Exercise 4

1 Time domain and Laplace domain

The following Laplace transforms are typical for process engineering applications.

1. Write the analytic expression for the time response $y = f(t)$ for the following signals.

$$F_1(s) = \frac{1}{s} \tag{1}$$

$$F_2(s) = \frac{1}{\tau_1 s + 1} \tag{2}$$

$$F_3(s) = \frac{1}{(\tau_1 s + 1)s} \tag{3}$$

$$F_4(s) = \frac{T_1 s + 1}{\tau_1 s + 1} \tag{4}$$

$$F_5(s) = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{5}$$

$$F_6(s) = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)s} \tag{6}$$

Hint: for F_6 you could use partial fraction decomposition.

Using table 3.1 in the textbook

$$f_1(t) = S(t)$$

$$f_2(t) = \frac{1}{\tau_1} e^{-t/\tau_1}$$

$$f_3(t) = 1 - e^{-t/\tau_1}$$

F_4 is not in the table => Partial frac decomposition

$$\frac{T_1 s + 1}{\tau_1 s + 1} = \alpha_0 + \frac{\alpha_1}{\tau_1 s + 1} \quad / \cdot \tau_1 s + 1$$

$$\frac{S}{\tau_1 s + 1} = \alpha_0 + \frac{\alpha_1}{\tau_1 s + 1}$$

$$T_1 s + 1 = \alpha_0 \tau_1 s + \alpha_0 + \alpha_1$$

$$\Rightarrow \begin{cases} T_1 = \alpha_0 \tau_1 \Rightarrow \alpha_0 = \dfrac{T_1}{\tau_1} \\ 1 = \alpha_0 + \alpha_1 \Rightarrow \alpha_1 = 1 - \alpha_0 = 1 - \dfrac{T_1}{\tau_1} = \dfrac{\tau_1 - T_1}{\tau_1} \end{cases}$$

$$\rightarrow F_4(s) = \frac{T_1}{\tau_1} + \frac{\tau_1 - T_1}{\tau_1} \cdot \frac{1}{\tau_1 s + 1}$$

$$\mathcal{L}\left(\frac{T_1}{\tau_1}\right) = \frac{T_1}{\tau_1} \cdot \delta(t)$$

$$\mathcal{L}\left(\frac{\tau_1 - T_1}{\tau_1} \cdot \frac{1}{\tau_1 s + 1}\right) = \frac{\tau_1 - T_1}{\tau_1} \cdot \frac{1}{\tau_1} \cdot e^{-t/\tau_1}$$

$$f_4(t) = \frac{T_1}{\tau_1} \delta(t) + \frac{\tau_1 - T_1}{\tau_1^2} e^{-t/\tau_1} = \begin{cases} \infty, & t = 0 \\ \dfrac{\tau_1 - T_1}{\tau_1^2} e^{-t/\tau_1}, & t \neq 0 \end{cases}$$

$$f_5(t) = \frac{1}{\tau_1 - \tau_2} \cdot \left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

$$f_6(t) = 1 + \frac{1}{\tau_1 - \tau_2} \cdot \left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

2. The Laplace transforms above may be considered as the signals $y(s)$ coming from a transfer function which takes an input $u(s)$, transforms it, and provides an output $y(s)$, as shown in Figure 1.

$$y(s) = G(s)u(s). \tag{7}$$

How do the transfer functions $G(s)$ corresponding to $F_1(s), \dots, F_6(s)$ look like, if:

(a) the input $u(s)$ is a unit step ($u(s) = 1/s$)? (If with an input $u(s) = 1/s$ we obtain an output $F_i(s)$, which is the transfer function $G_i(s)$?, for $i = 1, \dots, 6$)

(b) the input $u(s)$ is a unit impulse at time $t = 0$ ($u(s) = 1$)?

