See textbook page 62

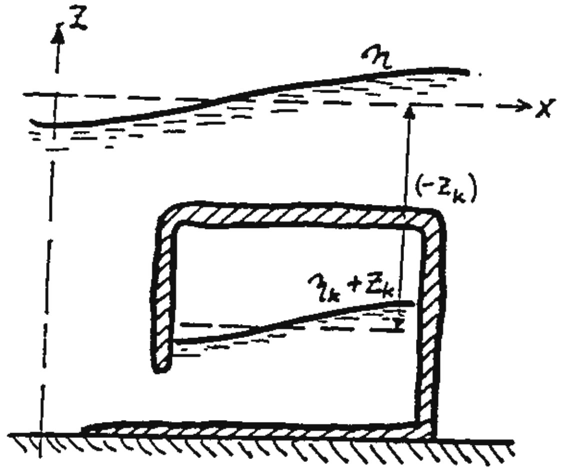

Fig. 4.5 [p. 62]

Fig. B.5. Submerged chamber for oscillating water column with equilibrium water level below the mean sea level.

$$\left[\frac{\partial\phi}{\partial t} + \frac{v^2}{2}\right]_{z=\eta_k+z_k} + g\eta_k + gz_k = C - \left[\frac{p_{tot}}{\rho}\right]_{z=\eta_k+z_k} = \tag{4.23}$$

$$= C - \frac{p_{air}}{\rho} = C - (-gz_k) - \frac{p_{atm} + p_k}{\rho}$$

Using $C = \frac{p_{atm}}{\rho}$ gives*)

Non-linear free-surface boundary conditions

*) We are going to derive the boundary condition [B25b] at the air-water interface. However, if we at this point linearise equations, the linearised version of the boundary condition may be derived simply as follows:

(4.23) $$\left[\frac{\partial\phi}{\partial t}\right]_{z=z_k} + \frac{p_k}{\rho} + g\eta_k = 0 \tag{4.25}$$ [B22a]

We have here neglected a term $v^2/2$ which is small of second order as compared to small first-order quantities like ϕ, η and their derivatives. The vertical velocity of the water surface is

$$(v_z)_{z=z_k} = \left[\frac{\partial\phi}{\partial z}\right]_{z=z_k} = \frac{\partial\eta_k}{\partial t} \tag{4.27}$$

Combination gives

$$-\frac{1}{\rho}\frac{\partial p_k}{\partial t} = \left[\frac{\partial^2\phi}{\partial t^2}\right]_{z=z_k} + g\frac{\partial\eta_k}{\partial t} = \left[\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right]_{z=z_k} \tag{4.28}$$ [B25a]

and, in particular, at the external free water surface

$$\left[\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right]_{z=0} = 0 \tag{4.29}$$ [B25]

(1) (1)

$$\left[\frac{\partial\phi}{\partial t} + \frac{v^2}{2}\right]_{z=\eta_k+z_k} + \frac{p_k}{\rho} + g\eta_k = 0$$

$$\boxed{g\eta_k + \frac{p_k}{\rho} + \left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\,\nabla\phi\cdot\nabla\phi\right]_{z=z_k+\eta_k} = 0} \qquad \text{[B22b]}$$

On the interface between water and air we have, in addition to the dynamic boundary condition (4.21) ~~[B22, 22b]~~, a kinematic boundary condition. Let (dx,dy) be so chosen that the fluid particle at the free surface in (x+dx, y+dy) at time t , arrives at time t + dt at free surface in (x,y) . See fig. B.6. It may be convenient to use the two-dimensional, horizontal gradient

$$\nabla_H\eta = \left[\frac{\partial\eta}{\partial x}, \frac{\partial\eta}{\partial y}\right]$$

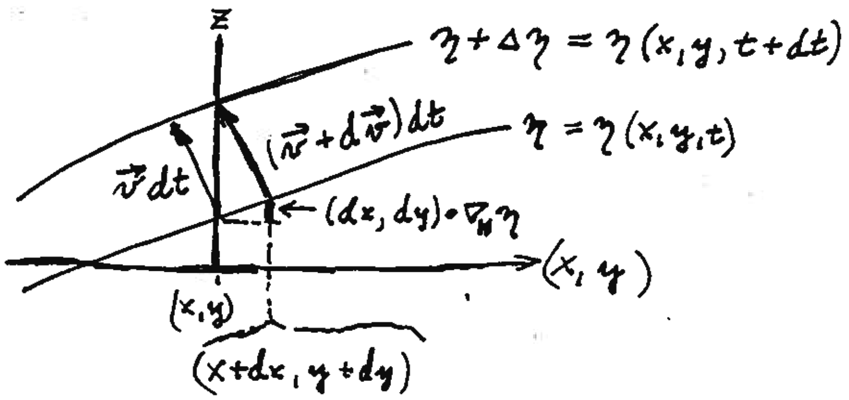

Fig. B.6. *Sketch of elevated free water surface at two different instants. (To derive the kinematic boundary condition.)*

The elevation of the free surface at (x,y) during time dt :

$$\frac{\partial\eta}{\partial t}\,dt = \Delta\eta = (dx,dy)\cdot\nabla_H\eta + (\vec{v} + d\vec{v})dt\cdot\vec{e}_z$$

$$= \frac{\partial\eta}{\partial x}\,dx + \frac{\partial\eta}{\partial y}\,dy + v_z dt$$

neglecting the second order infinitesimal term $dv_z dt$. Further $dx = -v_x dt = -\frac{\partial\phi}{\partial x}\,dt$, $dy = -v_y dt = -\frac{\partial\phi}{\partial y}\,dt$ and $v_z = \frac{\partial\phi}{\partial z}$.

Hence, we have the kinematic boundary condition

$$\boxed{\frac{\partial\eta}{\partial t} = \left[\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x} - \frac{\partial\eta}{\partial y}\frac{\partial\phi}{\partial y}\right]_{z=\eta}}$$

or

$$\boxed{\left[\frac{\partial\phi}{\partial z}\right]_{z=\eta} = \frac{\partial\eta}{\partial t} + \nabla_H\eta\cdot(\nabla_H\phi)_{z=\eta}} \qquad [F2]$$

On the water surface below the entrapped air, we have similarly

$$\boxed{\frac{\partial\eta_k}{\partial t} = \left[\frac{\partial\phi}{\partial z} - \nabla_H\eta_k\cdot\nabla_H\phi\right]_{z=z_k+\eta_k}}$$

Note that the time t enters explicitly only in the free-surface boundary conditions and not into the partial differential equation $\nabla^2\phi = 0$ and into the remaining boundary conditions. Thus, without a free surface the solution $\phi = \phi(x,y,z,t)$ could not represent a wave.

