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ON THE CONVERGENCE RATE OF OPERATOR SPLITTING FOR HAMILTON–JACOBI EQUATIONS WITH SOURCE TERMS*

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Abstract. We establish a rate of convergence for a semidiscrete operator splitting method applied to Hamilton–Jacobi equations with source terms. The method is based on sequentially solving a Hamilton–Jacobi equation and an ordinary differential equation. The Hamilton–Jacobi equation is solved exactly while the ordinary differential equation is solved exactly or by an explicit Euler method. We prove that the L^∞ error associated with the operator splitting method is bounded by $\mathcal{O}(\Delta t)$, where Δt is the splitting (or time) step. This error bound is an improvement over the existing $\mathcal{O}(\sqrt{\Delta t})$ bound due to Souganidis [*Nonlinear Anal.*, 9 (1985), pp. 217–257]. In the one-dimensional case, we present a fully discrete splitting method based on an unconditionally stable front tracking method for homogeneous Hamilton–Jacobi equations. It is proved that this fully discrete splitting method possesses a linear convergence rate. Moreover, numerical results are presented to illustrate the theoretical convergence results.

Key words. Hamilton–Jacobi equation, source term, viscosity solution, numerical method, operator splitting, error estimate, front tracking

AMS subject classifications. 70H20, 49H25, 74S30, 65M15, 35L60, 65M12

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1. Introduction. The purpose of this paper is to study the error associated with an operator splitting procedure for nonhomogeneous Hamilton–Jacobi equations of the form

$$
\begin{aligned}
&u_t + H(t,x,u,Du) = G(t,x,u) \quad \text{in } \; Q_T = \mathbb{R}^N \times (0,T),\\
&u(x,0) = u_0(x) \quad \text{in } \; \mathbb{R}^N,
\end{aligned}
\tag{1.1}
$$

where $u = u(x,t)$ is the scalar function that is sought, $u_0 = u_0(x)$ is a given initial function, H is a given Hamiltonian, and D denotes the gradient with respect to $x = (x_1, \ldots, x_N)$. Hamilton–Jacobi equations arise in a variety of applications, ranging from image processing, via mathematical finance, to the description of evolving interfaces (front propagation problems). In general, problems such as (1.1) do not possess classical solutions. In fact, solutions of (1.1) generically develop discontinuous derivatives in finite time even with a smooth initial condition. However, under quite general conditions they possess unique viscosity solutions [6].

It is well known that (homogeneous) Hamilton–Jacobi equations are closely related to (homogeneous) conservation laws. In the one-dimensional case, the notion of viscosity solutions of Hamilton–Jacobi equations is equivalent to the notion of entropy solutions (in the sense of Kružkov [32]) of scalar conservation laws; see [5, 24, 26, 30, 36]

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for details. In the multidimensional case ($d > 1$), this one-to-one correspondence no longer exists. Instead the gradient $p = Du$ satisfies (at least formally) a nonstrictly hyperbolic system of conservation laws; see [24, 27, 30, 36] for details. Exploiting this "correspondence" between Hamilton–Jacobi equations and conservation laws, many numerical methods have been developed to accurately capture solutions of Hamilton–Jacobi equations with discontinuous gradients: see [8, 37] for finite difference schemes of upwind type (see also [31]), [1, 29] for finite volume schemes, [39, 40] for ENO schemes, [35, 33] for central schemes, [4, 20] for finite element methods, [24] for relaxation schemes, and [13, 14, 26, 27, 49] for front tracking methods. Using operator splitting, it is also possible to use "homogeneous" Hamilton–Jacobi solvers as building blocks in numerical methods for nonhomogeneous problems. In the present context, operator splitting means "splitting off" or isolating the effect of the source term G. Operator splitting is particularly important when seeking to extend front tracking methods, which are based on solving Riemann problems for homogeneous equations, to problems involving source terms (see the discussion below).

Operator splitting for Hamilton–Jacobi equations, or more generally fully nonlinear second order partial differential equations [6], have been used by Souganidis [43], Barles and Souganidis [3], Sun [45], and Barles [2]. Among these, the paper by Souganidis [43] is the most relevant one for the present work. In that paper, general operator splitting formulas are analyzed and shown to converge to the unique viscosity solution of the governing Hamilton–Jacobi equation as the splitting step tends to zero. The generality in [43] allows for dimensional splitting as well as "splitting of" the source term as we do in the present paper.

In Barles and Souganidis [3], the authors consider fully nonlinear second order elliptic or parabolic partial differential equations and propose an abstract convergence theory for general (monotone, stable, and consistent) approximation schemes. This theory is then applied to splitting methods as well as many other types of numerical methods. In Barles [2], the author studies, among other things, splitting methods for nonlinear degenerate elliptic and parabolic equations arising in option pricing models. In Sun [45], the author studies a dimensional splitting method for a class of second order Hamilton–Jacobi–Bellman equations related to stochastic optimal control problems.

We now summarize the operator splitting procedure analyzed in this paper and state briefly the obtained theoretical result. To ease the presentation, let us for the moment consider the simplified nonhomogeneous Hamilton–Jacobi equation

$$u_t + H(Du) = G(u), \qquad u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N,\ t \in (0,T). \tag{1.2}$$

A presentation of the splitting procedure and the corresponding theoretical result in the general case (1.1) can be found in section 3. Let $v(x,t) = S(t)v_0(x)$ denote the unique viscosity solution of the homogeneous Hamilton–Jacobi equation

$$v_t + H(Dv) = 0, \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.3}$$

where $S(t)$ is the so-called solution operator associated with (1.3) at time t. Next, let $E(t)$ denote the explicit Euler operator, i.e., $v(x,t) = E(t)v_0(x)$ is defined by

$$v(x,t) = v_0(x) + t\,G(v_0(x)).$$

Our operator splitting method then takes the form

$$u(x, i\Delta t) \approx \big[S(\Delta t)E(\Delta t)\big]^i u_0(x), \tag{1.4}$$

where $\Delta t > 0$ is the splitting (or time) step and $i = 0, \dots, n$ with $n\Delta t = T$.

In this paper, we prove that this splitting approximation converges as $\Delta t \to 0$ to the unique viscosity solution of (1.2). More precisely, we prove that the L^∞ error associated with the time splitting (1.4) is of order Δt:

$$\max_{i=1,\dots,n} \left\| u(\cdot, i\Delta t) - \left[S(\Delta t)E(\Delta t)\right]^i u_0 \right\|_{L^\infty} \le K\Delta t \tag{1.5}$$

for some constant $K > 0$ depending on the data of the problem but not Δt.

In passing, we mention that the proof of (1.5) is inspired by an idea used in Langseth, Tveito, and Winther [34]. In that paper, the authors proved a linear L^1 convergence rate for operator splitting applied to one-dimensional scalar conservation laws with source terms. Having said this, we stress that our method of proof uses "pure" viscosity solution techniques and do not rely on the equivalence between the notions of viscosity [7] and entropy [32] solutions, which exists (only) in the one-dimensional homogeneous case.