Figure 1: Transfer function block

a) If $y = G \cdot u \Rightarrow G = \frac{y}{u} = \frac{y}{1/s} = y \cdot s$

$$G_1(s) = \frac{1}{s} \cdot s = 1$$

$$G_2(s) = \frac{1}{\tau_1 s + 1} \cdot s = \frac{1}{\tau_1 s + 1}$$

$$G_3(s) = \frac{1}{(\tau_1 s + 1)s} \cdot s = \frac{1}{\tau_1 s + 1}$$

$$G_4(s) = \frac{T_1 s + 1}{\tau_1 s + 1} \cdot s = \frac{(T_1 s + 1)s}{\tau_1 s + 1}$$

$$G_5(s) = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \cdot s = \frac{s}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$G_6(s) = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)s} \cdot s = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

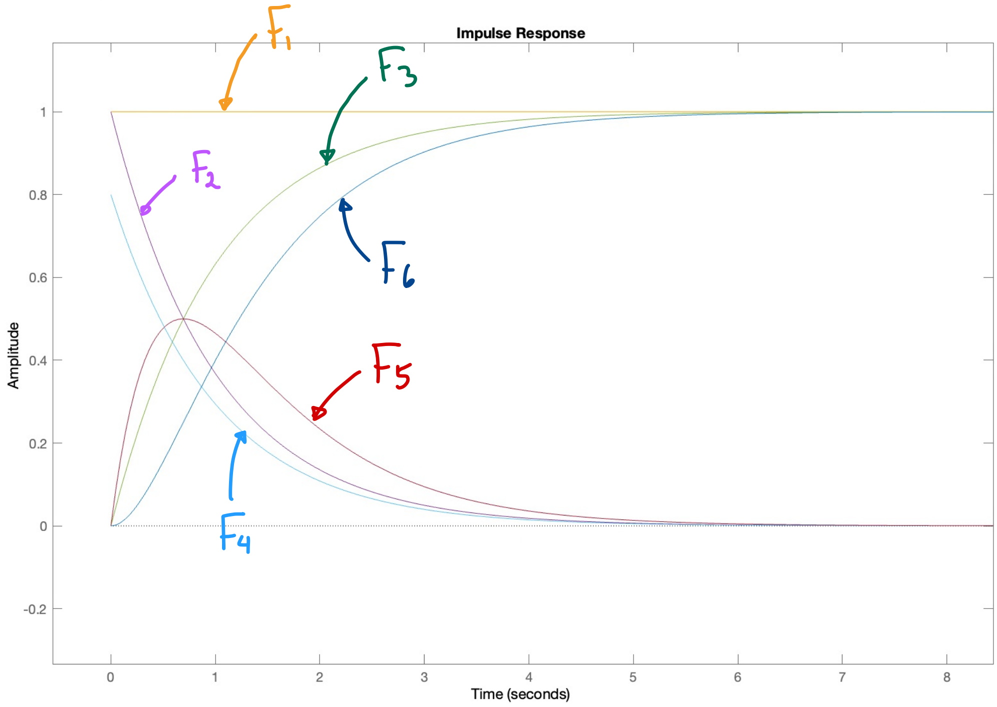
b) $u(s) = \mathcal{L}(\delta(t)) = 1$, as $y = G \cdot u = G \Rightarrow G = y$

We get that $G_i = f_i$

3. Plot the responses $y_1 = f_1(t), y_2 = f_2(t), \ldots, y_6 = f_6(t)$ (using Matlab/Simulink or alternatively by hand) for $\tau_1 = 1, \tau_2 = 0.5, T_1 = 0.2$

   - In Matlab, you can use the command `impulse`. Information about the usage can be found by typing `doc impulse` in the Matlab workspace.
   - Example: $F_1(s) = \frac{1}{s}$

```
>> s = tf('s')
>> F = 1/s
>> impulse(F)
```




```
s = tf('s');
tau1 = 1;
tau2 = 0.5;
T1 = 0.2;


F1 = 1/s;
impulse(F1)
hold on;

F2 = 1/(tau1*s + 1);
impulse(F2)

F3 = 1/(s*(tau1*s + 1));
impulse(F3);

F4 = (T1*s + 1)/(tau1*s + 1);
impulse(F4);

F5 = 1/((tau1*s + 1)*(tau2*s + 1));
impulse(F5);

F6 = 1/((tau1*s + 1)*(tau2*s + 1)*s);
impulse(F6);
```

2 Second order system (Part I – Open loop)

Two tanks are connected in series, as shown in Figure 2. The temperature T_0 is varying, but we would like to keep the exit temperature of the second tank T_2 at a constant value using a proportional controller. In this process, the temperature in the second tank is the measured variable, and the heat applied to the first tank is the manipulated variable (input).