(Remark: Since $\frac{\partial\eta}{\partial z} \equiv 0$ $\nabla_H\eta = \nabla\eta$ (Remember that $\eta = \eta(x,y,t)$)
Hence also: $\nabla_H\eta\cdot\nabla_H\phi = \nabla\eta\cdot\nabla\phi$)

Summary of basic equations for surface waves:

$$\nabla^2\phi = 0 \qquad \text{throughout the fluid}$$

Boundary conditions:

$$\frac{\partial\phi}{\partial n} = 0 \qquad \text{on fixed solid boundaries}$$

$$\frac{\partial\phi}{\partial n} = u_n \qquad \text{on moving solid boundaries}$$

On the free water surface $z = \eta$ we have (with constant atmospheric pressure):

$$g\eta + \left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right]_{z=\eta} = 0$$

$$\frac{\partial\eta}{\partial t} = \left[\frac{\partial\phi}{\partial z} - \nabla_H\eta\cdot\nabla_H\phi\right]_{z=\eta}$$

On the water surface $z = z_k + \eta_k$ (below entrapped air with dynamic pressure p_k) we have:

$$g\eta_k + \frac{p_k}{\rho} + \left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right]_{z=z_k+\eta_k} = 0$$

$$\frac{\partial\eta_k}{\partial t} = \left[\frac{\partial\phi}{\partial z} - \nabla_H\eta_k\cdot\nabla_H\phi\right]_{z=z_k+\eta_k}$$

(It can be shown that the two free-surface conditions may be combined into one in terms of ϕ

$$\left[\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right]\frac{p_k}{\rho} + \frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} + \frac{\partial v^2}{\partial t} + \frac{1}{2}\vec{v}\cdot\nabla v^2 = 0$$

[B25b]

on $z = z_k + \eta_k$ (Cf. C.C. Mei, 1983, p. 5))

After obtaining the solution for the velocity potential $\phi = \phi(x,y,z,t)$ we find the wave elevation $\eta = \eta(x,y,t)$ (and $\eta_k = \eta_k(x,y,t)$) and the hydrodynamic pressure $p = p(x,y,z,t)$ Using the Bernoulli equation - cf. eqs. (4.12) ~~[B14]~~ and (4.14) ~~[B15]~~ we obtain

$$p = -\rho\frac{\partial\phi}{\partial t} - \frac{\rho v^2}{2} = -\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right] \qquad (4.31)\ [F3]$$

See textbook page 63-64

B.4. LINEARISATION OF THE BASIC EQUATIONS

Assuming low amplitude waves we shall neglect products and powers of order two and higher of the variable ϕ and its derivatives. Then the free-surface boundary conditions (4.11) and [F2] [F1 - F2] simplify to

$$g\eta + \left[\frac{\partial\phi}{\partial t}\right]_{z=0} = 0 \qquad \frac{\partial\eta}{\partial t} = \left[\frac{\partial\phi}{\partial z}\right]_{z=0} = (v_z)_{z=0}$$

$$g\eta_k + \frac{p_k}{\rho} + \left[\frac{\partial\phi}{\partial t}\right]_{z=z_k} = 0 \qquad \frac{\partial\eta_k}{\partial t} = \left[\frac{\partial\phi}{\partial z}\right]_{z=z_k}$$

Differentiation of the left-hand equation and substitution into the right-hand equation result in

$$\left[\frac{\partial^2\phi}{\partial t^2} + g\,\frac{\partial\phi}{\partial z}\right]_{z=0} = 0 \qquad (4.29)\ [B25]$$

and

$$\left[\frac{\partial^2\phi}{\partial t^2} + g\,\frac{\partial\phi}{\partial z}\right]_{z=z_k} = -\frac{1}{\rho}\,\frac{\partial p_k}{\partial t}$$

The hydrodynamic pressure is given by the linearised equation obtained from [F3]

$$p = -\rho\frac{\partial\phi}{\partial t} \qquad (4.31)\ [B27]$$

Extensions to Section 4.5 (pp. 83-87)

Chapter S [Replaces previous chapter C]

J. Falnes:

LECTURES ON STATISTICS AND ENERGY TRANSPORT OF REAL OCEAN WAVES

1. INTRODUCTION

In the lectures on linear wave theory we analysed harmonical waves on the basis of the assumptions of negligible viscosity and of irrotational fluid motion. In particular, we considered particular solutions representing plane waves or circular waves. A harmonical plane wave may be called a "regular wave" as opposed to the real "irregular waves" of the ocean. The swells, that is travelling waves which have left their region of generation by winds, are closer to harmonical waves than the more irregular, locally wind-generated waves.

One reason for describing ocean waves by statistical methods is the difficulty of a deterministic solution of the Laplace equation with the proper linear and non-linear boundary conditions. Moreover, the waves are generated by winds, for which a deterministic description is very difficult. Thus, only statistical information is available for the boundary condition on the interface between the water and the (windy) air.

The wind waves have a stochastic nature, and, hence, a statistical modelling may be very useful. An irregular wave may be considered, at least approximately, as a superposition of harmonical waves of many different frequencies. Usually, only statistical information is, at best, available for the amplitude, the phase and the direction of propagation of each individual harmonical wave.

Wind-generated gravity waves on the sea surface have wave period typically in the region 1 to 30 seconds, with the most important energy content in the region 5-15 s. Waves of longer period may be generated by sub-sea avalanches or by earth quakes. Even longer periods (approximately 12 h and 24 h) are associated with the tidal waves generated by gravity interaction of the moon and the sun with the oceans moving along with our planet in its rotation. Surface waves of period less then about 10^{-1} s are capillary waves, for which the surface tension contributes more than gravity to the wave's potential energy. Since we are here considering waves of period in excess of 1 s, only, we shall throughout neglect surface tension.