As an easy by-product of our analysis, we also obtain an error estimate of the form (1.5) for a variant of (1.4) in which the Euler operator $E(t)$ is replaced by the exact solution operator associated with the ordinary differential equation

$$u_t = G(t, x, u), \qquad v(x, 0) = v_0(x), \qquad x \in \mathbb{R}^N,\ t > 0. \tag{1.6}$$

This error estimate is an improvement of an earlier estimate by Souganidis in [43]. In [43], an L^∞ error estimate of order $\sqrt{\Delta t}$ is obtained for a more general operator splitting procedure, which also includes source splitting. This low convergence rate reflects, of course, the lack of regularity of the viscosity solution and is the "usual" convergence rate obtained for (finite difference and viscous) approximate solutions of Hamilton–Jacobi equations; see [31, 36, 8].

In applications, the exact solution operator $S(t)$ must be replaced by a numerical method. In this paper, we consider the one-dimensional case and replace $S(t)$ by the unconditionally stable front tracking method described in Holden, Holden, and Høegh-Krohn [16] and Karlsen and Risebro [26]. We refer to Karlsen and Risebro [27] for extensions of the method to multidimensional Hamilton–Jacobi equations. Front tracking methods for Hamilton–Jacobi equations have also been developed by Glimm et al. [13, 14], Karlsen and Risebro [26, 27], and Vanderwoude [49]. Indeed one of the main points of the present work is the following simple one: Since front tracking methods are fast and accurate numerical methods for homogeneous Hamilton–Jacobi equations, we do not wish to complicate these methods by a direct inclusion of the source term, which would imply that one has to track curved fronts as well as solving nonhomogeneous Riemann problems. The tracking of curved fronts is more time consuming and difficult to program than the tracking of fronts whose paths are straight lines. In addition, solutions of nonhomogeneous Riemann problems are much more difficult to obtain than solutions of homogeneous Riemann problems. Note that for front tracking methods, a direct inclusion of the source term would not improve the accuracy of the scheme. Furthermore, as we can conclude from the (theoretical) results obtained in this paper, splitting off the source term does not deteriorate the accuracy either! In this context we also mention that for Hamilton–Jacobi equations in several space dimensions, front tracking as defined in [16, 26] and dimensional splitting give a family of numerical methods which are unconditionally stable in the sense that the there is no CFL condition associated with the methods; see [27]. However, for these methods error analysis is not available and we have therefore chosen not to discuss fully discrete splitting methods for multidimensional equations in this paper. Having said this, we do not hesitate to say that splitting off the source term also works very well in practice in several space dimensions. We would like to mention that the main results obtained in this paper also hold for weakly coupled systems of Hamilton–Jacobi equations. The details are presented in [23].

Although operator splitting methods have to some extent been studied and used as computation tools for Hamilton–Jacobi (and related) equations, we feel that these methods have not reached the same degree of popularity as they have for hyperbolic conservation laws. In fact, the first order dimensional splitting method was first introduced by Godunov [15] as a method for solving multidimensional conservation laws. Later this method was modified by Strang [44] to achieve formal second order accuracy. Rigorous convergence results (within the Kružkov framework of entropy solutions [32]) for dimensional splitting methods appeared two decades later with the paper by Crandall and Majda [9]; see also Holden and Risebro [18]. More recently, L^1 error estimates of order $\sqrt{\Delta t}$ were obtained independently by Teng [48] and Karlsen [25].

Splitting methods for scalar conservation laws with source terms have been analyzed by Tang and Teng [47], Langseth, Tveito, and Winther [34], and Tang [46]. We refer to Holden and Risebro [19] for conservation laws with a stochastic source term. It is worthwhile mentioning that Tang [46] deals with *stiff* source terms. More precisely, his work shows that for some classes of dissipative stiff source terms, it is possible to derive an error estimate which is independent of the "stiffness parameter." We pose as an interesting open research problem whether it is possible to obtain a similar result for operator splitting of Hamilton–Jacobi equations with stiff source terms.

Operator splitting methods for conservation laws with parabolic (diffusive) terms have been analyzed by Karlsen and Risebro [28] and Evje and Karlsen [12]; see also the lecture notes [11] (and the references therein) for a thorough discussion of viscous splitting methods and their applications. Finally, splitting methods for conservation laws with dispersive terms have been used very recently by Holden, Karlsen, and Risebro [17].

The rest of this paper is organized as follows. In section 2, we collect some useful results from the theory of viscosity solutions for Hamilton–Jacobi equations. In section 3, we provide a precise description of the operator splitting and state the main convergence results. In section 4, we give detailed proofs of the results stated in section 3. In section 5, we present and analyze a fully discrete operator splitting method for one-dimensional equations. Furthermore, we present numerical examples illustrating the theoretical results.

2. Preliminaries. We start by stating the definition of viscosity solutions as well as some results about existence, uniqueness, and regularity properties of such solutions. These results will be needed in the sections that follow. Precise statements and proofs (or references to proofs) of these results can be found in [42]; see also [43, 21, 22].

Let us introduce some notation. If U is a set, and $f : U \to \mathbb{R}$ is a bounded measurable function on U, then $\|f\| := \text{ess sup}_{x \in U} |f(x)|$. Furthermore let $BUC(U)$, $Lip(U)$, and $Lip_b(U)$ denote the spaces of bounded uniformly continuous functions, Lipschitz functions, and bounded Lipschitz functions on U, respectively. Finally, if $f \in Lip(U)$ we denote the Lipschitz constant of f by $\|Df\|$. Throughout this paper, we let C and γ denote generic constants. We shall mostly use γ in exponential form, i.e, $e^{\gamma t}$. The reason for having two generic constants is that we wish to avoid expressions like "Ce^{Ct}."

For $F \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$, we consider throughout this section the following general equation:

$$u_t + F(t,x,u,Du) = 0 \quad \text{in } \; Q_T, \tag{2.1}$$

with initial condition

$$u(x,0) = u_0(x) \quad \text{in } \; \mathbb{R}^N, \tag{2.2}$$

where $u_0 \in BUC(\mathbb{R}^N)$. Note that (1.1) is a special case of (2.1) and (2.2).

DEFINITION 2.1 (viscosity solution). *Let $F \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$.*

(1) *A function $u \in C(Q_T)$ is a viscosity subsolution of* (2.1) *if for every $\phi \in C^1(Q_T)$, if $u-\phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then*

$$\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \le 0.$$

(2) *A function $u \in C(Q_T)$ is a viscosity supersolution of* (2.1) *if for every $\phi \in C^1(Q_T)$, if $u-\phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then*

$$\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) \ge 0.$$

(3) *A function $u \in C(Q_T)$ is a viscosity solution of* (2.1) *if it is both a viscosity sub- and supersolution of* (2.1).

(4) *A function $u \in C(\bar{Q}_T)$ is a viscosity solution of the initial value problem* (2.1) *and* (2.2) *if u is a viscosity solution of* (2.1) *and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$.*

In order to have existence and uniqueness of solution to (2.1)–(2.2), we need further conditions on F. In this paper we assume the following standard conditions on F; see also Souganidis [43, 41].

(F1) For each $R > 0$, $F \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ is uniformly continuous on $[0,T]\times\mathbb{R}^N\times[-R,R]\times B_N(0,R)$, where $B_N(0,R) := \{x \in \mathbb{R}^N : |x| \le R\}$.

(F2) $\sup_{\bar{Q}_T} |F(t,x,0,0)| < \infty$.

