1 Step responses

1. Consider a system $Y(s) = G(s)U(s)$ where $U(s)$ is a unit step at $t = 0$ ($U(s) = \frac{1}{s}$). Prove that

   (a) $\lim_{t\to\infty} y(t) = \lim_{s\to 0} G(s)$ (Steady state gain)

   (b) $\lim_{t\to 0} y(t) = \lim_{s\to\infty} G(s)$ (Initial gain)

   (c) $\lim_{t\to 0} y'(t) = \lim_{s\to\infty} sG(s)$ (Initial slope)

   Hint: Use the relationship:

$$\mathcal{L}\{\frac{df}{dt}\} = \int_0^\infty \frac{df}{dt} e^{-st} dt = s\mathcal{L}\{f(t)\} - \lim_{t\to 0^+} [f(t)] \,. \tag{1}$$

   For a.) take the limit $s \to 0$.
   For b.) take the limit $s \to \infty$.
   For c.) use the statement from b.) ($\lim_{t\to 0} y(t) = \lim_{s\to\infty} G(s)$), and use the fact that differentiation in the time domain is the same as multiplication by s in the frequency domain.

a) I'm assuming that $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\left(\frac{dy}{dt}\right) = \int_0^\infty \frac{dy}{dt} e^{-st} dt = sY(s) - \lim_{t\to 0^+}[y(t)] = sY(s) - y(0)$$

However, taking $\lim_{s\to 0}$ gives another result

$$\lim_{s\to 0} \mathcal{L}\left(\frac{dy}{dt}\right) = \lim_{s\to 0}\int_0^\infty \frac{dy}{dt} e^{-st} dt = \int_0^\infty \frac{dy}{dt} dt = [y]_0^\infty = \lim_{t\to\infty} y(t) - y(0)$$

This should be equal to $\lim_{s\to 0}$ of the previous

$$\lim_{s\to 0}[sY(s) - y(0)] = \lim_{s\to 0}\left[\frac{\cancel{s}}{\cancel{s}} G(s)\right] - y(0) = \lim_{s\to 0} G(s) - y(0)$$

$Y(s) = G(s)U(s) = G(s)\cdot\frac{1}{s}$

Therefore:

$$\lim_{t\to\infty} y(t) - \cancel{y(0)} = \lim_{s\to 0} G(s) - \cancel{y(0)}$$

$$\underline{\underline{\lim_{t\to\infty} y(0) = \lim_{s\to 0} G(s)}} \quad \square$$

b) As we did in a)

$$\lim_{s\to\infty} \mathcal{L}\left(\frac{dy}{dt}\right) = \lim_{s\to\infty} \int_0^\infty \frac{dy}{dt} e^{-st}\, dt = \lim_{s\to\infty} \int_0^\infty \frac{dy}{dt} \left(\frac{1}{e^s}\right)^t dt = \int_0 \frac{dy}{dt} \cdot 0\, dt = 0$$

Which should be equal to:

$$\lim_{s\to\infty} \left[s\, Y(s) - y(0)\right] = \lim_{s\to\infty} \left[\cancel{s} \frac{1}{\cancel{s}} G(s)\right] - \underset{\lim_{t\to 0} y(t)}{\underset{\parallel}{y(0)}} = \lim_{s\to\infty} G(s) - \lim_{t\to 0} y(t)$$

Then

$$\lim_{s\to\infty} G(s) - \lim_{t\to 0} y(t) = 0$$

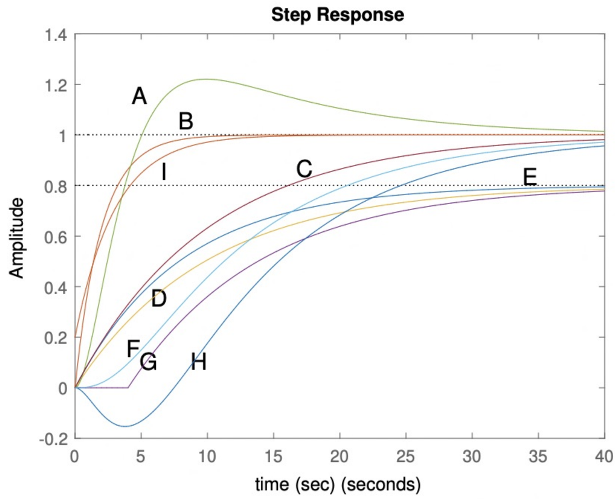
$$\underline{\underline{\lim_{t\to 0} y(t) = \lim_{s\to\infty} G(s) \quad \square}}$$

c) We know that $y'(t) = \frac{dy}{dt} = s \cdot Y(s)$, applying this to the result from b):

$$\underline{\underline{\lim_{t\to 0} y'(t) = \lim_{s\to\infty} sG(s) \quad \square}}$$

