A "MAXIMUM PRINCIPLE FOR SEMICONTINUOUS FUNCTIONS" APPLICABLE TO INTEGRO-PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We formulate and prove a non-local "maximum principle for semicontinuous functions" in the setting of fully nonlinear and degenerate elliptic integro-partial differential equations with integro operators of second order. Similar results have been used implicitly by several researchers to obtain comparison/uniqueness results for integro-partial differential equations, but proofs have so far been lacking.

1. INTRODUCTION

The theory of viscosity solutions (existence, uniqueness, stability, regularity etc.) for fully nonlinear degenerate second order partial differential equations is now highly developed [4, 5, 13, 15]. In recent years there has been an interest in extending viscosity solution theory to integro-partial differential equations (integro-PDEs henceforth) [1, 2, 3, 6, 8, 9, 10, 29, 30, 31, 32, 33, 34]. Such non-local equations occur in the theory of optimal control of Lévy (jump-diffusion) processes and find many applications in mathematical finance, see, e.g., [1, 2, 3, 9, 8, 10, 17] and the references cited therein. We refer to [18, 19] for a deep investigation of integro-PDEs in the framework of Green functions and regular solutions, see also [20].

In this paper we are interested in comparison/uniqueness results for viscosity solutions of fully nonlinear degenerate elliptic integro-PDEs on a possibly unbounded domain $\Omega \subset \mathbb{R}^N$. To be as general as possible, we write these equations in the form

$$F(x, u(x), Du(x), D^2u(x), u(\cdot)) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \times C^2_p(\Omega) \to \mathbb{R}$ is a given functional. Here $\mathbb{S}^N$ denotes the space of symmetric $N \times N$ real valued matrices, and $C^2_p(\Omega)$ denotes the space of $C^2(\Omega)$ functions with polynomial growth of order $p \geq 0$ at infinity. At this stage we simply assume that the non-local part of F is well defined on Ω.

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These equations are non-local as is indicated by the $u(\cdot)$-term in (1.1). A simple example of such an equation is

$$
\begin{aligned}
&-\varepsilon\Delta u + \lambda u \\
&-\int_{\mathbb{R}^N\setminus\{0\}} \left[u(x+z) - u(x) - zDu(x)\,\mathbf{1}_{|z|<1}\right] m(dz) = f(x) \quad \text{in} \quad \mathbb{R}^N,
\end{aligned} \tag{1.2}
$$

where $\lambda > 0$, $\varepsilon \geq 0$, $f : \mathbb{R}^N \to \mathbb{R}$ is uniformly continuous, and $m(dz)$ is a nonnegative Radon measure on $\mathbb{R}^N\setminus\{0\}$ (the so-called Lévy measure) with a singularity at the origin satisfying

$$
\int_{\mathbb{R}^N\setminus\{0\}} \left(|z|^2\,\mathbf{1}_{|z|<1} + |z|^p\,\mathbf{1}_{|z|\geq 1}\right) m(dz) < \infty. \tag{1.3}
$$

Note that the Lévy measure integrates functions with p-th order polynomial growth at infinity. In view of (1.3), a simple Taylor expansion of the integrand shows that u has to belong to $C^2_p(\mathbb{R}^N)$ for the integro operator in (1.2) to be well defined. From this we also see that the integro operator in (1.2) acts as a non-local second order term, and for that reason the "order" of the integro operator is said to be two. If $|z|^2$ in (1.3) is replaced by $|z|$, this changes the order of the integro operator from two to one, since then it acts just like a non-local first order term. Finally, if $|z|^2$ in (1.3) is replaced by 1 (i.e., $m(dz)$ is a bounded measure), then the integro operator in (1.2) is said to be bounded or of order zero. In the bounded case, the integro operator acts just like a non-local zero order term.

A significant example of a non-local equation of the form (1.1) is the non-convex Isaacs equations associated with zero-sum, two-player stochastic differential games (see, e.g., [16] for the case *without* jumps)

$$
\inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \left\{-\mathcal{L}^{\alpha,\beta}u(x) - \mathcal{I}^{\alpha,\beta}u(x) + f^{\alpha,\beta}(x)\right\} = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.4}
$$

where $\mathcal{A}$ and $\mathcal{B}$ are compact metric spaces and for any sufficiently regular ϕ,

(1.5)

$$
\begin{cases}
\mathcal{L}^{\alpha,\beta}\phi(x) = \mathrm{tr}\left[a^{\alpha,\beta}(x)D^2\phi\right] + b^{\alpha,\beta}(x)D\phi + c^{\alpha,\beta}(x)\phi, \\
a^{\alpha,\beta}(x) = \dfrac{1}{2}\sigma^{\alpha,\beta}(x)\sigma^{\alpha,\beta}(x)^\top \geq 0,\ \sigma^{\alpha,\beta}(x) \in \mathbb{R}^{N\times K},\ 1 \leq K \leq N, \\
\mathcal{I}^{\alpha,\beta}\phi(x) = \displaystyle\int_{\mathbb{R}^M\setminus\{0\}} \left[\phi(x+\eta^{\alpha,\beta}(x,z)) - \phi(x) - \eta^{\alpha,\beta}(x,z)D\phi(x)\,\mathbf{1}_{|z|<1}\right] m(dz).
\end{cases}
$$

Here tr and $^\top$ denote the trace and transpose of matrices. The Lévy measure $m(dz)$ is a nonnegative Radon measure on $\mathbb{R}^M\setminus\{0\}$, $1 \leq M \leq N$, satisfying a condition similar to (1.3), see (A0) and (A4) in Section 3. Also see Section 3 for the (standard) regularity assumptions on the coefficients, $\sigma^{\alpha,\beta}(x)$, $b^{\alpha,\beta}(x)$, $c^{\alpha,\beta}(x)$, and $\eta^{\alpha,\beta}(x,z)$. We remark that if $\mathcal{A}$ is a singleton, then equation (1.4) becomes the convex Bellman equation associated with optimal control of Lévy (jump-diffusion) processes over an infinite horizon (see, e.g., [29, 30] and the references therein). Henceforth we call equation (1.4) for the Bellman/Isaacs equation.

Rather general existence and comparison/uniqueness results for viscosity solutions of first order integro-PDEs (no local second order term) can be found in [33, 34, 31, 32], see also [9] for the Bellman equation associated with a singular control problem arising in finance.

Depending on the order of the integro operator (i.e., the assumptions on the singularity of the Lévy measure $m(dz)$ at the origin), the case of second order degenerate elliptic (or parabolic) integro-PDEs is more complicated. When the integro operator is of order zero (bounded), general existence and comparison/uniqueness results for (semicontinuous and unbounded) viscosity solutions are given in [1, 2, 3]. When the integro operator is of second order (i.e., the Lévy measure $m(dz)$ is unbounded near the origin as in (1.3)), the existence and uniqueness of unbounded viscosity solutions of systems of semilinear degenerate parabolic integro-PDEs is proved in Barles, Buckdahn, and Pardoux [6]. Pham [30] proved an existence result and a comparison principle among uniformly continuous (and hence at most linearly growing) viscosity sub- and supersolutions of parabolic integro-PDEs of the Bellman type (i.e. (1.4) with singleton $\mathcal{A}$). Motivated by singular stochastic control applications in finance, the papers [8, 10] provide existence and comparison results for non-local degenerate elliptic free boundary problems with state-constraint boundary conditions.

The main contribution of the present paper is to provide "non-local" versions of Proposition 5.1 in Ishii [22] (see also Proposition II.3 in Ishii and Lions [23]) and Theorem 1 in Crandall [11], which are properly adapted to integro-PDEs of the form (1.1). This "non-local maximum principle" is used to obtain comparison principles for semicontinuous viscosity sub- and supersolutions of (1.1). Although there exist already comparison results for some integro-PDEs with a second order integro operator, see [6, 30, 8, 10], they are all based on the by now standard approach that uses the maximum principle for semicontinuous functions [12, 13]. As we argue for in Section 2, it is in general not clear how to implement this approach for non-local equations. After all the maximum principle for semicontinuous functions [12, 13] is a local result! This was one of our motivations for writing this paper, which in contrast to [6, 30, 8, 10] advocates the use of original approach due to Jensen, Ishii, Lions [27, 22, 23] for proving comparison results for non-local equations. Although our main result (see Theorem 4.9) is not surprising, and the tools used in the proof are nowadays standard in the viscosity solution theory, it has not appeared in the literature before and in our opinion it seem to provide the "natural" framework for deriving general comparison results for fully nonlinear degenerate second order integro-PDEs. Moreover, we stress that our Theorem 4.9 cannot be derived directly from the maximum principle for semicontinuous functions [12, 13] (although it is well known that this can be done in the pure PDE case).

In addition to the main result mentioned above, our paper complements the existing literature [6, 30] (see also [8, 10]) on second order PDEs with integro operators of second order in the following ways: (i) Our formulation is abstract and more general, (ii) we consider only semicontinuous sub- and supersolutions, and (iii) we consider (slightly) more general integro operators (see Remark 6.1).

The remaining part of this paper is organized as follows: In Section 2 we discuss our main result (Theorem 4.9) in the simplest possible context of (1.2) and relate it to some of the existing literature on integro-PDEs. In Section 3 we first list our assumptions on the coefficients in the Bellman/Isaacs equation (1.4). Then we state and discuss a comparison theorem for these equations. In Section 4 we list the assumptions for the problem (1.1). Then we give two equivalent definitions of a viscosity solution for (1.1) and illustrate them on the Bellman/Isaacs equation. Finally we state our main result (Theorem 4.9). In Section 5 we list structure conditions on (1.1) implying, via Theorem 4.9, quite general comparison/uniqueness results for possibly unbounded domains. The structure conditions are illustrated on the Bellman/Isaacs equation. Section 6 contains the proof of the comparison theorem for the Bellman/Isaacs equation, and the proof of Theorem 4.9 is given in Section 7.

Although we only discuss elliptic integro-PDEs, it is not hard to formulate "parabolic" versions of our main results (for example Theorem 4.9), see [26].

We end this introduction by collecting some notations that will be used throughout this paper. If x belong to $U \subset \mathbb{R}^n$ and $r > 0$, then $B(x,r) = \{x \in U : |x| < r\}$. We use the notation $\mathbf{1}_U$ for the function that is 1 in U and 0 outside. By a modulus ω, we mean a positive, nondecreasing, continuous, sub-additive function which is zero at the origin. Let $C^n(\Omega)$ $n = 0, 1, 2$ denote the spaces of n times continuously differentiable functions on Ω. We let $USC(\Omega)$ and $LSC(\Omega)$ denote the spaces of upper and lower semicontinuous functions on Ω, and $SC(\Omega) = USC(\Omega) \cup LSC(\Omega)$. A lower index p denotes the polynomial growth at infinity, so $C^n_p(\Omega)$, $USC_p(\Omega)$, $LSC_p(\Omega)$, $SC_p(\Omega)$ consist of functions f from $C^n(\Omega)$, $USC(\Omega)$, $LSC(\Omega)$, $SC(\Omega)$ satisfying the growth condition

$$|f(x)| \leq C(1 + |x|^p) \quad \text{for all} \quad x \in \Omega.$$

Finally, in the space of symmetric matrices $\mathbb{S}^N$ we denote by $\leq$ the usual ordering (i.e. $X \in \mathbb{S}^N$, $0 \leq X$ means that X positive semidefinite) and by $|\cdot|$ the spectral radius norm (i.e. the maximum of the absolute values of the eigenvalues).

