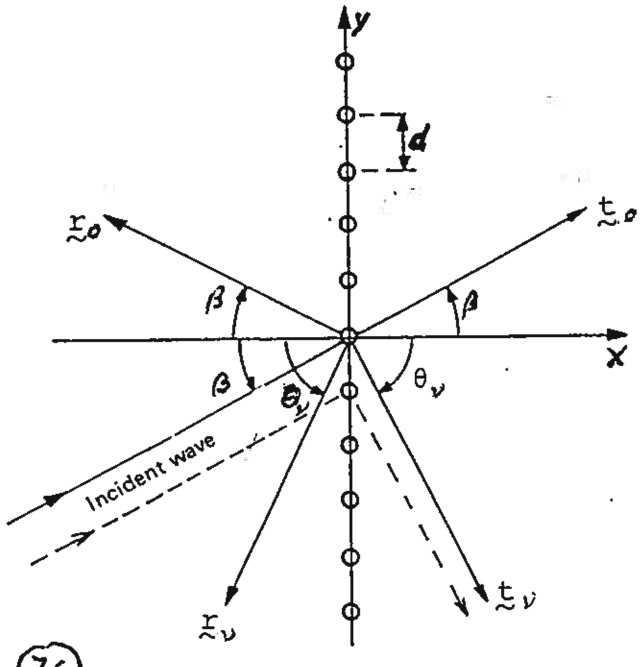


*Fig. K3. Rays showing the directions in which the radiated waves interfere constructively. The shown dashed ray path is* $\nu$ *wavelengths shorter than the parallel ray path just above.*

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For a real solution $\theta_\nu$ we require

$$|\sin\theta_\nu| \le 1$$

$$\nu = (\sin\beta - \sin\theta_\nu)d/\lambda$$

$$(\sin\beta - 1)d/\lambda < \nu < (\sin\beta + 1)d/\lambda$$

$$\nu_{min} = -[(1-\sin\beta)d/\lambda] \qquad \nu_{max} = [(1+\sin\beta)d/\lambda]$$

The bracket [] denotes integer part of. If

$\lambda/d > 1 + \sin|\beta|$ we have $\nu_{min} = \nu_{max} = 0$

meaning that there are no other solutions of [L32] than

$$\theta = \theta_o = \beta \quad \text{and} \quad \theta = \pi - \theta_o = \pi - \beta \qquad [L34]$$

The requirement

$$d < \frac{\lambda}{1+\sin|\beta|} \qquad [L35]$$

gives for normal incidence $(\beta = 0)$, that $d < \lambda$ .

The directions $(\theta_\nu, \pi-\theta_\nu)$ of rays of constructive interference, and also the "array radiation damping" $(\underset{\sim}{A}'d)$, depend on $\beta$ , the angle of wave incidence. However, we may impose the condition [L24a] (p. 70)

$$\underset{\sim}{\pi}_m = \underset{\sim}{\pi}\, e^{-ikmd\sin\beta}$$

on all modes of oscillation, even if an incident wave is absent. By imposing forced oscillations with this phase relationship, we generate plane waves radiated in the directions $\theta_\nu$ and $(\pi-\theta_\nu)$ . Cf. waves with arbitrary propagation direction in wide wave tanks produced by multi-element wave generators.

## K.4. THE ARRAY RADIATION DAMPING MATRIX

We have defined the array radiation damping matrix

$$\underset{\sim}{A}'d = \frac{\omega\rho D}{2k} \lim_{n\to\infty} \int_0^{2\pi} \underset{\sim}{a}(\theta)\tilde{a}^*(\theta) \frac{\sin^2\{n\ kd(\sin\theta-\sin\beta)/2\}}{n\ \sin^2\{kd(\sin\theta-\sin\beta)/2\}}\, d\theta$$

Noting that for $n \to \infty$ there are contributions to this integral only for the values $\theta = \theta_\nu$ and $\theta = \pi - \theta_\nu$ , we may write

$$\underset{\sim}{A}'d = \frac{\omega\rho D}{2k} \sum_{\nu=\nu_{min}}^{\nu_{max}} \left[ \underset{\sim}{a}(\theta_\nu)\tilde{a}^*(\theta_\nu)I(\theta_\nu) + \underset{\sim}{a}(\pi-\theta_\nu)\tilde{a}^*(\pi-\theta_\nu)I(\pi-\theta_\nu)\right]$$

where

$$I(\theta_\nu) = \lim_{n\to\infty} \int_{\theta_\nu-\epsilon}^{\theta_\nu+\epsilon} \frac{\sin^2\{n\ kd(\sin\theta-\sin\beta)/2\}}{n\ \sin^2\{kd(\sin\theta-\sin\beta)/2\}}\, d\theta \qquad [K37]$$

Here $\epsilon > 0$ . However, we may choose $\epsilon$ arbitrarily small provided $n$ is large enough. Using the series expansion

$$\sin\theta = \sin\theta_\nu + (\theta-\theta_\nu)\cos\theta_\nu + O\{(\theta-\theta_\nu)^2\}$$

we have

$$\sin\theta - \sin\beta = \sin\theta_\nu - \sin\beta + (\theta-\theta_\nu)\cos_\nu + O\{(\theta-\theta_\nu)^2\}$$

$$= -\nu \frac{2\pi}{kd} + (\theta-\theta_\nu)\cos\theta_\nu + O\{(\theta-\theta_\nu)^2\}$$

Because of the periodic behaviour of the integrand, we may drop the first term of this series. Hence

$$I(\theta_\nu) = \lim_{n\to\infty} \int_{\theta_\nu-\epsilon}^{\theta_\nu+\epsilon} n \frac{\sin^2\{n\ kd\ \cos\theta_\nu(\theta-\theta_\nu)/2 + \ldots\}}{n^2\sin^2\{kd\ \cos\theta_\nu(\theta-\theta_\nu)/2 + \ldots\}} d\theta$$

As new integration variable we take $x = \frac{1}{2}\, n\, kd\, (\theta-\theta_\nu)\cos\theta_\nu$ . Then

$$I(\theta_\nu) = \frac{2}{kd|\cos\theta_\nu|} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{2\pi}{kd|\cos\theta_\nu|} \qquad [L42]$$

Note that we have let $n\epsilon \to \infty$ . The integral converges rapidly when the integration region $|x| < \frac{1}{2}\, kd|\cos\theta_\nu|n\epsilon$ increases. Similarly we have

$$I(\pi-\theta_\nu) = \frac{2\pi}{kd|\cos(\pi-\theta_\nu|)} = \frac{2\pi}{kd|\cos\theta_\nu|} = I(\theta_\nu)$$

Note that since we have chosen $|\theta_\nu| < \pi/2$ we have $|\cos\theta_\nu| = \cos\theta_\nu$ .