2. For transfer functions $g_1 - g_9$ in Table 1, fill in: steady state gain, initial gain, and initial slope.

Transfer function	SS-gain	Initial gain	Initial slope	Conclusion
$g_1 = \frac{0.8}{8s+1}$	0,8	0	0,1	E
$g_2 = \frac{1}{2s+1}$	1,0	0	0,5	B
$g_3 = \frac{0.8}{10s+1}$	0,8	0	0,08	D
$g_4 = \frac{0.8}{10s+1} e^{-4s}$	0,8	0	0	G
$g_5 = \frac{15s+1}{(10s+1)(2s+1)^2}$	1,0	0	0	A
$g_6 = \frac{1}{(10s+1)(2s+1)^2}$	1,0	0	0	F
$g_7 = \frac{1}{10s+1}$	1,0	0	0,1	C
$g_8 = \frac{-5s+1}{(10s+1)(2s+1)^2}$	1,0	0	0	H
$g_9 = \frac{0.6s+1}{3s+1}$	1,0	0,2	∞	I



SS-gain: $g(0)$

Initial gain: $g(\infty)$

Initial slope: $\lim_{s\to\infty} (s \cdot G(s))$

Explanations on next page.

3. Figure 1 shows the step response of transfer functions $g_1 - g_9$. In the "Conclusion" column of Table 1 assign the step responses A-I in the time domain (shown in Figure 1) to the correct transfer functions $g_1 - g_9$.

The only graph corresponding to an initial gain of 0,2 is I => g_9 is I

There are two graphs with zeros, g_5 will overshoot because of the zero => g_5 = A

g_8 has a negative sign in "n(s)" => Inverse response => g_8 = H

g_4 is the only one with a delay => g_4 is G

Of the remainder with SS = 1,0, g_6 has more poles relative to zeros => Flatter response => g_6 is F

Of the two last transfer functions with SS = 1,0, g_2 has the smallest time constant, and will have the fastest response => g_2 is B and g_7 is C

Similarily, as only two transfer functions with SS = 0,8 remain, g_1 is E and g_3 is D

2 Tank System - Part two: Close loop

We consider the two tank system from Exercise 4, problem 2, and we assume that the results are available and can be re-used without deriving them. In the previous exercise, we have showed that the open loop relanshionship between $T_0(s)$, $Q(s)$ and $T_2(s)$ is

$$T_2(s) = \frac{1}{\tau_1 * s + 1} \frac{1}{\tau_2 * s + 1} (T_0 + k_Q Q(s)) \tag{2}$$

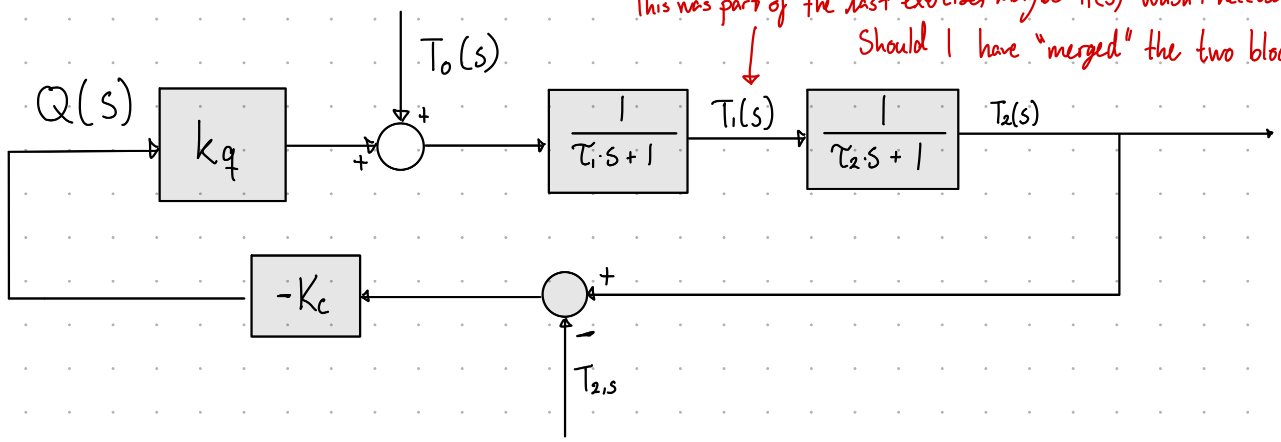
where, $\tau_1 = 5$ min and $\tau_2 = 30$ min, and $k_Q = 0.714$ K/kW. in the following, we assume $y(s) = T_2(s)$ [K], $u(s) = Q(s)$ [kW], and $d(s) = T_0(s)$ [K].