2. Discussion of main result

To explain the contribution of the present paper and put it in a proper perspective with regards some of the existing literature [6, 30, 8, 10], let us elaborate on a difficulty related to proving comparison/uniqueness results arising from the very notion of a viscosity solution. For illustrative purposes, we focus on the simple equation (1.2). The general case (1.1) will be treated in the sections that follow.

First of all, since the equation is non-local it is necessary to use a global formulation of viscosity solutions: A function $u \in USC(\mathbb{R}^N)$ is a viscosity subsolution of (1.2) if

$$\begin{aligned} & -\varepsilon\Delta\phi(x) + \lambda u(x) \\ & - \int_{\mathbb{R}^M\setminus\{0\}} \left[\phi(x+z) - \phi(x) - zD\phi(x)\,\mathbf{1}_{|z|<1}\right] m(dz) \leq f(x) \quad \text{in} \quad \mathbb{R}^N, \end{aligned} \tag{2.1}$$

for any $x \in \mathbb{R}^N$ and $\phi \in C^2_p(\mathbb{R}^N)$ such that x is a *global* maximum point for $u - \phi$. Note that (2.1) makes sense in view of (1.3) and the C^2_p regularity of ϕ. A viscosity supersolution $u \in LSC(\mathbb{R}^N)$ is defined similarly.

One can dispense with the growth restrictions on the test functions by replacing the definition of a viscosity subsolution (2.1) by the following equivalent one: A function $u \in USC_p(\mathbb{R}^N)$ is a viscosity subsolution of (1.2) if for any $\kappa > 0$

$$\begin{aligned} -\varepsilon\Delta\phi(x) + \lambda u(x) - \int_{0<|z|<\kappa} \left[\phi(x+z) - \phi(x) - zD\phi(x)\,\mathbf{1}_{|z|<1}\right] m(dz) & \\ - \int_{|z|\geq\kappa} \left[u(x+z) - u(x) - zD\phi(x)\,\mathbf{1}_{|z|<1}\right] m(dz) \leq f(x), & \end{aligned} \tag{2.2}$$

for any $x \in \mathbb{R}^N$ and $\phi \in C^2(\mathbb{R}^N)$ such that x is a global maximum point for $u-\phi$. We have a similar equivalent formulation of a viscosity supersolution $u \in LSC_p(\mathbb{R}^N)$. It is this second formulation that is used to prove comparison/uniqueness results for (1.2).

In the pure PDE setting ($m(dz) \equiv 0$), nowadays comparison principles are most effectively proved using the so-called maximum principle for semicontinuous functions [12, 13]. However, this result is not formulated in terms of test functions, but rather in terms of the second order semijets $\mathcal{J}^{2,+}$, $\mathcal{J}^{2,-}$, or more precisely their closures $\overline{\mathcal{J}}^{2,+}$, $\overline{\mathcal{J}}^{2,-}$ (see [13] for definitions of the semijets). Let u be a viscosity subsolution of (1.2). If $(q, X) \in \overline{\mathcal{J}}^{2,+}u(x)$, then by definition there exists a sequence of triples (x_k, q_k, X_k) such that $(q_k, X_k) \in \mathcal{J}^{2,+}u(x_k)$ for each k and

$$(x_k, u(x_k)) \to (x, u(x)), \qquad q_k \to q, \qquad X_k \to X, \qquad \text{as } k \to \infty, \tag{2.3}$$

and $u - \phi$ has a global maximum at $x = x_k$ for each k. According to a construction by Evans (see, e.g., [15, Proposition V.4.1]), for each k there is a C^2 function $\phi_k : \mathbb{R}^N \to \mathbb{R}$ such that

$$\phi_k(x_k) = u(x_k), \qquad D\phi_k(x_k) = q_k, \qquad D^2\phi_k(x_k) = X_k,$$

and $u - \phi_k$ has a global maximum at $x = x_k$. Applying (2.2) we thus get

$$\begin{aligned} -\varepsilon \operatorname{tr} X_k + \lambda u(x_k) - \int_{0<|z|<\kappa} \left[\phi_k(x_k + z) - \phi_k(x_k) - z q_k \mathbf{1}_{|z|<1}\right] m(dz) \\ - \int_{|z|\geq\kappa} \left[u(x_k + z) - u(x_k) - z q_k \mathbf{1}_{|z|<1}\right] m(dz) \leq f(x_k), \end{aligned} \tag{2.4}$$

for each k. In view of (2.3), in the pure PDE setting ($m(dz) \equiv 0$) one can send $k \to \infty$ in (2.4), the result being a formulation of the subsolution inequality (2.2) in terms of the elements (q, X) in $\overline{\mathcal{J}}^{2,+}u(x)$. A similar formulation (in terms of the elements in $\overline{\mathcal{J}}^{2,-}u(x)$) can be given for a supersolution u. Consequently, a comparison principle in the pure PDE case can then be deduced using the maximum principle for semicontinuous functions [12, 13].

The situation is less clear in the non-local case. When $m(dz) \neq 0$, we can easily send $k \to \infty$ in the second integral term in (2.4) thanks to $u \in USC_p(\mathbb{R}^N)$. To handle the first integral term, suppose for the moment that the sequence $\{\phi_k\}_{k=1}^{\infty} \subset C^2(\mathbb{R}^N)$ converges (say, uniformly on compact subsets of $\mathbb{R}^N$) to a limit $\overline{\phi}$ that belongs to $C^2(\mathbb{R}^N)$ and $D\overline{\phi}(x) = q$. It is then clear that

$$\begin{aligned} -\varepsilon \operatorname{tr} X + \lambda u(x) - \int_{0<|z|<\kappa} \left[\overline{\phi}(x + z) - \overline{\phi}(x) - z D\overline{\phi}(x) \mathbf{1}_{|z|<1}\right] m(dz) \\ - \int_{|z|\geq\kappa} \left[u(x + z) - u(x) - z q \mathbf{1}_{|z|<1}\right] m(dz) \leq f(x), \end{aligned} \tag{2.5}$$

where $(q, X) \in \overline{\mathcal{J}}^{2,+}u(x)$ and with a similar inequality for supersolutions. Now we could again prove comparison/uniqueness results using the maximum principle for semicontinuous functions [12, 13]. This approach to proving a comparison principle for viscosity solutions of integro-PDEs was first suggested by Pham [30], and later used in [9, 10]. Indeed, converted to our setting, Lemma 2.2 in [30] states that one can find a C^2 function $\overline{\phi}$ such that (2.5) holds and another C^2 function such that the corresponding inequality for a viscosity supersolution holds. However, there is no proof of this lemma in [30], and neither is it clear to us how to prove (2.5) in general. To be more precise, we do not know how to prove that the sequence $\{\phi_k\}_{k=1}^{\infty} \subset C^2(\mathbb{R}^N)$ has a limit point $\overline{\phi}$ that in general belongs to $C^2(\mathbb{R}^N)$. The C^2 requirement of such a limit is necessary if we want to make sense to (2.5) when the integro operator is of second order.

We will take a different approach to proving comparison/uniqueness results for integro-PDEs. Namely, following the original of Jensen [27], Ishii [22], Ishii and Lions [23], and Crandall [11], we establish some sort of "non-local" maximum principle for semicontinuous viscosity sub- and supersolutions of (1.1) (see Theorem 4.9), and then various comparison principles can be derived from this result. Let us illustrate our approach on (1.2).

Let u and v be respectively viscosity sub- and supersolutions of (1.2). A standard trick in viscosity solution theory for dealing with the low regularity of the solutions is the doubling of variables device. Instead of studying directly a global maximum point of $u(x) - v(x)$, we consider a global maximum point $(\bar{x}, \bar{y})$ of

$$u(x) - v(y) - \phi(x, y),$$

where ϕ is a suitable C^2 penalization term. Our main result (Theorem 4.9) applied to (1.2) yields the following: For any $\gamma \in (0, \frac{1}{2})$ there exists matrices $X, Y \in \mathbb{S}^N$ satisfying

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{1}{1-2\gamma} D^2\phi(\bar{x}, \bar{y}),$$

such that the following two inequalities hold:

$$\begin{aligned}
-\varepsilon \operatorname{tr} X + \lambda u(\bar{x}) - \int_{0<|z|<\kappa} & \left[\phi(\bar{x}+z, \bar{y}) - \phi(\bar{x}, \bar{y}) - zD_x\phi(\bar{x}, \bar{y})\, \mathbf{1}_{|z|<1}\right] m(dz) \\
- \int_{|z|\geq\kappa} & \left[u(\bar{x}+z) - u(\bar{x}) - zD_x\phi(\bar{x}, \bar{y})\, \mathbf{1}_{|z|<1}\right] m(dz) \leq f(\bar{x}), \\
-\varepsilon \operatorname{tr} Y + \lambda v(\bar{y}) - \int_{0<|z|<\kappa} & \left[\phi(\bar{x}, \bar{y}+z) - \phi(\bar{x}, \bar{y}) + zD_y\phi(\bar{x}, \bar{y})\, \mathbf{1}_{|z|<1}\right] m(dz) \\
- \int_{|z|\geq\kappa} & \left[v(\bar{y}+z) - v(\bar{y}) + zD_y\phi(\bar{x}, \bar{y})\, \mathbf{1}_{|z|<1}\right] m(dz) \geq f(\bar{y}).
\end{aligned}$$

The key point is that the C^2 penalization function $\phi(x, y)$ used in the doubling of variables device occupies the slots in the integro operator near the origin. Indeed, equipped with the above result, it is possible to derive comparison/uniqueness results for (1.2) as in, e.g., Pham [30].

In [6, Proof of Theorem 3.5], Barles, Buckdahn, and Pardoux used the maximum principle for semicontinuous functions [12, 13] and a result very much in the spirit of the above result (or Theorem 4.9) for proving uniqueness of viscosity solutions for parabolic integro-PDEs. However, the authors give no proof of such a result. We also stress that for the first order version of (1.2) ($\varepsilon = 0$), the above two inequalities come for free from the nature of the point $(\bar{x}, \bar{y})$ and the definition of a viscosity solution, see, e.g., [34, 9]. However, in the second order case ($\varepsilon > 0$), the proof of this result, or more generally Theorem 4.9, is more involved in the sense that it consists of adapting the chain of arguments developed by Jensen [27], Ishii [22], Ishii and Lions [23], and Crandall [11] to our non-local situation. The above result (or more generally Theorem 4.9) can be viewed as some sort of "non-local" maximum principle for *semicontinuous viscosity sub- and supersolutions*. It should be compared with the "local" maximum principle for *semicontinuous functions* [11, 12, 13].

