Exercise 3

Problem 1: Linearization of functions

A) Introduction to linearization

In this task you will learn how to linearize a quadratic function. Given the nonlinear function

$$f(x) = x^2 \tag{1}$$

Find a linear approximation of the nonlinear function on the form

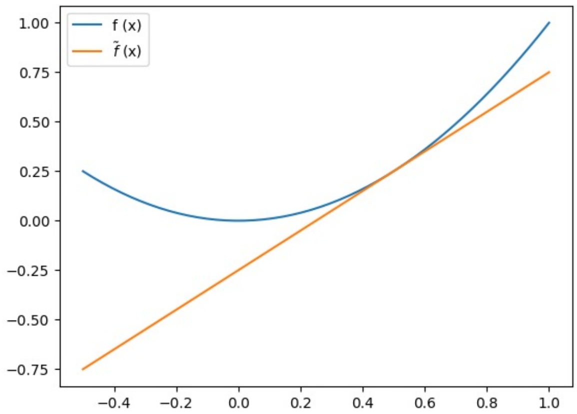
$$f(x) \simeq \tilde{f}(x) = \left.\frac{df}{dx}\right|_{x^*}(x - x^*) + f(x^*) \tag{2}$$

around the nominal point $x^* = 0.5$. Plot the nonlinear and the linearized function and compare. You can do this by hand or using Matlab. Is the linearized function a good approximation of the nonlinear function?

Note that Eq. 2 is a first order Taylor expansion around nominal point x^*. First order means that we only consider the first derivative terms and ignore higher order derivatives.

$$\tilde{f}(x) = 2x^*(x - x^*) + f(x^*)$$

Plotting yields:



We see that close to x^*, it is a good approximation, but the range where it fits is relatively small, so in general, it's a bad approximation

B) Linearization of a valve

Let us consider the valve shown in Figure 1. In this task you will apply what you learned in Task A to the general valve equation given by Eq. 3.

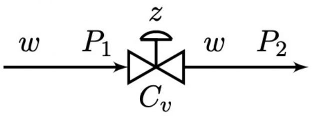


Figure 1: Valve

The flow w [$\mathrm{kg\,s^{-1}}$] through a valve is generally a nonlinear function of the pressures on the two sides of the valve (P_1, P_2) and the valve opening z. The most common form of the valve expression is

$$w\ [\mathrm{kg\,s^{-1}}] = C_v\, f(z)\, \sqrt{\rho\,(P_1 - P_2)} \tag{3}$$

Consider a case with

$$C_v = 0.032\,\mathrm{m}^2$$

and

$$f(z) = z^2 \quad \text{(quadratic valve characteristic)}$$

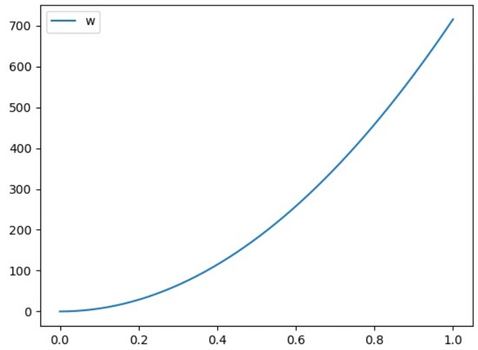
Here C_v [m^2] denotes a constant parameter, P_1 and P_2 denote the pressures in [$\mathrm{N\,m^{-2}}$], ρ is the fluid density at the inlet [$\mathrm{kg\,m^{-3}}$], and the variable z denotes the adjustable valve opening from 0 (closed) to 1 (fully open). Assume the density to be constant with pressure $\rho = 1000\,\mathrm{kg\,m^{-3}}$.

Tasks

For control, we often want to have approximate, linear relationships between the variables.

1. Plot w as a function of z, with $P_1 - P_2 = 5$ bar, and with z in the range $0-1$ (the actual range is 0 to 250 mm).
2. Linearize Eq. 3, that is, find the expression
$$\Delta w \approx K_1 \Delta P_1 + K_2 \Delta P_2 + K_3 \Delta z \tag{4}$$
where $\Delta w = w - w^*$, $\Delta P_1 = P_1 - P_1^*$, $\Delta P_2 = P_2 - P_2^*$ and $\Delta z = z - z^*$. The asterisk (*) indicates the value of the variable at the linearization point. Note that
$$K_3 = \left.\frac{\partial w}{\partial z}\right|_* = \left.\frac{\partial f}{\partial z}\right|_*$$
is the slope of the function plotted in the Task 1.
3. Evaluates the gains (K_1, K_2 and K_3) at two different points: ($P_1^* = 6\,\mathrm{bar}$, $P_2^* = 1\,\mathrm{bar}$, $z^* = 0.02$) and ($P_1^* = 6\,\mathrm{bar}$, $P_2^* = 1\,\mathrm{bar}$, $z^* = 0.96$).
4. Comment on what this gain variation may imply for control.