Tasks

1. Using a proportional controller to control $T_2(s)$ to the desired setpoint $T_{2,\text{set}}(s)$, we want to examine the closed loop performance properties of the system. The control law for a P-controller is $Q(s) = -K_C(T_2(s) - T_{2,\text{set}}(s))$

   (a) Draw the block diagram of the closed loop system.

This was part of the last exercise, maybe $T_1(s)$ wasn't necessary? Should I have "merged" the two blocks?



(b) Show that the closed loop transfer function for the setpoint response G_{cl} such that $T_2(s) = G_{cl} T_{2,\text{set}}$ can be written in the standard form for second order processes

$$G_cl(s) = \frac{1}{\tau^2 s^2 + 2\tau\zeta s + 1} K_{cl} \tag{3}$$

where,

$$\tau = \sqrt{\frac{\tau_1 \tau_2}{1 + k_Q K_C}} \tag{4a}$$

$$\zeta = \frac{1}{2} \frac{\tau_1 + \tau_2}{\sqrt{\tau_1 \tau_2}} \frac{1}{\sqrt{1 + k_Q K_C}} \tag{4b}$$

$$K_{cl} = \frac{k_Q K_C}{1 + k_Q K_C} \tag{4c}$$

Combining the given equations (2 and $Q = K_c(T_2 - T_{2,set})$)

$$\Rightarrow T_2 = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \left(T_0 - k_Q K_c (T_2 - T_{2,set})\right)$$ Solve for T_2

$$\left(1 + \frac{k_Q K_c}{(\tau_1 s + 1)(\tau_2 s + 1)}\right) T_2 = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \left(T_0 + k_Q K_c T_{2,set}\right)$$

$$T_2 = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \cdot \frac{1}{1 + \frac{k_Q K_c}{(\tau_1 s + 1)(\tau_2 s + 1)}} \left(T_0 + k_Q K_c T_{2,set}\right)$$

$$T_2 = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1) + k_Q K_c} \cdot \left(T_0 + k_Q K_c T_{2,set}\right)$$

This means that the transfer function for $T_2 = G_{cl} \cdot T_{2,set}$

$$\Rightarrow G_{cl} = \frac{k_Q K_c}{(\tau_1 s + 1)(\tau_2 s + 1) + k_Q K_c} = \frac{k_Q K_c}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2) s + k_Q K_c} \quad / \cdot \frac{\frac{1}{1 + k_Q K_c}}{\frac{1}{1 + k_Q K_c}}$$

$$G_{cl} = \frac{1}{\frac{\tau_1 \tau_2}{1 + k_Q K_c} s^2 + \frac{\tau_1 + \tau_2}{1 + k_Q K_c} s + 1} \cdot \frac{k_Q K_c}{1 + k_Q K_c}$$

Recognizing terms from the problem text $K_{cl} = \frac{k_Q K_c}{1 + k_Q K_c}$

$$\frac{\tau_1 \tau_2}{1 + k_Q K_c} = \left(\sqrt{\frac{\tau_1 \tau_2}{1 + k_Q K_c}}\right)^2 = \tau^2$$

$$\Rightarrow G_{cl} = \frac{1}{\tau^2 s^2 + \frac{\tau_1 + \tau_2}{1 + k_Q K_c} s + 1} K_c$$

Then $\frac{\tau_1 + \tau_2}{1 + k_Q K_c} = 2\tau\zeta$

$$\Rightarrow 2\zeta = \frac{\frac{\tau_1+\tau_2}{1+k_a K_c}}{\sqrt{\frac{\tau_1\tau_2}{1+k_a K_c}}} = \frac{\tau_1+\tau_2}{\sqrt{\tau_1\tau_2}} \cdot \frac{1}{\sqrt{1+k_a K_c}}$$

$$\zeta = \frac{1}{2} \cdot \frac{\tau_1+\tau_2}{\sqrt{\tau_1\tau_2}} \cdot \frac{1}{\sqrt{1+k_a K_c}}$$

Then, finally:

$$\underline{\underline{G_{cl} = \frac{1}{\tau^2 s^2 + 2\tau\zeta s + 1} \cdot K_{cl}}} \quad \square$$

(c) Fill in the missing values in Table 2. By using equations (4 a,b,c):

The steady-state gain is $\lim_{s\to 0} G_{cl}(s)$

The steady state offset (for a unit step) is $1 - \lim_{s\to 0} G_{cl}(s) = 1 - K_{cl}$