3. The Bellman/Isaacs equation

In this section we will give natural assumptions on the coefficients in the Bellman/Isaacs equation (1.4) that leads to comparison results for bounded semicontinuous viscosity sub- and supersolutions in $\mathbb{R}^N$. We state the comparison results, but postpone the proof to Section 6. We remark here that Pham [30] presents a comparison principle for uniformly continuous sub- and supersolutions of the parabolic Bellman equation. The results in this section can be seen as slight extensions of his result (to more general non-linearities, semicontinuous sub/super solutions and slightly more general integro operators), but the techniques are essentially the same. The purpose of this section is simply to provide an example where we may use our "non-local maximum principle" to obtain comparison results. Furthermore, the Bellman/Isaacs equation, under the assumptions stated below, will serve as examples in the abstract and more general theory developed in the sections that follow.

The following conditions are natural and standard for (1.4) in view of the connections to the theory of stochastic control and differential games (see, e.g., [15, 16, 28, 30]):

There are constants $K_1, K_x \geq 0$, a function $\rho \geq 0$, and a modulus of continuity ω, such that the following statements hold for every $x, y \in \mathbb{R}^N$, $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $z \in \mathbb{R}^M \setminus \{0\}$:

(A0) σ, b, c, f, η are continuous w.r.t. x, α, β and Borel measurable w.r.t. z; $\mathcal{A}, \mathcal{B}$ are compact metric spaces; and $m(dz)$ is a positive Radon measure on $\mathbb{R}^M \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^M \setminus \{0\}} \left(\rho(z)^2\, \mathbf{1}_{|z|<1} + \mathbf{1}_{|z|\geq 1}\right)\, m(dz) < \infty.$$

(A1) $|\sigma^{\alpha,\beta}(x) - \sigma^{\alpha,\beta}(y)| + |b^{\alpha,\beta}(x) - b^{\alpha,\beta}(y)| \leq K_1|x-y|$.

(A2) $|c^{\alpha,\beta}(x) - c^{\alpha,\beta}(y)| + |f^{\alpha,\beta}(x) - f^{\alpha,\beta}(y)| \leq \omega(|x-y|)$.

(A3) $|\eta^{\alpha,\beta}(x,z) - \eta^{\alpha,\beta}(y,z)| \leq \rho(z)|x-y|$.

(A4) $|\eta^{\alpha,\beta}(x,z)| \leq \rho(z)(1+|x|)$ and $|\eta^{\alpha,\beta}(x,z)|\, \mathbf{1}_{|z|<1} \leq K_x$.

(A5) $-c^{\alpha,\beta} \geq \lambda > 0$.

The Lévy measure $m(dz)$ may have a singularity at $z = 0$. As an example in $\mathbb{R}^N$, take $\rho(z) = |z|$ and $m(dz) = z^{-(N+1)-\delta}\, \mathbf{1}_{|z|<1} dz$ where $\delta \in (0,1)$. According to our definition, this integro operator has order 2. Furthermore, according to (A0), the Lévy measure $m(dz)$ integrates bounded functions away from the origin. Compared to Section 1 this means that $p = 0$ (see also (C1) in Section 4). Because of this, (A0), (A4), and a Taylor expansion of the integrand shows that the integro part of the Bellman/Isaacs equation (1.4) is well defined for $C_0^2(\mathbb{R}^N)$ functions, see [18, 19, 30].

We also remark that (A1) and (A2) imply

$$|\sigma^{\alpha,\beta}(x)| + |b^{\alpha,\beta}(x)| + |c^{\alpha,\beta}(x)| + |f^{\alpha,\beta}(x)| \leq C(1+|x|),$$

for some constant $C > 0$. It is the growth of f at infinity that determines the growth of the solutions at infinity, so if f is bounded so are the solutions.

Theorem 3.1. *Assume (A0) – (A5) hold, $f^{\alpha,\beta}$ is bounded uniformly in α and β, and $u, -v \in USC_0(\mathbb{R}^N)$. If u is a viscosity subsolution and v a viscosity supersolution of* (1.4)*, then $u \le v$ in $\mathbb{R}^N$.*

As an immediate consequence we have uniqueness of bounded viscosity solutions of (1.4). The notion of viscosity solutions will be defined in Section 4, and Theorem 3.1 will be proved in Section 6 using the abstract comparison result Theorem 5.2.

Remark 3.2 (Growth at infinity). If the integrability condition in (A0) is replaced by

$$\int_{\mathbb{R}^M\setminus\{0\}} \left(\rho(z)^2\,\mathbf{1}_{|z|<1} + (1+\rho(z))\,\mathbf{1}_{|z|\ge 1}\right)\, m(dz) < \infty,$$

and we drop the assumption that f is bounded, then the above assumptions leads to problems where the solutions may have linear growth at infinity. This case seems to be more difficult, and we do not know if (the modified) assumptions (A0) – (A5) are sufficient to have a comparison result. However, there are two special cases where we may have comparison results:

- If σ, b, and η are bounded, and c is constant, then a comparison result can be obtained by adapting the techniques of Ishii in [22] (Theorem 7.1).
- If λ is sufficiently large compared to $|D\sigma|$, $|D\eta|$, and $|Db|$, then we have a comparison result because of cancellation effects in the proof, cf. Pham [30] where this technique is used in the parabolic case.

If the above assumption are modified appropriately and λ is big enough, then we can have comparison results for solutions with arbitrary polynomial growth. For parabolic problems (see, e.g., [30]) we are always in this situation since we can have λ arbitrary large after an exponential-in-time scaling of the solution.

Remark 3.3. It is possible to consider Radon measures m depending on x, α, β under assumptions similar to those used in Soner [34] for first order integro-PDEs.

Remark 3.4. It is also possible to consider bounded domains Ω, but then we need a condition on the jumps so that the jump-process does not leave Ω.

4. The main result

In this section we state the general assumptions, give two equivalent definitions of viscosity solutions, and give our main result (Theorem 4.9). As we go along, we use the Bellman/Isaacs equation (1.4) for illustrative purposes.

For every $x, y \in \Omega, r, s \in \mathbb{R}, X, Y \in \mathbb{S}^N$, and $\phi, \phi_k, \psi \in C^2_p(\Omega)$ we will use the following assumptions on (1.1):

(C1) The function $(x, r, q, X) \mapsto F(x, r, q, X, \phi(\cdot))$ is continuous, and if $x_k \to x, D^n\phi_k \to D^n\phi$ locally uniformly in Ω for $n = 0, 1, 2$, and $|\phi_k(x)| \le C(1 + |x|^p)$ (C independent of k and x), then

$$F(x_k, r, q, X, \phi_k(\cdot)) \to F(x, r, q, X, \phi(\cdot)).$$

(C2) If $X \le Y$ and $\phi - \psi$ has a global maximum at x, then

$$F(x, r, q, X, \phi(\cdot)) \ge F(x, r, q, Y, \psi(\cdot)).$$

(C3) If $r \leq s$, then $F(x,r,q,X,\phi(\cdot)) \leq F(x,s,q,X,\phi(\cdot))$.

(C4) For every constant $C \in \mathbb{R}$, $F(x,r,q,X,\phi(\cdot)+C) = F(x,r,q,X,\phi(\cdot))$.

Example 4.1. The Bellman/Isaacs equation (1.4) satisfies conditions (C1) – (C4) with $\Omega = \mathbb{R}^N$ and $p = 0$ when assumptions (A0), (A4), and (A5) hold. For this equation

$$F(x,r,q,X,\phi(\cdot)) = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ -\mathrm{tr}\left[a^{\alpha,\beta}(x)\,X\right] - b^{\alpha,\beta}(x)\,q - c^{\alpha,\beta}(x)\,r + f^{\alpha,\beta}(x) - \int_{\mathbb{R}^M\setminus\{0\}} \left[\phi(x+\eta^{\alpha,\beta}(x,z)) - \phi(x) - \eta^{\alpha,\beta}(x,z)\,q\,\mathbf{1}_{|z|<1}\right] m(dz) \Big\}.$$

Definition 4.2. A locally bounded function $u \in USC(\Omega)$ ($u \in LSC(\Omega)$) is a *viscosity subsolution* (*viscosity supersolution*) of (1.1) if for every $x \in \Omega$ and $\phi \in C^2_p(\Omega)$ such that x is a global maximizer (global minimizer) for $u - \phi$,

$$F(x,u(x),D\phi(x),D^2\phi(x),\phi(\cdot)) \leq 0 \ (\geq 0).$$

We say that u is a *viscosity solution* of (1.1) if u is both a sub- and supersolution of (1.1).

Remark 4.3. Because we allow for singular integro terms (first or second order integro operators), to have a meaningful definition we use the test function ϕ (also) in the non-local argument.

Note that viscosity solutions according to this definition are continuous. Without changing the solutions, we may change this definition in the following two standard ways:

Lemma 4.4. *(i) If (C4) holds, we may assume that $\phi(x) = u(x)$ in Definition 4.2.*

(ii) If (C2) holds, we may replace global extremum by global strict extremum in Definition 4.2.

Proof. We only prove (ii) and here we only consider maxima. Assume $\phi \in C^2_p(\Omega)$ is such that $u-\phi$ has a global maximum at $x \in \Omega$. Pick a non-negative $\psi \in C^2(\Omega)$ with compact support such that $\psi|_{B(x,\delta)}(y) = |x-y|^4$ for some $0 < \delta < \mathrm{dist}(x,\partial\Omega)$. Now $u - (\phi+\psi)$ has a global strict maximum at x, and $D(\phi+\psi) = D\phi$ and $D^2(\phi+\psi) = D^2\phi$ at x. Since $\phi - (\phi+\psi) = -\psi$ also has a global maximum at x, by (C2) and the above considerations we have

$$\begin{aligned} &F(x,u(x),D(\phi+\psi)(x),D^2(\phi+\psi)(x),(\phi+\psi)(\cdot)) \\ &\quad \leq F(x,u(x),D\phi(x),D^2\phi(x),\phi(\cdot)) \ (\leq 0), \end{aligned}$$

and the proof is complete. □

The concept of a solution in Definition 4.2 is an extension of the classical solution concept.

Lemma 4.5. *(i) If (C2) holds, then a classical subsolution u of* (1.1) *belonging to $C^2_p(\Omega)$ is a viscosity subsolution of* (1.1).

(ii) A viscosity subsolution u of (1.1) *belonging to $C^2_p(\Omega)$ is a classical subsolution of* (1.1).