The array radiation damping matrix now becomes

$$\underset{\sim}{A}'d = \frac{\pi\omega\rho D}{k^2 d} \sum_\nu \frac{\underset{\sim}{a}(\theta_\nu)\tilde{a}^*(\theta_\nu) + \underset{\sim}{a}(\pi-\theta_\nu)\tilde{a}^*(\pi-\theta_\nu)}{\cos\theta_\nu} \qquad [L43]$$

$$= \frac{\omega\rho D}{2k^2 d} \sum_\nu \frac{\underset{\sim}{h}(\theta_\nu)\tilde{h}^*(\theta_\nu) + \underset{\sim}{h}(\pi-\theta_\nu)\tilde{h}^*(\pi-\theta_\nu)}{\cos\theta_\nu} \qquad [L43a]$$

where we have used the Kochin function vector

$$\underset{\sim}{h}(\theta) = \sqrt{2\pi}\ \underset{\sim}{a}(\theta)e^{i\pi/4}$$

For one group of $N$ oscillators, we have the two following alternative expressions for the radiation damping $(N\times N)$ matrix:

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$$\underset{\sim}{A} = \frac{\omega\rho D}{4\pi k} \int_0^{2\pi} \underset{\sim}{h}(\theta)\tilde{h}^*(\theta)\, d\theta = \frac{k}{16\pi J} \int_{-\pi}^{\pi} \underset{\sim}{\kappa}^*(\beta)\tilde{\kappa}(\beta)\, d\beta$$

where

$$J = \frac{\rho g^2 D}{4\omega} |A|^2$$ (4.136) ~~[B75]~~

We may also write the radiation damping matrix as

$$\underset{\sim}{A} = \frac{k}{16\pi} \int_0^{2\pi} \underset{\sim}{\kappa}^*(\theta\pm\pi)\tilde{\kappa}(\theta\pm\pi)d\theta$$

Also the array radiation damping matrix may be expressed in terms of the excitation vector

$$\underset{\sim}{A}'d = \frac{1}{8Jd} \sum_{\nu} \frac{\underset{\sim}{\kappa}^*(\theta_\nu-\pi)\tilde{\kappa}(\theta_\nu-\pi)+\underset{\sim}{\kappa}^*(-\theta_\nu)\tilde{\kappa}(-\theta_\nu)}{\cos\theta_\nu} \qquad [L44]$$

It is convenient to abbreviate eq. [L43a] as follows

$$\boxed{\underset{\sim}{A}'d = \frac{\omega\rho D}{2k^2 d} \sum_{\nu} \frac{\underset{\sim}{t}_\nu \tilde{t}_\nu^* + \underset{\sim}{r}_\nu \tilde{r}_\nu^*}{\cos\theta_\nu}} \qquad [T15a]$$

where

$$\underset{\sim}{t}_\nu \equiv \underset{\sim}{h}(\theta_\nu) \quad \text{and} \quad \underset{\sim}{r}_\nu \equiv \underset{\sim}{h}(\pi-\theta_\nu) \qquad [T16a]$$

(See fig. K.3.) These vectors may be partitioned (into vectors of two subspaces) as follows

$$\underset{\sim}{t}_{p,\nu} = \underset{\sim}{h}_p(\theta_\nu) \quad \text{and} \quad \underset{\sim}{r}_{p,\nu} = \underset{\sim}{h}_p(\pi-\theta_\nu)$$

$$\underset{\sim}{t}_{u,\nu} = \underset{\sim}{h}_u(\theta_\nu) \quad \text{and} \quad \underset{\sim}{r}_{u,\nu} = \underset{\sim}{h}_u(\pi-\theta_\nu) \qquad [T16]$$

For instance, the array radiation resistance matrix may be written:

$$\underset{\sim}{R}'d = \frac{\omega\rho D}{2k^2 d} \sum_{\nu} \frac{\underset{\sim}{t}_{u,\nu}\tilde{t}_{u,\nu}^* + \underset{\sim}{r}_{u,\nu}\tilde{r}_{u,\nu}^*}{\cos\theta_\nu} \qquad [T15]$$

Note that the array radiation damping matrix $\underset{\sim}{A}'d$ depends on $\beta$ .

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For $\beta = 0$ , corresponding to normal wave incidence, $\underset{\sim}{A}'d$ is just the radiation damping matrix for one group of oscillating water columns and bodies in a wave channel of width $d$ . Then, since $\sin\beta = 0$ we have

$$\theta_{-\nu} = -\theta_\nu \quad \text{and} \quad -\nu_{min} = \nu_{max} = [d/\lambda] \; .$$

Hence, for $\sin\beta = 0$ the radiation damping matrix for one group in the wave channel is

$$\underset{\sim}{A}'d = \frac{\omega\rho D}{k^2 d} \left\{ \frac{\underset{\sim}{t}_o \tilde{t}_o^* + \underset{\sim}{r}_o \tilde{r}_o^*}{2} + \sum_{\nu=1}^{\nu_{max}} \frac{\underset{\sim}{t}_\nu \tilde{t}_\nu^* + \underset{\sim}{r}_\nu \tilde{r}_\nu^*}{\cos\theta_\nu} \right\}$$

provided the group is symmetric with respect to the midplane of the channel, which means that $\underset{\sim}{t}_{-\nu} = \underset{\sim}{t}_\nu$ and $\underset{\sim}{r}_{-\nu} = \underset{\sim}{r}_\nu$ .

K.5. TWO-DIMENSIONAL RADIATION DAMPING MATRIX

For the 2-dimensional case we define Kochin functions, far-field coefficients, excitation volume flux and excitation force per unit width (in y direction). We denote those quantities by a prime (′).

The above analysis applies if we let $d \to 0$ , meaning that $\nu_{min} = \nu_{max} = 0$ . Further $\beta = 0$ or $\beta = \pi$ are the only possible angles of wave incidence. Introducing $\underset{\sim}{h}'(\theta) = \underset{\sim}{h}(\theta)/d$ we then have the 2-dimensional radiation damping matrix

$$\underset{\sim}{A}' = \frac{\omega\rho D}{2k^2}\left(\underset{\sim}{h}'(0)\tilde{h}'^*(0) + \underset{\sim}{h}'(\pi)\tilde{h}'^*(\pi)\right)$$

We may introduce the excitation parameters

$$\underset{\sim}{\kappa}(\beta) = \begin{bmatrix} \underset{\sim}{Q}(\beta) \\ \underset{\sim}{F}(\beta) \end{bmatrix} = \frac{\rho g D}{k}\begin{bmatrix} -\underset{\sim}{h}_p(\beta\pm\pi) \\ \underset{\sim}{h}_u(\beta\pm\pi) \end{bmatrix} A$$

Then we have the alternative expression

$$\underset{\sim}{A}' = \frac{1}{8J}\left(\underset{\sim}{\kappa}'^*(\pi)\tilde{\kappa}'(\pi) + \underset{\sim}{\kappa}'^*(0)\tilde{\kappa}'(0)\right) \qquad \text{[B79]}$$

with

$$J = \frac{\rho g^2 D}{4\omega}|A|^2 \qquad \text{(4.136) [B75]}$$

For comparison we restate here the corresponding three-demensional reciprocity relations

$$\underset{\sim}{A} = \frac{\omega\rho D}{4\pi k}\int_0^{2h} h(\theta)h^*(\theta)d\theta = \frac{k}{16\pi J}\int_{-\pi}^{\pi}\underset{\sim}{\kappa}^*(\beta)\tilde{\kappa}(\beta)d\beta$$

K.6. ARRAY OF HEAVING AXISYMMETRIC BODIES

Let us as a simple example consider an array of axisymmetric bodies constrained to oscillate in the heave mode only. Then there is only one oscillator per "group". For normal incidence ($\beta = 0$) this corresponds to one body centered in a wave channel of width d, as indicated in fig. K.4.