(F3) There is a constant $L > 0$ such that for $(t,x) \in \bar{Q}_T$, $p \in \mathbb{R}^N$, $r,s \in \mathbb{R}$, $|F(t,x,r,p) - F(t,x,s,p)| \le L|r-s|$.

(F4) For each $R > 0$ there is a constant $C_R > 0$ such that for $|r| \le R$, $x,y,p \in \mathbb{R}^N, t \in [0,T]$, $|F(t,x,r,p) - F(t,y,r,p)| \le C_R(1+|p|)|x-y|$.

(F5) Condition (F4) is to hold with the roles of x and t interchanged.

(F6) For each $R > 0$ there is a constant $M_R > 0$ such that for $|r|,|p|,|q| \le R$, $(x,t) \in \bar{Q}_T$, $|F(t,x,r,p) - F(t,x,r,q)| \le M_R|p-q|$.

Under conditions (F1)–(F4), there exists a unique viscosity solution $u \in Lip_b(\mathbb{R}^N)$ to the initial value problem (2.1)–(2.2). Furthermore, if $v \in Lip_b(\mathbb{R}^N)$ is another viscosity solution with initial value v_0, then there exists a constant γ such that

$$\|u(\cdot,t) - v(\cdot,t)\| \le e^{-\gamma t}\|u_0 - v_0\|. \tag{2.3}$$

Moreover, the following inequalities hold:

$$\|u(\cdot,t)\| \le e^{\gamma t}\left(\|u_0\| + Ct\right), \tag{2.4}$$

$$\|Du(\cdot,t)\| \le e^{\gamma t}\left(\|Du_0\| + Ct\right), \tag{2.5}$$

$$\|u(\cdot,t) - u_0\| \le t\, c\left(\|u_0\|, \|Du_0\|\right), \tag{2.6}$$

$$\|u(\cdot,t) - u(\cdot,s)\| \le |t-s| \;\; c\Big(\|u\|, \sup_{\tau\in[0,T]} \|Du(\cdot,\tau)\|\Big) \tag{2.7}$$

for some constants γ and C and some positive function $c \in C(\mathbb{R}^2)$.

Finally, we will need the following stability result.

PROPOSITION 2.1. *Let F be as above, and let K be a nonnegative constant. Assume that $u \in Lip_b(\bar{Q}_T)$ is the viscosity solution of* (2.1), *and $v \in Lip_b(\bar{Q}_T)$ is a viscosity solution of*

$$|v_t + F(t, x, v, Dv)| \le K \quad \text{in } Q_T. \tag{2.8}$$

Let L be the Lipschitz constant of F with respect to the third variable. Then for $0 \le s \le t \le T$,

$$e^{-Lt}\|u(\cdot,t) - v(\cdot,t)\| \le e^{-Ls}\|u(\cdot,s) - v(\cdot,s)\| + Ke^{-Ls}(t-s).$$

This is essentially Theorem V.2 (iii) in [7]; see also [22].

3. Statement of the results. We will study the convergence rate of operator splitting applied to the Hamilton–Jacobi equation (1.1), where $u_0 \in Lip_b(\mathbb{R}^N)$ and H, G satisfy conditions (F1)–(F6) with F replaced by H and G, respectively. These conditions correspond to the conditions Souganidis used for his more general splitting in [43]. See also [21, 22] for the precise statements.

First we will state an error bound for the splitting procedure when the ordinary differential equation is approximated by the explicit Euler method. To define the operator splitting, let $E(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ denote the Euler operator defined by

$$E(t,s)v_0(x) = v_0(x) + (t-s)G(s, x, v_0(x)) \tag{3.1}$$

for $0 \le s \le t \le T$ and $v_0 \in Lip_b(\mathbb{R}^N)$. Furthermore, let $S(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ be the solution operator of the Hamilton–Jacobi equation

$$\begin{aligned} &v_t + H(t, x, v, Dv) = 0 \quad \text{in } \mathbb{R}^N \times (s, T), \\ &v(x, s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.2}$$

where $v_0 \in Lip_b(\mathbb{R}^N)$. Note that S is well-defined on the time interval $[s, T]$, since (3.2) is essentially a special case of (1.1). More precisely, there exists a unique viscosity solution $v \in Lip_b(\mathbb{R}^N \times [s, T'])$ for any $T' > 0$.

The operator splitting solution $\{v(x, t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \le T$, is defined by

$$\begin{aligned} &v(x, t_i) = S(t_i, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})(x), \\ &v(x, 0) = v_0(x). \end{aligned} \tag{3.3}$$

Note that this approximate solution is defined only at discrete t-values. The first result in this paper states that the operator splitting solution, when (3.2) is solved exactly, converges linearly in Δt to the viscosity solution of (1.1).

THEOREM 3.1. *Let $u(x,t)$ be the viscosity solution of* (1.1) *on the time interval $[0, T]$ and $v(x, t_i)$ be the operator splitting solution* (3.3). *There exists a constant $C > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $i = 1, \dots, n$,*

$$\|u(\cdot, t_i) - v(\cdot, t_i)\| \le C\left(\|u_0 - v_0\| + \Delta t\right).$$

We will prove this theorem in the next section.

Our second theorem gives a convergence rate for operator splitting when the explicit Euler operator E is replaced by an exact solution operator $\bar{E}$. More precisely, let $\bar{E}(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$ be the exact solution operator of the ordinary differential equation

$$(3.4)\qquad \begin{aligned} &v_t = G(t,x,v) \quad \text{in } \mathbb{R}^N \times (s,T),\\ &v(x,s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned}$$

where $v_0 \in Lip_b(\mathbb{R}^N)$. Note that $\bar{E}$ is well-defined on the time interval $[s,T]$. In fact, the Lipschitz assumptions on G are sufficient for (3.4) to have a unique solution $v \in C^1([s,T']; Lip_b(\mathbb{R}^N))$ for any $T' > 0$.

Let us define the following operator splitting solution $\{\bar{v}(x,t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \le T$, by

$$(3.5)\qquad \begin{aligned} &\bar{v}(x,t_i) = S(t_i,t_{i-1})\bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})(x),\\ &\bar{v}(x,0) = v_0(x). \end{aligned}$$

The following result is a consequence of Theorem 3.1.

COROLLARY 3.1. *Let $u(x,t)$ be the viscosity solution of* (1.1) *on the time interval $[0,T]$ and $\bar{v}(x,t_i)$ be the operator splitting solution* (3.5)*. There exists a constant $C > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $i = 1,\dots,n$,*

$$\|u(\cdot,t_i) - \bar{v}(\cdot,t_i)\| \le C\left(\|u_0 - v_0\| + \Delta t\right).$$

We also prove the corollary in the next section.

Remark 3.2. Corollary 3.1 improves Theorem 4.1(b) in [43] for the splitting defined in (3.5). Note that the generality in [43] allows for a G function also depending on the gradient. The convergence rate $\mathcal{O}(\sqrt{\Delta t})$ is obtained for this more general operator splitting.

4. Proofs of Theorem 3.1 and Corollary 3.1. In this section, we provide proofs of Theorem 3.1 and Corollary 3.1, starting with the proof of Theorem 3.1. For orientation, let us mention that a longer preprint version of this paper (containing more detailed proofs) is available [22] (see also [21]).