K_C	Response time τ	Damping factor ζ	Closed loop gain K_{cl}	Steady state-offset
0	12,25	1,43	0	1
1	9,35	1,09	0,42	0,58
2	7,86	0,92	0,59	0,41
20	3,13	0,37	0,93	0,07
50	2,02	0,24	0,97	0,03
200	1,02	0,12	0,99	0,01

(d) Plot $y = \Delta T_2$ for a step in $\Delta T_{2,set} = 1$ for the values of K_C given in Table 2 (You can either use the enclosed Simulink file (twoTanks.mdl)), or the `step` command in Matlab. Example:

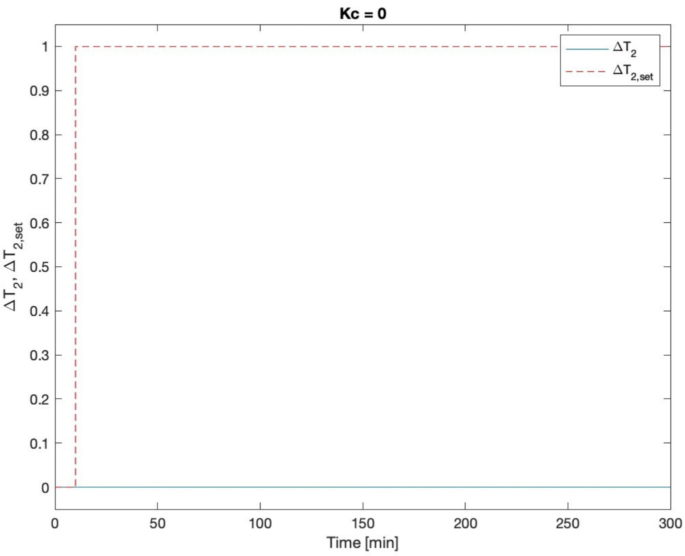
Listing 1: Matlab code for step response

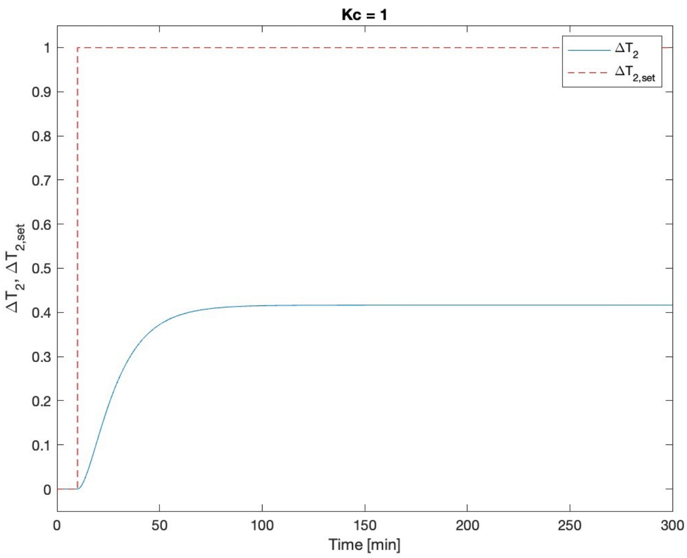
```
Kcl = ...
tau = ...
zeta = ...
s=tf('s')
Gcl = Kcl/(tau^2*s+2*tau*zeta*s+1);
step(Gcl)
```

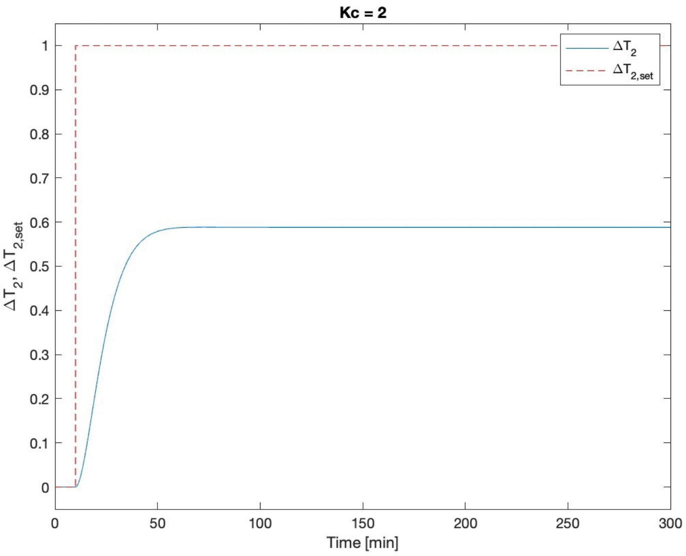
Note that the system never becomes unstable, no matter what values K_C takes. Why is that?