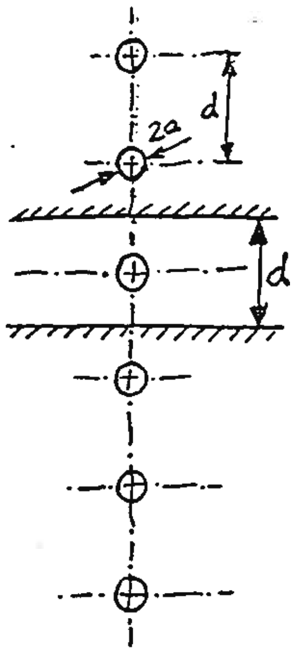


*Fig. K4. Array of axisymmetric bodies. With normal incidence ($\beta=0$) two parallel fixed planes, representing the walls of a wave channel of width $d$, may be introduced without disturbing the ideal-fluid motion.*

In the case of a single body (or $d \to \infty$) the radiation resistance is

$$R_{33} = (\omega\rho D/2k)\,|h_{s3}|^2$$

where the Kochin function $h_{s3}$ is independent of $\theta$. For the case of the array the Kochin function is

$$h_3(\theta) \approx h_{s3}$$

provided $2a \ll d$ and $kd \lesssim 4$ (Budal et al. 1979). Probably the approximation is reasonable good if $kd$ is not too close to a multiple of $2\pi$. If $kd \lesssim 2\pi/(1+\sin|\beta|)$ we have $\upsilon_{min} = \upsilon_{max} = 0$ and, hence, with $\theta_o \doteq \beta$, the array radiation resistance is

$$R'_{33}d = \frac{\omega\rho D}{2k^2 d}\,\frac{|h_3(\beta)|^2 + |h_3(\pi-\beta)|^2}{\cos\beta} \approx$$

$$R'_{33}d \approx \frac{\omega\rho D\ |h_{s3}|^2}{k^2 d\ \cos\beta} \qquad [K54]$$

For normal wave incidence ($\beta = 0$) this gives

$$R'_{33}d \approx \frac{2}{kd} R_{33}$$

Dynamic reflection by heaving slender bodies

Assuming that the diameter 2a of the body is very much smaller than one wavelenght $\lambda$, there is negligible wave scattering on the body. Then an incident wave that hits a row of fixed (non-moving) bodies, passes through the row as a tramsmitted wave which is essentially an undisturbed continuation of the incident wave.

Assume next the bodies perform heaving oscillation, all with the same amplitude and phase. Each body generates a circular wave, which interferes with the circular waves from the other bodies. At some distance from the infinitely long row of bodies the resultant wave is essentially a plane wave, provided the spacing d is less than a half wavelength. (Cf. problem ~~B.2.~~ 4.6) For symmetry reasons there are two equally large plane waves, one radiated in the positive x-direction and one radiated in the negative x-direction.

Assume that the incident wave has normal incidence ($\beta = 0$), and that the bodies are heaving with such a velocity amplitude $\hat{u}_3$ that the wave radiated in the positive x-direction, has the same magnitude as the transmitted wave. Further, if the phase of the radiated wave is arranged to be opposite to the phase of the transmitted wave, the two waves cancel each other. The wave radiated in the negative x-direction, opposite to the direction of propagation of the incident wave, also has the same magnitude as the transmitted wave and hence, as the incident wave. Consequently, the result is that the incident wave is, in effect, totally reflected by the row of heaving bodies, and therefore, no wave energy is absorbed by the bodies.

It is easily seen from eq. [L30] (nr. 75) -cf. also fig. ~~D.9~~ 6.3. - that the absorbed power P vanishes if the bodies are stationary, $\hat{u}_3 = 0$, and also if

$$\hat{u}_3 = 2\,\hat{u}_{3,opt} = \hat{F}_3/R'_{33}d$$

For this latter case where the heave velocity is just twice of the optimum velocity corresponding to maximum absorbed power (cf. eq. ~~[D199a]~~ (6.7)), we have "dynamic reflection" from the heaving slender (low-scattering) bodies. Note that the velocity has to be in phase with the excitation force.

The plane wave behind the row of oscillating bodies has almost disappeared, except for the local non-propagating disturbance close to the bodies. As far as concerns the plane wave, the line of bodies corresponds to a node of the resulting standing wave in front of the row. In contrast, for reflection at a fixed vertical wall there is an antinode at the plane of reflection. (Cf. problem ~~B.1~~ 4.5.)

Dynamic reflection is also possible with oblique wave incidence ($\beta \neq 0$). Then there is a lag of $(2\pi d/\lambda)\sin\beta$ in the phase of oscillation for adjacent bodies of the row. The angle of reflection equals the angle on incidence $\beta$ . In order to avoid radiation of additional waves in unwanted directions it is necessary that $d < \lambda/s$ , where $s = 1 + \sin|\beta|$ has a value between 1 and 2. Cf. eq. [L35] (p. 77).

Dynamic reflection can be experimentally demonstrated in a wave channel as indicated in fig. D.18 (p. D.54). The width of the channel is $d = 0.40$ m , and the heaving body is placed in the middle of the channel. Because of the totally reflecting side walls of the channel the experimental situation corresponds to normal incidence of waves on an infinitely long row of equidistant heaving bodies. (See fig. K4.) The diameter of the body is $2a = 0.15$ m. The wave behind the body, which is absorbed at the beach (A), is measured by a probe ($S_3$). In front of the body two probes ($S_1$ and $S^3$) are used since there are both an incident wave and a reflected wave. One probe is preferably placed in a node, the other in an antinode of the standing wave. All probes are placed sufficiently far (more than one wavelength) from the body in order to measure on waves which are essentially plane.