For the wind-generated gravity waves various characteristic times of different orders of magnitude are of interest. One is the wave period ($\sim 10^1$ s), and another is the duration of wave groups ($\sim 10^2$ s). Note that even the swells, due to a certain frequency spread of their spectrum, have their individual wave heights varying from one minute to the next. Still another characteristic time ($\sim 10^3$ s) is the duration of a sea wave state considered to be approximately statistically stationary. (The measurement of a certain sea state at a particular location usually lasts for 1024 s or 2048 s.) A wave measurement representative for a time interval $\sim 10^3$ s is representative for a corresponding horizontal length $\sim 10^4$ m along the sea surface. (The ratio between this length and the time interval is the group velocity v_g which is $\sim 10^1$ m/s for waves of period $T \sim 10^1$ s.) The characteristic time for the duration of a weather condition is typically 10^4 s to 10^5 s. (A new measurement of the sea state is usually made every 3 hours.) In some ocean regions there are seasonal variations in the wave climate. For instance, in the northern oceans of the world the waves are, in average, larger in the winter than in the summer. The corresponding characteristic time is a season or a year ($\sim 10^7$ s).

2. THE ENERGY SPECTRUM

In this section we shall consider the (Fourier) decomposition of an irregular wave into components of harmonical waves. Further we shall present definitions of wave spectra.

See page 83 in textbook

For a progressive harmonical plane wave the stored energy per unit surface

[B85]

$$E = E_k + E_p = 2E_k = 2E_p = \frac{\rho g}{2} \left| \hat{\eta}_f \right|^2 = \rho g \, \overline{\eta^2(t)} \qquad \text{(S1)} \quad (4.169)$$

according to linear wave theory. Still assuming linear theory the superposition principle is applicable, which means that a real sea state may be described in terms of components of harmonical waves. Correspondingly we may rewrite eq. (S1) as

$$E = \rho g \, \overline{\eta^2(t)} = \rho g \int_0^\infty S(f)df \qquad (4.170) \quad \text{(S2)}$$

where

$$\overline{\eta^2(t)} = \int_0^\infty S(f)df \equiv H_s^2/16 \qquad (4.171) \quad \text{(S3)}$$

Here $S(f)$ is termed the energy spectrum, or simply: the spectrum.

Various parameterisations have been proposed for A and B. One possible variant is

$$A = BH_s^2/4 \quad \text{and} \quad B = (5/4)f_p^4 \qquad (4.193) \quad (S27)$$

Here H_s is the "significant wave height" and f_p the "peak frequency" (the frequency for which S has its maximum). The corresponding wave period $T_p = 1/f_p$ is called the "peak period".

Another functional relation is the <u>JONSWAP spectrum</u>,

$$S(f) = (\alpha g^2/f^5) \exp\{-(5/4)(f_p/f)^4\} \gamma^s \qquad (S28)$$

which has four parameters. Here

$$s = \exp\{-(\frac{f}{f_p} - 1)^2/(2\sigma^2)\} \qquad (S29)$$

with

$$\sigma = \begin{cases} \sigma_a & \text{for} \quad f < f_p \\ \sigma_b & \text{for} \quad f > f_p \end{cases} \qquad (S30)$$

(Including the two different values of σ, the JONSWAP spectrum has five parameters). The parameters α, γ and σ are dimensionless. The "peakedness" parameter γ is typically in the range 1 to 7 with $\gamma \approx 3$ as a typical value. For $\gamma = 1$, the JONSWAP spectrum is equivalent to the PM spectrum. For $\gamma > 1$ the JONSWAP spectrum has a narrower bandwidth than the PM spectrum. From a series of wave measurements in the North Sea, the values

$$\sigma_a = 0.07, \quad \sigma_b = 0.09 \quad \text{and} \quad \gamma = 3.3 \qquad (S31)$$

have been adopted as typical average values. In practice these parameters may vary with the wave state (that is, with the values of H_s and f_p). Neglecting this relatively slight variation, the parameter α is proportional to the square of the significant wave height H_s, in accordance with eq. (S3). For average wave states we have $\alpha \sim 10^{-2}$.

For the direction-resolved spectrum we may write

$$s(f,\beta) = D(\beta,f)\ S(f) \qquad \text{(4.194) (S32)}$$

where

$$\int_{-\pi}^{\pi} D(\beta,f)d\beta = 1 \qquad \text{(4.195) (S33)}$$

One proposal for the directional distribution (neglecting its possible frequency dependence) is

$$D(\beta) = \begin{cases} (2/\pi)\cos^2(\beta - \beta_0) & \text{for } |\beta - \beta_0| < \pi/2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(4.196) (S34)}$$

where β_0 is the predominant angle of incidence.

The real-valued, non-negative function $s(f,\beta)$ provides information on the modulus of the complex valued function $A(\omega,\beta)$. From eq. (4.188) (S22) we have

$$|A(\omega,\beta)| = \left\{\frac{2s_\omega(\omega,\beta)}{\Delta\omega\Delta\beta}\right\}^{\frac{1}{2}} = \frac{1}{2\pi}\left\{\frac{2s(f,\beta)}{\Delta f\Delta\beta}\right\}^{\frac{1}{2}} \qquad \text{(4.197) (S35)}$$

That is, only the amplitudes of the harmonical waves are given, while the spectrum provides no information on the phase of the individual harmonical waves. Writing

$$A(\omega,\beta)=|A(\omega,\beta)|e^{i\psi(\omega,\beta)} \qquad \text{(4.198) (S36)}$$

the unknown phase ψ is assumed to be uniformly statistically

See textbook page 86

distributed in the interval $(-\pi,\pi)$. That is, its distribution function is

$$f(\psi) = \begin{cases} \frac{1}{2\pi} & \text{when} \quad -\pi < \psi \leq \pi \\ 0 & \text{otherwise} \end{cases} \qquad (4.199) \quad (S37)$$

When simulating a real irregular wave the amplitudes $|A(\omega,\beta)|$ are selected according to the appropriate spectrum $s(f,\beta)$ while the phases $\psi(f,\beta)$ are chosen as random numbers in the interval $-\pi < \psi \leq \pi$.

3. COLLECTION OF WAVE DATA

A vast quantity of information on ocean wave data has been collected during the last century by visual observation from ships and from coastal meteorological observations. The intention of such an observation is to quantify the wave height, the wave period and the direction from which the wave comes.

More reliable information has been obtained during the last few decades by means of physical measurements, most of which have been made at the sea surface. Alternatively, measurements can be made from below or above the surface, for instance by utilising laser or radar from above, or by using a pressure transducer or an inverted narrow-beam echo sounder at a submerged fixed position. More recently shore-based seismometers have been used to assess the offshore wave energy resource (A.C. Kibblewhite and E.P.M. Brown, The use of shore-based seismometers for wave energy resource assessment in New Zealand, OCEANS'91, Hawaii, 1-3 October 1991, Proceedings, Vol. I, pp. 375-379).

