Problem 1: Stability analysis

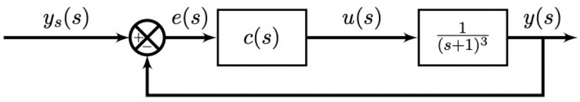


Figure 1: Closed loop control system

In this exercise you are to study the stability of the system given in Figure 1.

1. Assume that the system is controlled by a P controller, $c(s) = K_c$.
   The value of K_c must lie between an upper and a lower bound to guarantee stability.
   Determine the bounds for K_c:
   (a) Using the Routh-Hurwitz criterion (Seeborg, p. 192)
   (b) (Optional, only do this if you feel like it ◡̈) By calculating the poles of the feedback system

a) The characteristic equation is:

$$1 + loop = 0$$

$$1 + C \cdot \frac{1}{(s+1)^3} = 0$$

$$1 + K_c \cdot \frac{1}{(s+1)^3} = 0$$

$$(s+1)^3 + K_c = 0$$

$$s^3 + 3s^2 + 3s + 1 + K_c = 0$$

$$s^3 + 3s^2 + 3s + (1 + K_c) = 0$$

Test 1: Coefficients, all coefficient must have the same sign

$\Rightarrow$ As $1, 3, 3 > 0$, then $1 + K_c > 0$

$$\underline{K_c > -1}$$

Test 2: Set up Routh array:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

$$\begin{array}{c|cccc}
\text{Row} & & & & \\
1 & a_n & a_{n-2} & a_{n-4} & \cdots \\
2 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
3 & b_1 & b_2 & b_3 & \cdots \\
4 & c_1 & c_2 & \cdots & \\
\vdots & \vdots & & & \\
n+1 & z_1 & & &
\end{array}$$

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$\vdots$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

$$\vdots$$

We have $n=3 \Rightarrow 4$ rows, with $a_3=1$, $a_2=a_1=3$, $a_0=K_c+1$

$$b_1 = \frac{a_2 a_1 - a_3 a_0}{a_2} = \frac{9-K_c+1}{3} = \frac{8-K_c}{3}$$

$$b_2 = \frac{a_2 \cdot 0 - a_3 \cdot 0}{a_2} = 0$$

$$c_1 = \frac{b_1 \cdot a_0 - a_2 \cdot b_2}{b_1} = \frac{b_1 \cdot a_0}{b_1} = a_0$$

$$\begin{array}{|cc}
1 & 3 \\
3 & K_c+1 \\
\frac{8-K_c}{3} & 0 \\
K_c+1 &
\end{array}$$

If all values in the left column are positive, then the system is stable $\Rightarrow K_c+1>0 \Rightarrow K_c>-1$

$$\frac{8-K_c}{3} > 0 \Rightarrow K_c < 8$$

$$\Rightarrow \underline{\underline{-1 < K_c < 8}}$$

2. Assume PI control, $c(s) = K_c \frac{1+\tau_I s}{\tau_I s}$.

   (a) Determine the boundaries in which K_c and τ_I must be chosen to guarantee stability. Use the Routh-Hurwitz criterion.
   (b) Indicate the stability region in a diagram with K_c and τ_I as axes.
   (c) Calculate the SIMC parameters using $\tau_c = \theta$ (θ = effective time delay).
   (d) Show the point corresponding to the SIMC parameters in the diagram.
   (e) Is the SIMC tuning in the stable region? How much can K_c and τ_I change without rendering the system unstable?

a) The characteristic polynomial is:

$$1 + \text{loop} = 0$$

$$1 + K_c \frac{1+\tau_I s}{\tau_I s} \cdot \frac{1}{(s+1)^3} = 0$$

$$\tau_I s(s+1)^3 + K_c \tau_I s + K_c = 0$$

$$\tau_I s^4 + 3\tau_I s^3 + 3\tau_I s^2 + \tau_I s + K_c \tau_I s + K_c$$

$$\underbrace{\tau_I s^4}_{a_4} + \underbrace{3\tau_I s^3}_{a_3} + \underbrace{3\tau_I s^2}_{a_2} + \underbrace{(\tau_I + K_c \tau_I) s}_{a_1} + \underbrace{K_c}_{a_0}$$