The natural (eigen) frequency of the oscillating system can be changed by varying the mass of the counterweight (C in fig. D.18) (p. D54) and of the ballast within the body. At resonance, when the natural frequency is adjusted to the same frequency as the wave, the phase condition, $\gamma_3 = 0$ , is fulfilled. The friction loss in the pulley (P in fig. D.18) and the

p. D54 (p. 46)

viscous loss are so small that the other condition, P = 0 , for dynamic reflection is essentially fulfilled.

In fig. K.5 we present measurements taken when the body is adjusted to resonance with a period $T = 2\pi/\omega = 1.1$ s . The body, which has an equilibrium draught of 125 mm, has an oscillation amplitude of 4 cm while the incident wave amplitude is $|A| = 4$ mm . For dynamic reflection the latter amplitude is close to half the elevation amplitude in the antinode, as given by curve $S_1$ in fig. K.5. Due to the dynamic reflection there is almost calm sea behind the body. As appears from fig. K.5 the wave height is reduced by a factor of nearly 10 with respect to the wave height in the antinode. With respect to the incident wave the wave height is reduced by a factor of about 5.

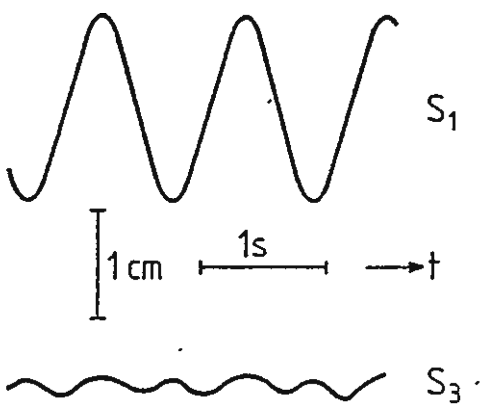


*Fig. K.5. Oscillograms of wave elevation when the conditions for dynamic reflection are essentially fulfilled. Wave period $T = 1.1$s . $S_1$: Recording from wave probe in antinode in front of the body. $S_3$: Recording from wave probe in progressive wave behind the body.*

However, it appears that the signal from probe $S_3$ contains mostly higher harmonics, and very little of the fundamental wave. Hence the reduction in wave height at the natural frequency of the body is even better than fig. K.5 indicates. The signal from the probe at the node of the standing wave has roughly the same magnitude and similar appearance as the signal from probe $S_3$ . The presence of higher harmonics is partly due to the wave maker. It is probable that the body oscillation also contributes to the generation of higher harmonics. The higher harmonic waves are relatively small. The linearity in the experiment is very good although the oscillation amplitude is a large as 4 cm.

Maximum absorbed power

The maximum absorbed power per body is

$$(P'd)_{max} = \frac{|F_3|^2}{8R_{33}'d} = \frac{kd}{2}\frac{|F_3|^2}{8R_{33}}$$

Neglecting the difference between the heave excitation force for a single body and for a body in the row ($F_{s3} \approx F_3$), this shows that the maximum absorbed power may be increased by a factor (kd/2) for each body is in a row compared to an isolated single body. We could term this factor the "interaction factor" since it is related to the hydrodynamic interaction between the bodies in the array. This interaction factor increases to $\pi$ when kd increases to $2\pi$ . (When $kd > 2\pi$ other rays ($\nu = \pm 1$) become important. They represent additional radiated power.) With

$$F_3 = F_3(\beta) = \frac{\rho g D}{2} A\, h_3(\beta \pm \pi) \approx \frac{\rho g D}{k} A\, h_{s3}$$

we obtain

$$(P'd)_{max} = \frac{kd}{2} \frac{|(\rho g D/k) A\, h_{s3}|^2}{8(\omega \rho D/2k)\,|h_{s3}|^2} = \frac{\rho g^2 D d |A|^2}{8\omega}$$

This is 50 % of the incident wave's power transport

$$Jd = \frac{\rho g^2 D}{4\omega} |A|^2 d$$

per heaving body of the array.

## K.7. MAXIMUM POWER ABSORPTION WITH ARRAYS

The absorbed power per group is

$$P'd = P'_e d - P'_r d$$

where

$$P'_e d = \frac{1}{4} (\tilde{\kappa}\underset{\sim}{\pi}^* + \tilde{\kappa}^* \underset{\sim}{\pi}) = \frac{1}{4} (\tilde{\pi}^* \underset{\sim}{\kappa} + \tilde{\kappa}^* \underset{\sim}{\pi})$$

and

$$P'_r d = \frac{1}{2} \tilde{\pi}(\underset{\sim}{A}'d)\underset{\sim}{\pi}^* = \frac{1}{2} \tilde{\pi}^* (\underset{\sim}{A}'d)^* \underset{\sim}{\pi} = \frac{1}{2} \tilde{\pi}^* (\tilde{A}'d)\underset{\sim}{\pi} \qquad \text{[T11a]}$$

We shall now discuss optimisation with no amplitude constraints. The optimum complex amplitude

$$\underset{\sim}{\pi}_o = \begin{bmatrix} \underset{\sim}{p}_o \\ \underset{\sim}{u}_o \end{bmatrix}$$

has to satisfy

$$(\underset{\sim}{A}'d)^* \underset{\sim}{\pi}_o = \frac{1}{2} \underset{\sim}{\kappa} \qquad \text{[T13a]}$$

giving

$$(P'd)_{max} = (P'_r d)_{opt} = \frac{1}{2} (P'_e d)_{opt} =$$

$$= \frac{1}{2} \tilde{\pi}_o (\underset{\sim}{A}'d) \underset{\sim}{\pi}_o^* = \frac{1}{4} \tilde{\kappa} \; \underset{\sim}{\pi}_o^* = \frac{1}{4} \tilde{\kappa}^* \underset{\sim}{\pi}_o$$

If the inverse matrix $(\underset{\sim}{A}'d)^{-1}$ exists, we have the unique optimum complex amplitude vector

$$\underset{\sim}{\pi}_o = \frac{1}{2} (\underset{\sim}{A}'^* d)^{-1} \underset{\sim}{\kappa}$$

and the maximum absorbed power per group

$$(P'd)_{max} = \frac{1}{8} \tilde{\kappa} (\underset{\sim}{A}'d)^{-1} \underset{\sim}{\kappa}^*$$

Note that $(\underset{\sim}{A}'d)$ as well as the excitation vector $\underset{\sim}{\kappa}$ depends on $\beta$ , the angle of wave incidence.