1.




```python
# B)

dP = 5 * 10 ** 5  # Pa
Cv = 0.032  # m2
rho = 1000  # kg/m3


def f(z):
    return z ** 2

def w(z):
    return Cv*f(z)*np.sqrt(dP*rho)


z_values = np.linspace(0, 1, 100)
w_values = w(z_values)


plt.plot(z_values, w_values, label='w')
plt.legend()
plt.show()
```

2. $$\Delta W = \tilde{f}(P_1, P_2, Z) = \left.\frac{\partial w}{\partial P_1}\right|_* \Delta P_1 + \left.\frac{\partial w}{\partial P_2}\right|_* \Delta P_2 + \left.\frac{\partial w}{\partial z}\right|_* \Delta z$$

$$K_1 = \left.\frac{\partial w}{\partial P_1}\right|_* = C_v f(z^*) \cdot \frac{1}{2} \rho \left(\rho (P_1^* - P_2^*)\right)^{-1/2}$$

$$\underline{\underline{K_1 = \frac{C_v f(z^*) \sqrt{\rho}}{2\sqrt{P_1^* - P_2^*}}}}$$

$$K_2 = \left.\frac{\partial w}{\partial P_2}\right|_* = C_v \cdot f(z^*) \cdot \frac{1}{2} \cdot (-\rho) \left(\rho (P_1^* - P_2^*)\right)^{-1/2}$$

$$\underline{\underline{K_2 = -\frac{C_v f(z^*) \sqrt{\rho}}{2\sqrt{P_1^* - P_2^*}} = -K_1}}$$

$$\underline{\underline{K_3 = \left.\frac{\partial w}{\partial z}\right|_* = C_v \cdot \frac{\partial f(z^*)}{\partial z} \cdot \sqrt{\rho (P_1^* - P_2^*)} = 2 \cdot C_v \cdot z^* \sqrt{\rho (P_1^* - P_2^*)}}}$$

3. $P_1^* = 6$ bar, $P_2^* = 1$ bar, $z^* = 0{,}02$

$\Rightarrow K_1 = 2{,}86 \cdot 10^{-7}$

$K_2 = -2{,}86 \cdot 10^{-7}$

$K_3 = 28{,}6$

$P_1^* = 6$ bar, $P_2^* = 1$ bar, $z^* = 0{,}96$

$\Rightarrow K_1 = 6{,}59 \cdot 10^{-4}$

$K_2 = -6{,}59 \cdot 10^{-4}$

$K_3 = 1373{,}8$

4. The non-linearity may affect control if not accounted for, at small z, increasing z has less effect than at large z (as shown in the plot in 1.), meaning that a poorly designed controller will oscillate easier farther away from the $*$-values.

Problem 2: Linearization and state space form

In an isothermal continuous stirred tank reactor (CSTR) with constant volume V, two reactions take place:

$$A \rightarrow B; \qquad r_1 = k_1\, c_A$$
$$B \rightarrow C; \qquad r_2 = k_2\, c_B$$

The data for the problem are given in Table 1.

Table 1: Reactor data (at nominal point)

Variable	Value	Description
c_{AF}	10 kmol/m^3	Feed concentration A
c_{BF}	0 kmol/m^3	Feed concentration B
c_{CF}	0 kmol/m^3	Feed concentration C
V	0.9 m^3	Reactor volume
q	0.1 m^3/min	Nominal feed flowrate (Input)
k_1	1 min^{-1}	Reaction constant reaction 1
k_2	1 min^{-1}	Reaction constant reaction 2