Test 1. all coefficients >0

$\Rightarrow \tau_I > 0$

$3\tau_I > 0 \Rightarrow \tau_I > 0$

$(\tau_I + K_c\tau_I) > 0$

$K_c > 0$

Test 2: Setting up routh array, $n=4 \Rightarrow 5$ rows

τ_I	$3\tau_I$	K_c
$3\tau_I$	$\tau_I + K_c\tau_I$	
$\frac{(8-K_c)\tau_I}{3}$	K_c	
$\frac{\frac{8-K_c}{3}(\tau_I+K\tau_I)-3K_c}{\frac{8-K_c}{3}}$	0	
K_c		

$$b_1 = \frac{a_3 a_2 - a_4 a_1}{a_3} = \frac{3\tau_I 3\tau_I - \tau_I(\tau_I + K_c\tau_I)}{3\tau_I}$$

$$= \frac{9\tau_I - \tau_I + K_c\tau_I}{3} = \frac{(8-K_c)\tau_I}{3}$$

$$b_2 = \frac{a_3 a_0 - a_4 \cdot 0}{a_3} = \frac{a_3}{a_3} a_0 = a_0 = K_c$$

$$b_3 = \frac{a_3 \cdot 0 - a_4 \cdot 0}{a_3} = 0$$

$$c_1 = \frac{b_1 a_1 - a_3 b_2}{b_1} = \frac{\frac{8-K_c}{3}\cancel{\tau_I}(\tau_I + K_c\tau_I) - 3\cancel{\tau_I}K_c}{\frac{8-K_c}{3}\cancel{\tau_I}}$$

$$c_2 = 0$$

$$d_1 = \frac{c_1 \cdot b_2 - b_1 \cdot \overset{0}{\overset{\shortparallel}{c_2}}}{c_1} = \frac{c_1 \cdot b_2}{c_1} = b_2 = K_c$$

Then, all elements in the left column >0

$\tau_I > 0$ ✓

"equal"

$3\tau_I > 0$

$\frac{(8-K_c)\tau_I}{3} > 0$, as $\tau_I > 0 \Rightarrow 8-K_c > 0 \Rightarrow K_c < 8$ ✓

$\frac{\frac{8-K_c}{3}(\tau_I+K\tau_I)-3K_c}{\frac{8-K_c}{3}} > 0$ ← Solving for τ_I

Can multiply by $\frac{8-K_c}{3}$ as $8-K_c > 0$

$K_c > 0$ ✓

$$\underline{\underline{0 < K_c < 8}}$$

$$\frac{8-K_c}{3}\tau_I(1+K_c) - 3K_c > 0$$

$$\underline{\underline{\tau_I > \frac{9K_c}{(1+K_c)(8-K_c)}}}$$

b) See plot under d)

c) The transfer function is $g = \frac{1}{(s+1)^3}$

Using the half rules for SIMC:

$$\tau_1 = \tau_{10} + \frac{\tau_{20}}{2} = 1 + \frac{1}{2} = 1,5$$

$$\theta = \theta_0 + \frac{\tau_{20}}{2} + \tau_{30} = \frac{1}{2} + 1 = 1,5$$

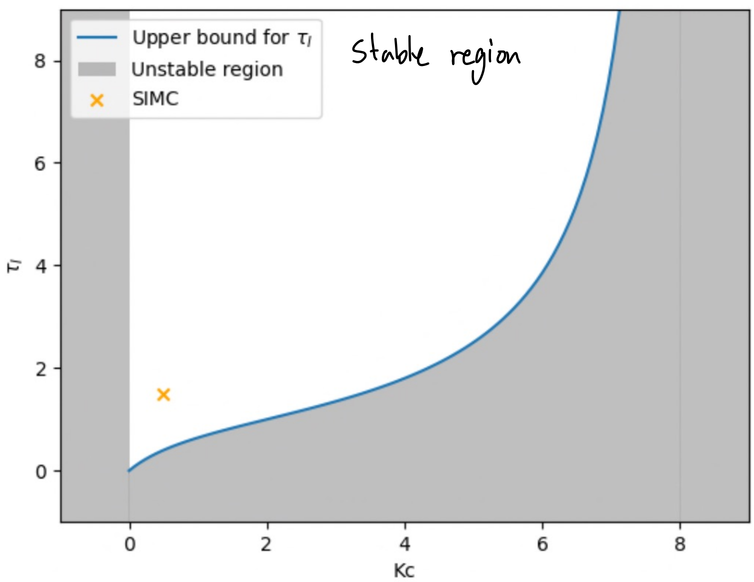
$$g \approx \frac{e^{-1,5s}}{1,5s+1}$$

Using SIMC rules: $K_c = \frac{1}{1} \cdot \frac{\tau_1}{\tau_c + \theta} = \frac{1,5}{2 \cdot 1,5} = 0,5$

$$\tau_I = \min\left(1,5,\ 4(1,5+1,5)\right) = 1,5$$

$$\underline{\underline{K_c = 0,5,\ \tau_I = 1,5}}$$

d) The plot for b) and d):



Found by iteration

e) SIMC is in the stable region. Keeping τ_I constant, $0 < K_c < 3,36$

Keeping K_c costant, $\tau_I > 0,83$

Problem 2: Poles and zeros, the complex plane

1. Consider the transfer function

$$g(s) = \frac{s-3}{s(s^2+4s+5)} \tag{1}$$

   (a) Determine the poles and zeros of $g(s)$

   (b) Show the locations of the poles and zeros on the complex plane.

2. Consider the transfer function

$$g(s) = \frac{1+2s}{1+10s}$$

   Now let $s = j\omega$, where j is defined as $j = \sqrt{-1}$.

   (a) For the values $\omega = 0.01s^{-1}, 0.05s^{-1}, 0.1s^{-1}, 0.2s^{-1}, 0.5s^{-1}, 1s^{-1}, 10s^{-1}$, set up a table which shows $g(j\omega)$ (the complex number) together with the absolute value of the gain $|g(j\omega)|$ and the phase angle $\angle g(s)$.

   (b) With the values you found for $|g(j\omega)|$ and $\angle g(s)$, use the attached template (bodetemplate) to plot by hand:
   - On the log-log scale, plot the value of $|g(j\omega)|$ as a function of ω.
   - On the semi-log scale, plot the $\angle g(j\omega)$ as a function of ω.
   - Include the asymptotes in your plots.

   (c) Draw by hand the Nyquist plot for $g(s)$, using the values (complex number) you obtained in the table.

1.a) There is one zero: $Z_1 = 3$

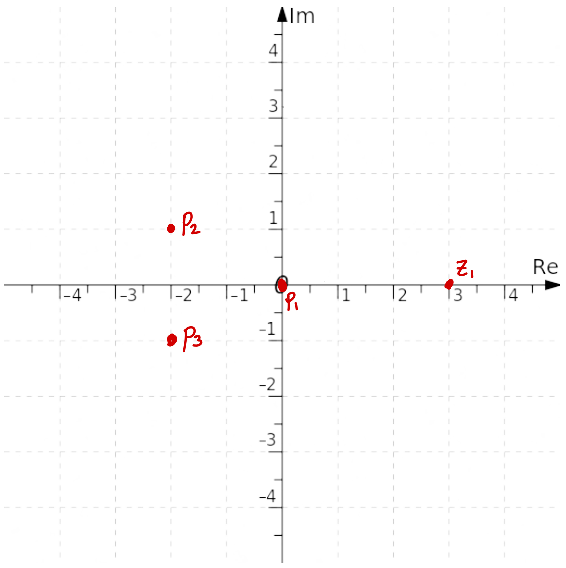
And 3 poles:

$P_1 = 0$

and the two solutions to $s^2+4s+5=0$, using the quadratic equation:

$P_2 = -2 + i$

$P_3 = 2 - i$



2 a) Inserting $s = j\omega$:

$$g = \frac{1+2j\omega}{1+10j\omega}$$

To get rid of complex number in denominator, multiply with conjugate

$$= \frac{1+2j\omega}{1+10j\omega} \cdot \frac{1-10j\omega}{1-10j\omega}$$

$$= \frac{1-8j\omega+20\omega^2}{1+100\omega^2}$$

$$= \underbrace{\frac{1+20\omega^2}{1+100\omega^2}}_{Re} + \underbrace{\frac{-8\omega}{1+100\omega^2}}_{Im} \cdot j$$

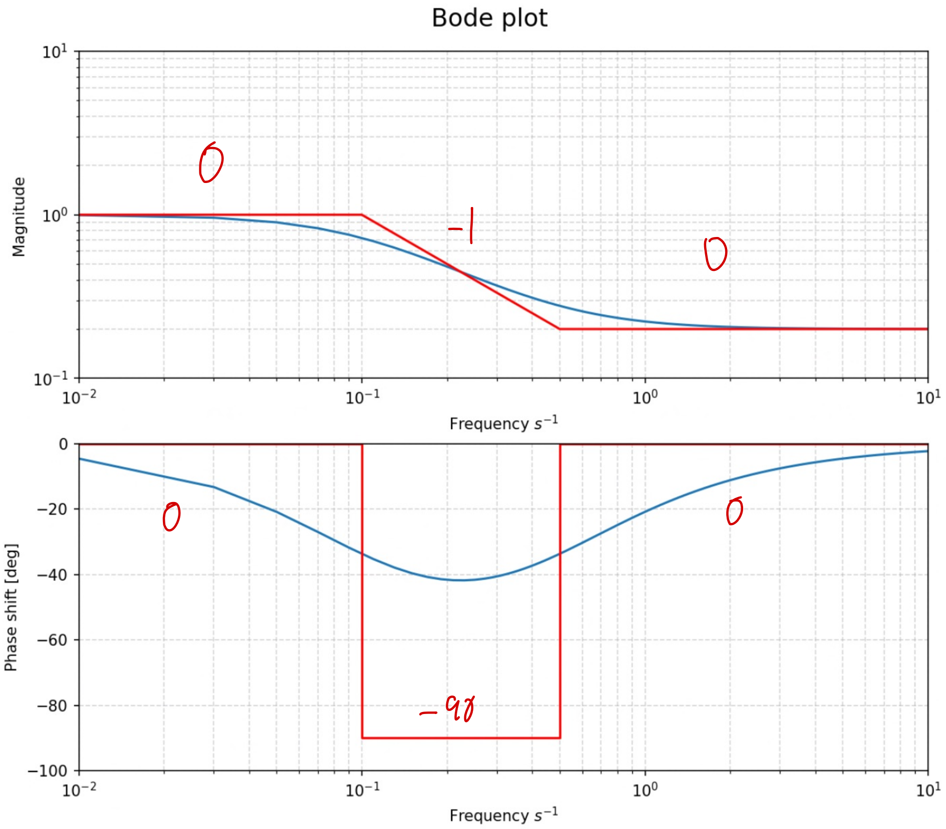
Then, $|g(j\omega)| = \sqrt{Re(j\omega)^2 + Im(j\omega)^2}$

$$\angle g(j\omega) = \arctan\left(\frac{Im(j\omega)}{Re(j\omega)}\right)$$

omega	Re(g)	Im(g)	angle_g	len_g
0.01	0.9921	-0.0792	-4.5648	0.9952
0.05	0.84	-0.32	-20.8545	0.8989
0.1	0.6	-0.4	-33.6901	0.7211
0.2	0.36	-0.32	-41.6335	0.4817
0.5	0.2308	-0.1538	-33.6901	0.2774
1	0.2079	-0.0792	-20.8545	0.2225
10	0.2001	-0.008	-2.2895	0.2002

b)

Asymptotes



Rule for asymptotic Bode-plot, L = k(Ts+1)/(τs+1)….. :

1. Start with low-frequency asymptote (s→0)
   (a) If constant (L(0)=k):
   Gain=k (slope=0)
   Phase=0°
   (b) If integrator (L=k'/s):
   Gain slope= -1 (on log-log plot). Need one fixed point, for example, gain=1 at ω=k'
   Phase: -90°.