Measurement of waves in shallow water can be made by means of a wave staff, a vertical pole fixed on the sea bed. This may be provided with a series of vertically displaced electrodes with which rising or falling sea water is making or breaking contact, respectively. Another method is to use resistance wires held vertically through the surface. A method which is less susceptible to biological fouling is to use electrically insulated wires. The fact that the electrical capacity between the sea water and the wire varies linearly with the depth of immersion, is utilised to measure the wave elevation.

To map the wave climate of the oceans it has been common to make wave measurements at the surface in deep water locations. A much used method is to use a moored float containing a vertical seeking accelerometer. If also the pitch and roll of the float are measured together with its orientation, data is also obtained on the direction of wave propagation. The float contains a battery and signal processing equipment together with a magnetic tape recorder and/or a radio transmitter. Direct radio transmission (VHF) is possible if the distance to land is less than 30 km. Otherwise radio transmission by satellite (ARGOS system) has become common in recent years.

When sufficient measured wave data are not available one have to rely on wave data computed from meteorological data on wind and/or air pressure. Many places such data have been recorded during several decades, giving a basis for longterm statistics. The method is named "hindcasting" which aims at obtaining wave spectra or other wave parameters from earlier measurements of wind and air pressure. Such wave data are less accurate than measured wave data. If possible the computational model for obtaining the hindcast data should be calibrated by means of measured wave data.

A record of the measured wave elevation may look like the graph shown in fig. S1, which shows the elevation of an irregular wave during five (zero-upcross) periods. A recording lasting for 1024 s or 2048 s contains several hundred wave periods. The total recording time divided by the number of zero-upcross periods, is the "average zero-upcross period"

$$T_z = \frac{1}{N} \sum_{i=1}^{N} T_i \ . \qquad \text{(S38)}$$

The traditional definition of the "significant wave height" $H_{1/3}$ is the average of the highest one third of the waves. Thus,

$$H_{1/3} = \frac{3}{N} \sum_{j=1}^{N/3} H_j \qquad \text{(S39)}$$

where the numbers j are those selected numbers among the numbers i for which the H_j values are larger than the remaining H_i values. In most practical cases this traditional definition agrees fairly well with the modern definition of significant wave height given in eq. (S3). As appears from fig. S1 there may be more than one wave crest between two successive wave upcrossings. Hence the average period T_c between wave crests may be shorter than T_z ,

$$T_c \leq T_z \qquad \text{(S40)}$$

(13)

(13)

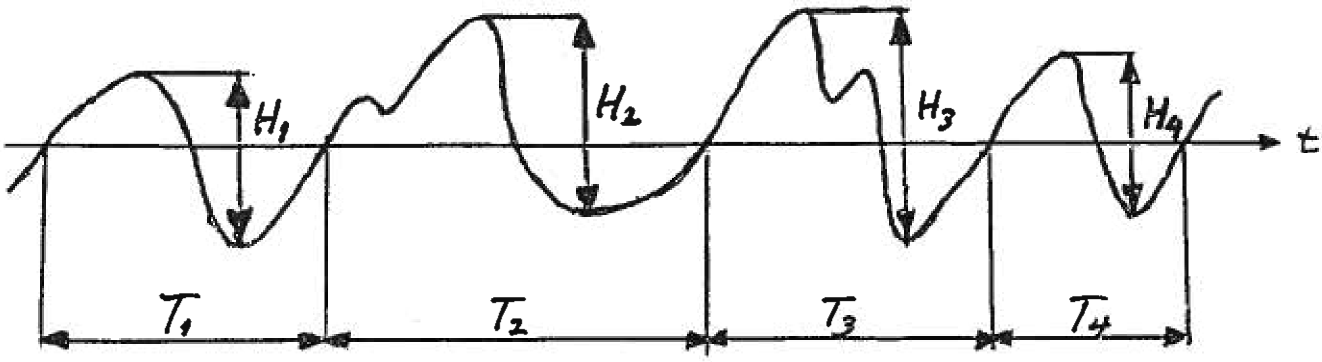

Fig. S1. A wave elevation record of four (zero-upcross) periods. The individual "zero-upcross" period T_i (i = 1, 2, 3, ...) is the time interval between two consequtive times when the wave elevation crosses zero in the upwards direction. The individual wave height H_i is defined as the vertical difference between the maximum elevation and the (following) minimum elevation within the same zero-upcross period.

For waves measured off the Norwegian west coast average values of T_c/T_z are in the region 0.4 to 0.6. Moreover T_z/T_p, where T_p is the peak period, is typically between 0.65 and 0.7.

In order to measure a wave state with a wave measuring buoy the wave elevation is recorded every 1 s (or 0.5 s). This time interval is called the sampling period T_{sampl} and its inverse $f_{sampl} = 1/T_{sampl}$ the sampling frequency. During a recording time T_{rec} which is 2048 s (or 1024 s), 2^n samples are collected, where $n = 11$ (or 10). From this time series the wave spectrum of the wave state is computed by FFT (fast Fourier transform). The frequency resolution of the obtained spectrum is $\Delta f = \Delta\omega/2\pi = 1/T_{rec}$. (Note that this is a discretised spectrum corresponding to a Fourier series of fundamental frequency Δf.) According to the sampling theorem, the highest frequency of the obtained spectrum is the Nyquist frequency $f_c = 1/(2\ T_{sampl})$ which is 0.5 Hz (or 1 Hz). Due to possible inaccuracies inherent in the measurements and in the analysis it may be convenient to smooth the obtained spectrum before presentation, at the cost of reducing the frequency resolution. For instance, if every eight neighbouring frequency components are averaged, the frequency resolution is reduced by a factor of 8. If $\Delta f = 1/1024$ Hz the frequency interval between the components in the smoothed spectrum will be 1/128 Hz. This reduced frequency resolution is tolerable in most cases. Smoothing reduces the uncertainty of the estimated spectrum.

It should be noted that FFT requires less computing time than other alternatives of analysis, such as using ordinary Fourier transform or computation of auto-correlation functions. However, FTT requires that the number of samples of the time series is a multiple power of 2.