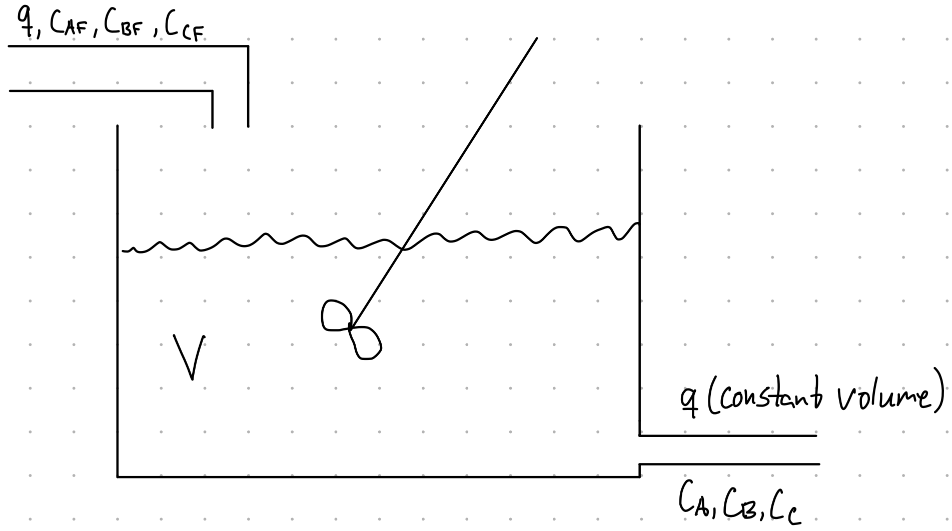
Tasks

1. Draw the process flowsheet. Consider the reactor has one flow in and one flow out. Write the variables from Table 1 in the figure.
2. Set up the dynamic component balances for the three components A, B and C.
3. Determine the steady state operating point corresponding to the nominal point.
4. The reactor compositions (Δc_A, Δc_B, Δc_C), and the input Δq change with time. The other variables can be assumed constant. The Δ symbol is used for indicating that we consider deviation variables, e.g. $\Delta q = q - q^*$. Linearize the system equations around the nominal steady state operating point. i.e. find linear equations for ${d\Delta C_A}/{dt}$, ${d\Delta C_B}/{dt}$, and ${d\Delta C_C}/{dt}$.
5. *Extra, will be covered in class*:
   Write the linear equations on state space form:

   $\frac{dx}{dt} = Ax + Bu$

   where:
   A and B are constant matrices,
   $x = [\Delta c_A, \Delta c_B, \Delta c_C]^T$
   $u = \Delta q$

1.



2. The balances are component in tank = component flowing in − component flowing out − ~~Comp. consumed~~ Created

$$\frac{d(V \cdot C_A)}{dt} = qC_{AF} - qC_A + r_A V = q(C_{AF} - C_A) - r_1 V \quad / V \text{ is constant, } r_1 = k_1 C_A$$

$$\frac{dC_A}{dt} = \frac{q}{V}(C_{AF} - C_A) - k_1 C_A$$

Similarily, for B and C

$$\frac{dC_B}{dt} = \frac{q}{V}(C_{BF} - C_B) + k_1 C_A - k_2 C_B \quad / \; C_B = 0 \quad (r_B = r_1 - r_2)$$

$$\underline{\underline{\frac{dC_B}{dt} = -\frac{q}{V} C_B + k_1 C_A - k_2 C_B}}$$

$$\underline{\underline{\frac{dC_C}{dt} = -\frac{q}{V} C_C + k_2 C_B}} \qquad (C_C = 0, \; r_C = r_2)$$

3. Steady state => no change with regards to time

$$\frac{dC_A}{dt} = \frac{q}{V}(C_{AF} - C_A^*) - k_1 C_A^* = 0$$

$$\Rightarrow \left(\frac{q}{V} + k_1\right) C_A^* = \frac{q}{V} C_{AF}$$

$$\underline{\underline{C_A^* = \frac{q/V \; C_{AF}}{q/V + k_1} = \frac{q \, C_{AF}}{q + k_1 V}}}$$