An important step in the proof of Theorem 3.1 is to introduce a suitable comparison function.

(a) *Introducing a comparison function.*

Before we can introduce the comparison function, we need an auxiliary result. For $0 \le s \le t \le T$, let $w(\cdot,t) = S(t,s)w_0$ denote the viscosity solution of the Hamilton–Jacobi equation (3.2) with initial condition w_0. For a given function $\psi \in C^1(\mathbb{R}^N \times [s,T])$, we introduce the function

$$q(x,t) := w(x,t) + \psi(x,t).$$

Assuming that w is C^1, it follows that q is a C^1 solution of the following initial value problem:

$$(4.1)\qquad \begin{aligned} &q_t + H(t,x,q-\psi,Dq-D\psi) = \psi_t \quad \text{in } \mathbb{R}^N \times (s,T),\\ &q(x,s) = w_0(x) + \psi(x,s) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

Moreover, this is still true if w and q are only required to be viscosity solutions of (3.2) and (4.1), respectively.

LEMMA 4.1. *Let w be a viscosity solution of* (3.2) *and* $\psi \in C^1(\mathbb{R}^N \times [s,T])$; *then* $q := w + \psi$ *is a viscosity solution of* (4.1).

Proof. Assume $\phi \in C^1(\mathbb{R}^N \times (s,T))$ and that $q - \phi$ has a local maximum at $(x_0, t_0) \in \mathbb{R}^N \times (s,T)$. This means that $w - (\phi - \psi)$ has a local maximum at (x_0, t_0). Since $(\phi - \psi)$ is a C^1 test function and w is by assumption a viscosity solution of (3.2), the definition of a viscosity subsolution yields

$$(\phi_t - \psi_t)(x_0, t_0) + H(t_0, x_0, (q - \psi)(x_0, t_0), (D\phi - D\psi)(x_0, t_0)) \leq 0,$$

where we replaced $w(x_0, t_0)$ by $(q - \psi)(x_0, t_0)$. The inequality holds for any test function ϕ and for any local maximum of $q - \phi$. Therefore q is a viscosity subsolution of (4.1). Similarly one can show that q is a viscosity supersolution of (4.1). ∎

Let j be such that $1 \leq j \leq n$. Recall that to compute the operator splitting solution v at time $t_j = j\Delta t$, we do j steps. In each step we first apply the Euler operator E for a time interval of length Δt. Then we use the resulting function as an initial condition for problem (3.2) which is also solved for a time interval of length Δt. The main step in the proof of Theorem 3.1 is to estimate the error between u and v for one single time interval of length Δt. Hence we are interested in estimating

$$\|u(\cdot, t_i) - S(t_i, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})\|, \qquad i = 1, \dots, n,$$

where $v(x, 0) = v_0(x)$.

Now fix $i = 1, \dots, n$, and define the function $\zeta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ as follows:

$$\zeta(x,t) := S(t, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})(x).$$

Observe that

$$\zeta(x, t_i) = v(x, t_i).$$

To estimate the difference between $u(\cdot, t_i)$ and $v(\cdot, t_i)$, we need to introduce the comparison function $q^\delta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ defined by

$$q^\delta(x,t) = \zeta(x,t) + \psi^\delta(x,t), \tag{4.2}$$

where $\psi^\delta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ is defined by

$$\psi^\delta(x,t) = -(t_i - t) \int_{\mathbb{R}^N} \eta_\delta(z) G(t_{i-1}, x - z, v(x - z, t_{i-1}))\, dz. \tag{4.3}$$

Here $\eta_\delta(x) := \frac{1}{\delta^N}\eta(\frac{x}{\delta})$, where η is the standard mollifier satisfying

$$\eta \in C_0^\infty(\mathbb{R}^N \times [0,T]), \quad \eta(x) = 0 \text{ when } |x| > 1, \quad \int_{\mathbb{R}^N} \eta(x)\, dx = 1.$$

The introduction of the function q^δ is inspired by the comparison function used in [34]. For each $x \in \mathbb{R}^N$, we see that $q^\delta(x, t_i) = v(x, t_i)$ and we will later show that

$$q^\delta(x, t_{i-1}) \to v(x, t_{i-1}) \quad \text{as} \quad \delta \to 0.$$

The difference

$$u(\cdot, t_i) - v(\cdot, t_i) = u(\cdot, t_i) - q^\delta(\cdot, t_i)$$

will be estimated by deriving a bound on the difference

$$u(\cdot,t) - q^\delta(\cdot,t) \qquad \forall t \in [t_{i-1}, t_i].$$

To this end, observe that q^δ is a viscosity solution to

$$q_t^\delta + H(t,x,q^\delta - \psi^\delta, Dq^\delta - D\psi^\delta) = \psi_t^\delta \quad \text{in } \mathbb{R}^N \times (t_{i-1}, t_i), \tag{4.4}$$

$$q^\delta(x,t_{i-1}) = \zeta(x,t_{i-1}) + \psi^\delta(x,t_{i-1}) \quad \text{in } \mathbb{R}^N. \tag{4.5}$$

This is a consequence of Lemma 4.1 since $\psi^\delta \in C^\infty(\mathbb{R}^N \times [t_{i-1}, t_i])$. Now we proceed by deriving a priori estimates for u, v, ψ^δ, and q^δ that are independent of Δt.

(b) *A priori estimates for u, v, ψ^δ, and q^δ.*

We start by analyzing S and E. Let $w \in Lip_b(\mathbb{R}^N)$. Assume that

$$\max\left\{ \sup_{0\le s\le t\le T} \|E(t,s)w\|, \sup_{0\le s\le t\le T} \|S(t,s)w\| \right\} < \infty. \tag{4.6}$$

For $0 \le s \le t \le T$, let $\bar{w}(x,t-s) = S(t,s)w(x)$. This function is a viscosity solution of (3.2) on $[0,T-s]$ when $H(t,x,r,p)$ is replaced by $H(\tau+s,x,r,p)$. The initial condition is $\bar{w}(x,0) = w(x)$. Applying (2.4)–(2.6) to $\bar{w}$ and then using $S(t,\tau+s)w(x) = \bar{w}(x,\tau)$, we get the following estimates:

$$\|S(t,s)w\| \le e^{\gamma(t-s)}(\|w\| + C(t-s)), \tag{4.7}$$

$$\|D\{S(t,s)w\}\| \le e^{\gamma(t-s)}\left(\|Dw\| + C(t-s)\right), \tag{4.8}$$

$$\|S(t,s)w - w\| \le (t-s)c(\|w\|, \|Dw\|), \tag{4.9}$$

for some constants γ and C, and some positive function $c \in C(\mathbb{R}^2)$.

Let us turn to E. The following estimates are consequences of the definition (3.1) of E and the properties of G and w:

$$\|E(t,s)w\| \le (1+\gamma(t-s))\|w\| + C(t-s), \tag{4.10}$$

$$\|D\{E(t,s)w\}\| \le (1+\gamma(t-s))\|Dw\| + C(t-s), \tag{4.11}$$

$$\|E(t,s)w - w\| \le (t-s)(C + \gamma\|w\|). \tag{4.12}$$

Now we see that assumption (4.6) holds. Just replace $t-s$ by T in expressions (4.7) and (4.10).