It will be convenient to use the following alternative formulation of the optimisation problem

$$(P'd)_{max} = \frac{1}{4} (-\tilde{Q}^* \quad \tilde{F}^*) \begin{bmatrix} -\underset{\sim}{p}_o \\ \underset{\sim}{u}_o \end{bmatrix}$$

where the optimum complex amplitudes must satisfy

$$(\underset{\sim}{A}'d) \begin{bmatrix} -\underset{\sim}{p}_o \\ \underset{\sim}{u}_o \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\underset{\sim}{Q} \\ \underset{\sim}{F} \end{bmatrix}$$

The wave power transport per group, associated with the incident wave, is

$$Jd \cos\beta = \frac{\rho g^2 Dd}{4\omega} |A|^2 \cos\beta$$

We define the relative absorbed power

$$E = \frac{P'd}{Jd \cos\beta}$$

Further,

$$\epsilon \equiv E_{max} = \frac{(P'd)_{max}}{Jd \cos\beta} = \frac{\frac{1}{4} (-\tilde{Q}^* \; \tilde{F}^*) \begin{bmatrix} -\underset{\sim}{p}_o \\ \underset{\sim}{u}_o \end{bmatrix}}{(\rho g^2 Dd/2\omega) |A|^2 \cos\beta} =$$

$$\frac{\omega}{\rho g^2 Dd |A|^2 \cos\beta} \frac{\rho g D A^*}{k} (\tilde{h}_p^*(\beta \pm \pi) \; \tilde{h}_u^*(\beta \pm \pi)) \begin{bmatrix} -\underset{\sim}{p}_o \\ \underset{\sim}{u}_o \end{bmatrix} =$$

$$\boxed{\epsilon = E_{max} = \tilde{h}^*(\beta \pm \pi) \; \underset{\sim}{v} \equiv \tilde{h}_o^* \; \underset{\sim}{v}} \qquad [T24]$$

(90)

where

(90)

$$\underset{\sim}{v} = \frac{\omega}{Agkd\ \cos\beta} \begin{bmatrix} -\underset{\sim}{p}_o \\ \underset{\sim}{u}_o \end{bmatrix} \qquad [T23a]$$

The optimum complex amplitudes have to be chosen such that the vector $\underset{\sim}{v}$ satisfies

$$(\underset{\sim}{A}'d)\ \underset{\sim}{v} = \frac{\omega}{Agkd\ \cos\beta} \frac{1}{2} \begin{bmatrix} -\underset{\sim}{Q} \\ \underset{\sim}{F} \end{bmatrix}$$

that is,

$$\frac{\omega\rho D}{2k^2 d} \sum_{\nu} \frac{\underset{\sim}{t}_\nu \tilde{t}_\nu^* + \underset{\sim}{r}_\nu \tilde{r}_\nu^*}{\cos\theta_\nu}\ \underset{\sim}{v} = \frac{\omega}{2Agkd\ \cos\beta} \frac{\rho g D A}{k} \begin{bmatrix} \underset{\sim}{h}_p(\beta\pm\pi) \\ \underset{\sim}{h}_u(\beta\pm\pi) \end{bmatrix}$$

or

$$\boxed{\sum_{\nu} c_\nu^{-1} (\underset{\sim}{t}_\nu \tilde{t}_\nu^* + \underset{\sim}{r}_\nu \tilde{r}_\nu^*)\ \underset{\sim}{v} = \underset{\sim}{h}(\beta\pm\pi) \equiv \underset{\sim}{h}_o} \qquad [T25a]$$

where

$$c_\nu \equiv \frac{\cos\theta_\nu}{\cos\beta} \qquad [T22]$$

For the case with "<u>incidence-reflection symmetry</u>", that is, when

$$\underset{\sim}{h}(\beta\pm\pi) = \underset{\sim}{h}(\pi-\beta) \quad \text{or} \quad \underset{\sim}{h}_o = \underset{\sim}{r}_o \qquad [T26]$$

we have formally equal results to those given by Falnes (1984, Applied Ocean Research, Vol. 6, pp 16-22). However, here those results have now been extended to the case where there are oscillating water columns in addition to the oscillating bodies.

When there are only OWC-s or only oscillating bodies $\underset{\sim}{A}'d$ is real ($\underset{\sim}{A}'d = \underset{\sim}{G}'d$ or $\underset{\sim}{A}'d = \underset{\sim}{R}'d$ , respectively). In those cases we may write

$$\sum_{\nu} c_\nu^{-1} (\underset{\sim}{t}_\nu \tilde{t}_\nu^* + \underset{\sim}{r}_\nu \tilde{r}_\nu^*)\ \underset{\sim}{v} = \underset{\sim}{h}_o \qquad [T48]$$

and, hence, the results obtained for the case without "incidence-reflection symmetry" are valid. However, when there are

(91) K24 (91)

both OWC-s and oscillating bodies we cannot make the replacement where [T48] is obtained from [T25a] (p. 91).

## K.8. THE CASE WITH ONLY ONE OSCILLATOR PER GROUP

In the group we have only one OWC or only one body which is constrained to oscillate in one mode only. The condition for optimum is now

$$\sum_{\nu} c_{\nu}^{-1} \left(|t_{\nu}|^2 + |r_{\nu}|^2\right) v = h_o$$

$$v = \frac{h_o}{\sum_{\nu} c_{\nu}^{-1}\left(|t_{\nu}|^2 + |r_{\nu}|^2\right)}$$

giving

$$\epsilon = E_{max} = h_o^* v = \frac{|h_o|^2}{\sum_{\nu} c_{\nu}^{-1}\left(|t_{\nu}|^2 + |r_{\nu}|\right)^2}$$

If the interspace $d$ is short enough, that is, if

$$d < \frac{2\pi/k}{1 + \sin|\beta|}$$

we have $\nu_{min} = \nu_{max} = 0$, and with $c_o^{-1} = \frac{\cos\beta}{\cos\theta_o} = 1$

this gives the maximum absorbed relative power

$$E_{max} = \frac{|h_o|^2}{|t_o|^2 + |r_o|^2} = \frac{|h(\beta \pm \pi)|^2}{|h(\beta)|^2 + |h(\pi - \beta)|^2}$$

If the system has a vertical symmetry plane normal to the y axis (cf. fig. K.6), we have "incidence-reflection "symmetry, $h_o = r_o$ , which gives

$$E_{max} = \frac{1}{1 + |t_o/r_o|^2}$$

(Note that with normal incidence ($\beta$=0) this is valid even if there is no vertical symmetry plane normal to the y axis.)