$$\frac{dC_B}{dt} = -\frac{q}{V} C_B^* + k_1 C_A^* - k_2 C_B^* = 0$$

$$\left(\frac{q}{V} + k_2\right) C_B^* = k_1 C_A^*$$

$$\underline{\underline{C_B^* = \frac{k_1 V}{q + k_2 V} C_A^*}}$$

$$\frac{dC_C}{dt} = -\frac{q}{V} C_C^* + k_2 C_B^* = 0$$

$$\frac{q}{V} C_C^* = k_2 C_B^*$$

$$\underline{\underline{C_C^* = \frac{k_2 V}{q} C_B^*}}$$

4. Using the results from 2.:

$$\frac{dC_A}{dt} = \frac{q}{V}(C_{AF} - C_A) - k_1 C_A = f$$

$$\frac{dC_B}{dt} = -\frac{q}{V} C_B + k_1 C_A - k_2 C_B = g$$

$$\frac{dC_C}{dt} = -\frac{q}{V} C_C + k_2 C_B = h$$

$$\underline{\underline{\frac{d\Delta C_A}{dt} = \left.\frac{\partial f}{\partial q}\right|_* \Delta q + \left.\frac{\partial f}{\partial C_A}\right|_* \Delta C_A = \frac{C_{AF} - C_A^*}{V} \Delta q - \left(\frac{q^*}{V} + k_1\right) \Delta C_A}}$$

$$\underline{\underline{\frac{d\Delta C_B}{dt} = \left.\frac{\partial g}{\partial q}\right|_* \Delta q + \left.\frac{\partial g}{\partial C_A}\right|_* \Delta C_A + \left.\frac{\partial g}{\partial C_B}\right|_* \Delta C_B = -\left(\frac{C_B^*}{V}\right) \Delta q + k_1 \Delta C_A - \left(\frac{q^*}{V} + k_2\right) \Delta C_B}}$$

$$\underline{\underline{\frac{d\Delta C_C}{dt} = \left.\frac{\partial h}{\partial q}\right|_* \Delta q + \left.\frac{\partial h}{\partial C_B}\right|_* \Delta C_B + \left.\frac{\partial h}{\partial C_C}\right|_* \Delta C_C = -\left(\frac{C_C^*}{V}\right) \Delta q + k_2 \Delta C_B - \left(\frac{q^*}{V}\right) \Delta C_C}}$$

5. $X = \begin{bmatrix} \Delta C_A \\ \Delta C_B \\ \Delta C_C \end{bmatrix}$

$U = [\Delta q]$

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccc} \Delta C_A & \Delta C_B & \Delta C_C \end{array}$$

$$A = \begin{bmatrix} -\left(\frac{q^*}{V}+k_1\right) & 0 & 0 \\ k_1 & -\left(\frac{q^*}{V}+k_2\right) & 0 \\ 0 & k_2 & -\frac{q^*}{V} \end{bmatrix} \begin{array}{l} d\Delta C_A/dt \\ d\Delta C_B/dt \\ d\Delta C_C/dt \end{array}$$

$$\Delta q$$

$$B = \begin{bmatrix} \frac{C_{AF}-C_A^*}{V} \\ -\frac{C_B^*}{V} \\ -\frac{C_C^*}{V} \end{bmatrix} \begin{array}{l} dC_A/dt \\ dC_B/dt \\ dC_C/dt \end{array}$$

$$\Rightarrow \frac{dx}{dt} = \begin{bmatrix} -\left(\frac{q^*}{V}+k_1\right) & 0 & 0 \\ k_1 & -\left(\frac{q^*}{V}+k_2\right) & 0 \\ 0 & k_2 & -\frac{q^*}{V} \end{bmatrix} X + \begin{bmatrix} \frac{C_{AF}-C_A^*}{V} \\ -\frac{C_B^*}{V} \\ -\frac{C_C^*}{V} \end{bmatrix} U$$

Problem 3: Laplace transformations

The Laplace transform is a variable transformation from time t [s] as the independent variable to the complex variable s [s^{-1}]. For a function $f(t)$ (which is usually zero for $t < 0$) it is defined as

$$F(s) = \mathcal{L}\{f(t)\} \triangleq \int_0^\infty f(t)\, e^{-st}\, dt$$

1. Show that:

   (a) For a constant a we have: $\mathcal{L}\{a \cdot f(t)\} = a \cdot \mathcal{L}\{f(t)\}$

   (b) For $f(t=0) = 0$, it holds (which is the most important property for us!)

   $$\mathcal{L}\left\{\frac{df}{dt}\right\} = s \cdot \mathcal{L}\{f(t)\}$$

a) $\mathcal{L}(a \cdot f(t)) = \int_0^\infty a \cdot f(t) e^{-st} dt = a \int_0^\infty f(t) e^{-st} dt = a \cdot \mathcal{L}(f(t))$ $\square$

b) $\mathcal{L}\left(\frac{df}{dt}\right) = \int_0^\infty \underbrace{\frac{df}{dt} e^{-st}}_{u' \cdot v} dt = \underbrace{\left[f(t) \cdot e^{-st}\right]_0^\infty}_{u \cdot v} - (-s) \int_0^\infty \underbrace{f(t) e^{-st}}_{u\ v'} dt$