The volume of the first tank is $V_1 = 100\ell$ and for the second tank is $V_2 = 600\ell$ We assume perfect level control and perfect mixing. At the nominal operating point the feed flow is $q = 20\ell/\text{min}$, and the inlet temperature is $T_0 = 60°\text{C}$. The specific heat capacity of the liquid can be assumed to be $C_v = 4200$ J/kgK, and the density of the liquid $\rho = 1000\text{kg/m}^3$.

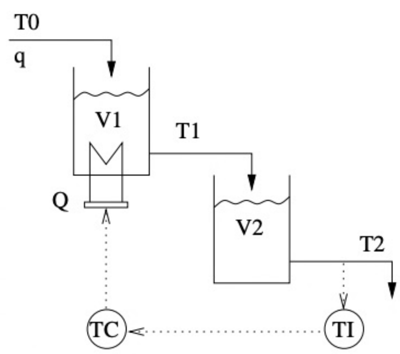


Figure 2: Two connected tanks

We are going to apply the following procedure:

1. Model the process
2. Linearize the model (deviation variables)
3. Laplace transform to obtain a transfer function
4. Algebraic operations in the Laplace domain
5. Draw the block diagram

In this exercise, we assume that the temperature loop is *not* closed (open loop).

1. Formulate the energy balances for the tank system and derive the two differential equations for the temperatures T_1 and T_2.

From (11.13) in Skogestad:

$$m \cdot \frac{dh}{dt} = W_{in}(h_{in} - h) - W_{out}(h_{out} - h) + Q + \cancel{W_s} - (p_{ex} - p)\cancel{\frac{dV}{dt}} + V\cancel{\frac{dp}{dt}}$$

For both tanks, $W_s = 0$, $\frac{dV}{dt} = 0$ and pressure is assumed constant $\Rightarrow \frac{dp}{dt} = 0$

$$m = \rho V, \quad W = \rho \cdot q, \quad h = h_{ref} + \int_{T_{ref}}^{T} C_p(T)\,dT = C_p(T - T_{ref}) = C_p T$$

Set $h_{ref} = 0$, and $C_p = \text{const}$ $\quad T_{ref} = 0$

We end up with

$$\rho V \cdot \frac{d(C_p T)}{dt} = \rho V C_p \frac{dT}{dt} = \rho q_{in} C_p (T_{in} - T) - \rho q_{out} (\cancel{T_{out} - T}) + Q$$

Perfect LC $\Rightarrow q_{in} = q_{out} = q$, Perfect mixing $\Rightarrow T_{out} = T$, We have a liquid $\Rightarrow C_p \approx C_v$

$$\underline{\rho V C_v \frac{dT}{dt} = \rho q C_v (T_{in} - T) + Q}$$

For the first Tank:

$$\rho V_1 C_v \frac{dT_1}{dt} = \rho q C_v (T_0 - T_1) + Q$$

On standard form:

$$\underline{\underline{\frac{dT_1}{dt} = \frac{q}{V_1}(T_0 - T_1) + \frac{Q}{\rho V_1 C_v}}}$$

For tank 2, the process is simular, however, there is no Q

$$\Rightarrow \underline{\underline{\frac{dT_2}{dt} = \frac{q}{V_2}(T_1 - T_2)}}$$

2. Linearize the model and write it in the state-space form:

$$\frac{dx}{dt} = Ax + Bu + Ed \tag{8}$$

where $x = [\Delta T_1, \Delta T_2]^T$, $d = \Delta T_0$ and $u = \Delta Q$. A, B and E are constant matrices.