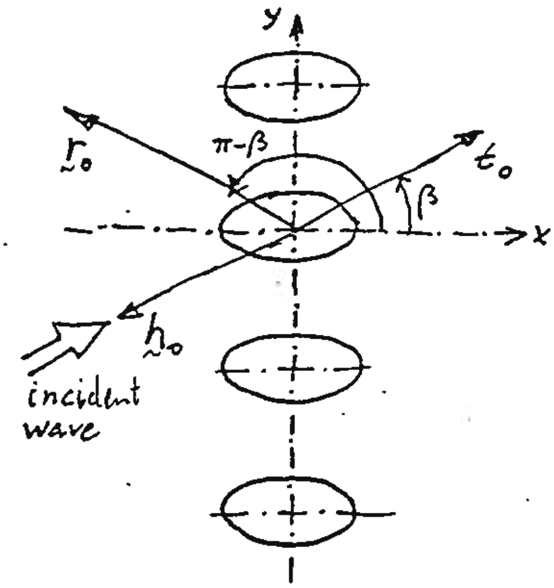


*Fig. K6. An array where each group has a symmetry plane normal to the y-axis, has "incidence-reflection" symmetry. Then the Kochin functions $\underset{\sim}{r}_o$ and $\underset{\sim}{h}_o$ are equal.*

If there is also a vertical symmetry plane normal to the x axis, then $|t_o| = |r_o|$ , and hence

$$\epsilon = E_{max} = \frac{1}{2}$$

Thus, such a symmetric one-oscillator system cannot absorb more than 50 percent of the incident wave power.

If the system cannot radiate a wave on the "transmitting" side (to the right in the figure), then $t_o = h(\beta) = 0$ which gives $E_{max} = 1$ . Thus in this case 100 % of the incident wave power may be absorbed, if the optimum complex amplitude is achieved for the oscillator. If $|t_o| << |r_o|$ this situation is closely approached, $E_{max} \approx 1$ as, for instance, with the Salter duck (cf. fig. K.7) or with a large OWC structure having its mouth on one side (cf. fig. K.8).

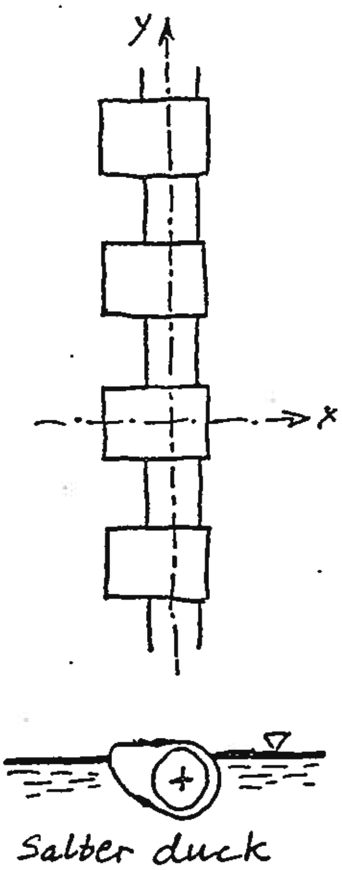


*Fig. K7. Top view and side view of Salter "ducks" which can perform pitching oscillation relative to a common cylindrical spine.*

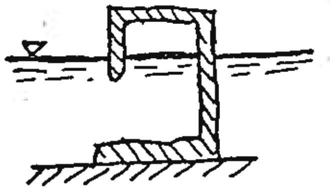

*Fig. K8. Side view of OWC structure standing on the sea bottom.*

If the OWC structure is small (in the y-direction) compared to the wavelength, the condition $|t_o| << |r_o|$ is not well satisfied, even if the OWC has its mouth to one side only.

If there is a reflecting wall behind the one-oscillator array (cf. fig. K9), it is ensured that $t_o = h(\beta) = 0$ (and that $t_\upsilon = 0$ for all $\upsilon$). Hence, with optimum oscillation we have 100 percent absorption of the incident wave, if

$$d < \frac{\lambda}{1+\sin|\beta|}$$

even if the oscillating system by itself is symmetric with respect to a plane normal to the x axis.

95



95

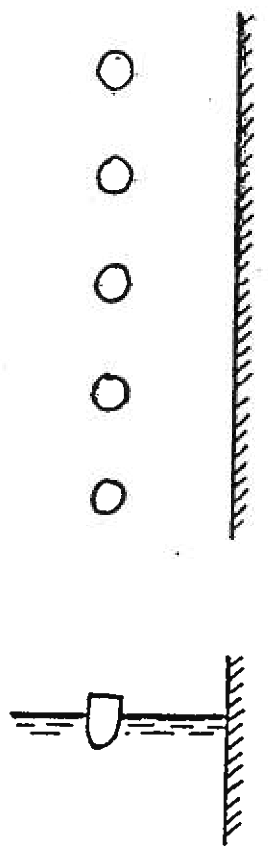

*Fig. K9. Top view and side view of an array of oscillating bodies parallel to a reflecting vertical wall.*

Note that the reflecting wall has a significant influence on the array radiation damping $\Delta' d$ (= $R'd$ for one-mode oscillating bodies, or = $G'd$ for the case of one-OWC-array). The radiation damping will be strongly frequency dependent, in particular, when the distance between the array and the reflecting wall is large.

For a symmetric array, which radiates with equal strength in both directions, one of the rays is reflected, and then it interferes with the other ray, constructively or destructively, according to the phase difference obtained by return after reflection.

## K9. THE CASE WITH TWO OSCILLATORS PER GROUP

Next we consider a two-oscillator array, for instance, a row of symmetric bodies as shown in fig. K.10 (6.1), where the bodies oscillate in a symmetric mode (heave) and an antisymmetric mode (surge or pitch).

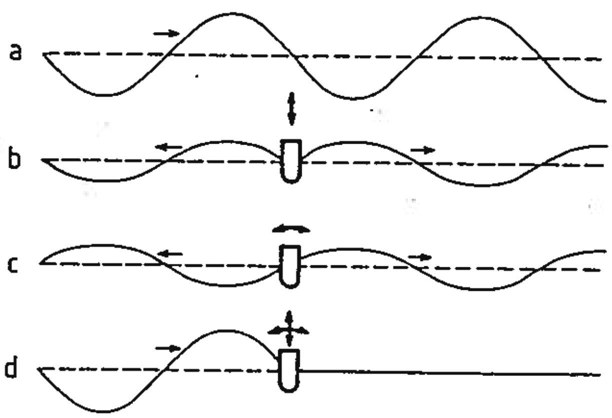


Fig. 6.1

*Fig. K10. Total wave power absorption by means of a two-dimensional body or of one row of bodies oscillating in a symmetric mode and an antisymmetric mode.*
*a. Incident wave.*
*b. Symmetric wave radiation by heaving oscillation.*
*c. Antisymmetric wave radiation by pitching (or surging) oscillation.*
*d. Superposition of the radiated waves on the incident wave results in complete absorption of the incident wave.*

(97) (97)

However, let us first summarise results obtained for the symmetric one-oscillator array. If the system oscillates in one mode only, with symmetric or antisymmetric wave radiation (cf. fig. K.10 b or c), the optimum case with 50 % power absorption corresponds to the following situation. As a result of interference between the incident wave and the radiated wave a resulting downstream wave has an amplitude which is half that of the original incident wave, and hence the associated downstream power transport is 25 % of the incident power transport. An equally large wave is radiated in the upstream direction. Thus, of the incident power 50 % is absorbed, 25 % is reflected and 25 % is transmitted.