Next we introduce an alternative definition of viscosity solutions that is needed for proving comparison and uniqueness results. For every $\kappa \in (0,1)$, assume that we have a function

$$F_\kappa : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \times SC_p(\Omega) \times C^2(\Omega) \to \mathbb{R}$$

satisfying the following list of assumptions for every $\kappa \in (0,1), x, y \in \Omega, r, s \in \mathbb{R}, q \in \mathbb{R}^N, X, Y \in \mathbb{S}^N, u, -v \in USC_p(\Omega), w \in SC_p(\Omega)$, and $\phi, \phi_k, \psi, \psi_k \in C^2_p(\Omega)$:

(F0) $F_\kappa(x,r,q,X,\phi(\cdot),\phi(\cdot)) = F(x,r,q,X,\phi(\cdot))$.

(F1) The function F in (F0) satisfies (C1).

(F2) If $X \le Y$ and both $u - v$ and $\phi - \psi$ have global maxima at x, then
$$F_\kappa(x,r,q,X,u(\cdot),\phi(\cdot)) \ge F_\kappa(x,r,q,Y,v(\cdot),\psi(\cdot)).$$

(F3) The function F in (F0) satisfies (C3).

(F4) For all constants $C_1, C_2 \in \mathbb{R}$,
$$F_\kappa(x,r,q,X,w(\cdot)+C_1,\phi(\cdot)+C_2) = F_\kappa(x,r,q,X,w(\cdot),\phi(\cdot)).$$

(F5) If $\psi_k \to w$ a.e. in Ω and $|\psi_k(x)| \le C(1+|x|^p)$ (C independent of k and x), then
$$F_\kappa(x,r,q,X,\psi_k(\cdot),\phi(\cdot)) \to F_\kappa(x,r,q,X,w(\cdot),\phi(\cdot)).$$

Remark 4.6. If (F0) – (F4) hold, then (C1) – (C4) hold.

Example 4.7. For the Bellman/Isaacs equation (1.4),

$$\begin{aligned} F_\kappa(x,r,q,X,u(\cdot),\phi(\cdot)) = \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\Big\{ &-\mathrm{tr}\left[a^{\alpha,\beta}(x)\,X\right] - b^{\alpha,\beta}(x)\,q - c^{\alpha,\beta}(x)\,r \\ &+ f^{\alpha,\beta}(x) - B_\kappa^{\alpha,\beta}(x,q,\phi(\cdot)) - B^{\alpha,\beta,\kappa}(x,q,u(\cdot))\Big\}, \end{aligned}$$

where

$$\begin{aligned} &B_\kappa^{\alpha,\beta}(x,q,\phi(\cdot)) \\ &= \int_{B(0,\kappa)\setminus\{0\}} \left[\phi(x+\eta^{\alpha,\beta}(x,z)) - \phi(x) - \eta^{\alpha,\beta}(x,z)\,q\right]\, m(dz), \\ &B^{\alpha,\beta,\kappa}(x,q,u(\cdot)) \\ &= \int_{\mathbb{R}^M\setminus B(0,\kappa)} \left[u(x+\eta^{\alpha,\beta}(x,z)) - u(x) - \eta^{\alpha,\beta}(x,z)\,q\,\mathbf{1}_{|z|<1}\right]\, m(dz). \end{aligned}$$

If $\kappa < 1$ and conditions (A0), (A4), and (A5) hold, then (F0) – (F5) hold for (1.4) with $\Omega = \mathbb{R}^N$ and $p = 0$.

Lemma 4.8 (Alternative definition)**.** *Assume (F0), (F2), (F4), and (F5) hold. $u \in USC_p(\Omega)$ ($u \in LSC_p(\Omega)$) is a viscosity subsolution (viscosity supersolution) of (1.1) if and only if for every $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that x is a global maximizer (global minimizer) for $u - \phi$,*

$$F_\kappa(x,u(x),D\phi(x),D^2\phi(x),u(\cdot),\phi(\cdot)) \le 0 \ (\ge 0) \quad \textit{for every } \kappa \in (0,1).$$

Proof. The proof follows [31], see also [6].

If. Let $u - \phi$ have a global maximum at x for some $\phi \in C^2_p(\Omega)$. Using (F0), (F2) and the assumptions of the lemma we have

$$F(x,u(x),D\phi(x),D^2\phi(x),\phi(\cdot)) = F_\kappa(x,u(x),D\phi(x),D^2\phi(x),\phi(\cdot),\phi(\cdot))$$

$$\leq F_\kappa(x, u(x), D\phi(x), D^2\phi(x), u(\cdot), \phi(\cdot)) \leq 0.$$

Only if. Let $\phi \in C^2(\Omega)$ be such that $u - \phi$ has a global maximum at x. By an argument similar to the one in the proof of Lemma 4.4 with (F4) replacing (C4), we can assume that $(u-\phi)(x) = 0$. Pick a sequence of $C^2_p(\Omega)$ functions $\{\phi_\varepsilon\}_\varepsilon$ such that $u \leq \phi_\varepsilon \leq \phi$ and $\phi_\varepsilon \to u$ a.e. as $\varepsilon \to 0$. It follows that $u - \phi_\varepsilon$ and $\phi_\varepsilon - \phi$ also have global maxima at x. The last maximum implies that at x, $D(\phi_\varepsilon - \phi) = 0$ and $D^2(\phi_\varepsilon - \phi) \leq 0$. By (F2), (F0), and Definition 4.2 we have

$$\begin{aligned} &F_\kappa(x, u(x), D\phi(x), D^2\phi(x), \phi_\varepsilon(\cdot), \phi(\cdot)) \\ &\qquad \leq F_\kappa(x, u(x), D\phi_\varepsilon(x), D^2\phi_\varepsilon(x), \phi_\varepsilon(\cdot), \phi_\varepsilon(\cdot)) \\ &\qquad = F(x, u(x), D\phi_\varepsilon(x), D^2\phi_\varepsilon(x), \phi_\varepsilon(\cdot)) \leq 0. \end{aligned}$$

Since $\phi_\varepsilon \to u$ a.e., sending $\varepsilon \to 0$ in the above inequality and using (F5) yields the "$\leq$" inequality in the lemma, and the *only if* part is proved. □

We have now come to our main theorem. It is this theorem that should replace the maximum principle for semicontinuous functions [12, 13] when proving comparison results for integro-PDEs with first or second order integro operators.

Theorem 4.9. *Let $u, -v \in USC_p(\Omega)$ satisfy $u(x), -v(x) \leq C(1 + |x|^2)$ and solve in the viscosity solution sense*

$$F(x, u, Du, D^2u, u(\cdot)) \leq 0 \quad \textit{and} \quad G(x, v, Dv, D^2v, v(\cdot)) \geq 0,$$

where F and G satisfy (C1) – (C4). Let $\phi \in C^2(\Omega \times \Omega)$ and $(\bar{x}, \bar{y}) \in \Omega \times \Omega$ be such that

$$(x, y) \mapsto u(x) - v(y) - \phi(x, y)$$

has a global maximum at $(\bar{x}, \bar{y})$. Furthermore, assume that in a neighborhood of $(\bar{x}, \bar{y})$ there are continuous functions $g_0 : \mathbb{R}^{2N} \to \mathbb{R}, g_1, g_2 : \mathbb{R}^N \to \mathbb{S}^N$ with $g_0(\bar{x}, \bar{y}) > 0$, satisfying

$$D^2\phi \leq g_0(x, y) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} g_1(x) & 0 \\ 0 & g_2(y) \end{pmatrix}.$$

If, in addition, for each $\kappa \in (0, 1)$ there exist F_κ and G_κ satisfying (F0) – (F5), then for any $\gamma \in (0, \frac{1}{2})$ there are two matrices $X, Y \in \mathbb{S}^N$ satisfying

$$(4.1)\quad -\frac{g_0(\bar{x}, \bar{y})}{\gamma} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} - \begin{pmatrix} g_1(\bar{x}) & 0 \\ 0 & g_2(\bar{y}) \end{pmatrix} \leq \frac{g_0(\bar{x}, \bar{y})}{1 - 2\gamma} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

such that

$$(4.2)\quad F_\kappa(\bar{x}, u(\bar{x}), D_x\phi(\bar{x}, \bar{y}), X, u(\cdot), \phi(\cdot, \bar{y})) \leq 0,$$

$$(4.3)\quad G_\kappa(\bar{x}, v(\bar{y}), -D_y\phi(\bar{x}, \bar{y}), Y, v(\cdot), -\phi(\bar{x}, \cdot)) \geq 0.$$

The proof of Theorem 4.9 is given in Section 7. We underline that the key point in Theorem 4.9 is the validity of the inequalities (4.2) and (4.3). The proof of Theorem 4.9 shows that $(D_x\phi(\bar{x}, \bar{y}), X) \in \overline{\mathcal{J}}^{2,+}u(\bar{x})$ and $(-D_y\phi(\bar{x}, \bar{y}), Y) \in \overline{\mathcal{J}}^{2,+}v(\bar{y})$. This information alone would in the pure PDE case, under certain (semi)continuity assumptions on the equations, imply that the viscosity solution inequalities hold. In the non-local case, the situation is more delicate and we refer to Section 2 for a discussion of this point.

The technical assumption $u(x), -v(x) \leq C(1 + |x|^2)$ is an artifact of the method of proof, and it does not seem so easy to remove. However, in applications this condition does not create any difficulties. The assumptions on the test-function is satisfied in most practical cases. For test-functions like

$$\phi(x, y) = \frac{1}{\delta}|x-y|^q, \qquad \delta > 0,\ q > 0,$$

the assumptions hold for all $(x, y) \in \mathbb{R}^{2N}$ when $q \geq 2$ and for all $(x, y) \in \{x \neq y\}$ otherwise.

Finally, we remark that it is possible to have a result without the restriction on $D^2\phi$. Such a result (Lemma 7.4) is actually used to prove Theorem 4.9. But this result is indirect in the sense that it is not the function $(x, y) \mapsto u(x)-v(y)-\phi(x, y)$ that is considered directly but rather its "sup-convoluted" version $(x, y) \mapsto u^\varepsilon(x) - v_\varepsilon(y) - \phi(x, y)$. In fact, this was the original approach to uniqueness of viscosity solutions for second order PDEs, cf. Jensen, Ishii, Lions [27, 22, 23].

5. A general comparison theorem

In this section we use Theorem 4.9 to prove a general comparison result for non-local equations of the form (1.1) where $\Omega \subset \mathbb{R}^N$ is a possibly unbounded domain. We need two additional assumptions on the equation, and we state them for the F_κ functions.