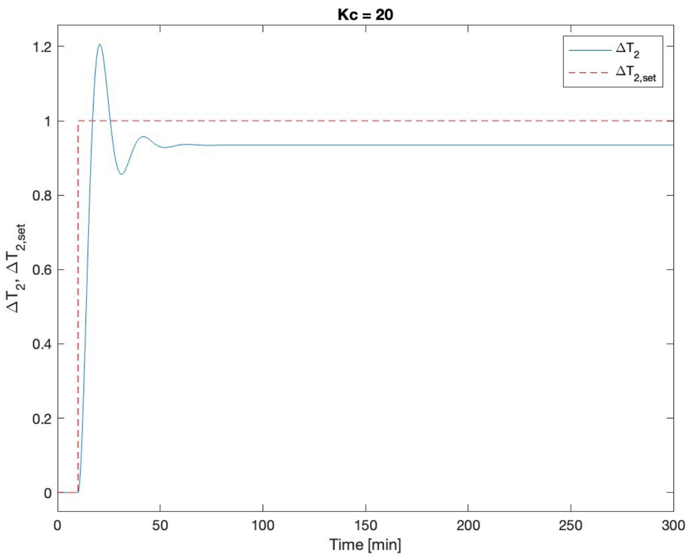
Because that there is no negative signs in the poles in the denominator of the transfer function. (All poles are less than 0)

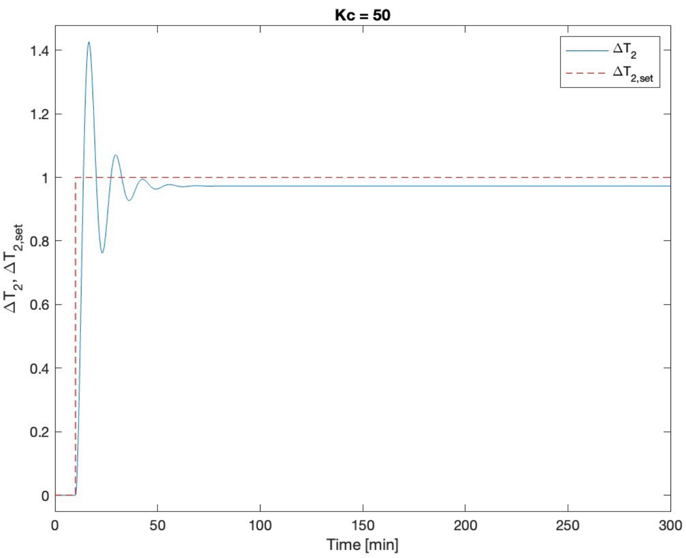
$\tau > 0$ and $\zeta > 0$

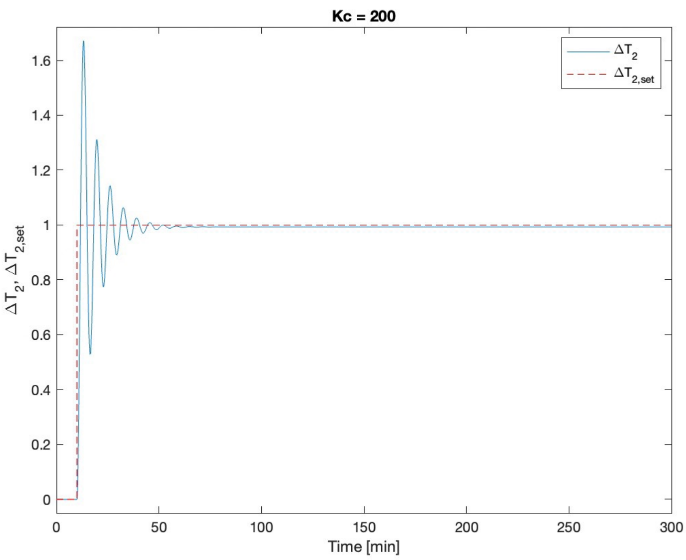

Kc = 0
ΔT_2
$\Delta T_{2,set}$
ΔT_2, $\Delta T_{2,set}$
Time [min]


Kc = 1
ΔT_2
$\Delta T_{2,set}$
ΔT_2, $\Delta T_{2,set}$
Time [min]


Kc = 2
ΔT_2
$\Delta T_{2,set}$
ΔT_2, $\Delta T_{2,set}$
Time [min]


Kc = 20
ΔT_2
$\Delta T_{2,set}$
ΔT_2, $\Delta T_{2,set}$
Time [min]


Kc = 50
ΔT_2
$\Delta T_{2,set}$
ΔT_2, $\Delta T_{2,set}$
Time [min]