4. WAVE SPECTRUM PARAMETERS

We define the j-th order moment m_j of the spectrum as

$$m_j = \int_0^\infty f^j S(f)df \qquad \text{(S41)}$$

Various wave parameters may be defined on the basis of these moments. For instance, in accordance with eq. (S3), we have

$$H_s = H_{mo} \equiv 4\sqrt{m_0} \qquad \text{(S42)}$$

which is an estimate of the traditionally defined significant wave height $H_{1/3}$. Further,

$$T_{m_0\,1} \equiv \frac{m_0}{m_1} \quad \text{and} \quad T_{m_0\,2} \equiv \left\{ \frac{m_0}{m_2} \right\}^{\frac{1}{2}} \qquad \text{(S43)}$$

are estimates of the average zero-upcross period T_z, and

$$T_{m_2\,4} \equiv \left\{ \frac{m_2}{m_4} \right\}^{\frac{1}{2}} \qquad \text{(S44)}$$

is an estimate of the average crest period T_c. Moreover,

$$T_{pc} \equiv \frac{m_{-2}\; m_1}{{m_0}^2} \qquad \text{(S45)}$$

is an estimate of the peak period T_p. The estimates $T_{m_0\,1}$ and $T_{m_0\,2}$ are typically a few percent larger and smaller, respectively, compared to T_z. The estimate T_{pc} may exceed T_p by 10 percent, and $T_{m_2\,4}$ may exceed T_c by 30 percent.

Longuet-Higgin's spectrum-width parameter is defined as

$$\epsilon = \left\{1 - \frac{m_2^{\,2}}{m_0 \; m_4}\right\}^{\frac{1}{2}} \tag{S46}$$

For a very broad spectrum, $m_4 \to \infty$ we have $\epsilon \to 1$. On the other hand, for a very narrow spectrum, peaked at $\omega = \omega_p$ we have $m_j \to m_0 \omega_p^j$ and hence $\epsilon \to 0$. For measured wave spectra we expect to find ϵ between 0 and 1. In most cases the value of ϵ is found between 0.4 and 0.5.

In recent years another bandwidth parameter

$$\nu = \left\{\frac{m_0 \, m_2}{m_1^{\,2}} - 1\right\}^{\frac{1}{2}} \tag{S47}$$

is favoured. This is the normalised "radius of gyration" of the spectrum about its mean frequency

$$f_{10} = m_1 / m_0 = 1/T_{m01} \tag{S48}$$

The "moment of inertia" about the axis $f = 0$ is m_2 and about the axis f_{10} (the "centre of gravity") it is $m_2 - f_{10}^2 m_0$. Thus the "radius of gyration" is $\{m_2 / m_0 - f_{10}^2\}^{\frac{1}{2}}$ and if it is normalised with respect to f_{10} eq. (S47) results. For a regular wave $\nu = 0$, while it can be shown that $\nu = 0.425$ for a PM spectrum. For a JONSWAP spectrum with σ and γ given by eq. (S31) we have $\nu = 0.39$.

5. WAVE ENERGY TRANSPORT

From linear wave theory for a regular wave it is known that, on deep water, the wave energy transport is (cf. eq. 4.136)

[B75]

$$J = \frac{\rho g^2}{2\omega}\,\frac{1}{2}\,|A|^2 = \frac{\rho g^2}{2\omega}\,\overline{\eta^2(t)} = \frac{\rho g^2}{4\pi f}\,\overline{\eta^2(t)} \qquad \text{(S49)}$$

for a harmonical wave. For an irregular wave with elevation variance

$$\overline{\eta^2(t)} = \int_0^\infty S(f)df = m_0 = (H_s/4)^2 \qquad \text{(S50)}$$

(4.171)

the wave energy transport is accordingly

$$J = \frac{\rho g^2}{2}\int_0^\infty \frac{S(f)}{\omega}\,df = \frac{\rho g^2}{4\pi}\,m_{-1} = \frac{\rho g^2}{64\pi}\,\frac{m_{-1}}{m_0}\,16\,m_0 \qquad \text{(S51)}$$

Thus, we have

$$J = \frac{\rho g^2}{2\omega_J}\,\frac{H_s^2}{16} = \frac{\rho g^2}{64\pi}\,T_J\,H_s^2 \qquad \text{(S52)}$$

where

$$T_J = \frac{2\pi}{\omega_J} = \frac{1}{f_J} = T_{m-10} \equiv \frac{m_{-1}}{m_0} \qquad \text{(S53)}$$

is the "energy-transport period" or simply "energy period", which may exceed the zero-upcross period T_z by about 20 percent. Defining the numerical coefficient

$$\alpha_J = \frac{\rho g^2}{64\pi} = \frac{1020\cdot 9.81^2}{64\pi} = 488\,\frac{W/m}{s\ m^2} = 0.49\,\frac{kW/m}{s\ m^2} \qquad \text{(S54)}$$

the wave energy transport is

$$J = \alpha_J T_J H_s^2 \tag{S55}$$

Note from linear wave theory that for a harmonical wave we have

$$J = \alpha_0 T H^2 \tag{S56}$$

where $\alpha_0 = 2\alpha_J$.

Average values for the wave power transport off the Norwegian coast, as presented in fig. S2, have been computed from available wave data (Torsethaugen 1990). These average values for the wave power transport agree fairly well with previous results north of 62° N. However, in the North Sea (south of 62° N) the updated values in fig. S2 are about one half of previously published values. For many places in the sea, wave data including directional information are obtained by hindcasting. The hindcast model is calibrated by comparison with measured waves in the locations indicated in fig. S2.