2. Break frequencies (order from large T to small T):

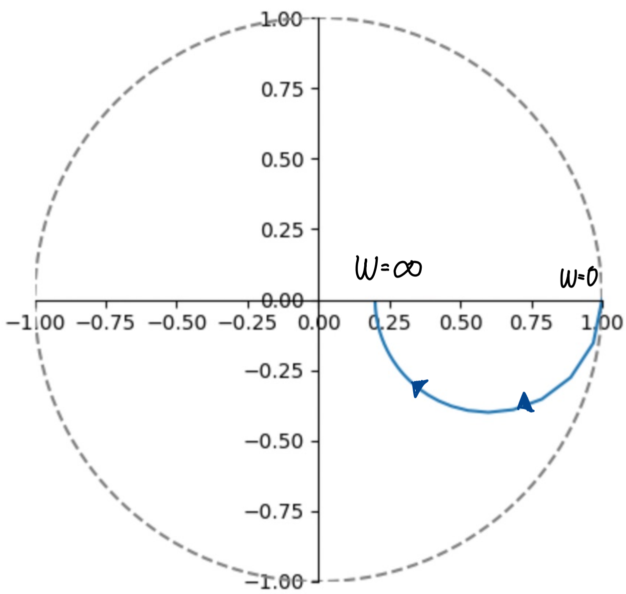
	Change in gain slope	Change in phase
ω=1/T (zero)	+1	+90° (-90° if T negative)
ω=1/τ (pole)	-1	-90° (+90° if τ negative)

3. Time delay, $e^{-\theta s}$. Gain: no effect, Phase contribution: -ωθ [rad] (-1 rad = -57° at ω=1/θ)

$$g(s) = \frac{1+2s}{1+10s}$$

	Gain	Phase
$w = \frac{1}{2} = 0{,}5$	+1	+90
$w = \frac{1}{10} = 0{,}1$	-1	-90

c)



Problem 3: Windup

Windup is an important problem for PID controllers in practice. It introduces a nonlinear effect caused by the actuators limitations, i.e. a valve cannot be more than fully open or closed, or a motor cannot exceed its speed limit. If the controller output reaches the actuator limits, the feedback path is broken and control is lost (i.e. the process is in open loop). In this exercise two anti-windup implementations are presented:

1. series (simple implementation)
2. back-calculation

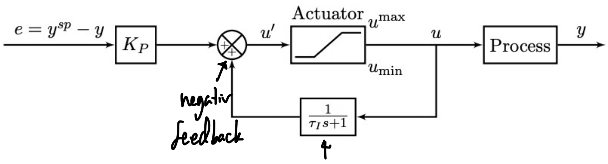


Figure 2: Simple PID-controller with antiwindup. Series implementation

Simple PI controller with anti-windup

Figure 2 shows a simple implementation of a PI controller with anti-windup. This is a special case of back-calculation with no tunable parameter for the anti-windup.

1. What can you say about the feedback path?
2. Find the transfer function $C(s)$ without actuator saturation (i.e. $u' = u$), such that
$$u'(s) = C(s)e(s) \tag{2}$$
3. Find the transfer functions $C(s)$ and $D(s)$ with actuator saturation at $u = u^{\max}$, such that
$$u'(s) = C(s)e(s) + D(s)u(s)^{\max} \tag{3}$$

1. We have a negative feedbackloop with an actuator preventing u from being larger than u_{max} or less than u_{min}

2. The transfer function to u' is:

$$K_p \cdot e(s) + u(s) \cdot \frac{1}{\tau_I s+1} = u'(s)$$

Setting $u(s) = u'(s)$ gives:

$$K_p \cdot e(s) + u' \cdot \frac{1}{\tau_I s+1} = u'$$

$$K_p \cdot e(s) = \left(1 - \frac{1}{\tau_I s+1}\right) u'$$

$$K_p \cdot e(s) = \left(\frac{\tau_I s + \cancel{1} - \cancel{1}}{\tau_I s+1}\right) u'$$

$$K_p \cdot e(s) = \frac{\tau_I s}{\tau_I s+1} u'$$

$$\underline{u' = K_p \frac{\tau_I s+1}{\tau_I s} e(s)} \Rightarrow \underline{\underline{C(s) = K_p \cdot \frac{\tau_I s+1}{\tau_I s}}}$$