For every $\kappa \in (0, 1), x, y \in \Omega, r \in \mathbb{R}, p \in \mathbb{R}^N, X, Y \in \mathbb{S}^N, u, -v \in USC_p(\Omega)$, and $\phi \in C^2(\Omega)$ the following statements hold:

(F6)

There is a $\lambda > 0$ such that if $s \leq r$, then

$$F_\kappa(x, r, p, X, u(\cdot), \phi(\cdot)) - F_\kappa(x, s, p, X, u(\cdot), \phi(\cdot)) \geq \lambda(r - s).$$

(F7)

For any $\delta, \varepsilon > 0$, define

$$\phi(x, y) = \frac{1}{\delta}|x-y|^2 - \varepsilon(|x|^2 + |y|^2).$$

If $u(x), -v(x) \leq C(1 + |x|^2)$ in Ω, and $(\bar{x}, \bar{y}) \in \Omega \times \Omega$ is such that

$$(x, y) \mapsto u(x) - v(y) - \phi(x, y)$$

has a global maximum at $(\bar{x}, \bar{y})$, then for any $\kappa > 0$ there are numbers $m_{\kappa,\delta,\varepsilon}$ satisfying $\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\lim_{\kappa\to 0} m_{\kappa,\delta,\varepsilon} = 0$ and a modulus ω such that

$$F_\kappa\left(\bar{y}, r, \frac{1}{\delta}(\bar{x}-\bar{y}) - \varepsilon\bar{y}, Y, v(\cdot), -\phi(\bar{x}, \cdot)\right) - F_\kappa\left(\bar{x}, r, \frac{1}{\delta}(\bar{x}-\bar{y}) + \varepsilon\bar{x}, X, u(\cdot), \phi(\cdot, \bar{y})\right)$$
$$\leq \omega\left(|\bar{x}-\bar{y}| + \frac{1}{\delta}|\bar{x}-\bar{y}|^2 + \varepsilon(1 + |\bar{x}|^2 + |\bar{y}|^2)\right) + m_{\kappa,\delta,\varepsilon},$$

for every X, Y satisfying

(5.1)
$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{4}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\varepsilon\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Condition (F7) is a version for non-local equations (in an unbounded domain) of the standard condition (3.14) in [13]. The inequality (5.1) corresponds to the second inequality in (4.1) with $\gamma = 1/4$.

Example 5.1. If (A0) – (A5) are satisfied, then (F6) and (F7) are satisfied for the Bellman/Isaacs equation (1.4) when F_κ is defined as in Example 4.7. We will show this in the next section.

Theorem 5.2. *Assume that for every $\kappa \in (0,1)$ there exists F_κ satisfying (F0) – (F7), that $u, -v \in USC_p(\overline{\Omega})$ are bounded from above, and that for every $z \in \partial\Omega, x \in \Omega$,*

$$u(z) \geq u(x) - \omega_0(|x-z|) \quad \text{and} \quad v(z) \leq v(x) + \omega_0(|x-z|), \tag{5.2}$$

where ω_0 is a modulus.

If u and v are respectively sub- and supersolutions of (1.1)*, and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.*

Remark 5.3. The above result implies uniqueness of bounded viscosity solutions on a possibly unbounded domain Ω. The "ω_0 condition" means that the semicontinuous viscosity sub- and supersolutions u and v are uniformly semicontinuous up to the boundary. Any viscosity solution satisfying this condition attains its boundary values uniformly continuously.

Proof. Define $\Psi(x,y) = u(x) - v(y) - \phi(x,y)$, where ϕ is defined in (F7). By standard arguments there is a point $(\bar{x},\bar{y}) \in \overline{\Omega} \times \overline{\Omega}$ (depending on δ and ε) such that Ψ attains its supremum over $\Omega \times \Omega$ here. Define $\sigma := \sup_{\Omega\times\Omega} \Psi = \Psi(\bar{x},\bar{y})$. Note that for any $x \in \Omega$, $(u-v)(x) - 2\varepsilon|x|^2 \leq \sigma$. So, obviously we are done if we can prove that $\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\lim_{\kappa\to 0}\sigma \leq 0$.

To prove this, we will first derive a positive upper bound for σ. We may assume $\sigma > 0$ since otherwise any positive upper bound trivially holds. Since u and $-v$ are bounded from above, we have the following bounds

$$\varepsilon(|\bar{x}|^2 + |\bar{y}|^2) \leq \omega_1(\varepsilon) \quad \text{and} \quad \frac{1}{\delta}|\bar{x}-\bar{y}|^2 \leq \omega_2(\delta), \tag{5.3}$$

where $\omega_i, i = 1,2$, are moduli not depending on any of the parameters κ, ε, δ. These are standard results, see, e.g., [13, Lemma 3.1] for the proofs. Now, either (i) $(\bar{x},\bar{y}) \in \partial(\Omega\times\Omega)$, or (ii) $(\bar{x},\bar{y}) \in \Omega\times\Omega$. In case (i), (5.2) and $u \leq v$ on $\partial\Omega$ implies that $u(\bar{x}) - v(\bar{y}) \leq \omega_0(|\bar{x}-\bar{y}|)$, and hence

$$\sigma \leq \omega_0(|\bar{x}-\bar{y}|) - \frac{1}{\delta}|\bar{x}-\bar{y}|^2 - \varepsilon(|\bar{x}|^2+|\bar{y}|^2) \leq \omega_0(|\bar{x}-\bar{y}|) =: I.$$

In case (ii), we apply Theorem 4.9 to find matrices $X, Y \in \mathbb{S}^N$, satisfying (4.1), such that (4.2) and (4.3) hold. Since $\sigma > 0$ implies that $u(\bar{x}) \leq v(\bar{y})$, subtracting the above inequalities and using (F6) and (F7) yield

$$\begin{aligned}\lambda(u(\bar{x}) - v(\bar{y})) &\leq F_\kappa(\bar{y}, v(\bar{y}), -D_y\phi(\bar{x},\bar{y}), Y, v_\varepsilon(\cdot), -\phi(\bar{x},\cdot)) \\ &\qquad - F_\kappa(\bar{x}, v(\bar{y}), D_x\phi(\bar{x},\bar{y}), X, u^\varepsilon(\cdot), \phi(\cdot,\bar{y})) \\ &\leq \omega\left((|\bar{x}-\bar{y}| + \frac{1}{\delta}|\bar{x}-\bar{y}|^2) + \varepsilon(1+|\bar{x}|^2+|\bar{y}|^2)\right) + m_{\kappa,\delta,\varepsilon} =: II.\end{aligned}$$

So we have $\sigma \leq u(\bar{x}) - v(\bar{y}) \leq II/\lambda$. To complete the proof, we combine cases (i) and (ii) to obtain the following upper bound on σ,

$$\sigma \leq \max(I, II/\lambda),$$

where by (5.3) I and II only depends on $\kappa, \varepsilon, \delta$, and

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\lim_{\kappa\to 0}\sigma \leq 0.$$

□

Remark 5.4. The case when the solutions have linear growth at infinity seems to be more difficult, and we do not know the optimal conditions for having a comparison result in this case. However, there are two special cases where we may have a comparison result:

- If F_κ, F are uniformly continuous in all variables, then a comparison result can be obtained by adapting the techniques of Ishii in [22, Theorem 7.1].
- If λ is sufficiently large, then we have a comparison result due to "cancellation effects" in the proof.

If λ is big enough, then we can handle arbitrary polynomial growth in the solutions by slightly changing assumption (F7). For parabolic problems we are always in this situation since we can have λ arbitrary large after an "exponential-in-time" scaling of the solution.

6. Proof of comparison for Bellman/Isaacs equation, Theorem 3.1

As an application of the general results presented in the previous sections, we prove in this section a comparison result for semicontinuous sub- and supersolutions of the elliptic Bellman/Isaacs equation (Theorem 3.1).

In view of Examples 4.1 and 4.7 and the abstract comparison result in the previous section (Theorem 5.2), Theorem 3.1 follows if we can verify that (F0) – (F7) hold for the functions F_κ defined in Example 4.7. The only difficult part is to show that (F7) holds, so we restrict our discussion to this condition. In the pure PDE case this is proved by Ishii [22]. Although not stated as such, in the integro-PDE case this is essentially proved in [30] for uniformly continuous u, v (see also [34, 8, 10]). To give the reader some ideas how this is done (for semicontinuous u, v), we consider briefly the integro operator of the Bellman/Isaacs equation (1.4). According to Example 4.7, it can be decomposed into $B_\kappa^{\alpha,\beta}$ and $B^{\alpha,\beta,\kappa}$, and thanks to (A0) and (A4), the $B_\kappa^{\alpha,\beta}$ term goes to zero as $\kappa \to 0$. Let us now consider the other term. For (F7) to be satisfied, it is necessary that

$$\begin{aligned}(6.1)\qquad &-B^{\alpha,\beta,\kappa}\left(\bar{y}, \frac{1}{\delta}(\bar{x}-\bar{y}) - \varepsilon\bar{y}, v(\cdot)\right) + B^{\alpha,\beta,\kappa}\left(\bar{x}, \frac{1}{\delta}(\bar{x}-\bar{y}) + \varepsilon\bar{x}, u(\cdot)\right)\\ &\qquad \leq \omega\left(|\bar{x}-\bar{y}| + \frac{1}{\delta}|\bar{x}-\bar{y}|^2 + \varepsilon(1+|\bar{x}|^2+|\bar{y}|^2)\right) + m_{\kappa,\delta,\varepsilon}.\end{aligned}$$

Let us write $B^{\alpha,\beta,\kappa} = B_1^{\alpha,\beta,\kappa} + B_2^{\alpha,\beta}$, where $B_1^{\alpha,\beta,\kappa}$ is the part where z is integrated over the set $\kappa \leq |z| \leq 1$ and $B_2^{\alpha,\beta}$ is the part where z is integrated over the set $|z| \geq 1$. The part of (6.1) corresponding to $B_1^{\alpha,\beta,\kappa}$ can be handled as follows. If we let $\psi(x,y) = u(x) - v(y) - \phi(x,y)$, then a simple calculation shows that the integrand of this part equals

$$\begin{aligned}&\psi(\bar{x}+\eta^{\alpha,\beta}(\bar{x},z), \bar{y}+\eta^{\alpha,\beta}(\bar{y},z)) - \psi(\bar{x},\bar{y})\\ &\qquad + \frac{1}{\delta}|\eta^{\alpha,\beta}(\bar{x},z) - \eta^{\alpha,\beta}(\bar{y},z)|^2 + \varepsilon(|\eta^{\alpha,\beta}(\bar{x},z)|^2 + |\eta^{\alpha,\beta}(\bar{y},z)|^2).\end{aligned}$$