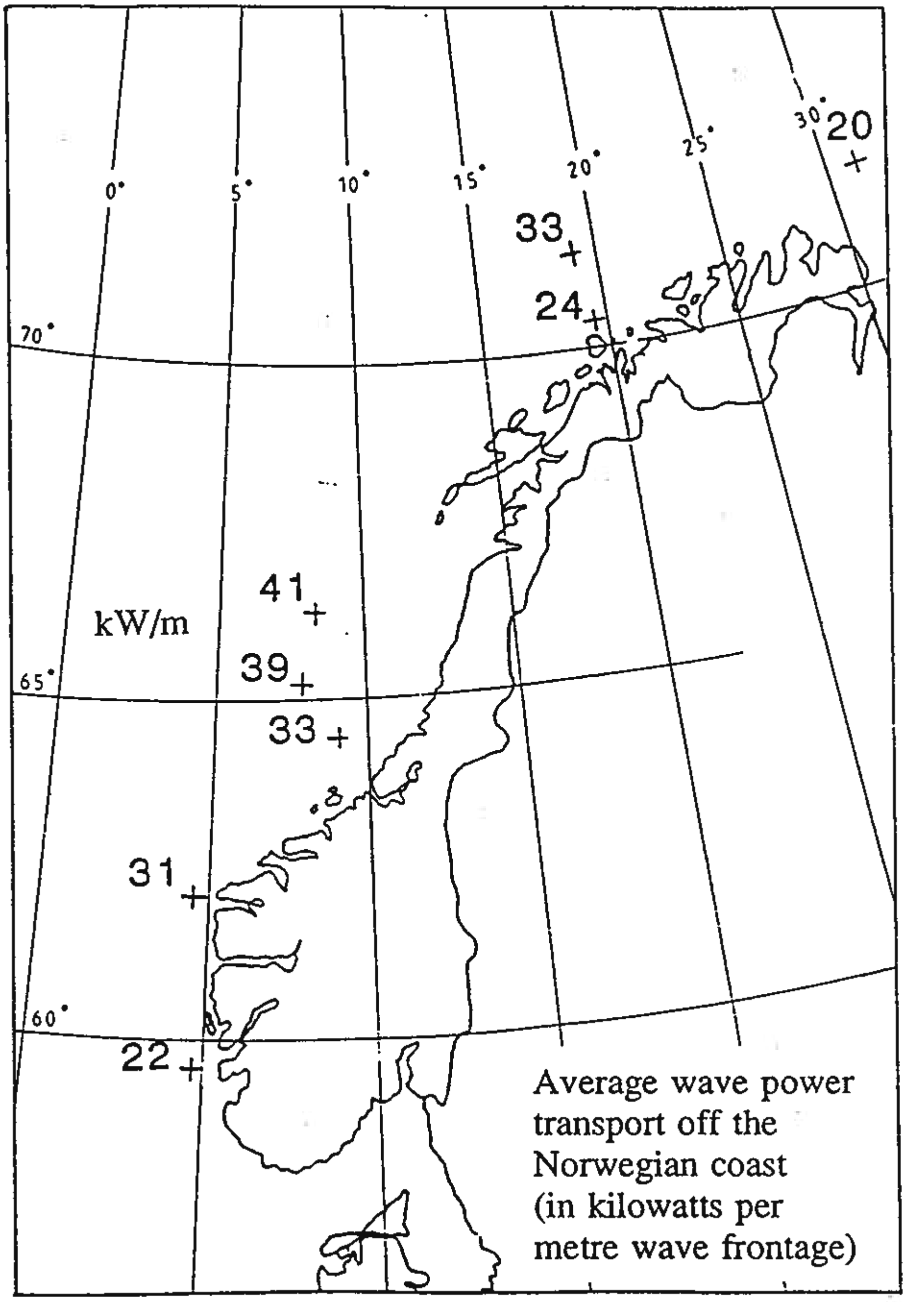

Fig. S2. Average wave power transport J (in kW/m) off the Norwegian coast. The crosses (+) indicate locations where waves have been measured. [K. Torsethaugen, "Bølgedata for vurdering av bølgekraft". NHL-report No. STF60 A90120, 1990-12-20]

(20)

(20)

For the location Utsira (59°30′ N, 4°82′ E) the characteristic times obtained from the measurements are as follows: $T_c = 3.2$ s, $T_{m_{02}} = 5.4$ s, $T_z = 5.8$ s, $T_{m_{01}} = 6.1$ s, $T_J = T_{m_{-10}} = 7.3$ s, $T_p = 8.9$ s and $T_{pc} = 10.0$ s .

The above average values for the wave energy transport are based on the direction-integrated spectrum $S(\omega)$. Thus J is the power (in kW) which passes an envisaged vertical cylinder of diameter 1 m. The power passing an envisaged vertical strip of width 1 m with its normal pointing in the direction θ is

$$J_\theta = \frac{\rho g^2}{4\pi} \int_0^\infty \int_{\beta_1}^{\beta_2} \frac{s(f,\beta)}{f} \cos(\beta-\theta) \, df \, d\beta \qquad \text{(S57)}$$

where $\beta_1 = \theta - \pi/2$ and $\beta_2 = \theta + \pi/2$. Thus only wave components contribute, for which the group velocity has a positive component in the direction which makes an angle θ with the x axis. Note that J_θ may be expected to have a particularly high value if θ is chosen close to the angle of incidence of the prevailing wave.

Average values for J and for the best values of J_θ are shown in fig. S3 on the map of western Europe and the adjoining seas.

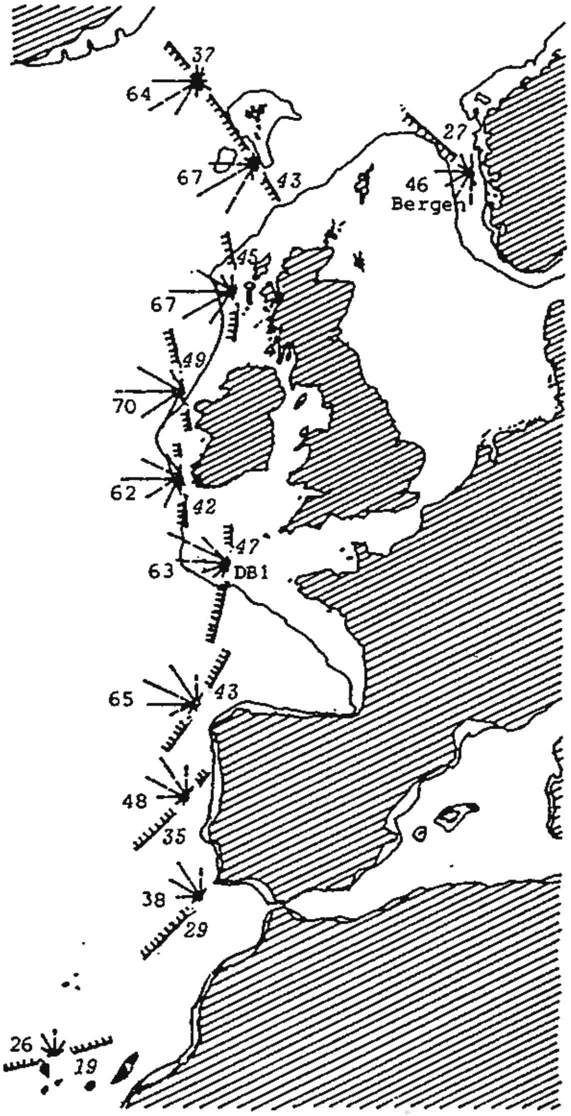

Fig. S3. Average wave energy transport for western Europe based on hindcasting. Wave roses show mean power from each 30° sector. The larger numbers show energy transport J (in kW/m) while smaller numbers (printed in italics) show the energy transport J_θ crossing lines (⊥⊥⊥⊥⊥⊥⊥) whose direction θ maximizes J_θ at the particular location. [D. Mollison, Wave climate and the wave power resource, Proc. IUTAM Symposium, Lisbon, Portugal, 1985, pp. 133-156].