3. Now, we have:

$$u'(s) = K_p \cdot e(s) + \frac{1}{\tau_I s + 1} u_{max}$$

$$\Rightarrow \underline{\underline{C(s) = K_p, \; D(s) = \frac{1}{\tau_I s + 1}}}$$

PI controller with back-calculation and input tracking

For processes with a large integral time, the simple implementation can be slow to follow the real actuator value. In this cases, the difference between the actuator and the controller output (u') is fed to the integral action with a gain of $\frac{1}{\tau_T}$. The result is that controller output tracks the actuator with a time constant τ_T. This scheme is called back-calculation, and the feedback path from the actuator is called input tracking.

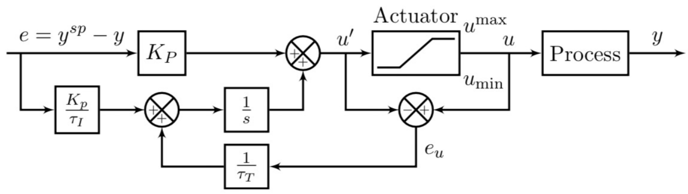


Figure 3: PID Controller with antiwindup. Back-calculation implementation

1. Find the transfer function $C(s)$ without actuator saturation (i.e. $u' = u$), such that
$$u'(s) = C(s)e(s) \tag{4}$$

2. Find the transfer functions $C(s)$ and $D(s)$ with actuator saturation at $u = u^{\max}$, such that
$$u'(s) = C(s)e(s) + D(s)u^{+max}(s) \tag{5}$$

1. The transfer function to u' is:

$$u' = K_p \cdot e + \left((u - u') \cdot \frac{1}{\tau_T} + \frac{K_p}{\tau_I} e\right) \cdot \frac{1}{s}$$

If $u = u'$, then

$$u' = K_p \cdot e + \left(0 \cdot \frac{1}{\tau_T} + \frac{K_p}{\tau_I} e\right) \cdot \frac{1}{s}$$

$$= K_p \cdot e + \frac{K_p}{\tau_I s} e$$

$$\underline{u' = K_p \cdot \frac{\tau_I s + 1}{\tau_I s} \cdot e} \quad \Rightarrow \underline{\underline{C(s) = K_p \cdot \frac{\tau_I s + 1}{\tau_I s}}}$$

2. Inserting u_{max} gives:

$$u' = K_p \cdot e + \left((u^{max} - u') \cdot \frac{1}{\tau_T} + \frac{K_p}{\tau_I} e \right) \cdot \frac{1}{s}$$

$$u' = K_p \cdot \frac{\tau_I s + 1}{\tau_I s} \cdot e + \frac{u^{max}}{\tau_T s} - \frac{u'}{\tau_T s}$$

$$u'\left(1 + \frac{1}{\tau_T s}\right) = K_p \cdot \frac{\tau_I s + 1}{\tau_I s} e + \frac{u^{max}}{\tau_T s}$$

$$u'\left(\frac{\tau_T s + 1}{\tau_T s}\right) = K_p \cdot \frac{\tau_I s + 1}{\tau_I s} e + \frac{u^{max}}{\tau_T s}$$

$$\underline{u' = K_p \cdot \frac{\tau_T(\tau_I s + 1)}{\tau_I(\tau_T s + 1)} \cdot e + \frac{1}{\tau_T s + 1} \cdot u^{max}}$$

$$\Rightarrow \underline{\underline{C(s) = K_p \cdot \frac{\tau_T(\tau_I s + 1)}{\tau_I(\tau_T s + 1)}, \quad D(s) = \frac{1}{\tau_T s + 1}}}$$

3. *Extra.* In what case are the two implementations identical?

From the expressions above, we see that they are equal when $u = u^{max}$