If the oscillation amplitude is now doubled, there is no downstream wave, since the incident wave is on the downstream side totally cancelled by the radiated wave. However, the upstream radiated wave is also doubled, which means that the incident wave is effectively reflected. No power is absorbed by the oscillator, since now the radiated power equals the excitation power. Thus the one-oscillator array now acts as a dynamic reflector.

If we have two parallel rows of this kind it is, in principle, possible to absorb all incident wave power, when the downstream row acts as a dynamic reflector as an alternative to a reflecting wall. All incident wave power must then be absorbed by the front (or upstream) row.

With a symmetric one-body array oscillating in a symmetric mode and an antisymmetric mode it is, in principle, possible to eliminate the resulting downstream wave and also the resulting radiated upstream wave, as indicated in fig. K10. As a result of such a wave interference, 100 % of the incident wave power must be absorbed by the two-oscillator array.

Consider now a symmetric one-body array constrained to move in two modes only, the surge mode (no. 1) which is antisymmetric and the heave mode (no. 3) which is symmetric. Thus,

$$t_{1,o} = -r_{1,o} \quad \text{and} \quad t_{3,o} = r_{3,o}$$

Then the vectors

$$\underset{\sim}{t}_o = \begin{bmatrix} -r_{1,o} \\ r_{3,o} \end{bmatrix} \quad \text{and} \quad \underset{\sim}{r}_o = \begin{bmatrix} r_{1,o} \\ r_{3,o} \end{bmatrix}$$

are evidently linearly independent (provided $r_{1,o}\, r_{3,o} \neq 0$). Hence 100 % wave power absorption is possible with this system.

Consider, for simplicity, the case with incidence-reflection symmetry. Eqs. [T26] and [T25] (2) gives (1)(1.91)

$$\underset{\sim}{h}_o = \underset{\sim}{r}_o \quad \text{or} \quad \underset{\sim}{h}(\beta \pm \pi) = \underset{\sim}{h}(\pi - \beta)$$

and

$$\sum_{\nu} c_\nu (\underset{\sim}{t}_\nu \tilde{t}^*_\nu + \underset{\sim}{r}_\nu \tilde{r}^*_\nu)\underset{\sim}{v} = \underset{\sim}{h}_o = \underset{\sim}{r}_\nu$$

$$\underset{\sim}{r}_o(\tilde{h}^*_o \underset{\sim}{v} - 1) + \underset{\sim}{t}_o(\tilde{t}^*_o \underset{\sim}{v}) + \sum_{\nu \neq 0}{}' c_\nu^{-1}(\underset{\sim}{r}_\nu(\tilde{r}^* \underset{\sim}{v}) + \underset{\sim}{t}_\nu(\tilde{r}^*_\nu \underset{\sim}{v})) = 0 \qquad [T32]$$

Now if $d < \frac{\lambda}{1+\sin|\beta|}$

the last term vanishes and we have

$$\underset{\sim}{r}_o(\underset{\sim}{t}_o(\tilde{r}^*_o \underset{\sim}{v} - 1) + \underset{\sim}{t}_o(\tilde{t}^*_o \underset{\sim}{v}) = 0$$

If the vectors $\underset{\sim}{r}_o$ and $\underset{\sim}{t}_o$ are linearly independent this requires that

$$\tilde{r}_o^* \underset{\sim}{v} = 1 \quad \text{and} \quad \tilde{t}_o^* \underset{\sim}{v} = 0$$

Then

$$\epsilon = E_{max} = \tilde{h}_o^* \underset{\sim}{v} = \tilde{r}_o^* \underset{\sim}{v} = 1$$

which means that 100 % of the incident wave is absorbed. The condition $\tilde{t}_o^* \underset{\sim}{v} = 0$ means that the ray transmitted in direction $\theta_o = \beta$ has a vanishing amplitude.

Two rows of point absorbers

A point absorber is a wave absorbing oscillator for which the horizontal extension is much smaller than a wavelength. It is typically a heaving body or an OWC with isotropic wave radiation, that is, its Kochin function $h_s$ is independent of the angle $\theta$ . Let its maximum horizontal diameter be 2a (ka << 1).

Two rows of point absorbers are placed along the lines $x = \pm c/2$ as indicated in fig. K.11. We shall use the assumption

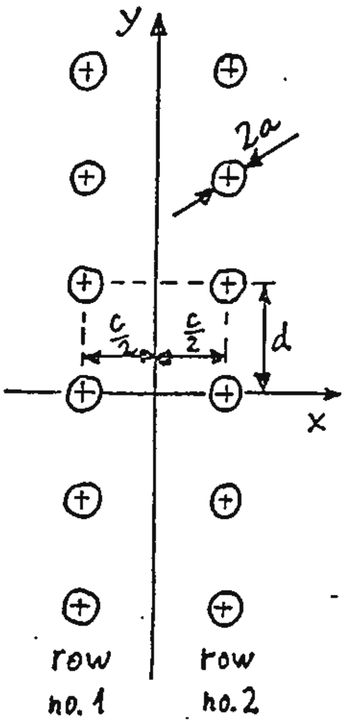


*Fig. K11. Top view of two parallel rows of slender heaving bodies.*

that the Kochin function is not changed when placing point absorbers together in an array. This is a reasonable approximation provided $a \ll c$ and $a \ll d$ and provided $kd$ is not too close to a multiple of $2\pi$ (which would give a strong interference effect in the y direction). Since the point absorbers of group no. 0 are displaced a distance $c/2$ from the origin, the Kochin function vector for group no. 0 is (cf. eqs. [~~122a~~ L21a (p. 70)] and ~~[P165]~~ (7.187) - (7.190))

$$\underset{\sim}{h}(\theta) = h_s \begin{bmatrix} e^{-ik(c/2)\sin\theta} \\ e^{ik(c/2)\cos\theta} \end{bmatrix} \equiv h_s \begin{bmatrix} e^{-i\zeta(\theta)} \\ e^{i\zeta(\theta)} \end{bmatrix}$$

where

$$\zeta(\theta) \equiv (kc/2)\cos\theta = -\zeta(\pi-\theta) \qquad [T8]$$

Remember that

$$\underset{\sim}{t}_o = \underset{\sim}{h}(\beta) = h_s \begin{bmatrix} e^{-i\zeta(\beta)} \\ e^{i\zeta(\beta)} \end{bmatrix} \quad \text{and} \quad \underset{\sim}{r}_o = \underset{\sim}{h}(\pi-\beta) = h_s \begin{bmatrix} e^{i\zeta(\beta)} \\ e^{-i\zeta(\beta)} \end{bmatrix}$$