The figure is reproduced from page 148 of the paper
Mollison, D.: "Wave climate and the wave power resource". *Proceedings IUTAM Symposium on Hydrodynamics of Ocean Wave-Energy Utilization,* (edited by D.V. Evans and A.F. de Falcão) Lisbon, Portugal, 8-11 July 1985, Springer-Verlag, pp 133-156, 1986.

6. SHORT-TERM WAVE STATISTICS

A wave recording gives a series of pairs (H_i, T_i) of individual wave heights H_i and zero-upcross periods T_i (fig. S1). In short-term wave statistics the wave height and the period are considered as stochastic variables. To a reasonable degree of accuracy the cumulative probability distribution for the wave height has been found to follow the Rayleigh probability curve. That is, the probability that the wave height H does not exceed a chosen wave height h is

$$P(H \leq h) = 1 - \exp\left\{-2(h/H_{m0})^2\right\} \qquad \text{(S58)}$$

Thus, the probability that H is larger than h is $\exp\{-2(h/H_{m0})^2\}$. For instance, the probability that a wave has a height larger than the significant wave height H_{m0} is $e^{-2} = 0.135$ and the probability that an individual wave height is $2H_{m0}$ is only $\exp\{-2 \cdot 2^2\} = e^{-8} = 3.4 \cdot 10^{-4}$

Moreover, it is possible to show that

$$\frac{H_{1/3}}{H_{mo}} = 1.0011 \approx 1.00 \qquad \text{(S62)}$$

Thus if the individual wave heights obey the statistical Rayleigh distribution there is a very good agreement between the traditional and modern definitions of the significant wave height.

(23)

(23)

The statistical distribution of the zero-upcross periods has been difficult to describe in simple terms.

It is observed that a high wave is often preceded or succeeded by another wave which is higher than average. This effect is not very noticeable in the case of locally generated wind waves, but very marked with narrow-band spectra, such as is the case when the wave state is dominated by swells.

In the derivation of the general formula for group velocity, $v_g = d\omega/dk$ one may consider a wave with a narrow spectrum. This wave is a superposition of several harmonical waves with relatively closely spaced frequencies. The resulting wave may be interpreted as a "carrier wave" with a central frequency within the spectrum (propagating at the phase velocity) and with a slowly modulated amplitude (an envelope propagating at the group velocity). The square of this modulated amplitude (envelope) represents the wave power transport averaged over the individual wave periods. It is regrettable that modulation of the amplitude thus results in a variation with time of the available wave energy. An example is shown in fig. S4. If the energy taken up by a wave power converter could be stored during about 100 seconds, this would result in a more even delivery of useful power from the wave power plant.

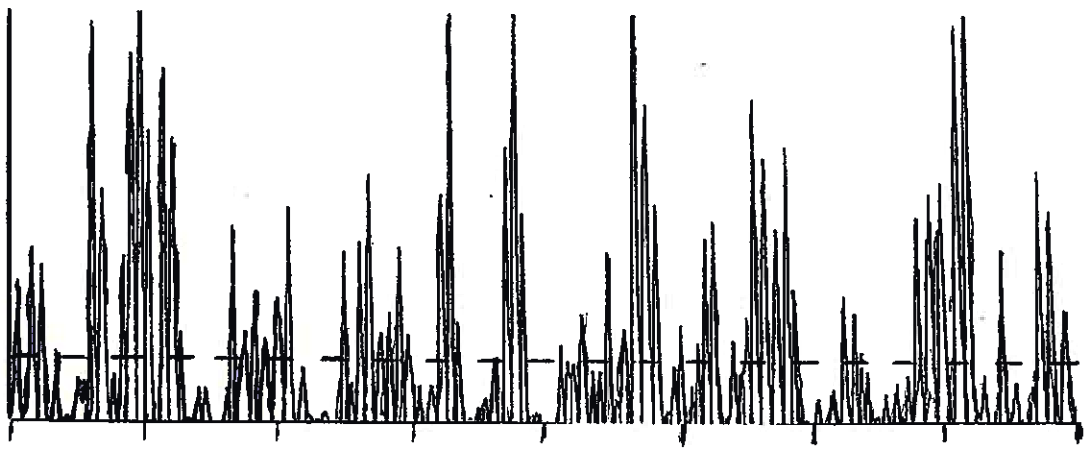

Fig. S4. The plot shows η^2 *(the square of the wave elevation) versus time during a typical record lasting for 8 minutes. The mean value, indicated by the horizontal dashed line, is a measure of the wave power transport J , averaged over the 8 minutes. The signal has been clipped at 7 times the mean value and would have exceeded this level 5 times during the 8 minute run. [S. Salter, World progress in wave energy - 1988. Journal of Ambient Energy, Vol. 10, No. 1, Jan. 1989, pp. 3-24.]*

In order to design the necessary energy storage system in the wave power converter statistical information is required on the duration of wave groups and on the time interval between wave groups. These times may vary significantly. Some theoretical studies have been made of this kind of statistics. [See pp. 143-157 in M.J. Tucker, Waves in Ocean Engineering. Measurement, Analysis, Interpretation. (Ellis Horwood, 1991)]. Here we shall only state some results which give estimates for average values in the case of a narrow spectrum. Consider a wave group where a number N of individual waves (zero-upcross waves) have an elevation exceeding a level ρ. The corresponding duration of a group defined by the chosen level is N times the estimated zero-upcross time $T_{m20} = (m_0/m_2)^{\frac{1}{2}}$. The statistical average of N is

$$\bar{N} = \left\{(\nu^{-2}+1)/2\pi\right\}^{\frac{1}{2}} \frac{m_0^{\frac{1}{2}}}{\rho} \qquad \text{(S63)}$$

where ν is the spectrum width parameter given by eq. (S27). As was to be expected, it is seen that the average group duration is longer, if the spectrum is narrower, that is if ν is smaller. Moreover the duration is shorter if a higher level ρ is used for defining the "group". The mean interval $\bar{G}$ between groups is (in terms of numbers of average zero-upcross waves)

$$\bar{G} = \bar{N} \exp\{\rho^2/2m_0\} \qquad \text{(S64)}$$

The formulas for $\bar{N}$ and $\bar{G}$ have been derived by Longuet-Higgins (1984) [Statistical properties of wave groups in a random sea. Phil. Trans. Roy. Soc. A.312, pp. 219-250]. In his derivation he assumed that the spectrum was truncated to the limits