Since ψ has a global maximum at $(\bar{x},\bar{y})$ the two first terms are non-positive, so by (A0), (A3), and (A4), we get

$$-B_1^{\alpha,\beta,\kappa}\left(\bar{y}, \frac{1}{\delta}(\bar{x}-\bar{y}) - \varepsilon\bar{y}, v(\cdot)\right) + B_1^{\alpha,\beta,\kappa}\left(\bar{x}, \frac{1}{\delta}(\bar{x}-\bar{y}) + \varepsilon\bar{x}, u(\cdot)\right)$$

$$\leq C\left(\frac{1}{\delta}|\bar{x}-\bar{y}|^2+\varepsilon(|\bar{x}|^2+|\bar{y}|^2)\right),$$

for some constant C. To handle the part of (6.1) corresponding to $B_2^{\alpha,\beta,\kappa}$, we introduce

$$M_{\delta,\varepsilon} := \sup_{x,y\in\mathbb{R}} \psi(x,y) \qquad \text{and} \qquad M := \sup_{x\in\mathbb{R}}(u-v),$$

and remark that $\lim_{\varepsilon\to 0}\lim_{\delta\to 0} M_{\delta,\varepsilon} = M$. Then it follows that

$$\begin{aligned}
&-B_2^{\alpha,\beta,\kappa}\left(\bar{y},\frac{1}{\delta}(\bar{x}-\bar{y})-\varepsilon\bar{y},v(\cdot)\right)+B_2^{\alpha,\beta,\kappa}\left(\bar{x},\frac{1}{\delta}(\bar{x}-\bar{y})+\varepsilon\bar{x},u(\cdot)\right)\\
&=\int_{|z|\geq 1}\left[\left(u(\bar{x}+\eta^{\alpha,\beta}(\bar{x},z))-u(\bar{x})\right)-\left(v(\bar{y}+\eta^{\alpha,\beta}(\bar{y},z))-v(\bar{y})\right)\right]m(dz)\\
&=\int_{|z|\geq 1}\Big[\underbrace{u(\bar{x}+\eta^{\alpha,\beta}(\bar{x},z))-v(\bar{y}+\eta^{\alpha,\beta}(\bar{y},z))}_{g(\bar{x},\bar{y},z)}-M_{\delta,\varepsilon}+\phi(\bar{x},\bar{y})\Big]\,m(dz).
\end{aligned}$$

The last equality follows from the definition of $M_{\delta,\varepsilon}$ since $(\bar{x},\bar{y})$ is a maximum point of ψ by assumption. As we have seen before (see (5.3)), $|\bar{x}-\bar{y}|\to 0$ as $\delta\to 0$ and $|\bar{x}|,|\bar{y}|$ are bounded as long as $\varepsilon>0$ is kept fixed, and since $g(\bar{x},\bar{y},z)$ is upper semicontinuous in x and y this leads to

$$\limsup_{\delta\to 0} g(\bar{x},\bar{y},z)\leq M.$$

An other application of (5.3) shows that $\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\phi(\bar{x},\bar{y})=0$. So sending first $\delta\to 0$ (taking limit superior) and then $\varepsilon\to 0$, we see that the above integrand and hence also the integral (by Lebesgue dominated convergence theorem), is upper bounded by 0. We can conclude that there is an upper bound $m_{\delta,\varepsilon}$ of the difference in the $B_2^{\alpha,\beta,\kappa}$ terms such that $\lim_{\varepsilon\to 0}\lim_{\delta\to 0} m_{\delta,\varepsilon}=0$, and the proof of (6.1) is complete.

Remark 6.1. The trick of dividing $B^{\alpha,\beta,\kappa}$ into two terms [10] allows one to consider more general Lévy measures than in [30]. In [30] it is required that

$$\int_{\mathbb{R}^M\setminus\{0\}}\rho(z)^2\,m(dz)$$

is finite, while we assume the weaker condition (A0).

7. Proof of the main result, Theorem 4.9

The outline of the proof of Theorem 4.9 is as follows. First we regularize the sub- and supersolutions using the ε-sup and ε-inf convolutions, thereby yielding approximate sub- and supersolutions of the original equations that are twice differentiable a.e. Using the classical maximum principle, we derive for these approximate sub- and supersolutions an analogous result to Theorem 4.9. In this result (Lemma 7.4) the lower bounds in the matrix inequality corresponding to (4.1) depends on the regularization parameter ε. A transformation of these matrices (Lemma 7.7) leads to new matrices satisfying (4.1) independently of ε. Furthermore, the viscosity solution inequalities for the approximate sub- and supersolutions are still satisfied with these new matrices. We can then go to the limit along a subsequence of $\varepsilon\to 0$ and obtain Theorem 4.9.

The first part of this approach corresponds to the original approach of Jensen [27], giving the first general uniqueness results for viscosity solutions of second order PDEs. Actually, we follow the more refined approach of Ishii [22] and Ishii and Lions [23]. In the second part, the key ingredient is a matrix lemma of Crandall [11]. We remark that our approach deviates from the by now standard approach based on the maximum principle for semicontinuous functions [12, 13]. As explained in Section 2, it appears that a "local" approach based on the maximum principle for semicontinuous functions is not straightforward to implement for the non-local equation (1.1).

We start by defining the sup and inf convolutions and stating some of their properties.

Definition 7.1. Let $f \in USC(\Omega)$ satisfy $f(x) \le C(1+|x|^2)$ in Ω and $0 < \varepsilon < (2C)^{-1}$. The sup convolution f^ε is defined as

$$f^\varepsilon(x) = \sup_{y\in\Omega} \left(f(y) - \frac{|x-y|^2}{\varepsilon} \right).$$

Let $f \in LSC(\Omega)$ satisfy $f(x) \ge -C(1+|x|^2)$ in Ω and $0 < \varepsilon < (2C)^{-1}$. The inf convolution f_ε is defined as

$$f_\varepsilon(x) = \inf_{y\in\Omega} \left(f(y) + \frac{|x-y|^2}{\varepsilon} \right).$$

Lemma 7.2. *Let $f \in USC(\Omega)$ satisfy $f(x) \le C(1+|x|^2)$ in Ω and $0 < \varepsilon < (2C)^{-1}$.*

(i) $f^\varepsilon(x) \le 2C(1+|x|^2)$ and $f^\varepsilon(x) + \frac{1}{\varepsilon}|x|^2$ is convex and locally Lipschitz in Ω.

(ii) $f \le f^\varepsilon \le f^{\bar\varepsilon}$ for $0 < \varepsilon \le \bar\varepsilon < (2C)^{-1}$ and $f^\varepsilon \to f$ pointwise as $\varepsilon \to 0$.

(iii) Let $\varepsilon < (4C)^{-1}$ and define $C_f(x) := \big(4C(1+|x|^2) - 2f(x)\big)^{1/2}$. If $x \in \Omega$ is such that $\operatorname{dist}(x,\partial\Omega) > \varepsilon^{1/2}C_f(x)$, then there exists $\bar x \in \Omega$ such that $|x-\bar x| \le \varepsilon^{1/2}C(x)$ and

$$f^\varepsilon(x) = f(\bar x) - \frac{1}{\varepsilon}|x-\bar x|^2.$$

Since $f_\varepsilon = -(-f)^\varepsilon$, we immediately get the corresponding properties for the inf-convolution. We refer to [4, 15] for proofs of results like those in Lemma 7.2. Now if f is a function satisfying the assumptions of Lemma 7.2, we define

$$\Omega_\varepsilon^f = \left\{ x \in \Omega \ : \ \operatorname{dist}(x,\partial\Omega) > \varepsilon^{1/2}C_f(x) \right\},$$

where $C_f(x)$ is defined in Lemma 7.2. Moreover, let τ_h denote the shift operator defined by

$$(\tau_h\phi)(x) = \phi(x+h)$$

for any function ϕ and $x, x+h$ in the domain of definition of ϕ.

Lemma 7.3. *Assume that (C3) holds, $u, -v \in USC_p(\Omega)$ satisfy $u(x), -v(x) \le C(1+|x|^2)$, and $0 < \varepsilon < (4C)^{-1}$.*

(a) If u is a viscosity subsolution of (1.1), then u^ε solves

$$F_\varepsilon(x, u^\varepsilon(x), Du^\varepsilon(x), D^2u^\varepsilon(x), u^\varepsilon(\cdot)) \le 0 \quad in \quad \Omega_\varepsilon^u,$$

in the viscosity solution sense, where

$$F_\varepsilon(x,r,p,X,\phi(\cdot)) = \inf_{|x-y|\le C_u(x)\varepsilon^{1/2}} F(y,r,p,X,\tau_{x-y}\phi(\cdot)).$$

(b) If v is a viscosity supersolution of (1.1)*, then v_ε solves*

$$F^\varepsilon(x, v_\varepsilon(x), Dv_\varepsilon(x), D^2 v_\varepsilon(x), v_\varepsilon(\cdot)) \geq 0 \quad in \quad \Omega_\varepsilon^{-v},$$

in the viscosity solution sense, where

$$F^\varepsilon(x, r, p, X, \phi(\cdot)) = \sup_{|x-y| \leq C_{-v}(x)\varepsilon^{1/2}} F(y, r, p, X, \tau_{x-y}\phi(\cdot)).$$

Proof. We only prove (a), the proof of (b) is similar. Let $\phi \in C_p^2(\Omega)$ and $\bar{x} \in \Omega_\varepsilon^u$ be such that $u^\varepsilon - \phi$ has a global maximum at $\bar{x}$. According to Lemma 7.2 (iii) there is a $\bar{y} \in \Omega$ such that $|\bar{x} - \bar{y}| \leq C_u(\bar{x})\varepsilon^{1/2}$ and $u^\varepsilon(\bar{x}) = u(\bar{y}) - \frac{1}{\varepsilon}|\bar{x} - \bar{y}|^2$. Now it is not so difficult to see that $y \mapsto (u - \tau_{\bar{x}-\bar{y}}\phi)(y)$ has a global maximum at $\bar{y}$ (cf. [22, Proof of Proposition 4.2]). Since u is a viscosity subsolution of (1.1),

$$F(\bar{y}, u(\bar{y}), D(\tau_{\bar{x}-\bar{y}}\phi)(\bar{y}), D^2(\tau_{\bar{x}-\bar{y}}\phi)(\bar{y}), (\tau_{\bar{x}-\bar{y}}\phi)(\cdot)) \leq 0.$$

Since $|\bar{x} - \bar{y}| \leq C_u(\bar{x})\varepsilon^{1/2}$, $D^n(\tau_{\bar{x}-\bar{y}}\phi)(\bar{y}) = D^n\phi(\bar{x})$ for $n = 1, 2$, and $u^\varepsilon(\bar{x}) \leq u(\bar{y})$, it follows using (C3) and the above inequality that

$$F_\varepsilon(\bar{x}, u^\varepsilon(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x}), \phi(\cdot)) \leq 0,$$

and the proof is complete □

Now we have come to one of the main technical results in this paper. It is a version for integro-PDEs of Proposition 5.1 in Ishii [22] (see also Proposition II.3 in Ishii and Lions [23]).