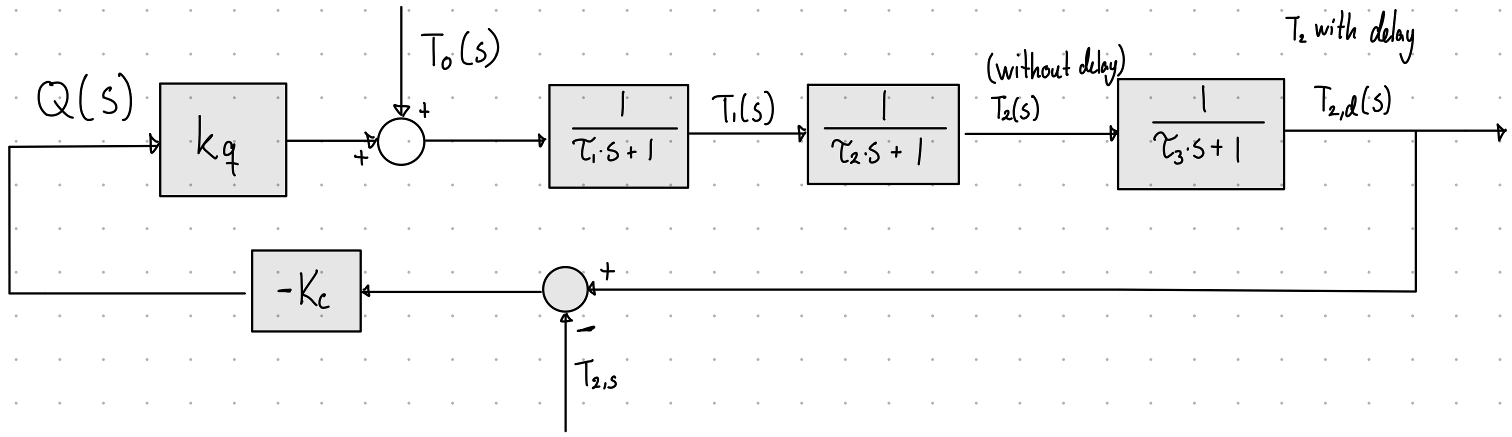

Kc = 200
ΔT_2
$\Delta T_{2,set}$
ΔT_2, $\Delta T_{2,set}$
Time [min]

2. In reality, the system nay become unstable for large K_C, because of e.g. additional measurement dynamics or time delays. Assume that the temperature measurement has additional first order dynamics, with a time constant $\tau_3 = 0.3$ min.

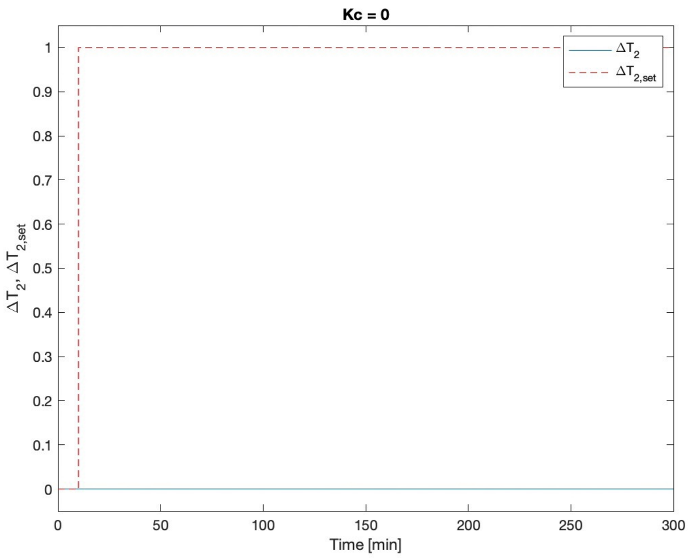
   (a) Draw the corresponding block diagram.

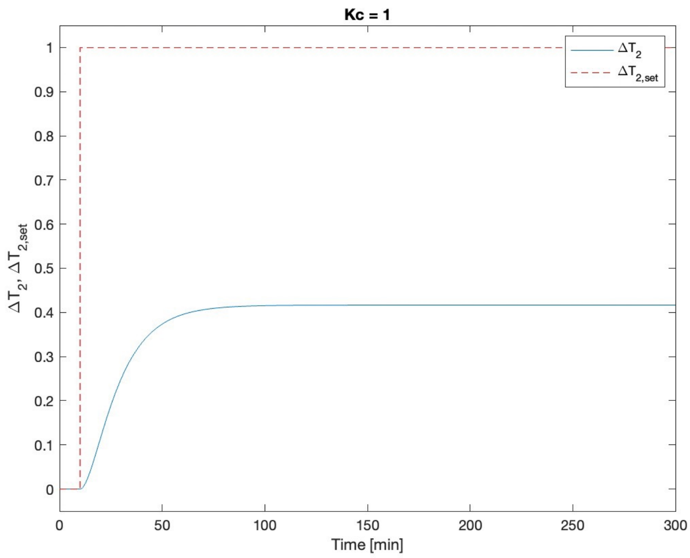
Additional measurement $\Rightarrow$ new $G(s) = \frac{1}{0{,}3s+1} \cdot G_a(s)$