Introducing

$$\Delta T_0 = T_0 - T_0^*$$
$$\Delta T_1 = T_1 - T_1^*$$
$$\Delta T_2 = T_2 - T_2^*$$
$$\Delta Q = Q - Q^*$$

as q, V_1, V_2, C_v are constant, we only need these variables

Linearing the differential equations using the 1st order taylor expansion

$$\frac{dT_1}{dt} = f_1 \,, \quad \frac{dT_2}{dt} = f_2$$

$$\frac{d\Delta T_1}{dt} \approx \left.\frac{\partial f_1}{\partial T_0}\right|_* \Delta T_0 + \left.\frac{\partial f_1}{\partial T_1}\right|_* \Delta T_1 + \left.\frac{\partial f_1}{\partial T_2}\right|_* \Delta T_2 + \left.\frac{\partial f_1}{\partial T}\right|_* \Delta Q$$

$$\underline{\frac{d\Delta T_1}{dt} = \frac{q}{V_1}\Delta T_0 - \frac{q}{V_1}\Delta T_1 + \cancel{0 \, \Delta T_2} + \frac{\Delta Q}{\rho V_1 C_v}}$$

And

$$\frac{d\Delta T_2}{dt} \approx \left.\frac{\partial f_2}{\partial T_0}\right|_* \Delta T_0 + \left.\frac{\partial f_2}{\partial T_1}\right|_* \Delta T_1 + \left.\frac{\partial f_2}{\partial T_2}\right|_* \Delta T_2 + \left.\frac{\partial f_2}{\partial Q}\right|_* \Delta Q$$

$$= \cancel{0 \cdot \Delta T_0} + \frac{q}{V_2}\Delta T_1 - \frac{q}{V_2}\Delta T_2 + \cancel{0 \cdot \Delta Q}$$

$$\underline{\frac{d\Delta T_2}{dt} = \frac{q}{V_2}\Delta T_1 - \frac{q}{V_2}\Delta T_2}$$

With $x = \begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix}$, $u = [\Delta Q]$, $d = [\Delta T_0]$

$$A = \begin{array}{cc} \begin{array}{cc} \Delta T_1 & \Delta T_2 \end{array} & \\ \begin{bmatrix} -\frac{q}{V_1} & 0 \\ \frac{q}{V_2} & -\frac{q}{V_2} \end{bmatrix} & \begin{array}{c} \frac{d\Delta T_1}{dt} \\ \frac{d\Delta T_2}{dt} \end{array} \end{array}, \quad B = \begin{array}{cc} \Delta Q & \\ \begin{bmatrix} \frac{1}{\rho V_1 C_v} \\ 0 \end{bmatrix} & \begin{array}{c} \frac{d\Delta T_1}{dt} \\ \frac{d\Delta T_2}{dt} \end{array} \end{array}, \quad E = \begin{array}{cc} \Delta T_0 & \\ \begin{bmatrix} \frac{q}{V} \\ 0 \end{bmatrix} & \begin{array}{c} \frac{d\Delta T_1}{dt} \\ \frac{d\Delta T_2}{dt} \end{array} \end{array}$$

Then, finally:

$$\underline{\underline{\frac{dx}{dt} = \begin{bmatrix} -\frac{q}{V_1} & 0 \\ \frac{q}{V_2} & -\frac{q}{V_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{\rho V_1 C_v} \\ 0 \end{bmatrix} u + \begin{bmatrix} \frac{q}{V} \\ 0 \end{bmatrix} d}}$$

3. Take the Laplace transform of the linearized differential equation of the first tank to show that the transfer function from $T_0(s)$ and $Q(s)$ to $T_1(s)$ is

$$T_1(s) = g_1(s)\left(T_0(s) + k_Q Q(s)\right) \tag{9}$$

where

$$g_1(s) = \frac{1}{\tau_1 s + 1}. \tag{10}$$

What is the value of τ_1 and K_Q?