Lemma 7.4. *Let $u, -v \in USC_p(\Omega)$ satisfy $u(x), -v(x) \leq C(1 + |x|^2)$, and solve in the viscosity solution sense*

$$F(x, u(x), Du(x), D^2u(x), u(\cdot)) \leq 0 \quad and \quad G(x, v(x), Dv(x), D^2v(x), v(\cdot)) \geq 0,$$

where F, G satisfy (C1) – (C4). For $0 < \varepsilon < (4C)^{-1}$, let $\phi \in C_p^2(\Omega \times \Omega)$ and $(\bar{x}, \bar{y}) \in \Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$ be such that

$$(x, y) \mapsto u^\varepsilon(x) - v_\varepsilon(y) - \phi(x, y)$$

has a global maximum over $\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$ at $(\bar{x}, \bar{y})$. Then there exist two matrices $X, Y \in \mathbb{S}^N$ satisfying

$$-\frac{2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\phi(\bar{x}, \bar{y}), \tag{7.1}$$

such that

$$F_\varepsilon(\bar{x}, u^\varepsilon(\bar{x}), D_x\phi(\bar{x}, \bar{y}), X, \phi(\cdot, \bar{y})) \leq 0, \tag{7.2}$$

$$G^\varepsilon(\bar{y}, v_\varepsilon(\bar{y}), -D_y\phi(\bar{x}, \bar{y}), Y, -\phi(\bar{x}, \cdot)) \geq 0. \tag{7.3}$$

Remark 7.5. Compared with Ishii [22, Proposition 5.1], the main feature of Lemma 7.4 is the inclusion of the penalization function $\phi(\cdot, \cdot)$ in the non-local slots in (7.2) and (7.3).

Remark 7.6. The condition $u(x), -v(x) \leq C(1 + |x|^2)$ in Lemma 7.4 is necessary for u^ε and v_ε to be well-defined according to Definition 7.1.

Proof. 1. Let $w(x, y) = u^\varepsilon(x) - v_\varepsilon(y)$. By Lemma 7.2 w is locally Lipschitz continuous and semi-convex in $\Omega \times \Omega$. By Alexandroff's theorem, w is twice differentiable a.e. in $\Omega \times \Omega$ (cf. [13, 15]).

2. By the assumptions, $w-\phi$ has a global maximum over $\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$ at $(\bar{x},\bar{y})$. By (C4), we may assume that $w=\phi$ at $(\bar{x},\bar{y})$ by adding a constant to ϕ if necessary. Furthermore, $(x,y)\mapsto w(x,y)-\phi(x,y)-\delta|(x,y)-(\bar{x},\bar{y})|^4$ has a strict maximum over $\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$ at $(\bar{x},\bar{y})$ for every $\delta>0$, and this maximum takes the value 0.

3. The crucial step in this proof is the application of Jensen's lemma, see Lemmas 3.10 and 3.15 in Jensen [27] or Lemma 5.3 in Ishii [22]. Pick a $r>0$ such that $B((\bar{x},\bar{y}),r)\subset \Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$, then by 1 and 2 we may apply Jensen's lemma to $w-\phi-\delta|(x,y)-(\bar{x},\bar{y})|^4$ on $B((\bar{x},\bar{y}),r)$. By this Lemma there are sequences $\{(x_k,y_k)\}_k\subset B((\bar{x},\bar{y}),r)$ and $\{(p_k,q_k)\}_k\subset \mathbb{R}^N\times\mathbb{R}^N$ such that (i) $(x_k,y_k)\to(\bar{x},\bar{y})$ and $(p_k,q_k)\to(0,0)$ as $k\to\infty$, (ii) w is twice differentiable at (x_k,y_k), and (iii) the function

$$(x,y)\mapsto w(x,y)-\phi(x,y)-\delta|(x,y)-(\bar{x},\bar{y})|^4-(p_k,q_k)\cdot(x,y)$$

attains its maximum over $B((\bar{x},\bar{y}),r)$ at (x_k,y_k). Note that for notational reasons, we suppress the dependence on δ in x_k,y_k,p_k,q_k.

4. Now, let $\bar{\phi}_{k,\delta}(x,y)=\phi(x,y)+\delta|(x,y)-(\bar{x},\bar{y})|^4+(p_k,q_k)\cdot(x,y)+C_{k,\delta}$ for some constant $C_{k,\delta}$, and note that by 3, $w-\bar{\phi}_{k,\delta}$ attains its maximum over $B((\bar{x},\bar{y}),r)$ at (x_k,y_k). Hence the differentiability of w implies that at (x_k,y_k), $Dw=D\bar{\phi}_{k,\delta}$ and $D^2w\le D^2\bar{\phi}_{k,\delta}$. Finally, choose $C_{k,\delta}$ such that $(w-\bar{\phi}_{k,\delta})(x_k,y_k)=0$.

5. Pick a non-negative function $\theta\in C^\infty(\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v})$ which is 1 in $B((\bar{x},\bar{y}),r/2)$ and 0 outside of $B((\bar{x},\bar{y}),r)$. Now define

$$\phi_{k,\delta}=\theta\,\bar{\phi}_{k,\delta}+(1-\theta)\,\phi.$$

Obviously $\phi_{k,\delta}\in C_p^2(\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v})$, and we claim that $w-\phi_{k,\delta}$ has a global maximum at (x_k,y_k). This follows since by 2, $w\le\phi$ in $\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$, and by 4, $w\le\bar{\phi}_{k,\delta}$ in $B((\bar{x},\bar{y}),r)$ and $w=\bar{\phi}_{k,\delta}$ at (x_k,y_k).

6. There exists a function $\psi_{k,\delta}\in C_p^2(\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v})$ such that $D^n\psi_{k,\delta}(x_k,y_k)=D^nw(x_k,y_k)$ for $n=0,1,2$ and $w\le\psi_{k,\delta}\le\phi_{k,\delta}$ in $\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$. In particular, $\psi_{k,\delta}-\phi_{k,\delta}$ (also) attains its maximum over $\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$ at (x_k,y_k).

To prove the above claim we consider separately the following cases: (i) $D^2w=D^2\phi_{k,\delta}$ at (x_k,y_k) and (ii) $D^2w<D^2\phi_{k,\delta}$ at (x_k,y_k) (note that trivially $D^2w\le D^2\phi_{k,\delta}$ at (x_k,y_k)). In case (i) we simply set $\psi_{k,\delta}=\phi_{k,\delta}$. In case (ii) we pick a $\bar{\phi}\in C^2(\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v})$ such that $D^n\bar{\phi}=D^nw$ at (x_k,y_k) for $n=0,1,2$, and $w-\bar{\phi}\le 0$ in $\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v}$. This can be done by a construction of Evans, see e.g. [15, Proposition V.4.1]. It follows that at (x_k,y_k), $\bar{\phi}=\phi_{k,\delta}$, $D\bar{\phi}=D\phi_{k,\delta}$, and $D^2\bar{\phi}<D^2\phi_{k,\delta}$. This means that we can find a $\bar{\delta}>0$ such that $\bar{\phi}<\phi_{k,\delta}$ in the ball $B((x_k,y_k),\bar{\delta})$. Now we define $\psi_{k,\delta}$ in the following way:

$$\psi_{k,\delta}=\theta\,\bar{\phi}+(1-\theta)\,\phi_{k,\delta},$$

where $\theta\in C^\infty(\Omega_\varepsilon^u \times \Omega_\varepsilon^{-v})$ is non-negative, 1 in $B((x_k,y_k),\bar{\delta}/2)$, and 0 outside of $B((x_k,y_k),\bar{\delta})$. This function θ is not to be confused with the θ in 5.

7. By 6, $(w-\psi_{k,\delta})(x,y_k)$ has a maximum over Ω_ε^u at x_k, and $(w-\psi_{k,\delta})(x_k,y)$ has a minimum over Ω_ε^{-v} at y_k. Hence Lemma 7.3 yields

$$F_\varepsilon(x_k,u^\varepsilon(x_k),D_x\psi_{k,\delta}(x_k,y_k),D_x^2\psi_{k,\delta}(x_k,y_k),\psi_{k,\delta}(\cdot,y_k))\le 0,$$
$$G^\varepsilon(y_k,v_\varepsilon(y_k),-D_y\psi_{k,\delta}(x_k,y_k),-D_y^2\psi_{k,\delta}(x_k,y_k),-\psi_{k,\delta}(x_k,\cdot))\ge 0.$$

By the properties of $\psi_{k,\delta}$, (C2), and since $\psi_{k,\delta}-\phi_{k,\delta}$ has its global maximum at (x_k,y_k), we get

$$F_\varepsilon(x_k,u^\varepsilon(x_k),Du^\varepsilon(x_k),X_{k,\delta},\phi_{k,\delta}(\cdot,y_k))\le 0,$$

$$G^\varepsilon(y_k, v_\varepsilon(y_k), Dv_\varepsilon(y_k), Y_{k,\delta}, -\phi_{k,\delta}(x_k, \cdot)) \geq 0,$$

where $X_{k,\delta} = D^2u^\varepsilon(x_k)$ and $Y_{k,\delta} = D^2v_\varepsilon(y_k)$.

8. Since $w - \phi_{k,\delta}$ has a maximum at (x_k, y_k) and by the semi-convexity of w, see Lemma 7.2 (i), the following inequality holds

$$-\frac{2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_{k,\delta} & 0 \\ 0 & -Y_{k,\delta} \end{pmatrix} \leq D^2\phi_{k,\delta}(x_k, y_k).$$

Furthermore, if $0 < \delta < 1$ and ε fixed then this inequality implies that $-CI \leq X_{k,\delta}, Y_{k,\delta} \leq CI$ for some constant $C > 0$. The set of such matrices is compact by Lemma 5.3 in Ishii [22], so we may pick a convergent subsequence, also denoted by $\{X_{k,\delta}, Y_{k,\delta}\}_k$, converging to some $X_\delta, Y_\delta \in \mathbb{S}^N$. By the above inequality and since $D^2\phi_{k,\delta}(x_k, y_k) \to D^2\phi(\bar{x}, \bar{y})$ as $k \to \infty$, we see that the limits X_δ, Y_δ satisfy (7.1).

9. The next step of the proof is to send $k \to \infty$ (along the subsequence in 8) in the inequalities at the end of 7, and conclude by continuity of all arguments and (C1) that

$$F_\varepsilon(\bar{x}, u^\varepsilon(\bar{x}), D_x\phi(\bar{x}, \bar{y}), X_\delta, \phi(\cdot, \bar{y}) + \delta\theta(\cdot, \bar{y})|(\cdot, \bar{y}) - (\bar{x}, \bar{y})|^4) \leq 0,$$
$$G^\varepsilon(\bar{y}, v_\varepsilon(\bar{y}), -D_y\phi(\bar{x}, \bar{y}), Y_\delta, -\phi(\bar{x}, \cdot) - \delta\theta(\bar{x}, \cdot)|(\bar{x}, \cdot) - (\bar{x}, \bar{y})|^4) \geq 0.$$

Let us verify the assumptions of (C1). First note that $|\phi_{k,\delta}(x, y)| \leq C(1+|x|^p+|y|^p)$ with C independent of δ, k. This bound follows from the definition of $\phi_{k,\delta}$ (see 4 and 5) since $\phi \in C_p^2$. Then we claim that $D^n\phi_{k,\delta}(\cdot, \bar{y}) \to D^n(\phi(\cdot, \bar{y}) + \delta|(\cdot, \bar{y}) - (\bar{x}, \bar{y})|^4)$ locally uniformly for $n = 0, 1, 2$ as $k \to \infty$ (and similarly for $\phi_{k,\delta}(\bar{x}, \cdot)$). By its definition (see 5) it is enough to check this for $\bar{\phi}_{k,\delta}$ in $B((\bar{x}, \bar{y}), r)$. But by the definition of $\bar{\phi}_{k,\delta}$ (see 4) this easily follows since by 6, $p_k, q_k \to 0$, and by 2, $C_{k,\delta} = \sup(w - \bar{\phi}_{k,\delta}) \to \sup(w - \phi - \delta|(x, y) - (\bar{x}, \bar{y})|^4) = 0$.