$$0.5\ f_p < f < 1.5\ f_p \tag{S65}$$

where f_p is the peak frequency of the spectrum. If we choose the level $\rho = H_{m0}/2 = 2m_0^{\frac{1}{2}}$ the formulas for $\bar{N}$ and $\bar{G}$ give

$$\bar{N}/\bar{G} = e^{-2} = 0.135 \tag{S66}$$

and

$$\bar{N} = 0.20\ (\nu^{-2} + 1)^{\frac{1}{2}} \tag{S67}$$

For the relatively broad PM spectrum $\nu = 0.425$, but $\nu = 0.180$ if the spectrum is filtered (or truncated) in accordance with the frequency pass band (S65). Then the formula (S67) gives $\bar{N} = 1.13$ which is an underestimate, since the true value is in the region of 1.25. Eq. (S66) indicates that less than 14 percent of the individual wave heights exceed the significant wave height H_{m0}. This also agrees with the Rayleigh distribution (S58).

Problem S4. (Significant wave heights)

Assuming that individual wave heights statistically obey the Rayleigh distribution, show that

$$H_{1/3}/H_{mo} = 1.0011$$

given that $\mathrm{erf}\{(\ln 3)^{\frac{1}{2}}\} = 0.861743$ and $\mathrm{erf}(\infty) = 1$, where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\{-t^2\}dt$$

is the error function. Further, given that $\mathrm{erf}\{(\ln 10)^{\frac{1}{2}}\} = 0.968120$ find the statistical value for $H_{1/10}/H_{mo}$ where $H_{1/10}$ is the average of the highest one tenth of the individual wave heights.

[Hint: The probability that a wave height does not exceed H is

$$F(H) = 1 - \exp\{-2(H/H_{mo})^2\}$$

and the corresponding probability density is $f(H) = dF(H)/dH$. According to definition we have

$$H_{1/3} = \frac{1}{1/3} \int_a^\infty Hf(H)dH$$

where a is determined by

$$\int_a^\infty f(H)dH = 1/3]$$

Problem S.4. Solution

Significant wave heights.

$$F(H) = 1 - \exp\{-2(H/H_{mo})^2\}$$

$$f(H) = \frac{dF(H)}{dH} = \frac{4H}{H_{mo}^2} \exp\{-2(H/H_{mo})^2\}$$

$$\frac{1}{3} = \alpha = \int_a^\infty f(H)\,dH = \int_a^\infty \frac{dF(H)}{dH}\,dH = F(\infty) - F(a) =$$

$$= 1 - F(a) = \exp\{-2(a/H_{mo})^2\}$$

$$2\left(\frac{a}{H_{mo}}\right)^2 = \ln\left(\frac{1}{\alpha}\right) = \ln 3$$

$$\frac{a}{H_{mo}} = \left\{\frac{1}{2}\ln 3\right\}^{1/2} = 0{,}741152 \quad (\text{for } \alpha = 1/3)$$

$$H_\alpha = \frac{\int_a^\infty H f(H)\,dH}{\int_a^\infty f(H)\,dH} = \frac{1}{\alpha}\int_a^\infty H f(H)\,dH =$$

$$= \frac{1}{\alpha}\int_a^\infty \frac{4H^2}{H_{mo}^2} \exp\left\{-2\left(\frac{H}{H_{mo}}\right)^2\right\} dH$$

$$\frac{2H^2}{H_{mo}^2} = \left(\frac{\sqrt{2}H}{H_{mo}}\right)^2 = t^2 \qquad t = \sqrt{2}\,\frac{H}{H_{mo}} \qquad dH = \frac{H_{mo}}{\sqrt{2}}\,dt$$

$$H_\alpha = \frac{1}{\alpha}\,\frac{H_{mo}}{\sqrt{2}} \int_{\sqrt{2}a/H_{mo}}^\infty 2t^2 \exp\{-t^2\}\,dt$$

Partial integration: $\int 2t^2 \exp\{-t^2\} = -t\exp\{-t^2\} + \int \exp\{-t^2\}\,dt + C$

$$H_\alpha/H_{mo} = \frac{1}{\alpha\sqrt{2}}\sqrt{2}(a/H_{mo}) \exp\{-2a^2/H_{mo}^2\} + \frac{1}{\alpha\sqrt{2}} \int_{\sqrt{2}a/H_{mo}}^\infty \exp\{-t^2\}\,dt$$

$$= \frac{a}{H_{mo}} + \frac{1}{\alpha\sqrt{2}} \int_{\sqrt{2}a/H_{mo}}^\infty \exp\{-t^2\}\,dt$$

$$\frac{H_\alpha}{H_{mo}} = \frac{a}{H_{mo}} + \frac{1}{\alpha\sqrt{2}}\,\frac{\sqrt{\pi}}{2}\left\{\mathrm{erf}(\infty) - \mathrm{erf}(\sqrt{2}\,a/H_{mo})\right\}$$

$$= \frac{a}{H_{mo}} + \frac{1}{\alpha}\sqrt{\frac{\pi}{8}}\left\{1 - \mathrm{erf}(\sqrt{2}\,a/H_{mo})\right\}$$

For $\alpha = 1/3$ $\quad a/H_{mo} = \left\{\frac{1}{2}\ln 3\right\}^{1/2} = 0.741152$

$$\frac{H_{1/3}}{H_{mo}} = \frac{a}{H_{mo}} + 3\sqrt{\frac{\pi}{8}}\left\{1 - \mathrm{erf}(\sqrt{\ln 3})\right\}$$

$$= 0.741152 + 3\sqrt{\frac{\pi}{8}}\left\{1 - 0.861743\right\} = 1.001071$$

$$\frac{H_{1/3}}{H_{mo}} = 1.0011$$

For $\alpha = 1/10$ $\quad a/H_{mo} = \left\{\frac{1}{2}\ln 10\right\}^{1/2} = 1.072983$

$$\frac{H_{1/10}}{H_{mo}} = \frac{a}{H_{mo}} + 10\sqrt{\frac{\pi}{8}}\left\{1 - \mathrm{erf}(\sqrt{\ln 10})\right\}$$

$$= 1.072983 + 10\sqrt{\frac{\pi}{8}}\left\{1 - 0.968120\right\} = 1.272759$$

$$\frac{H_{1/10}}{H_{mo}} = 1.273$$