If $d < \lambda/(1+\sin|\beta|)$ the array radiation damping matrix is,

$$\underset{\sim}{A}'d = \frac{\omega\rho D}{2k^2 d} \frac{\underset{\sim}{t}_o \tilde{\underset{\sim}{t}}_o^* + \underset{\sim}{r}_o \tilde{\underset{\sim}{r}}_o^*}{\cos\beta}$$

$$\underset{\sim}{A}'d = \frac{\omega\rho D|h_s|^2}{2k^2 d\,\cos\beta} \begin{bmatrix} 1 + 1 & e^{-2i\zeta(\beta)} + e^{i2\zeta(\beta)} \\ e^{i2\zeta(\beta)} + e^{-i2\zeta(\beta)} & 1 + 1 \end{bmatrix} =$$

$$= \frac{\omega\rho D|h_s|^2}{k^2 d\,\cos\beta} \begin{bmatrix} 1 & \cos 2\zeta(\beta) \\ \cos 2\zeta(\beta) & 1 \end{bmatrix} \qquad [T40]$$

The condition for optimum is

$$(\underset{\sim}{t}_o \tilde{\underset{\sim}{t}}_o^* + \underset{\sim}{r}_o \tilde{\underset{\sim}{r}}_o^*)\underset{\sim}{v} = \underset{\sim}{h}_o = \underset{\sim}{r}_o$$

When $\underset{\sim}{t}_o$ and $\underset{\sim}{r}_o$ are linearly independent, that is when

$$0 \neq \begin{vmatrix} e^{-i\zeta(\beta)} & e^{i\zeta(\beta)} \\ e^{i\zeta(\beta)} & e^{-i\zeta(\beta)} \end{vmatrix} = e^{-i2\zeta(\beta)} - e^{i2\zeta(\beta)} = -2i\sin\zeta(\beta)$$

this gives

$$\tilde{t}_o^* \underset{\sim}{v} = 0 \quad \text{and} \quad \tilde{r}_o^* \underset{\sim}{v} = 1$$

$$h_s^*(e^{-i\zeta(\beta)}v_1 + e^{-i\zeta(\beta)}v_2) = 0 \qquad h_o^*(e^{-i\zeta(\beta)}v_1 + e^{i\zeta(\beta)}v_2) = 1$$

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$$\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{i}{2h_s^* \sin 2\zeta(\beta)} \begin{bmatrix} e^{-i\zeta(\beta)} \\ -e^{i\zeta(\beta)} \end{bmatrix} \qquad [T24]$$

We see here that we have equally large amplitude in both rows, even if all power is absorbed by the front row, while the rear row acts as a dynamic reflector. For the special case where

$$c \cos\beta = \frac{\pi}{2k} = \frac{\lambda}{4}$$

that is $\zeta = \pi/4$ we have

$$\underset{\sim}{v} = \frac{h_s}{2|h_s|^2} \begin{bmatrix} e^{i\pi/4} \\ e^{-i\pi/4} \end{bmatrix} = \frac{\underset{\sim}{r}_o}{2|h_s|^2} = \frac{\underset{\sim}{h}_o}{2|h_s|^2}$$

which means that the oscillating velocity/(air pressure) is in phase with the excitation force/(volume flux) with this special separation between the two rows. In this special case $\underset{\sim}{A}'d$ is a diagonal matrix, since $\cos 2\zeta(\beta) = 0$ .

For other separations between the rows the optimum velocity /(air pressure) is larger by the factor $1/\sin 2\zeta(\beta)$ and the optimum oscillation state vector is not in phase with the excitation vector.

We notice that the optimum state vector becomes infinite when $c \cos\beta = \lambda/2$ , that is $\zeta = \pi//2$ . Then $\underset{\sim}{t}_o$ and $\underset{\sim}{r}_o$ are linearly dependent, and $\underset{\sim}{A}'d$ is a singular matrix. In this case it is not possible to absorb more than 50 % of the incident wave power. The two rows cannot cooperate to radiate an antisymmetric wave, but only a symmetric wave.

Problem K1. (Radiation resistance of heaving point absorber)

An axisymmetric buoy of diameter 2a is placed in the middle of a wave channel of width d. For the heave mode the excitation force coefficient is denoted as $(f_3'd)$ and the radiation resistance as $(R_{33}'d)$. However, we denote these parameters $f_3$ and $R_{33}$ respectively, if the buoy is placed in open water instead of in a wave channel. The purpose of this problem is to derive a simple one-term expression for $(R_{33}'d)$ in terms of $R_{33}$, $|f_3'd|$, $|f_3|$, d and k, where k is the angular repetency of the wave. This one-term expression (which involves multiplication and division, but no addition or subtraction) is valid only if kd is less than a certain number. Which number? (Derive the expression by considering the wave interference associated with maximum wave-power absorption in the case of unconstrained heave amplitude, that is, in the case of a sufficiently small incident wave). If the numbers (a/d) and (kd) are not too large, the "point-absorber" approximation $(f_3'd) \approx f_3$ is valid. Show that in this case we have $(R_{33}'d) \approx (2/kd)R_{33}$.

# Problem K1. Solution

Maximum absorbed power $P_{max} = \dfrac{|F_e|^2}{8 R_r}$ Cf. eq. ~~[A79](p. A22)~~ (3.45) or (6.10)

Axisymmetric heaving body in wave channel. Symmetric wave generation. Extract a maximum of energy from plane incident wave by maximum destructive wave interference. Generation of cross waves not useful for this wave generation. Cross waves not generated if $kd < \pi$. Cross waves of even order, that is, symmetric cross waves, not generated if $kd < 2\pi$.

From the symmetry of the wave generation we know that a maximum of 50 percent of the incident wave energy may be extracted by the heaving axisymmetric body in the middle of the wave channel

$$\frac{1}{2} J d = \frac{1}{2} \frac{\rho g^2 D}{4\omega} |A|^2 d = P_{max} = \frac{|\hat{F}_{e3}|^2}{8 R'_{33} d} = \frac{|f'_3 d|^2 |A|^2}{8 R'_{33} d}$$

With the body in open sea, we have similarly

$$\frac{\lambda}{2\pi} J = J/k = \frac{\rho g^2 D}{4\omega k} |A|^2 = P_{max} = \frac{|f_3|^2 |A|^2}{8 R_{33}}$$

By dividing the first expression by the second, we find

$$\frac{1}{2} kd = \left| \frac{f'_3 d}{f_3} \right|^2 \frac{R_{33}}{R'_{33} d}$$

$$\underline{R'_{33} d = \frac{2}{kd} \left| \frac{f'_3 d}{f_3} \right|^2 R_{33}}$$

This is valid for $\underline{kd < 2\pi}$ (not only for $kd < \pi$, because cross waves of odd order cannot be generated by ~~the~~ the symmetric body in the middle of the wave channel. If $f'_3 d \approx f_3$ we get

$$R'_{33} d \approx \frac{2}{kd} R_{33} \qquad \text{(q.e.d.)}$$



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