Note that we usually drop the Δ symbol when we are working in the Laplace domain (transfer functions). Nonetheless, all variables are given as deviation from the linearization point. For example, $\mathcal{L}\{\Delta T_0(t)\} = T_0(s)$.

$$\frac{d\Delta T_1}{dt} = \frac{q}{V_1}\Delta T_0 - \frac{q}{V_1}\Delta T_1 + \frac{\Delta Q}{\rho V_1 C_v} \qquad \left[\mathcal{L}\left(\frac{df}{dt}\right) = sf(s)\right]$$

$$sT_1(s) = \frac{q}{V_1}T_0(s) - \frac{q}{V_1}T_1(s) + \frac{1}{\rho V_1 C_v}\cdot Q(s)$$

$$\left(s + \frac{q}{V_1}\right)T_1(s) = \frac{q}{V_1}T_0(s) + \frac{1}{\rho V_1 C_v}Q(s)$$

$$T_1(s) = \frac{1}{s + \frac{q}{V_1}}\left(\frac{q}{V_1}T_0(s) + \frac{1}{\rho V_1 C_v}Q(s)\right)$$

$$= \frac{q/V_1}{s + q/V_1}\left(T_0(s) + \frac{1}{q/\cancel{V_1}}\cdot\frac{1}{\rho \cancel{V_1} C_v}Q(s)\right)$$

$$= \frac{1}{\frac{V_1}{q}s + 1}\left(T_0(s) + \frac{1}{\rho q C_v}Q(s)\right) \implies \underline{\underline{\tau_1 = \frac{V_1}{q} = \frac{100\ L}{20\ L/min} = 5\ min}}, \quad k_q = \frac{1}{\rho q C_v} = \frac{1}{\underbrace{1\ \cancel{kg}/\cancel{dm^3}}_{=1000\ kg/m^3} \cdot 20\ \cancel{dm^3}/min \cdot 4200\ J/\cancel{kg}\cdot K} = \frac{1}{84000}\ \frac{\cancel{min}\cdot K}{J}\cdot\frac{60\ s}{\cancel{min}}$$

$$\underline{\underline{= \frac{1}{\tau_1 s + 1}\left(T_0(s) + k_Q Q(s)\right)}}, \text{ which is what we wanted to prove}$$

$$\underline{\underline{k_q = 0{,}714\cdot 10^{-3}\ \frac{K}{W} = 0{,}714\ \frac{K}{kW}}}$$

4. Algebraic operations:

- Show that
$$T_2(s) = g_2(s)T_1(s) \tag{11}$$
Where
$$g_2(s) = \frac{1}{\tau_2 s + 1} \tag{12}$$
- Find the overall transfer functions $h_1(s)$ and $h_2(s)$, such that (still without control)
$$T_2(s) = h_1(s)T_0(s) + h_2(s)Q(s). \tag{13}$$

From $\frac{d\Delta T_2}{dt} = \frac{q}{V_2}\Delta T_1 - \frac{q}{V_2}\Delta T_2$, Laplace transform yields:

$$s\,T_2(s) = \frac{q}{V_2}\left(T_1(s) - T_2(s)\right)$$

$$\left(s + \frac{q}{V_2}\right) T_2(s) = \frac{q}{V_2} T_1(s)$$

$$T_2(s) = \frac{q/V_2}{s + q/V_2} T_1(s) = \frac{1}{\frac{V_2}{q}s + 1} T_1(s) \quad \Rightarrow \quad \frac{V_2}{q} = \tau_2$$

$$\underline{\underline{T_2(s) = \frac{1}{\tau_2 s + 1} T_1(s)}} \quad \square$$

Inserting $T_1(s)$ into $T_2(s)$:

$$T_2(s) = g_2\left[g_1\left(T_0(s) + k_q \cdot Q(s)\right)\right]$$
$$= g_1 \cdot g_2 \cdot T_0(s) + k_q g_1 g_2 \cdot Q(s)$$

This means that:

$$\underline{\underline{h_1(s) = g_1 \cdot g_2 = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}}}$$

I was told in the exercise session that I should have recognized this earlier: $k_q = \frac{1}{q/V_1} \cdot \frac{1}{\rho V_1 C_v} = \frac{V_1/q}{\rho V_1 C_v} = \frac{\tau_1}{\rho V_1 C_v}$ even though this should not matter much.

$$\Rightarrow \underline{\underline{h_2(s) = k_q g_1 g_2 = \frac{\tau_1}{(\tau_1 s + 1)(\tau_2 s + 1)\rho V_1 C_v}}}$$

5. Using the transfer functions $g_1(s)$, $g_2(s)$ and k_Q, draw the block diagram of the process.