LEMMA 4.2. *We have that $\max_{1\le i\le n}\|v(\cdot,t_i)\|$ is bounded independently of Δt. Moreover, there are constants γ and C such that for every $1 \le i \le n$,*

$$\|v(\cdot,t_i)\| \le e^{\gamma t_i}(\|v_0\| + t_i C), \tag{4.13}$$

$$\|Dv(\cdot,t_i)\| \le e^{\gamma t_i}\left(\|Dv_0\| + t_i C\right). \tag{4.14}$$

Proof. Assume that v is bounded; then successive use of expressions (4.7) and (4.10) yields (4.13), and similarly (4.14) follows from (4.8) and (4.11). In (4.13), replace t_i by T and we see that the assumption holds. ∎

From the definition (4.3) of ψ^δ, we easily see that the following lemma holds.

LEMMA 4.3. *There is a constant C such that for every $1 \le i \le n$ and $t \in [t_{i-1}, t_i]$,*

$$\|\psi^\delta(\cdot,t)\| \le (t_i - t)\left(C + \gamma\|v(\cdot,t_{i-1})\|\right), \tag{4.15}$$

$$\|D\psi^\delta(\cdot,t)\| \le (t_i - t)\left(C + \gamma\|Dv(\cdot,t_{i-1})\|\right). \tag{4.16}$$

Now we can prove a corresponding result for q^δ.

LEMMA 4.4. *There are constants* γ *and* C *such that for every* $1 \le i \le n$ *and* $t \in [t_{i-1}, t_i]$,

$$\|q^\delta(\cdot, t)\| \le e^{\gamma \Delta t} \left(\|v(\cdot, t_{i-1})\| + C\Delta t\right), \tag{4.17}$$

$$\|Dq^\delta(\cdot, t)\| \le e^{\gamma \Delta t} \left(\|Dv(\cdot, t_{i-1})\| + C\Delta t\right), \tag{4.18}$$

$$\|q^\delta(\cdot, t) - v(\cdot, t_{i-1})\| \le C\Delta t. \tag{4.19}$$

Proof. We give only the proof of (c), since the other statements are easy consequences of expressions (4.7), (4.8), (4.10), (4.11), and Lemma 4.3. Note that by the definition (4.2) of q^δ,

$$\begin{aligned}\|q^\delta(\cdot, t) - v(\cdot, t_{i-1})\| \le \|S(t, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1}) - E(t_i, t_{i-1})v(\cdot, t_{i-1})\| \\ + \|E(t_i, t_{i-1})v(\cdot, t_{i-1}) - v(\cdot, t_{i-1})\| + \|\psi^\delta\|.\end{aligned}$$

By Lemmas 4.3 and 4.2 we can find a constant independent of t, i and Δt such that

$$\|\psi^\delta\| \le C\Delta t.$$

Similarly we use (4.12) and Lemma 4.2 to show that

$$\|E(t_i, t_{i-1})v(\cdot, t_{i-1}) - v(\cdot, t_{i-1})\| \le C\Delta t,$$

where the constant is independent of i and Δt. By estimate (4.9) we get

$$\|S(t, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1}) - E(t_i, t_{i-1})v(\cdot, t_{i-1})\| \le C\Delta t.$$

Here we used that $c(\|E(t_i, t_{i-1})v(\cdot, t_{i-1})\|, \|D\{E(t_i, t_{i-1})v(\cdot, t_{i-1})\}\|) \le C$ by (4.10), (4.11), and Lemma 4.2 for a constant C independent of $i, n, \Delta t$. Now (4.19) follows. ∎

As a consequence of these results, we get the following bounds:

$$\begin{aligned}\max_{1\le i\le n} \|v(\cdot, t_i)\|, \sup_{[t_{i-1}, t_i]} \|\psi^\delta(\cdot, t)\|, \sup_{[t_{i-1}, t_i]} \|q^\delta(\cdot, t)\|, \sup_{[0,T]} \|u(\cdot, t)\| \le C, \\ \max_{1\le i\le n} \|Dv(\cdot, t_i)\|, \sup_{[t_{i-1}, t_i]} \|D\psi^\delta(\cdot, t)\|, \sup_{[t_{i-1}, t_i]} \|Dq^\delta(\cdot, t)\|, \sup_{[0,T]} \|Du(\cdot, t)\| \le C\end{aligned} \tag{4.20}$$

for some constant C independent of i, n, and Δt. This enables us to pick global Lipschitz constants for H and G that are independent of Δt; see also [21, 22]. We are now in a position to prove Theorem 3.1.

(c) *The proof of Theorem* 3.1.

We prove Theorem 3.1 by applying Proposition 2.1 to u and q^δ. Let us start by deriving an inequality of the form (2.8) from (4.4) satisfied by the comparison function q^δ.

Let ϕ be a C^1 function and assume that $q^\delta - \phi$ has a local maximum point in (t, x). Then by the definition of viscosity subsolution and (4.4) we get

$$\phi_t(x, t) + H(t, x, q^\delta(x, t) - \psi^\delta(x, t), D\phi(x, t) - D\psi^\delta(x, t)) \le \psi_t^\delta(x, t). \tag{4.21}$$

Now we estimate $\psi_t^\delta(x,t)$ and $H(t,x,q^\delta(x,t)-\psi^\delta(x,t), D\phi(x,t)-D\psi^\delta(x,t))$ as follows:

$$
\begin{aligned}
&|\psi_t^\delta(x,t) - G(t_{i-1}, x, q^\delta(x,t))| \\
&\qquad = \left| \int_{\mathbb{R}^N} \eta_\delta(z) G(t_{i-1}, x-z, v(x-z, t_{i-1}))dz - G(t_{i-1}, x, q^\delta(x,t)) \right| \\
&\qquad \le \int_{\mathbb{R}^N} \eta_\delta(z) |G(t_{i-1}, x-z, v(x-z, t_{i-1})) - G(t_{i-1}, x-z, q^\delta(x-z,t))| dz \\
&\qquad\quad + \int_{\mathbb{R}^N} \eta_\delta(z) |G(t_{i-1}, x-z, q^\delta(x-z,t)) - G(t_{i-1}, x, q^\delta(x-z,t))| dz \\
&\qquad\quad + \int_{\mathbb{R}^N} \eta_\delta(z) |G(t_{i-1}, x, q^\delta(x-z,t)) - G(t_{i-1}, x, q^\delta(x,t))| dz \\
&\qquad \le C(\Delta t + \delta).
\end{aligned}
$$

Here we used the Lipschitz continuity of G and Lemmas 4.2 and 4.4. Using this estimate and the Lipschitz continuity of G in t, we see that

$$
\begin{aligned}
\psi_t^\delta(x,t) &\le G(t,x,q^\delta(x,t)) + |G(t_{i-1},x,q^\delta(x,t)) - G(t,x,q^\delta(x,t))| \\
&\qquad + |{\psi^\delta}_t(x,t) - G(t_{i-1}, x, q^\delta(x,t))| \\
&\le G(t,x,q^\delta(x,t)) + C(\Delta t + \delta).
\end{aligned} \tag{4.22}
$$