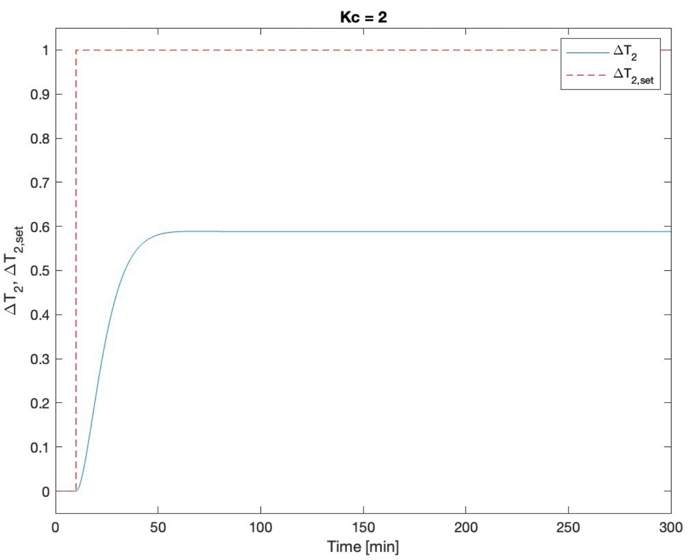
New block diagram:

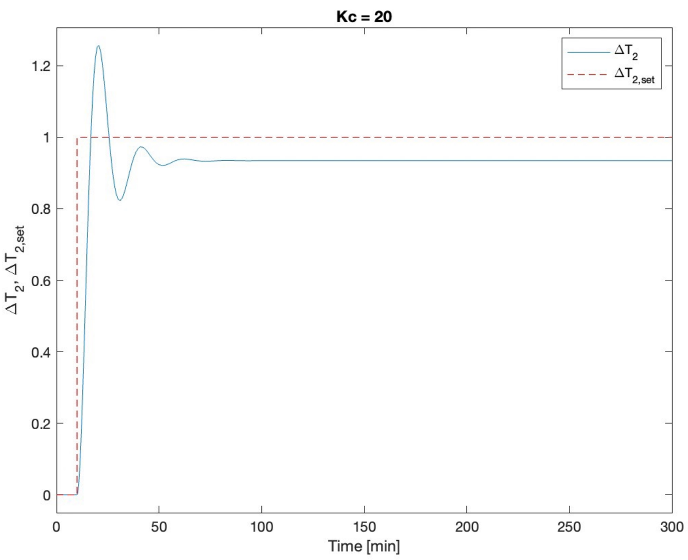


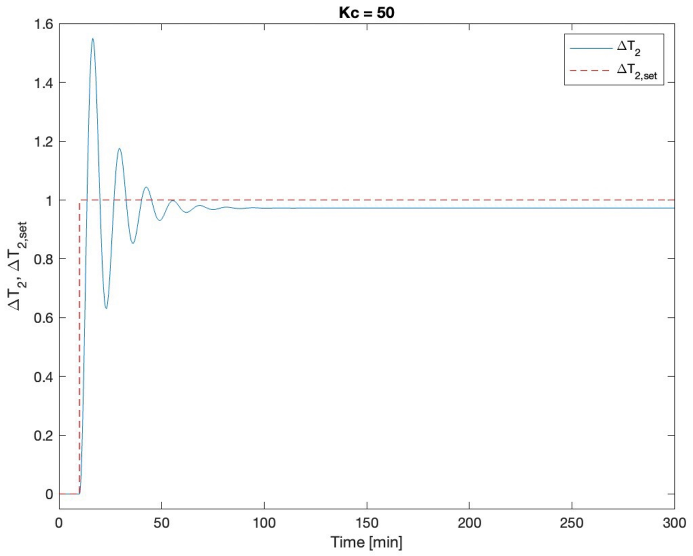
   (b) In Simulink, simulate the time response for controller gain values of $K_C = 0, 1, 2, 20, 50, 200$ kW/K. In the simulink file `twoTanks.mdl` you need to modify the measurement transfer function from 1 to $\frac{1}{0.3s+1}$, and you need to modify the controller gain.