$= 0 - 0 + s \cdot \mathcal{L}(f(t)) = s \cdot \mathcal{L}(f(t))$

$\Rightarrow \mathcal{L}\left(\frac{df}{dt}\right) = s \cdot \mathcal{L}(f(t))$ $\square$

2. Derive the Laplace transformation $F(s)$ for

(a) $f(t) = 1(t)$ (unit step function: $f(t) = 0$ for $t < 0$, and $f(t) = 1$ for $t \geq 0$)

(b) $f(t) = e^{\alpha t}$

(c) i. $f(t) = \delta(t - t_0)$ (unit impulse function at $t = t_0$)

ii. What is the Laplace transform for a unit impulse at $t_0 = 0$?

(d) $f(t) = \left\{ \begin{array}{ll} 0, & t < 0 \\ t, & t \geq 0 \end{array} \right\}$

(e) $f(t) = \left\{ \begin{array}{ll} 0, & t < 0 \\ \sin \omega t, & t \geq 0 \end{array} \right\}$ (Hint: Use Euler's formula)

a) $$F(s) = \mathcal{L}(1(t)) = \int_0^\infty \underbrace{1(t)}_{t \geq 0} e^{-st} dt = \int_0^\infty e^{-st} dt = \left[-\frac{1}{s} e^{-st}\right]_0^\infty = \frac{1}{s}$$

$$\underline{\underline{F(s) = \frac{1}{s}}}$$

b) Assuming $\alpha \neq s$

$$F(s) = \mathcal{L}(e^{\alpha t}) = \int_0^\infty e^{\alpha t} e^{-st} dt = \int_0^\infty e^{(\alpha - s)t} dt = \frac{1}{\alpha - s} \left[e^{(\alpha - s)t}\right]_0^\infty = -\frac{1}{\alpha - s} = \frac{1}{s - \alpha}$$

$$\underline{\underline{F(s) = -\frac{1}{\alpha - s} = \frac{1}{s - \alpha}}}$$

c) i) $\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$, it has the identity that: $\int_{-\infty}^{\infty} f(x) \cdot \delta(x) dx = f(0)$

$$\Rightarrow \underline{\underline{F(s) = \int_0^\infty \delta(t - t_0) \cdot e^{-st} dt = e^{-st_0}}}$$

$t - t_0 = 0 \Rightarrow t = t_0$

ii) $$\underline{\underline{F(s) = e^{-s \cdot 0} = 1}}$$

d)

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty \overset{u}{t} \overset{v'}{e^{-st}} dt = \left[\overset{u \cdot v}{-\frac{t}{s} e^{-st}}\right]_0^\infty - \left(-\frac{1}{s}\right) \int_0^\infty 1 \cdot e^{-st} dt$$

$f(t) = t \;\; \forall \; t \geq 0$

$$F(s) = 0 - 0 + \frac{1}{s} \left[-\frac{1}{s} e^{-st}\right]_0^\infty = -\frac{1}{s^2}(0 - 1) = \frac{1}{s^2}$$

$$\underline{\underline{F(s) = \frac{1}{s^2}}}$$

e) $F(s) = \int_0^\infty \sin(wt)\, e^{-st}\, dt$, Eulers formula: $\sin(wt) = \frac{e^{iwx} - e^{-iwx}}{2i}$

$= \frac{1}{2i}\int_0^\infty e^{iwt} e^{-st}\, dt - \frac{1}{2i}\int_0 e^{-iwt} e^{-st}\, dt$

$= \frac{1}{2i} \cdot \mathcal{L}\left(e^{iwt}\right) - \frac{1}{2i} \cdot \mathcal{L}\left(e^{-iwt}\right)$, $\alpha = iw$ or $\alpha = -iw$

$= \frac{1}{2i}\left(\frac{1}{s-iw} - \frac{1}{s+iw}\right)$

$= \frac{1}{2i}\left(\frac{\cancel{s}+iw-(\cancel{s}-iw)}{(s-iw)(s+iw)}\right)$

$= \frac{1}{\cancel{2i}}\left(\frac{\cancel{2i}w}{s^2-(iw)^2}\right)$

$F(s) = \frac{w}{s^2+w^2}$