We get the following estimate for H:

$$
\begin{aligned}
&H(t,x,q^\delta(x,t) - \psi^\delta(x,t), D\phi(x,t) - D\psi^\delta(x,t)) \\
&\qquad \ge H(t,x,q^\delta(x,t), D\phi(x,t)) - C|\psi^\delta(x,t)| - C|D\psi^\delta(x,t)| \\
&\qquad \ge H(t,x,q^\delta(x,t), D\phi(x,t)) - C\Delta t,
\end{aligned} \tag{4.23}
$$

where we have used the Lipschitz continuity of H and Lemmas 4.2 and 4.3. Substituting (4.22) and (4.23) into (4.21), we get

$$
\phi_t(x,t) + H(t,x,q^\delta(x,t), D\phi(x,t)) - G(t,x,q^\delta(x,t)) \le C(\Delta t + \delta).
$$

In a similar way we can show that if $\bar{\phi}$ is C^1 and $q^\delta - \bar{\phi}$ has a local minimum in (x,t), then

$$
\bar{\phi}_t(x,t) + H(t,x,q^\delta(x,t), D\bar{\phi}(x,t)) - G(t,x,q^\delta(x,t)) \ge -C(\Delta t + \delta).
$$

This means that q^δ satisfies

$$
|q_t^\delta(x,t) + H(t,x,q^\delta(x,t), Dq^\delta(x,t)) - G(t,x,q^\delta(x,t))| \le C(\Delta t + \delta)
$$

in the viscosity sense.

Now we can apply Proposition 2.1 to u and q^δ. Let $\tau \in [t_{i-1}, t_i]$ and let L be the Lipschitz constant of G with respect to v. Then

$$
e^{L(t_{i-1}-\tau)} \|u(\cdot,\tau) - q^\delta(\cdot,\tau)\| \le \|u(\cdot,t_{i-1}) - q^\delta(\cdot,t_{i-1})\| + C(\Delta t + \delta)\Delta t. \tag{4.24}
$$

Next, observe that

$$
\begin{aligned}
|v(x,t_j) - q^\delta(x,t_j)| &= |v(x,t_j) - E(t_{j+1},t_j)v(\cdot,t_j)(x) - \psi^\delta(x,t_j)| \\
&= |\Delta t\, G(t_j,x,v(x,t_j)) + \psi^\delta(x,t_j)| \\
&\le \Delta t \int_{\mathbb{R}^N} \eta_\delta(z) \Big| G(t_j,x,v(x,t_j)) \\
&\qquad\qquad - G(t_j,x-z,v(x-z,t_j)) \Big|\, dz \\
&\le \Delta t\, \delta C \|Dv(\cdot,t_j)\| + \Delta t\, \delta C,
\end{aligned}
\tag{4.25}
$$

where the last estimate follows from the triangle inequality and the Lipschitz continuity of G and $v(\cdot,t_j)$. Since $Dv(\cdot,t_j)$ is bounded uniformly in j and Δt, we have that

$$\big\|v(\cdot,t_j) - q^\delta(\cdot,t_j)\big\| \le C\Delta t \delta$$

for some constant C. By (4.24), (4.25), and Lemma 4.2, we get

$$
\begin{aligned}
\|u(\cdot,t_i) - v(\cdot,t_i)\| &= \big\|u(\cdot,t_i) - q^\delta(\cdot,t_i)\big\| \\
&\le e^{L\Delta t} \|u(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| + C\Delta t(\Delta t + \delta).
\end{aligned}
\tag{4.26}
$$

Since $i = 1,\dots,n$ was arbitrary, successive use of (4.26) gives

$$
\begin{aligned}
\|u(\cdot,t_j) - v(\cdot,t_j)\| &\le e^{Lt_j} \|u_0 - v_0\| + C(\Delta t + \delta)t_j \\
&\le C(\|u_0 - v_0\| + \Delta t + \delta), \quad \text{for } j = 1,\dots,n,
\end{aligned}
\tag{4.27}
$$

where C does not depend on Δt. Now we are done since sending $\delta \to 0$ in inequality (4.27) produces the desired result.

(d) *The proof of Corollary* 3.1.

We end this section by giving the proof of Corollary 3.1. To this end, we need Theorem 3.1 and the following estimate:

$$\|v(x,t_i) - \bar{v}(x,t_i)\| \le C\Delta t, \qquad i = 1,\dots,n, \tag{4.28}$$

where C is a constant depending on G, H, T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, and $\|Dv_0\|$ but not Δt. Equipped with (4.28), we get, for every $i = 1,\dots,n$,

$$
\begin{aligned}
\|u(\cdot,t_i) - \bar{v}(\cdot,t_i)\| &\le \|u(\cdot,t_i) - v(\cdot,t_i)\| + \|v(\cdot,t_i) - \bar{v}(\cdot,t_i)\| \\
&\le C(\|u_0 - v_0\| + \Delta t),
\end{aligned}
$$

and we can conclude that Corollary 3.1 holds.

It remains to show (4.28). First note that $\big\|\bar{E}(t_j,t)\bar{v}(\cdot,t_j)\big\|$ and $\|\bar{v}(\cdot,t_j)\|$ can be bounded independently of Δt by arguments similar to those used in the proof of Lemma 4.2. This means that we can find a Lipschitz constant for $G(t,x,\cdot)$ that is independent of Δt (and x,t). Using the same arguments as when estimating the local truncation error for the Euler method we find that

$$\big|E(t_i,t_{i-1})v(x,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(x,t_{i-1})\big| \le e^{\gamma\Delta t} |v(x,t_{i-1}) - \bar{v}(x,t_{i-1})| + C\Delta t^2$$

for constant γ and C that are independent of Δt. Now using this and (2.3), we find that

$$
\begin{aligned}
\|v(\cdot,t_i) - \bar{v}(\cdot,t_i)\| &= \big\|S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - S(t_i,t_{i-1})\bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})\big\| \\
&\le e^{\gamma\Delta t} \big\|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - \bar{E}(t_i,t_{i-1})\bar{v}(\cdot,t_{i-1})\big\| \\
&\le e^{\gamma\Delta t}\Big(\|v(\cdot,t_{i-1}) - \bar{v}(\cdot,t_{i-1})\| + C\Delta t^2\Big).
\end{aligned}
\tag{4.29}
$$

Since $\bar{v}(x,0) = v_0(x)$, repeated use of inequality (4.29) gives (4.28). This completes the proof.

5. A fully discrete splitting method for one-dimensional equations. In this section we describe a fully discrete operator splitting method that actually possesses a linear convergence rate. There are not many numerical methods that are likely to produce linear convergence, since numerical methods for Hamilton–Jacobi equations are usually based on numerical methods for conservation laws. Most methods for conservation laws (even "higher order" methods) have an L^1 convergence rate of 1/2 (or less). Roughly speaking, this translates to a L^∞ convergence rate for the Hamilton–Jacobi equations of 1/2. Therefore the linear error contribution $\mathcal{O}(\Delta t)$ (see Theorem 3.1) coming from the temporal splitting is swamped up by the method-dependent error, unless one uses a method that possesses a convergence rate of at least 1 for the Hamilton–Jacobi equation (3.2). The only methods likely to achieve this are translations of front tracking methods for conservation laws. Since these methods are first order (or higher [38]) only in the one-dimensional case, this section is entirely devoted to one-dimensional equations.

The front tracking method we shall use here was first proposed by Dafermos [10] and later shown to be a viable method for conservation laws by Holden, Holden, and Høegh-Krohn [16]. An extension of this method to Hamilton–Jacobi equations was studied in [26].

Without modification it applies to the initial value problem for the scalar conservation law

$$p_t + H(p)_x = 0,$$

which is equivalent (see the discussion in section 1) to the Hamilton–Jacobi equation

$$u_t + H\left(u_x\right) = 0, \qquad u(x,0) = u_0(x). \tag{5.1}$$

The Riemann problem for this is the case where

$$u_0(x) = u_0(0) + \begin{cases} p_l x & \text{for } x < 0, \\ p_r x & \text{for } x \geq 0, \end{cases} \tag{5.2}$$

where p_l and p_r are constants. We now briefly describe the solution of (5.2). Let $H_{\smile}\left(p; p_l, p_r\right)$ denote the lower convex envelope of H between p_l and p_r, i.e.,

(5.3)
$$H_{\smile}\left(p; p_l, p_r\right) = \sup\Big\{G(p) \;\Big|\; G'' \geq 0 \text{ and } G(p) \leq H(p) \text{ for } p \text{ between } p_l \text{ and } p_r\Big\}.$$

Similarly, let $H_{\frown}\left(p; p_l, p_r\right)$ denote the upper concave envelope of H between p_l and p_r. Also let

$$\tilde{H}\left(p; p_l, p_r\right) = \begin{cases} H_{\smile}\left(p; p_l, p_r\right) & \text{if } p_l \leq p_r, \\ H_{\frown}\left(p; p_l, p_r\right) & \text{if } p_l > p_r. \end{cases}$$

Note that $\tilde{H}'(p)$ is monotone between p_l and p_r; hence we can define its inverse and set

(5.4)
$$p(x,t) = \begin{cases} p_l & \text{for } x < t\min\left\{\tilde{H}'(p_l), \tilde{H}'(p_r)\right\}, \\ \left(\tilde{H}'\right)^{-1}\left(\frac{x}{t}\right) & \text{for } t\min\left\{\tilde{H}'(p_l), \tilde{H}'(p_r)\right\} \le x < t\max\left\{\tilde{H}'(p_l), \tilde{H}'(p_r)\right\}, \\ p_r & \text{for } x \ge t\max\left\{\tilde{H}'(p_l), \tilde{H}'(p_r)\right\}. \end{cases}$$

Then the viscosity solution of the Riemann problem (5.2) is given by (see [26])

$$u(x,t) = u_0(0) + xp(x,t) - tH(p(x,t)). \tag{5.5}$$

Note that in the case where H is convex, this formula can be derived from the Hopf–Lax formula for the solution to (5.1).

Note that the above construction (5.4) and (5.5) requires only that H is Lipschitz continuous, not differentiable. Exploiting this, let δ be a small positive number and set

$$H^\delta(p) = H(i\delta) + (p - i\delta)\,\frac{H((i+1)\delta) - H(i\delta)}{\delta} \quad \text{for } i\delta \le p < (i+1)\delta. \tag{5.6}$$

If H is Lipschitz continuous, then H^δ is piecewise linear and Lipschitz continuous. Furthermore, also $\tilde{H}^\delta$ will be piecewise linear and $((\tilde{H}^\delta)')^{-1}$ will be piecewise constant. Now set u^δ to be the viscosity solution of the Riemann problem for the equation

$$u_t^\delta + H^\delta\left(u_x^\delta\right) = 0.$$

From (5.5) we then see that u^δ will be piecewise linear. The discontinuities in u_x^δ will move with constant speed in the (x,t) plane.

This construction can be extended to more general initial values. Assume that $u_0^\delta(x)$ is a continuous piecewise linear function such that

$$\lim_{\delta\to 0}\left\|u_0^\delta - u_0\right\| = 0. \tag{5.7}$$

Then one can solve the initial Riemann problems located at the discontinuities of u_{0x}^δ according to (5.5). At some $t_1 > 0$, two of these discontinuities will interact, thereby defining a new Riemann problem at the interaction point. This can now be solved and the process repeated. Note that this amounts to solving the initial value problem for the conservation law

$$p_t^\delta + H^\delta\left(p^\delta\right)_x = 0, \qquad p^\delta(x,0) = u_{0x}^\delta(x).$$

In [16] it was shown that this yields a piecewise constant function $p^\delta(x,t)$, which is constant on a finite number of polygons in the (x,t) plane. Let $u^\delta(x,t)$ denote the result of applying (5.5) at each interaction of discontinuities. From [26], we have the following lemma.

LEMMA 5.1. *The piecewise linear function $u^\delta(x,t)$ is the viscosity solution of*

$$u_t^\delta + H^\delta\left(u_x^\delta\right) = 0, \quad u^\delta(x,0) = u_0^\delta(x). \tag{5.8}$$

Now we can state our main result.

THEOREM 5.1. *Let $u(x,t)$ be the viscosity solution of*

$$u_t + H(u_x) = G(x,t,u), \quad u(x,0) = u_0(x). \tag{5.9}$$

Let S^δ be the solution operator for (5.8)*, and let*

$$v^\delta(x,t) = S^\delta(t_i,t_{i-1})\, E(t_i,t_{i-1})\, v^\delta(\cdot,t_{i-1}) \quad \textit{for } t \in (t_{i-1},t_i], \tag{5.10}$$

with

$$v^\delta(x,0) = u_0(j\Delta x) + (x - j\Delta x)\frac{u_0((j+1)\Delta x) - u_0(j\Delta x)}{\Delta x} \quad \textit{for } x \in [j\Delta x,(j+1)\Delta x].$$

Then there is a constant K, depending only on $\|u_0\|$, $\|u_{0,x}\|$, H, G and T, such that

$$\left\|u(\cdot,t) - v^\delta(\cdot,t)\right\| \le K(\delta + \Delta t + \Delta x) \qquad \forall t \in (0,T). \tag{5.11}$$

Proof. Let w^δ denote the viscosity solution of

$$w_t^\delta + H^\delta\left(w_x^\delta\right) = G\left(t,x,w^\delta\right), \quad w^\delta(x,0) = u_0(x). \tag{5.12}$$

Then Theorem 3.1 and the fact that w^δ is Lipschitz continuous in time ensures the existence of a suitable constant K such that

$$\left\|w^\delta(\cdot,t) - v^\delta(\cdot,t)\right\| \le K\left(\left\|v^\delta(\cdot,0) - u_0\right\| + \Delta t\right). \tag{5.13}$$

By the definition of $v^\delta(x,0)$ and since $u_0 \in Lip_b(\mathbb{R})$,

$$\left\|v^\delta(\cdot,0) - u_0\right\| \le K\Delta x. \tag{5.14}$$

Also, from Proposition 1.4 in [42], we find that

$$\left\|u(\cdot,t) - w^\delta(\cdot,t)\right\| \le K \sup_{|p|\le L}\left|H(p) - H^\delta(p)\right| \le K\delta, \tag{5.15}$$

since we assume that H is locally Lipschitz. The result now follows from (5.13) and (5.15). ∎

Remark 5.2. If H and u_0 are twice continuously differentiable, then the estimates (5.14) and (5.15) can be replaced by

$$\left\|v^\delta(\cdot,0) - u_0\right\| \le K\Delta x^2 \quad \text{and} \quad \left\|u(\cdot,t) - w^\delta(\cdot,t)\right\| \le K\delta^2,$$

respectively. Thus the final error estimate (5.11) is found to be

$$\left\|u(\cdot,t) - v^\delta(\cdot,t)\right\| \le K\left(\delta^2 + \Delta x^2 + \Delta t\right). \tag{5.16}$$

Therefore, if H and u_0 are C^2, then δ and Δx can be chosen much larger than Δt without loss of accuracy.

Example 5.1. We now illustrate the above result with a concrete example and test the operator splitting method (5.10) on the initial value problem

$$u_t + \frac{1}{3}(u_x)^3 = u, \quad u(x,0) = \begin{cases} \sin(\pi x) & \text{for } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

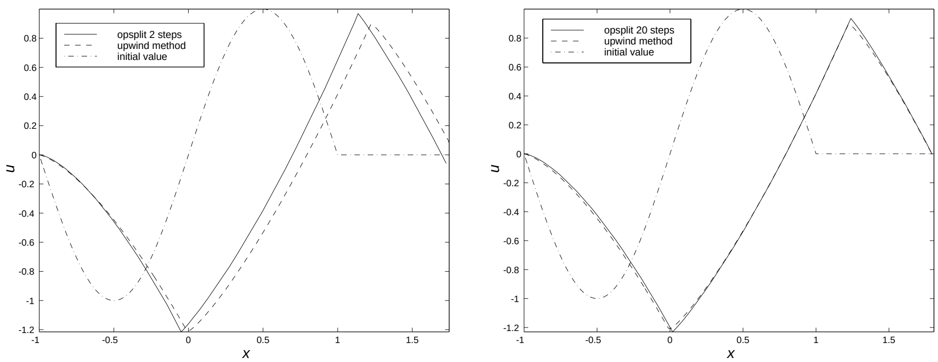

FIG. 1. *Left:* $u(x, 1/2)$ *with* $\Delta t = 0.25$*, right:* $u(x, 1/2)$ *with* $\Delta t = 0.025$.

TABLE 1
Convergence of operator splitting applied to (5.17).

	$100 \times$ relative L^∞ error		
#steps	$\Delta x = 0.04$	$\Delta x = 0.02$	$\Delta x = 0.01$
1	41.2	38.4	39.9
2	22.8	23.2	23.2
4	11.3	14.5	11.8
8	6.2	7.4	5.9
16	3.3	3.0	2.9
32	1.6	1.8	1.4

The approximate solution operators are front tracking for the Hamilton–Jacobi equation

$$u_t + \frac{1}{3}(u_x)^3 = 0$$

and Euler's method for the ordinary differential equation $u_t = u$. Figure 1 shows the approximate solution found using $\Delta x = 0.02$ and $\delta = 2\Delta x$, as well as the upwind approximation (5.18) with the same Δx. To the left we see the approximation $u(x, 1/2)$ obtained by two splitting steps, i.e., $\Delta t = 0.25$, and to the right we have used $\Delta t = 0.025$. To check the convergence rate (5.11), we compared the splitting approximations with a difference approximation on a fine grid. We used the upwind stencil

$$u_j^{i+1} = (1 + \Delta t)u_j^i - \frac{\Delta t}{3}\left(\frac{u_j^i - u_{j-1}^i}{\Delta x}\right)^3 \tag{5.18}$$

with (hopefully) self-explanatory notation. For the reference solution we used $\Delta x = 1/250$. In Table 1, we list the percentage relative L^∞ error for three difference sequences of approximations: $\Delta x = 0.04$, $\Delta x = 0.02$, and $\Delta x = 0.01$. In all cases $\delta = 2\Delta x$. We compared the approximations at $t = 1/2$. In the left column are the number of splitting steps ($\Delta t = 1/2\#$steps) and in the other columns we show the errors. From this table we see that the numerical convergence rate is linear in all three cases, confirming (5.11).

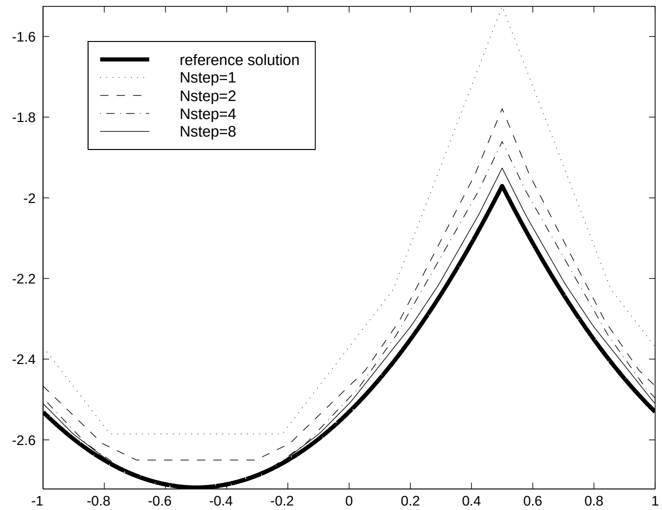

FIG. 2. *Approximate solutions of* (5.19) *at* $t = 1$ *with* $\Delta t = 1/\mathrm{Nstep}$ *and* $\mathrm{Nstep} = 1, 2, 4, 8$.

Example 5.2. As another example where we test the convergence rate (5.16), we compute approximate solutions of the initial value problem

$$u_t + \frac{1}{2}(u_x)^2 = u, \quad u(x,0) = \sin(\pi x). \tag{5.19}$$

As a reference solution, we have used the Engquist–Osher (or generalized upwind) scheme

$$u_j^i = u_j^i(1+\Delta t) - \frac{1}{2}\left(\min\left(\frac{u_{j+1}^i - u_j^i}{\Delta x}, 0\right)^2 + \max\left(\frac{u_j^i - u_{j-1}^i}{\Delta x}, 0\right)^2\right)$$

with $\Delta x = 1/2000$ (special millennium value). We compared the approximations at $t = 1$. In Figure 2 we show the approximate solutions with 1, 2, 4, and 8 steps as well as the reference solution at $t = 1$. Also, instead of the splitting described above, one can use the Strang splitting

$$u(\cdot, i\Delta t) \approx [E(\Delta t/2)S(\Delta t)E(\Delta t/2)]^i\, u_0.$$

This gives formal second order convergence, and one would expect it to be better than the Godunov splitting in practice. To take advantage of (5.16), we set

$$\Delta t = 1/\#\mathrm{steps}, \quad \Delta x = \sqrt{\Delta t/25}, \quad \text{and} \quad \delta = \sqrt{\Delta t/10}$$

as parameters for the front tracking scheme. In Table 2 we list the results. From this we see that in both cases the convergence rate is linear, but Strang splitting gives a much smaller error.

Acknowledgment. We thank an anonymous referee for drawing our attention to reference [46].

Table 2
Convergence of Godunov and Strang splitting.

100 × relative L^∞ error		
#steps	Godunov	Strang
1	18.80	3.32
2	7.46	1.73
4	4.04	0.93
8	1.67	0.48
16	0.80	0.21
32	0.48	0.10
64	0.19	0.05

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