10. The final step is to send $\delta \to 0$. Because X_δ, Y_δ satisfy (7.1), we have compactness as in 8, so we pick a subsequence $\delta \to 0$ such that the matrices converge to some $X, Y \in \mathbb{S}^N$. Of course X, Y still satisfy (7.1). Furthermore, by continuity of all arguments and (C1) we conclude that the inequalities in 9 become (7.2) and (7.3) as $\delta \to 0$ along this subsequence. □

The next result is a matrix lemma due to Crandall [11].

Lemma 7.7. *Let $X, Y \in \mathbb{S}^N$ satisfy*

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{7.4}$$

Then for $\gamma \in (0, \frac{1}{2})$, $(I - \gamma X)$ and $(I + \gamma Y)$ are invertible, and if

$$X^\gamma = X(I - \gamma X)^{-1} \quad \text{and} \quad Y_\gamma = Y(I + \gamma Y)^{-1}$$

then

$$X \leq X^\gamma \leq Y_\gamma \leq Y \tag{7.5}$$

and

$$-\frac{1}{\gamma}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X^\gamma & 0 \\ 0 & -Y_\gamma \end{pmatrix} \leq \frac{1}{1 - 2\gamma}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{7.6}$$

Using Lemmas 7.4 and 7.7, we now prove a version of Theorem 4.9 where the F/G-formulation is used instead of the F_κ/G_κ-formulation. Theorem 4.9 is an easy consequence of this result (see below).

Lemma 7.8. *Let $u, -v \in USC_p(\Omega)$ satisfy $u(x), -v(x) \leq C(1+|x|^2)$ and solve in the viscosity solution sense*

$$F(x,u,Du,D^2u,u(\cdot)) \leq 0 \quad \text{and} \quad G(x,v,Dv,D^2v,v(\cdot)) \geq 0,$$

where F, G satisfies (C1) – (C4). Let $\phi \in C^2_p(\Omega \times \Omega)$ and $(\bar{x}, \bar{y}) \in \Omega \times \Omega$ be such that

$$(x,y) \mapsto u(x) - v(y) - \phi(x,y)$$

has a global strict maximum at $(\bar{x}, \bar{y})$. Furthermore, assume that in a neighborhood of $(\bar{x}, \bar{y})$ there are continuous functions $g_0 : \mathbb{R}^{2N} \to \mathbb{R}, g_1, g_2 : \mathbb{R}^N \to \mathbb{S}^N$ with $g_0(\bar{x}, \bar{y}) > 0$, satisfying

$$D^2\phi \leq g_0(x,y) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} g_1(x) & 0 \\ 0 & g_2(y) \end{pmatrix}.$$

Then for each $\gamma \in (0, \frac{1}{2})$ there exist matrices $X, Y \in \mathbb{S}^N$ satisfying

$$-\frac{g_0(\bar{x},\bar{y})}{\gamma} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} - \begin{pmatrix} g_1(\bar{x}) & 0 \\ 0 & g_2(\bar{y}) \end{pmatrix} \leq \frac{g_0(\bar{x},\bar{y})}{1-2\gamma} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{7.7}$$

such that

$$F(\bar{x}, u(\bar{x}), D_x\phi(\bar{x},\bar{y}), X, \phi(\cdot,\bar{y})) \leq 0, \tag{7.8}$$

$$G(\bar{x}, v(\bar{y}), -D_y\phi(\bar{x},\bar{y}), Y, -\phi(\bar{x},\cdot)) \geq 0. \tag{7.9}$$

Remark 7.9. Compared with Crandall [11, Theorem 1], the main feature in Lemma 7.8 is the inclusion of the inequalities (7.8) and (7.9). In the pure PDE case, under certain (semi)continuity assumptions on the equation (1.1), the corresponding inequalities come for free. We refer to Section 2 for a discussion of this point.

Proof. For all sufficiently small $\varepsilon > 0$, $(x,y) \mapsto u^\varepsilon(x) - v_\varepsilon(y) - \phi(x,y)$ has a global maximum at some point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$, and as $\varepsilon \to 0$, $(x_\varepsilon, y_\varepsilon) \to (\bar{x}, \bar{y})$, $u^\varepsilon(x_\varepsilon) \to u(\bar{x})$, $v_\varepsilon(y_\varepsilon) \to v(\bar{y})$. Moreover, we may find a $\varepsilon' > 0$ and a $r > 0$ such that for all $\varepsilon < \varepsilon'$, (i) $(x_\varepsilon, y_\varepsilon) \in B((\bar{x},\bar{y}), r)$, (ii) $B((\bar{x},\bar{y}), r) \subset \Omega^u_\varepsilon \times \Omega^{-v}_\varepsilon$, and (iii) $g_0 > 0$ in $B((\bar{x},\bar{y}), r)$.

By Lemma 7.4 there exist two matrices $X, Y \in \mathbb{S}^N$ satisfying

$$-\frac{2}{\varepsilon} I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\phi(x_\varepsilon, y_\varepsilon),$$

and furthermore

$$F_\varepsilon(x_\varepsilon, u^\varepsilon(x_\varepsilon), D_x\phi(x_\varepsilon,y_\varepsilon), X, \phi(\cdot, y_\varepsilon)) \leq 0,$$
$$G_\varepsilon(y_\varepsilon, v_\varepsilon(y_\varepsilon), -D_y\phi(x_\varepsilon,y_\varepsilon), Y, -\phi(x_\varepsilon, \cdot)) \geq 0.$$

By the assumptions, we may rewrite the left hand side of the above matrix inequality as follows,

$$\begin{pmatrix} \tilde{X} & 0 \\ 0 & -\tilde{Y} \end{pmatrix} \leq \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where

$$\tilde{X} = \frac{1}{g_0(x_\varepsilon, y_\varepsilon)} (X - g_1(x_\varepsilon)) \qquad \text{and} \qquad \tilde{Y} = \frac{1}{g_0(x_\varepsilon, y_\varepsilon)} (Y + g_2(y_\varepsilon)).$$

These two matrices satisfies the assumptions of Lemma 7.7, so we can conclude that inequalities corresponding to (7.5) and (7.6) hold. Now define

$$\bar{X} = g_0(x_\varepsilon, y_\varepsilon)\tilde{X}^\gamma + g_1(x_\varepsilon) \qquad \text{and} \qquad \bar{Y} = g_0(x_\varepsilon, y_\varepsilon)\tilde{Y}_\gamma - g_2(y_\varepsilon).$$

The conclusions of Lemma 7.7 can then be written as follows,

$$X \leq \bar{X}, \qquad \bar{Y} \leq Y,$$

and

(7.10)

$$-\frac{g_0(x_\varepsilon, y_\varepsilon)}{\gamma}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} \bar{X} & 0 \\ 0 & -\bar{Y} \end{pmatrix} - \begin{pmatrix} g_1(x_\varepsilon) & 0 \\ 0 & g_2(y_\varepsilon) \end{pmatrix} \leq \frac{g_0(x_\varepsilon, y_\varepsilon)}{1-2\gamma}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

By degenerate ellipticity (C2),

$$F_\varepsilon(x_\varepsilon, u^\varepsilon(x_\varepsilon), D_x\phi(x_\varepsilon, y_\varepsilon), \bar{X}, \phi(\cdot, y_\varepsilon)) \leq 0, \tag{7.11}$$

$$G^\varepsilon(y_\varepsilon, v_\varepsilon(y_\varepsilon), -D_y\phi(x_\varepsilon, y_\varepsilon), \bar{Y}, -\phi(x_\varepsilon, \cdot)) \geq 0. \tag{7.12}$$

By continuity of g_0, g_1, g_2 on $B((\bar{x}, \bar{y}), r)$, we see from (7.10) that $\{\bar{X}\}_{\varepsilon>0}$ and $\{\bar{Y}\}_{\varepsilon>0}$ are compact in $\mathbb{S}(\mathbb{R}^N)$. Hence we may pick subsequences converging as $\varepsilon \to 0$ to limit matrices (still) called $\bar{X}$ and $\bar{Y}$. Moreover, sending $\varepsilon \to 0$ in (7.10) along such a subsequence gives the matrix inequality (7.7). Passing to the limit $\varepsilon \to 0$ along the same subsequence in (7.11) and (7.12) we obtain (7.8) and (7.9) using (C1) and continuity. The proof is complete. □

We are will now prove Theorem 4.9.

Proof of Theorem 4.9. After an application of Lemma 4.9, this proof is similar to the proof of Lemma 4.8. First note that we may assume that the maximum is strict and that the maximal value is 0. Then pick a sequence of $C_p^2(\Omega \times \Omega)$ functions $\{\phi_\varepsilon\}_{\varepsilon>0}$ such that $u(x) - v(y) \leq \phi_\varepsilon(x, y) \leq \phi(x, y)$ and $\phi_\varepsilon(x, y) \to u(x) - v(y)$ (pointwise) in $\Omega \times \Omega$. Note that $\phi_\varepsilon - \phi$ has a global maximum at $(\bar{x}, \bar{y})$, and hence $D(\phi_\varepsilon - \phi)(\bar{x}, \bar{y}) = 0$ and $D^2(\phi_\varepsilon - \phi)(\bar{x}, \bar{y}) \leq 0$. In particular, it follows that $(x, y) \mapsto u(x) - v(y) - \phi_\varepsilon(x, y)$ satisfies the assumptions in Lemma 7.8, so we have matrices $X, Y \in \mathbb{S}^N$ satisfying (7.7) (which equals (4.1)) and

$$F(\bar{x}, u(\bar{x}), D_x\phi_\varepsilon(\bar{x}, \bar{y}), X, \phi_\varepsilon(\cdot, \bar{y})) \leq 0,$$
$$G(\bar{y}, v(\bar{y}), -D_y\phi_\varepsilon(\bar{x}, \bar{y}), Y, -\phi_\varepsilon(\bar{x}, \cdot)) \geq 0.$$

Applying (F0) and then (F2) to the above inequalities for F and G yield

$$F_\kappa(\bar{x}, u(\bar{x}), D_x\phi(\bar{x}, \bar{y}), X, \phi_\varepsilon(\cdot, \bar{y}), \phi(\cdot, \bar{y})) \leq 0,$$
$$G_\kappa(\bar{y}, v(\bar{y}), -D_y\phi(\bar{x}, \bar{y}), Y, -\phi_\varepsilon(\bar{x}, \cdot), -\phi(\bar{x}, \cdot)) \geq 0.$$

Using (F5), (F4), and sending $\varepsilon \to 0$, yield (4.2) and (4.3). The proof is complete. □

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