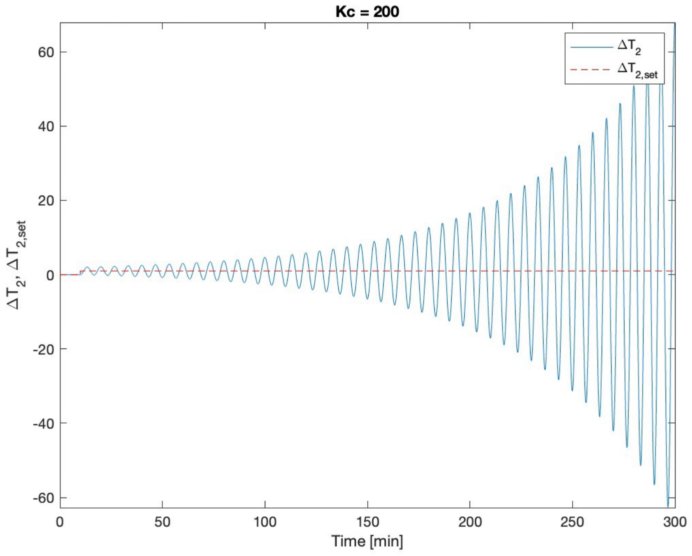






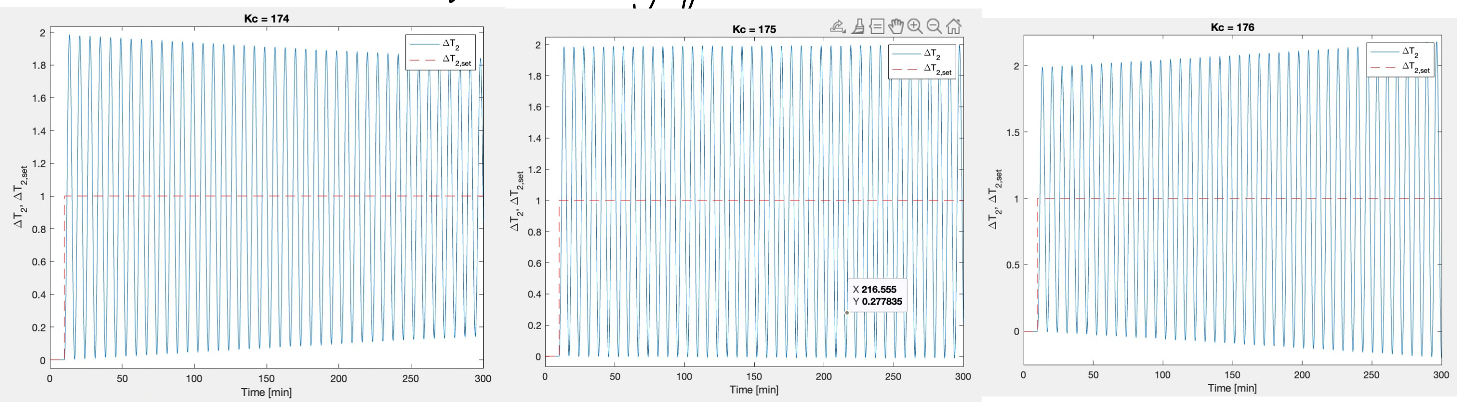






(c) Use the simulations to find out at which value of K_C the system becomes unstable.

Trial and error, we get instability for $K_c = 175$ or 176



3 Block diagrams

Consider the closed loop system, Figure 2, which can be described by the following equations in the time domain:

$$50\frac{dx}{dt} + x(t) = 2u(t) \quad (5)$$

$$y(t) = x(t-5) \quad (6)$$

$$\tau_I \frac{du}{dt} = y_s - y \quad (7)$$

Write the correct transfer functions into the blocks in Figure 2.

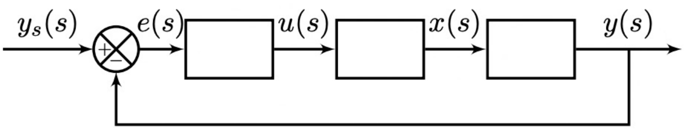


Figure 2: Fill in the correct transfer functions

Laplace transforming using tables in the book

(5) becomes: $50 s X(s) + X(s) = 2\,U(s) \Rightarrow X(s) = \frac{2}{50s+1} U(s)$

(6) becomes (using the shift theorem): $Y(s) = e^{-5s} X(s)$

(7) becomes $\tau_I\, s \cdot U(s) = Y_s(s) - Y(s) \Rightarrow U(s) = \frac{1}{\tau_I s}\left(Y_s(s) - Y(s)\right)$

We get the following block diagram: