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# ON THE RATE OF CONVERGENCE OF APPROXIMATION SCHEMES FOR BELLMAN EQUATIONS ASSOCIATED WITH OPTIMAL STOPPING TIME PROBLEMS.

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Abstract. We provide estimates on the rate of convergence for approximation schemes for Bellman equations associated with optimal stopping of controlled diffusion processes. These results extend (and slightly improve) resent results by Barles & Jakobsen to the more difficult time dependent case. The added difficulties are due to the presence of boundary conditions (initial conditions!) and the new structure of the equation which is now a parabolic variational inequality. The method presented is purely analytic and rather general and is based on earlier work by Krylov and Barles & Jakobsen. As applications we consider so-called control schemes based on the dynamic programming principle and finite difference methods (though not in the most general case). In the optimal stopping case these methods are similar to the Brennan & Schwartz scheme. A simple observation allow us to obtain the optimal rate 1/2 for the finite difference methods, and this is an improvement over previous results by Krylov and Barles & Jakobsen. Finally, we present an idea that allow us to improve all the above mentioned results in the linear case. In particular, we are able to handle finite difference methods with variable diffusion coefficients without the reduction of order of convergence observed by Krylov in the non-linear case.

## 1. Introduction

Optimal stopping time problems for controlled diffusion processes have been considered in great generality by using the dynamic programming principle approach and viscosity solution methods. The value-functions of such problems turn out to be the unique viscosity solution of the associated Bellman equations under natural conditions on the data. We refer to the book by Fleming and Soner [11] for optimal control problems and to the article by Pham [23] for optimal stopping time problems. For a detailed presentation of this notion of solution, see the User's guide [6].

In order to compute the value-function of such problems, many numerical schemes have been devised. In this paper we concentrate on so-called control schemes based on the dynamic programming principle and finite difference schemes. For the analysis of control schemes, we refer for instance to Capuzzo-Dolcetta [5], Falcone [8], Capuzzo-Dolcetta & Falcone [9], Menaldi [22], and Camilli & Falcone [4]. While finite difference methods have been considered by for instance Crandall & Lions [7], Souganidis [24], Kushner & Dupuis [18], and Krylov [16, 17]. The main focus of the above references are pure control problems. Numerical methods, and in particular finite difference methods, for optimal stopping time problems have been analyzed in for instance Glowinski, Lions & Trémolière [12], Wilmott, Dewynne & Howison [25] and Jaillet, Lamberton & Lapeyre [19]. We also mention that the convergence

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of these schemes can be obtained using the theory of Barles & Souganidis [2] stating roughly that any "stable, monotone, and consistent" scheme will converge.

In this paper we will consider the problem of finding error bounds for approximation schemes for Bellman equations. Krylov [16, 17] was the first to solve this problem. He developed a new method combining both analytic (PDE) arguments and probabilistic ones, thereby obtaining results for finite difference schemes. These results were then extended by Barles & Jakobsen [1] to a rather general class of monotone approximation schemes. This was done using a modified, purely analytical version of Krylov's method which to the author's opinion is much simpler. Included in this class is control schemes in the general case, and finite difference methods in the case of *constant diffusion coefficients*. This last restriction was not present in [17] at the cost of a reduced rate of convergence.

Using similar techniques this paper extends and improves Barles & Jakobsen [1] in the following ways:

- We treat the more difficult time-dependent case. The added difficulties are mainly due to (i) the presence boundary values (initial values), and (ii) the new structure of the problem which is now a parabolic variational inequality. The first difficulty is handled using techniques introduced by Krylov in [17]. The second one requires new estimates and to some extent new estimation techniques.
- A simple observation allow us to improve the results of [1] for finite difference schemes from 1/3 to get the optimal rate 1/2. The idea is to use a more refined consistency condition than [1], see condition (C4).
- We present an idea that allow us to get stronger results in the linear case.

Furthermore, it extends and improves Krylov [16, 17] in the following ways:

- We treat optimal stopping as well as optimal control in the time-dependent case. This leads to an obstacle problem with an associated variational inequality (the Bellman equation).
- We obtain for finite difference schemes the optimal rate 1/2, as opposed to 1/3 in [16] (constant coefficients), and 1/27 in [17] (variable coefficients). However, in general we need *constant diffusion coefficients* which was not the case in [17].
- In the linear case we have stronger results, which includes the case of variable diffusion coefficients. The rate obtained here is 1/2.
- We treat a larger class of monotone schemes (as in [1]) which includes control schemes.
- Our method (as in [1]) is purely analytic and to the author's opinion simpler than Krylov's method.

Now let us be more specific. We will consider the following type of Bellman initial value problem arising in a finite horizon, discounted stochastic optimal stopping and control problem.

$$G(t,x,u_t,u,Du,D^2u) := \min\big\{u_t + g(t,x,u,Du,D^2u), u - f(t,x)\big\} = 0 \quad \text{in} \quad Q_T := (0,T]\times\mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

with

$$g(t,x,r,p,M) = \inf_{\vartheta\in\Theta}\Big\{ -\frac{1}{2}\operatorname{tr}[a^\vartheta(t,x)M] - b^\vartheta(t,x)p - c^\vartheta(t,x)r - d^\vartheta(t,x)\Big\}.$$

where $u_0 \in C_b(\mathbb{R}^N)$, $f \in C_b(Q_T)$ and $a \geq 0$, $b$, $c$, $d$ are continuous functions defined on $Q_T \times \Theta$ with values respectively in the space $S(N)$ of symmetric $N\times N$ matrices,

$\mathbb{R}^N$ and $\mathbb{R}$. $\Theta$, the space of controls, is a compact metric space. We also make the natural assumption that $f(0,x) \le u_0(x)$ in $\mathbb{R}^N$.

Under suitable extra assumptions on $u_0$, $a$, $b$, $c$, $d$ and $f$, the solution of the equation which is also the value-function of the associated stochastic stopping and control problem, is bounded and uniformly continuous, see [20, 21, 23]. Furthermore it is expected to be Hölder continuous if $\sigma$, $b$, $c$, $d$, and $f$ satisfy suitable regularity properties. If $f \le \min_{Q_T} u$ then equation (1.1) becomes the HJB equation associated with an optimal control problem with no stopping:

$$u_t + g(t,x,u,Du,D^2u) = 0 \quad \text{in} \quad Q_T. \tag{1.3}$$

We will consider one-step in time approximation schemes of the following type:

$$\begin{aligned} &\widetilde{G}(h,t,x,u_h(t,x),[u_h]_{t,x}) \\ &:= \min\Big\{S\big(h,t,x,u_h(t,x),[u_h]_{t,x}\big), u_h(t,x) - f(t,x)\Big\} = 0 \\ &\qquad \text{in} \quad \widetilde{Q}_T := [\Delta t, T] \times \mathbb{R}^N, \end{aligned} \tag{1.4}$$

$$u_h(t,x) = g_h(t,x) \quad \text{in} \quad [0,\Delta t) \times \mathbb{R}^N, \tag{1.5}$$

where $h = (\Delta t, \Delta x)$, $\Delta t$ is the time step, $M\Delta t \le T$, $\Delta x$ is some small parameter which measures typically the $x$-mesh size, $u_h$ is the approximation of $u$ and the solution of the scheme, $[u_h]_{t,x}$ is a function defined at $(t,x)$ from $u_h$, $g_h$ is the initial data for the scheme, and finally $S$ and $\widetilde{G}$ denote the approximations of the (1.3) and (1.1) respectively.

A one-step in time scheme means that the solution at time $t$ depends on the solution at time $t - \Delta t$ ($u_h(t,x)$ depends on $u_h(t-\Delta t,\cdot)$). Implicit and explicit schemes are allowed. Moreover note that the function $u_h$ is defined at every point in $\overline{Q}_T$. For $u_h$ to be well-defined for every $t$, we need to specify initial data on the entire strip $[0,\Delta t) \times \mathbb{R}^N$.

We give a brief outline of the techniques used here. They are based on a tricky idea of Krylov: Consider the solution $u^\varepsilon$ of the following perturbed version of (1.1)

$$\begin{aligned} \min\Big\{ &u_t^\varepsilon(t,x) + \inf_{\substack{s\in(0,\varepsilon^2)\\ |e|\le\varepsilon}} g(t+s,x+e,u^\varepsilon(t,x),Du^\varepsilon(t,x),D^2u^\varepsilon(t,x)), \\ &u^\varepsilon(t,x) - f(t,x)\Big\} = 0 \qquad \text{in} \quad Q_T^\varepsilon := (-\varepsilon^2, T] \times \mathbb{R}^N, \end{aligned} \tag{1.6}$$

$$u^\varepsilon(-\varepsilon^2, x) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where the coefficients (except $f(t,x)$!) have been appropriately extended to $t > T$ (to be equal to their values at $t = T$). Regularize $u^\varepsilon$ by mollification, and use concavity of $F$ in $u$, $Du$, $D^2u$ to prove that the resulting function denoted by $u_\varepsilon$ is a (smooth) subsolution of (1.1) in $Q_T$. Now, if we can prove precise bounds on $\|u - u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$ and the derivatives of $u_\varepsilon$, we get "half the result", namely an upper estimate of $u - u_h$. To see this, one just has to plug $u_\varepsilon$ into the scheme and use the consistency condition in addition to some comparison properties for the scheme. The details follow in the next section.

The other estimate (a lower estimate of $u - u_h$) is then obtained by interchanging the role of the scheme and the equation in the above argument. This leads us to introduce the solution $u_h^\varepsilon$ of the perturbed version of the scheme (1.4)

$$\min\Big\{\inf_{\substack{s\in(0,\varepsilon^2)\\ |e|\le\varepsilon}} S\big(h,t+s,x+e,u_h^\varepsilon(t,x),[u_h^\varepsilon]_{t+s,x+e}\big),u_h^\varepsilon-f(t,x)\Big\}=0 \quad \text{in} \quad \widetilde{Q}_T^\varepsilon:=[\Delta t-\varepsilon^2,T]\times\mathbb{R}^N, \tag{1.8}$$

$$u_h^\varepsilon(t,x)=g_h(t+\varepsilon^2,x)\quad\text{in}\quad[-\varepsilon^2,\Delta t-\varepsilon^2)\times\mathbb{R}^N, \tag{1.9}$$

with appropriately extended coefficients. The difficulties with this procedure lead to restrictions in the class of schemes that can be considered. See [1] for a further discussion of these methods.

Let us give some examples to what kind of schemes our abstract result can handle. First consider the simple one space-dimensional HJB equation associated with an optimal control problem:

$$u_t+\inf_{\vartheta\in\Theta}\Big\{-\frac{1}{2}(\sigma^\vartheta)^2(t,x)u_{xx}-d^\vartheta(t,x)\Big\}=0\quad\text{in}\quad[0,T]\times\mathbb{R}. \tag{1.10}$$

For this equation we will consider (i) so-called control schemes:

(1.11)

$$u_h(t+\Delta t,x)=u_h(t,x)$$
$$+\sup_{\vartheta\in\Theta}\left\{\frac{1}{2}\left[u_h\left(t,x+\sigma^\vartheta(t,x)\sqrt{\Delta t}\right)+u_h\left(t,x-\sigma^\vartheta(t,x)\sqrt{\Delta t}\right)\right]+\Delta t d^\vartheta(t,x)\right\},$$

and (ii) finite difference schemes:

(1.12)

$$u_h(t+\Delta t,x)=u_h(t,x)$$
$$+\sup_{\vartheta\in\Theta}\left\{\frac{a^\vartheta}{2}\frac{\Delta t}{\Delta x^2}\Big(u_h(t,x+\Delta x)-2u_h(t,x)+u_h(t,x-\Delta x)\Big)+\Delta t\; d^\vartheta(t,x)\right\}.$$

However, in the case of finite difference schemes we will have to assume that $(\sigma^\vartheta)^2=a^\vartheta$ does not depend on $(t,x)$, see Section 3.

Now consider optimal stopping of a controlled diffusion. In a simple one space-dimensional case the Bellman equation take the following form:

$$\min\Big\{u_t+\inf_{\vartheta\in\Theta}\Big\{-\frac{1}{2}(\sigma^\vartheta)^2(t,x)u_{xx}-d^\vartheta(t,x)\Big\},u-f(t,x)\Big\}=0\quad\text{in}\quad[0,T]\times\mathbb{R}.$$

For this problem we will consider schemes of the type:

$$u_h(t+\Delta t,x)=\max\big\{S(\Delta t)u_h(t,x),f(t+\Delta t,x)\big\},$$

where $S(\Delta t)$ denotes the (formal) solution operator associated to some approximation scheme for (1.10). This is really a two step procedure:

(1) Determine the intermediate function $\bar{u}_h$ such that
$$\bar{u}_h(t+\Delta t,x)=S(\Delta t)u_h(t,x).$$
(2) Calculate $u_h(t+\Delta t,x)=\max\big\{\bar{u}_h(t+\Delta t,x),f(t+\Delta t,x)\big\}$.

In this paper we will give results for case when $S(\Delta t)$ is associated to the finite difference scheme (1.12), and indicate how to obtain similar results for the scheme (1.11). In the case of a pure stopping problem, i.e. no control – $\Theta$ is a singleton, this scheme is related to the so-called Brennan & Schwartz algorithm used for the pricing of American options, see [3, 19].

This paper is organized as follows: In Section 2 we state and prove the main result giving the rate of convergence for approximation schemes. This result is then applied to explicit finite difference schemes and control schemes in Sections 3 and 4 respectively. The linear case is considered in Section 5, and finally, the Appendices contain the proofs of some technical results.

## 2. The main result

In this section we state and prove the main result of this paper, a result giving the rate of convergence for certain approximation schemes for (1.1). We start by introducing the norms and spaces we will use in this paper. First, we define the norm denoted by $|\cdot|$ as follows: for any integer $m \geq 1$ and any $z = (z_i)_i \in \mathbb{R}^m$, we set $|z|^2 = \sum_{i_1=1}^m z_i^2$. We identify $N_1 \times N_2$ matrices with $\mathbb{R}^{N_1 \times N_2}$ vectors. For such matrices, $|M|^2 = \mathrm{tr}[M^T M]$ where $M^T$ denotes the transpose of $M$. Let $I \subset [0, \infty)$ be an interval. Let $N_1, N_2$ be nonnegative integers, and $f : I \times \mathbb{R}^N \to \mathbb{R}^{N_1 \times N_2}$ be a function, then we define the following (semi) norms:

$$|f|_0 = \sup_{(t,x)\in I\times\mathbb{R}^N} |f(t,x)|,$$

$$[f]_{,\delta} = \sup_{\substack{t\in I, x,\bar{x}\in\mathbb{R}^N \\ x\neq\bar{x}}} \frac{|f(t,x) - f(t,\bar{x})|}{|x-\bar{x}|^\delta}, \qquad [f]_{\delta/2,} = \sup_{\substack{t,\bar{t}\in I, x\in\mathbb{R}^N \\ t\neq\bar{t}}} \frac{|f(t,x) - f(\bar{t},x)|}{|t-\bar{t}|^{\delta/2}},$$

furthermore $[f]_\delta = [f]_{,\delta} + [f]_{\delta/2,}$ and $|f|_\delta = |f|_0 + [f]_\delta$. By $\mathcal{C}^\delta(\overline{Q}_T)$ we denote the set of functions $f : \overline{Q}_T \to \mathbb{R}$ with finite norm $|f|_\delta$. We denote by $D^i f$ the vector of the $i$-th order partial derivatives of $f$ with respect to $x$. Finally, throughout this paper we denote by $C$ constants independent of $t$, $x$, $h = (\Delta t, \Delta x)$, and $\varepsilon$.

We state the assumptions on the coefficients in the Bellman equation (1.1):

**(A)** (Conditions on data) For any $\vartheta \in \Theta$, $c^\vartheta \leq 0$ and $a^\vartheta \equiv \sigma^\vartheta \sigma^{\vartheta T}$ for some $N \times P$ matrix-valued function $\sigma^\vartheta$. Moreover $f(0,x) \leq u_0(x)$ in $\mathbb{R}^N$ and there exist $M > 0$ and $\delta \in (0,1]$ such that $\quad |\sigma^\vartheta|_1, |b^\vartheta|_1, |c^\vartheta|_\delta, |d^\vartheta|_\delta, |f|_\delta, |u_0|_\delta \leq M \quad$ for any $\vartheta \in \Theta$.

See Remark 2.2 about the condition $c^\vartheta \leq 0$. The next result states that under assumption (A), we have existence and uniqueness in $\mathcal{C}^\delta(\overline{Q}_T)$ of viscosity solutions of (1.1) & (1.2).

**Theorem 2.1.** *Assume (A) holds.*
*(a) There exist a unique viscosity solution of* (1.1) *and* (1.2) *in* $\mathcal{C}^\delta(\overline{Q}_T)$.
*(b) If* $\Delta t > 0$ *and* $u, -v \in USC(\overline{Q}_T)$ *are viscosity solutions of* $G[u] \leq 0$ *and* $G[v] \geq -k$ *in* $\widetilde{Q}_T$, *where* $k \geq 0$ *is a constant, then*

$$u - v \leq C\left(\sup_{[0,\Delta t)\times\mathbb{R}^N} |u-v| + k\right) \quad \text{in} \quad Q_T.$$

This result is now more or less classical, see e.g. Pham [23] Proposition 3.3 and Remark 2 page 11. We remark that the $x$-regularity in (a) follows from Theorem A.1 in the Appendix, and part (b) would follow from an easy modification in the proof that Theorem.

We state the assumptions on the scheme (1.4):

**(C1)** (Monotonicity) For every $h > 0$, $(t,x) \in \overline{Q}_T$, $r \in \mathbb{R}$, $m, m_0 \geq 0$ and bounded functions $u, v$ such that $u \leq v$ the following holds:

$$S(h,t,x,r+m+m_0 t,[v+m+m_0 t]_{t,x}) \geq m_0 + S(h,t,x,r,[u]_{t,x}).$$

**(C2)** (Regularity) For every $h > 0$ and $\phi \in C_b(\overline{Q}_T)$, $(t, x) \mapsto S(h, t, x, \phi(t, x), [\phi]_{t,x})$ is bounded and continuous in $\overline{Q}_T$ and the function $r \mapsto S(h, t, x, r, [\phi]_{t,x})$ is uniformly continuous for bounded $r$, uniformly with respect to $(t, x) \in \widetilde{Q}$.

To state the next assumption, we use a sequence of mollifiers $(\rho_\varepsilon)_\varepsilon$ defined as follows

$$\begin{cases} \rho_\varepsilon(t, x) = \dfrac{1}{\varepsilon^{N+2}}\rho(\dfrac{t}{\varepsilon^2}, \dfrac{x}{\varepsilon}) \quad \text{where} \quad \rho \in C^\infty(Q_T) \quad \text{is nonnegative,} \\ \qquad\qquad \text{have unit mass, and support in } (0, 1) \times B(0, 1). \end{cases} \tag{2.1}$$

**(C3)** (Concavity) For any $v \in C_b(\overline{Q}^\varepsilon_T)$, $\Delta t, \Delta x > 0$, $(t, x) \in \widetilde{Q}^\varepsilon$,

$$\begin{aligned} \int_{\overline{Q}_T} & S(h, t, x, v(t - s, x - e), [v(\cdot - s, \cdot - e)]_{t,x})\rho_\varepsilon(s, e)dsde \\ & \leq S(h, t, x, (v * \rho_\varepsilon)(t, x), [v * \rho_\varepsilon]_{t,x}). \end{aligned}$$

**(C4)** (Consistency) There exists integers $n, m, k_i, \bar{k} > 0$, $i = 1, 2, \ldots, n$ such that for every smooth $\phi$, $\Delta t, \Delta x \geq 0$, and $(t, x) \in \overline{Q}_T$:

$$\begin{aligned} & \left|\phi_t + g(t, x, \phi, D\phi, D^2\phi) - S(h, t, x, \phi(t, x), [\phi]_{t,x})\right| \\ & \leq C\left(\sum_{i=1}^{n} |D^i\phi|_0 \Delta x^{k_i} + |(\partial_t)^m\phi|_0 \Delta t^{\bar{k}}\right). \end{aligned}$$

**(C5)** (Commutation with translations) For any $\Delta t, \Delta x \geq 0$ small enough, $0 \leq \varepsilon \leq 1$, $(t, y) \in \widetilde{Q}_T$, $r \in \mathbb{R}$, $v \in C_b(\widetilde{Q}^\varepsilon_T)$, $0 \leq s, |e|^2 \leq \varepsilon^2$, we have

$$S(h, t, y, r, [v]_{t-s,y-e}) = S(h, t, y, r, [v(\cdot - s, \cdot - e)]_{t,y}).$$

Condition (C1) is a monotonicity condition stating that $S(h, t, x, r, [u]^h_x)$ is non-decreasing in $r \in \mathbb{R}$ and non-increasing in $[u]^h_x$ for bounded (possibly discontinuous) functions $u$ equipped with the usual partial ordering. Furthermore, this condition implies that the approximation contains a term approximating $u_t$ (the $m_0$-term). Condition (C3) is satisfied by Jensen's inequality if $S$ is concave in $r$ and $[u]^h_x$. Condition (C4) implies that smooth solutions of the scheme (1.4) will converge towards the solution of equation (1.1). Finally, it is easy to see that that (C2) – (C5) also hold for $\widetilde{G}$, and that (C1) holds for $\widetilde{G}$ when $m_0 = 0$.

**Remark 2.2.** Condition (C1) implies that $c^\vartheta \leq 0$ which was already assumed in (A). This is not a restriction because it can always be achieved via a transformation of the solution $u$ of (1.1), $v := e^{\sup_\vartheta |c^\vartheta|_0 t}u$. A similar thing can also be done directly on the solution $u_h$ of the scheme (1.4), $v_h := R(t, \Delta t)u_h$, where $R(t, \Delta t)$ is a suitable rational approximation of $e^{\sup_\vartheta |c^\vartheta|_0 t}$. (In Appendix B a related technique is used.)

**Remark 2.3.** Condition (C4) implies that $S$ is an implicit scheme. For an explicit scheme this condition would look like

$$\left|\left[\phi_t + g(s, x, \phi, D\phi, D^2\phi)\right]_{s=t-\Delta t} - S(h, t, x, \phi(t, x), [\phi]_{t,x})\right| \leq \ldots.$$

Note the shift in time! However the analysis is essentially the same, so we will only do the proofs in the implicit case.

Condition (C1) and (C2) imply a comparison result for uniformly continuous solutions of (1.4):

**Lemma 2.4.** *Assume (C1), (C2), and $u, v \in C_b(\overline{Q}_T)$ are uniformly continuous.*

*(a) If $\widetilde{G}[u] \leq 0$ and $\widetilde{G}[v] \geq 0$ in $\widetilde{Q}_T$, and $u \leq v$ in $[0, \Delta t) \times \mathbb{R}^N$, then*

$$u \leq v \quad in \quad \widetilde{Q}_T.$$

*(b) If* $\widetilde{G}[u] \leq 0$ *and* $\widetilde{G}[v] \geq -k$ *in* $\widetilde{Q}_T$*, where* $k \geq 0$ *is a constant, then*

$$u - v \leq \sup_{[0,\Delta t]\times\mathbb{R}^N} |u - v| + (1+t)k \quad \text{in} \quad \overline{Q}_T.$$

*Proof.* Assume (a) holds, then (b) follows since by (C1) $v + \sup_{[0,\Delta t]\times\mathbb{R}^N} |u-v| + (1+t)k$ is a supersolution of the scheme (1.4).

We prove (a) by assuming $m := \sup_{\widetilde{Q}_T}(u - v) > 0$ and deriving a contradiction. Consider $m_\varepsilon := \sup_{\widetilde{Q}_T}(u - v - \varepsilon(t - \Delta t))$ for $\varepsilon > 0$. It is obvious that $m \geq m_\varepsilon \to m$ as $\varepsilon \to 0$. We assume $\varepsilon > 0$ is so small that $m_\varepsilon > 0$. Let $\{t_n, x_n\}_n$ be a sequence in $\widetilde{Q}_T$ such that $\delta_n := u(t_n, x_n) - v(t_n, x_n) - \varepsilon(t_n - \Delta t) \to m_\varepsilon$ as $n \to \infty$. For $n$ large enough $\delta_n > 0$ (obviously), and $t_n - \Delta t > \rho > 0$ for some number $\rho$. If $t_n - \Delta t > \rho > 0$ did not hold for large $n$ and small $\rho$, there would exist a subsequence $t_{n_k} \to \Delta t$. By uniform continuity $|\delta_{n_k} - u(\Delta t, x_{n_k}) - v(\Delta t, x_{n_k})| \to 0$, which contradicts $m > 0$ since $0 \geq u(\Delta t, x_{n_k}) - v(\Delta t, x_{n_k}) \to m$. Using the fact that $u$ and $v$ are sub- and supersolutions we get

$$\begin{aligned}
0 &\geq \widetilde{G}(h, t_n, x_n, u(t_n, x_n), [u]^h_{t_n,x_n}) - \widetilde{G}(h, t_n, x_n, v(t_n, x_n), [v]^h_{t_n,x_n}) \\
&\geq \widetilde{G}(h, t_n, x_n, v(t_n, x_n) + \varepsilon(t_n - \Delta t) + \delta_n, [v + \varepsilon(t_n - \Delta t) + m_\varepsilon]^h_{t_n,x_n}) \\
&\quad - \widetilde{G}(h, t_n, x_n, v(t_n, x_n), [v]^h_{t_n,x_n}) \\
&\geq \varepsilon \min\{1, t_n - \Delta t\} - \omega(m_\varepsilon - \delta_n) ,
\end{aligned}$$

where $\omega(t) \to 0$ when $t \to 0^+$ is given by (C2). The second inequality is due to the monotonicity of $\widetilde{G}$ (C1), while the third inequality follows from both assumptions (C1) and (C2) (uniform continuity in the 4th variable) which yield:

$$\begin{aligned}
&S(h, t_n, x_n, v(t_n, x_n) + \varepsilon(t_n - \Delta t) + \delta_n, [v + \varepsilon(t_n - \Delta t) + m_\varepsilon]^h_{t_n,x_n}) \\
&\geq S(h, t_n, x_n, v(t_n, x_n), [v]^h_{t_n,x_n}) - \omega(\delta_n - m_\varepsilon) + \varepsilon.
\end{aligned}$$

Letting $n \to \infty$ now yields the contradiction $\varepsilon \min\{1, \rho\} \leq 0$. □

Uniqueness of uniformly continuous solutions of (1.4) is a consequence of the previous lemma.

In our approach, we need the solution of (1.8) to exist, to have a suitable regularity and to be close to the solution of (1.4). Since we are unable to prove that such results follow from (C1) – (C5), we need the following additional assumption:

**Assumption 2.5** (Perturbed Scheme)**.** *For any* $\Delta t, \Delta x > 0$ *sufficiently small and* $0 \leq \varepsilon \leq 1$*, there is a unique* $u^\varepsilon_h \in \mathcal{C}^\delta(Q^\varepsilon_T)$ *which is the solution of* (1.8) *in* $\widetilde{Q}^\varepsilon_T$*, and satisfies* $|u^\varepsilon_h|_\delta \leq C$ *and* $|u_h(t,x) - u^\varepsilon_h(t,x)| \leq C\varepsilon^\delta$ *in* $\overline{Q}_T$*.*

Note that $u^0_h = u_h$ is the solution of the scheme (1.4) in $\widetilde{Q}_T$. In particular this assumption yields existence, uniqueness, and regularity results for solutions of (1.4). We will check this assumption for each application.

Now we state the main result which says that the scheme (1.4) converges to the viscosity solution of (1.1) with given *a priori* error estimate.

**Theorem 2.6** (The Rate of Convergence)**.** *Assume (A), (C1) – (C5), and Assumption 2.5 hold, let* $u \in \mathcal{C}^\delta(\overline{Q}_T)$ *be the viscosity solution of* (1.1) *&* (1.2)*, and let* $u_h \in \mathcal{C}^\delta(\overline{Q}_T)$ *be the solution of the scheme* (1.4) *&* (1.5)*. Then if* $\Delta t, \Delta x \geq 0$ *are sufficiently small*

$$|u - u_h|_0 \leq C \left( \sup_{[0,\Delta t)\times\mathbb{R}^N} |u - g_h| + \Delta x^{\gamma_x} + \Delta t^{\gamma_t} \right),$$

*where, using the constants defined in (C4),*

$$\gamma_x := \min_{i=1,\dots,n} \left\{ \frac{k_i}{i} \right\} \quad \textit{and} \quad \gamma_t := \frac{\bar{k}}{2m}.$$

We proceed to prove Theorem 2.6, noting that the proof we give is, up-to adjustments to take care of the time dependence and peculiar form of the problem, the same as the corresponding proof in Barles & Jakobsen [1]. We follow essentially Krylov [17] in the way we handle the time dependence. The proof consist of two bounds which are proved separately. First we derive a lower bound for the difference $u - u_h$, using mostly properties of the equation (1.1), and then an upper bound using mainly properties of the scheme (1.4).

*Proof of the lower bound.* 1. We first consider the perturbed Bellman equation (1.6). Existence and properties of the solutions of (1.6) are given by

**Lemma 2.7.** *Assume that (A) hold and let* $0 \le \varepsilon \le 1$*. Then there is a unique* $u^\varepsilon \in \mathcal{C}^\delta(Q_T^\varepsilon)$ *which is the viscosity solution of* (1.6) *&* (1.7)*, and satisfies* $|u^\varepsilon|_\delta \le C$ *and* $|u^\varepsilon(t,x) - u(t,x)| \le C\varepsilon^\delta$ *in* $\overline{Q}_T$*.*

The proof of this result is given in the appendix.

2. Because of the definition of equation (1.6), the following inequality hold in the viscosity sense for every $s \in (0, \varepsilon^2)$ and $|e| \le \varepsilon$

$$\begin{aligned} &G(t+s, x+e, u_t^\varepsilon(t,x), u^\varepsilon(t,x), Du^\varepsilon(t,x) D^2 u^\varepsilon(t,x)) \\ &\ge -|f(t,x) - f(t+s, x+e)| \ge -[f]_\delta \varepsilon^\delta \quad \text{in} \quad Q_T^\varepsilon. \end{aligned}$$

After a change of variables, this implies that $u^\varepsilon(t-s, x-e)$ is an approximate subsolution of (1.1) in $Q_T$ for every $s \in (0, \varepsilon^2)$ and $|e| \le \varepsilon$.

3. We regularize $u^\varepsilon$ and define $u_\varepsilon := u^\varepsilon * \rho_\varepsilon$, where $\{\rho_\varepsilon\}_\varepsilon$ are the standard mollifiers defined in (2.1). Note that $u_\varepsilon$ is only well-defined on $\overline{Q}_T$ and not on all of $Q_T^\varepsilon$. We have

**Lemma 2.8.** *The function* $u_\varepsilon$ *satisfy* $G[u_\varepsilon] \ge -C\varepsilon^\delta$ *in* $Q_T$ *in the viscosity sense.*

The proof is given after the proof of Theorem 2.6.

4. By properties of mollifiers, $u_\varepsilon \in C^\infty(Q_T)$ with $|\partial_t^m u_\varepsilon|_0 \le C(\varepsilon^2)^{\delta/2-m}$ and $|D^n u_\varepsilon|_0 \le C\varepsilon^{\delta-n}$. By consistency (C4) we then have in $\widetilde{Q}_T$

$$\begin{aligned} &G(t, y, \partial_t u_\varepsilon(t,y), u_\varepsilon(t,y), Du_\varepsilon(t,y), D^2 u_\varepsilon(t,y)) \\ &\le \widetilde{G}(h, t, y, u_\varepsilon(t,y), [u_\varepsilon]_{t,y}) + C\left(\sum_{i=1}^n |D^i u_\varepsilon|_0 \Delta x^{k_i} + |(\partial_t)^m u_\varepsilon|_0 \Delta t^{\bar{k}}\right), \end{aligned}$$

and using Lemma 2.8 we deduce that

$$\widetilde{G}(h, t, y, u_\varepsilon(t,y), [u_\varepsilon]_{t,y}) \ge -C\left(\varepsilon^\delta + \sum_{i=1}^n \varepsilon^{\delta-i} \Delta x^{k_i} + \varepsilon^{\delta-2m} \Delta t^{\bar{k}}\right).$$

5. By comparison, Lemma 2.4 (b), we see that in $\overline{Q}_T$

$$u_h - u_\varepsilon \le C\left(\sup_{[0,\Delta t]\times\mathbb{R}^N} |u_\varepsilon - u_h| + \varepsilon^\delta + \sum_{i=1}^n \varepsilon^{\delta-i} \Delta x^{k_i} + \varepsilon^{\delta-2m} \Delta t^{\bar{k}}\right).$$

6. The properties of mollifiers and the uniform boundedness in $\mathcal{C}^\delta(\overline{Q}_T^\varepsilon)$ of $\{u^\varepsilon\}_\varepsilon$ imply $|u^\varepsilon(t,x) - u_\varepsilon(t,x)| \le C\varepsilon^\delta$ in $\overline{Q}_T$. Moreover from Lemma 2.7 it follows that $|u(t,x) - u^\varepsilon(t,x)| \le C\varepsilon^\delta$ in $\overline{Q}_T$, so we can conclude that $|u - u_\varepsilon|_0 \le C\varepsilon^\delta$.

7. Now choose $\varepsilon$ to be

$$\varepsilon = \max_{i=1,\dots,n} \left\{ \Delta x^{k_i/i}, \Delta t^{\bar{k}/2m} \right\},$$

which makes the $\varepsilon^\delta$-term equal to the biggest of the other terms in 5. Then by 5 and 6 we have

$$u_h - u \le C\left(\sup_{[0,\Delta t]\times\mathbb{R}^N}|u-u_h| + \Delta t^{\gamma_t} + \Delta x^{\gamma_x}\right) \quad \text{in} \quad \overline{Q}_T.$$

This concludes the proof of the lower bound. □

*Proof of the upper bound.* We follow exactly the same method as for the upper bound, interchanging the role of the equation and the scheme.

1. Let $u_h^\varepsilon$ be the $\mathcal{C}^\delta(\overline{Q}_T^\varepsilon)$ solution of the scheme (1.8) provided by Assumption 2.5. From the scheme (1.8), by performing the change of variables $(\tau,y)=(t+s,x+e)$, and using (C5), we see that for all $s\in(0,\varepsilon^2)$, $|e|\le\varepsilon$

$$\begin{aligned}&\widetilde{G}(h,\tau,y,u_h^\varepsilon(\tau-s,y-e),[u_h^\varepsilon(\cdot-s,\cdot-e)]_{\tau,y})\\ &\ge -|f(\tau,y)-f(\tau+s,y+e)| \ge -[f]_\delta\varepsilon^\delta \quad \text{in} \quad \widetilde{Q}_T.\end{aligned}$$

2. Let $\rho_\varepsilon$ be the mollifier defined in (2.1). Multiplying the above inequality by $\rho_\varepsilon(s,e)$, integrating with respect to $(s,e)$, and using that fact that $\rho_\varepsilon * \min\{g,h\}\le \min\{\rho_\varepsilon * g, \rho_\varepsilon * h\}$ and (C3) yield

$$\begin{aligned}-C\varepsilon^\delta &\le \int_{Q_T}\rho_\varepsilon(s,e)\widetilde{G}(h,\tau,y,u_h^\varepsilon(\tau-s,y-e),[u^\varepsilon(\cdot-s,\cdot-e)]_{\tau,y})dsde\\ &\le \widetilde{G}(h,\tau,y,(u_h^\varepsilon*\rho_\varepsilon)(\tau,y),[u_h^\varepsilon*\rho_\varepsilon]_{\tau,y}).\end{aligned}$$

3. Because of the properties of $u_h^\varepsilon$ given in Assumption 2.5 and the properties of mollifiers, $u_{h\varepsilon}:=u_h^\varepsilon*\rho_\varepsilon\in C^\infty(Q_T)$ with $|\partial_t^m u_{h\varepsilon}|_0\le C(\varepsilon^2)^{\delta/2-m}$ and $|D^n u_{h\varepsilon}|_0\le C\varepsilon^{\delta-n}$. Using (C4) we have in $\widetilde{Q}_T$

$$\begin{aligned}\widetilde{G}(h,t,x,u_{h\varepsilon}(t,x),[u_{h\varepsilon}]_{t,x}) &\le G(t,x,\partial_t u_{h\varepsilon},u_{h\varepsilon},Du_{h\varepsilon},D^2u_{h\varepsilon})\\ &+C\left(\sum_{i=1}^n|D^iu_{h\varepsilon}|_0\Delta x^{k_i}+|(\partial_t)^m u_{h\varepsilon}|_0\Delta t^{\bar{k}}\right).\end{aligned}$$

4. By 2. and 3. we have that $G(t,x,\partial_t u_{h\varepsilon},u_{h\varepsilon},Du_{h\varepsilon},D^2u_{h\varepsilon})\ge -C(\varepsilon^\delta+\sum_{i=1}^n\varepsilon^{\delta-i}\Delta x^{k_i}+\varepsilon^{\delta-2m}\Delta t^{\bar{k}})$ in $\widetilde{Q}_T$. So by the comparison principle for (1.1) (Theorem 2.1), the following inequality holds in $\overline{Q}_T$

$$u-u_{h\varepsilon}\le \sup_{[0,\Delta t)\times\mathbb{R}^N}|u-u_{h\varepsilon}|+C\left(\varepsilon^\delta+\sum_{i=1}^n\varepsilon^{\delta-i}\Delta x^{k_i}+\varepsilon^{\delta-2m}\Delta t^{\bar{k}}\right).$$

5. Again by the properties of mollifiers and the $\mathcal{C}^\delta(\overline{Q}_T^\varepsilon)$ regularity of $u_h^\varepsilon$ we get that $|u_{h\varepsilon}(t,x)-u_h^\varepsilon(t,x)|\le C\varepsilon^\delta$ in $\overline{Q}_T$. Moreover by Assumption 2.5 it follows that $|u_h(t,x)-u_h^\varepsilon(t,x)|\le C\varepsilon^\delta$ in $\overline{Q}_T$. All in all we conclude that $|u_h-u_{h\varepsilon}|_0\le C\varepsilon^\delta$.

6. Choosing $\varepsilon$ as we did in the proof of the lower bound and using 4 and 5 yield

$$u-u_h\le \sup_{[0,\Delta t)\times\mathbb{R}^N}|u-u_h|+C(\Delta x^{\gamma_x}+\Delta t^{\gamma_t}) \quad \text{in} \quad \overline{Q}_T.$$

This completes the proof of Theorem 2.6. □

*Proof of Lemma 2.8.* The proof relies on the following lemma which states that a finite convex combination of supersolutions of (1.1) is still a supersolution of (1.1).

**Lemma 2.9.** *Assume (A) holds, $\{u^i\}_{i=1}^n\subset C_b(\overline{Q}_T)$ is a set of viscosity supersolutions of* (1.1)*, and $\{\lambda_i\}_{i=1}^n$ is a set of non-negative numbers such that $\sum_{i=1}^n\lambda_i=1$. Then $\sum_{i=1}^n\lambda_iu^i$ is a viscosity supersolution of* (1.1)*.*

We will not prove this result, since its proof is almost identical to the proof of Lemma A.3 in [1]. Now let $Q_\delta^{s,e} := (s+[0,\delta))\times(e+[-\delta/2,\delta/2)^N)$, $\bar\rho_\varepsilon(s,e;\delta) = \int_{Q_\delta^{s,e}}\rho_\varepsilon$, and

$$I_\delta(t,x) := \sum_{\substack{(s,e)\in\\ \delta\mathbb{Z}\times\delta\mathbb{Z}^N}} u^\varepsilon(t-s,x-e)\bar\rho_\varepsilon(s,e;\delta).$$

Note that $\sum_{(s,e)\in\delta\mathbb{Z}\times\delta\mathbb{Z}^N}\bar\rho_\varepsilon(s,e;\delta)=1$, and that by a standard argument $I_\delta$, obtained through a discretization of the convolution integral, converges uniformly to $u_\varepsilon$. Furthermore, $I_\delta$ is a finite convex combination of supersolutions of a version of (1.1) where $d^\vartheta, f$ is replaced by $d^\vartheta+C\varepsilon^\delta, f+C\varepsilon^\delta$. Lemma 2.9 then yields that $I_\delta$ itself is a viscosity supersolution of this equation, and using the stability result for viscosity solutions of second order PDEs (Lemma 6.1 in [6]), this is still true for the limit function $u_\varepsilon$ obtained by taking $\delta\to\infty$. This concludes the proof. □

## 3. Application 1: Finite Difference Schemes

In this section we consider a class of finite difference schemes for the Bellman equation (1.1). This class is a subclass of monotone (and for simplicity explicit) finite difference schemes. It has been discussed for instance in Kushner & Dupuis [18], see also Fleming & Soner [11]. We assume that (A) holds with $\delta=1$ (for simplicity), that $a^\vartheta$ is independent of $(t,x)$ (a real restriction), and that the following two conditions hold for every $\vartheta\in\Theta$:

$$a_{ii}^\vartheta - \sum_{j\neq i}|a_{ij}^\vartheta| \geq 0, \quad i=1,\dots,N, \tag{3.1}$$

$$\frac{\Delta t}{\Delta x^2}\sum_{i=1}^N\Big\{a_{ii}^\vartheta - \sum_{j\neq i}|a_{ij}^\vartheta| + \Delta x|b_i^\vartheta|_0\Big\} + \Delta t|c^\vartheta|_0 \leq 1. \tag{3.2}$$

Assumption (3.1) is standard [18, 11] and states that $a^\vartheta$ has to be *diagonally dominant*. Assumption (3.2) is the CFL condition for the explicit scheme (3.3). The two conditions together will make our scheme (3.3) below monotone.

Now to define the finite difference schemes, we will introduce notation for the relevant differencing operators. Let $\{e_i\}_{i=1}^N$ be the standard basis in $\mathbb{R}^N$, and define

$$\begin{aligned}
\Delta_{x_i}^{\pm}w(t,x) &= \pm\frac{1}{\Delta x}\{w(t,x\pm\Delta x e_i)-w(t,x)\},\\
\Delta_{x_i}^{2}w(t,x) &= \frac{1}{\Delta x^2}\{w(t,x+\Delta x e_i)-2w(t,x)+w(t,x-\Delta x e_i)\},\\
\Delta_{x_ix_j}^{+}w(t,x) &= \frac{1}{2\Delta x^2}\{2w(t,x)+w(t,x+\Delta x e_i+\Delta x e_j)+w(t,x-\Delta x e_i-\Delta x e_j)\}\\
&\quad -\frac{1}{2\Delta x^2}\{w(t,x+\Delta x e_i)+w(t,x-\Delta x e_i)+w(t,x+\Delta x e_j)+w(t,x-\Delta x e_j)\},\\
\Delta_{x_ix_j}^{-}w(t,x) &= \frac{-1}{2\Delta x^2}\{2w(t,x)+w(t,x+\Delta x e_i-\Delta x e_j)+w(t,x-\Delta x e_i+\Delta x e_j)\}\\
&\quad +\frac{1}{2\Delta x^2}\{w(t,x+\Delta x e_i)+w(t,x-\Delta x e_i)+w(t,x+\Delta x e_j)+w(t,x-\Delta x e_j)\}.
\end{aligned}$$

Let $b^+ = \max\{b, 0\}$ and $b^- = (-b)^+$. Note that $b = b^+ - b^-$. For each $x$, $t$, $p_i^\pm$, $A_{ii}$, $A_{ij}^\pm$, $i, j = 1, \dots, N$, let

$$\widetilde{g}(t, x, r, p_i^\pm, A_{ii}, A_{ij}^\pm) = \inf_{\vartheta\in\Theta} \Big\{ \sum_{i=1}^N \Big[ -\frac{a_{ii}^\vartheta}{2} A_{ii} + \sum_{j\neq i} \Big( -\frac{a_{ij}^{\vartheta+}}{2} A_{ij} + \frac{a_{ij}^{\vartheta-}}{2} A_{ij} \Big) \\ -b_i^{\vartheta+}(t,x)p_i^+ + b_i^{\vartheta-}(t,x)p_i^- \Big] - c^\vartheta(t,x)r - d^\vartheta(t,x) \Big\}.$$

Let $u_h$ denote the solution of the schemes, then the scheme can be stated as follows:

$$\min\left\{ \frac{u_h(t+\Delta t, x) - u_h(t,x)}{\Delta t} \right. \\ + \widetilde{g}\Big(t, x, u_h(t,x), \Delta_{x_i}^\pm u_h(t,x), \Delta_{x_i}^2 u_h(t,x), \Delta_{x_i x_j}^\pm u_h(t,x)\Big), \\ \left. u_h(t+\Delta t, x) - f(t+\Delta t, x) \right\} = 0, \tag{3.3}$$

for any $(t, x) \in \{t_1, t_2, \dots, t_{N_t}\} \times \Delta x \mathbb{Z}^N$.

We proceed to derive an equivalent scheme to (3.3) which will have similarities with a discrete dynamical programming principle. This new scheme will be better suited to proving existence, regularity and continuous dependence results. Define the following "one step transition probabilities"

$$p^\vartheta(t, x, x) = 1 - \frac{\Delta t}{\Delta x^2} \sum_{i=1}^N \Big\{ a_{ii}^\vartheta - \sum_{j\neq i} |a_{ij}^\vartheta| + \Delta x |b_i^\vartheta(t,x)| \Big\},$$

$$p^\vartheta(t, x, x \pm \Delta x e_i) = \frac{\Delta t}{\Delta x^2} \Big\{ \frac{a_{ii}^\vartheta}{2} - \sum_{j\neq i} \frac{|a_{ij}^\vartheta|}{2} + \Delta x b_i^{\vartheta\pm}(t,x) \Big\},$$

$$p^\vartheta(t, x, x + \Delta x e_i \pm \Delta x e_j) = \frac{\Delta t}{\Delta x^2} \frac{a_{ij}^{\vartheta\pm}}{2},$$

$$p^\vartheta(t, x, x - \Delta x e_i \pm \Delta x e_j) = \frac{\Delta t}{\Delta x^2} \frac{a_{ij}^{\vartheta\mp}}{2},$$

and $p^\vartheta(t, x, y) = 0$ for all other $y$. Note that by (3.1) and (3.2), $0 \le p^\vartheta(t, x, y) \le 1$ and $\sum_{z\in\Delta x\mathbb{Z}^N} p^\vartheta(t, x, x+z) = 1$ for all $\vartheta, x, y$. A simple but tedious calculation now shows that that (3.3) can be written in the following way:

(3.4)

$$u_h(t+\Delta t, x) = \max\Bigg\{ f(t+\Delta t, x), \\ \sup_{\vartheta\in\Theta} \Big\{ \sum_{z\in\Delta x\mathbb{Z}^N} p^\vartheta(t, x, x+z) u_h(t, x+z) + \Delta t c^\vartheta(t,x) u_h(t,x) + \Delta t d^\vartheta(t,x) \Big\} \Bigg\}.$$

Note that we have multiplied one term in the maximum by $\Delta t$, see (3.3).

Let us check conditions (C1) – (C5). We start by defining precisely what we mean by $S$ and $[\cdot]_{t,x}$. For $\phi \in C_b(\mathbb{R}^N)$, set $[\phi]_{t,x}(\cdot) := \phi(t - \Delta t, x + \cdot)$, and define $S$ by

$$S(\Delta x, t, y, r, [\phi]_{t,x}) = \frac{r - [\phi]_{t,x}(0)}{\Delta t} + \inf_{\vartheta\in\Theta} \Bigg\{ -c^\vartheta(t-\Delta t, y)[\phi]_{t,x}(0) - d^\vartheta(t - \Delta t, y) \\ - \frac{1}{\Delta t} \Big[ \sum_{z\in\Delta x\mathbb{Z}^N} p^\vartheta(t-\Delta t, y, y+z)[\phi]_{t,x}(z) - [\phi]_{t,x}(0) \Big] \Bigg\}.$$

It is easy to see that $S$ defines a scheme which is equivalent to (3.3).

**Proposition 3.1.** *Assume (A),* (3.1), (3.2) *hold. Then the scheme* (3.3) *satisfy (C1) – (C5), where for any smooth $\phi$, (C4) takes the form*

$$\Big|\big[\phi_t + g(s,x,\phi,D\phi,D^2\phi)\big]_{s=t-\Delta t} - S(h,t,x,\phi(t,x),[\phi]_{t,x})\Big| \\ \leq C\left(|D^2\phi|_0\Delta x + |D^4\phi|_0\Delta x^2 + |\phi_{tt}|_0\Delta t\right).$$

*Proof.* Condition (C1) holds by (3.1) and (3.2), and (C2) holds by the regularity of the data, see (A). (C3) holds with because for any function $g(x,\vartheta)$,

$$\rho_\varepsilon * \inf_{\vartheta\in\Theta} g(\cdot,\vartheta)(x) \leq \rho_\varepsilon * g(\cdot,\vartheta)(x) \implies \rho_\varepsilon * \inf_{\vartheta\in\Theta} g(\cdot,\vartheta)(x) \leq \inf_{\vartheta\in\Theta} \rho_\varepsilon * g(\cdot,\vartheta)(x).$$

Taylor expansion of $\phi$ yields (C4). Finally (C5) holds since, for any bounded, continuous function $\phi$, $[\phi]_{t-s,x-e} = [\phi(\cdot - s,\cdot - e)]_{t,x}$. ☐

We proceed to proving existence, uniqueness, regularity, and *a priori* estimates for (3.3) in order to eventually prove Assumption 2.5. We start by the *a priori* estimates. Let $v$ be a solution of (3.3) with coefficients $(a^\vartheta, b^\vartheta, c^\vartheta, d^\vartheta, f)$, then for $t, t-n\Delta t\in[0,T]$, $n\in\mathbb{N}$

$$|v(t,\cdot)|_0 \leq |f|_0 + e^{n\Delta t C_0}\Big(|v(t-n\Delta t,\cdot)|_0 + n\Delta t\sup_{\vartheta\in\Theta}|d^\vartheta|_0\Big), \tag{3.5}$$

where $C_0 := \sup_\Theta |c^\vartheta|_0$. If $v$ is bounded then

$$[v(t,\cdot)]_{,1} \leq [f]_{,1} + e^{n\Delta t(C_0+C_1)}\Big([v(t-n\Delta t)]_{,1} + n\Delta t\sup_{\vartheta\in\Theta}\big\{|v|_0[c^\vartheta]_{,1} + [d^\vartheta]_{,1}\big\}\Big), \tag{3.6}$$

where $C_1 := \sup_\Theta\{\sum_{i=1}^N([b_i^{\vartheta+}]_{,1} + [b_i^{\vartheta-}]_{,1})\}$. Let $w$ be a solution of (3.4) with coeficients $(a^\vartheta,\bar b^\vartheta,\bar c^\vartheta,\bar d^\vartheta,\bar f)$ (yes $a^\vartheta$, not $\bar a^\vartheta$!). If $v$ is both bounded and $x$-Lipschitz continuous, then

$$|v(t,\cdot)-w(t,\cdot)|_0 \leq |f-\bar f|_0 + e^{n\Delta t C_0}\Big(|v(t-n\Delta t,\cdot) - w(t-n\Delta t,\cdot)|_0 \\ + n\Delta t\sup_\Theta\Big[2[v]_{,1}\sum_{i=1}^N|b_i^\vartheta - \bar b_i^\vartheta|_0 + |v|_0|c^\vartheta - \bar c^\vartheta|_0 + |d^\vartheta - \bar d^\vartheta|_0\Big]\Big). \tag{3.7}$$

These estimates are easy to prove using (3.4) and the following simple lemma:

**Lemma 3.2.** *Let $a^0,b,c,d\geq 0$. Then $a^n = d + e^{cn}(a^0+nb)$ solve*

$$a^{n+1} \leq \max\{d,(1+c)a^n + b\}.$$

We will only prove (3.6). The two other proofs are similar but easier.

*Proof of* (3.6). Let $t>0$ be such that $t+\Delta t\in(0,T]$. Using (3.4) and the inequality $\sup\{\cdots\}-\sup\{\dots\}\le\sup\{\cdots-\dots\}$ we see that

$$\begin{aligned}
&v(t+\Delta t,x)-v(t+\Delta t,y)\\
&\le\max\Bigg\{\sup_{\Theta}\Bigg\{\sum_{z\in\Delta x\mathbb{Z}^N}p^\vartheta(t,x,x+z)\big(v(t,x+z)-v(t,y+z)\big)\\
&\qquad+\sum_{z\in\Delta x\mathbb{Z}^N}v(t,y+z)\big(p^\vartheta(t,x,x+z)-p^\vartheta(t,y,y+z)\big)\\
&\qquad+\Delta t\big(c^\vartheta(t,x)v(t,x)-c^\vartheta(t,y)v(t,y)\big)+\Delta t\big(d^\vartheta(t,x)-d^\vartheta(t,y)\big)\Bigg\},\\
&\qquad f(t+\Delta t,x)-f(t+\Delta t,y)\Bigg\}.
\end{aligned}$$

By the definition of $p^\vartheta$ we have $\sum_{z\in\Delta x\mathbb{Z}^N}p^\vartheta(t,x,x+z)(v(t,x+z)-v(t,y+z))\le[v(t,\cdot)]_{,1}|x-y|$. Furthermore since

$$p^\vartheta(t,x,x)-p^\vartheta(t,y,y)=-\frac{\Delta t}{\Delta x}\sum_{i=1}^N(|b_i^\vartheta(t,x)|-|b_i^\vartheta(t,y)|),$$

$$p^\vartheta(t,x,x\pm\Delta xe_i)-p^\vartheta(t,y,y\pm\Delta xe_i)=\frac{\Delta t}{\Delta x}(b_i^{\vartheta\pm}(t,x)-b_i^{\vartheta\pm}(t,y)),$$

we see that

$$\begin{aligned}
&\sum_{z\in\Delta x\mathbb{Z}^N}v(t,y+z)(p^\vartheta(t,x,x+z)-p^\vartheta(t,y,y+z))\\
&\le\Delta t\sum_{i=1}^N\Big[(b_i^{\vartheta+}(t,x)-b_i^{\vartheta+}(t,y))\Delta_{x_i}^+v(t,x)+(b^{\vartheta-}(t,x)-b^{\vartheta-}(t,y))\Delta_{x_i}^+v(t,x)\Big].
\end{aligned}$$

Estimating the $c^\vartheta$-terms and combining all the above estimates yield

$$\begin{aligned}
\frac{v(t+\Delta t,x)-v(t+\Delta t,y)}{|x-y|}&\le\max\Bigg\{[f]_{,1}\,,\ \Delta t\sup_{\vartheta\in\Theta}\big\{|v|_0[c^\vartheta]_{,1}+[d^\vartheta]_{,1}\big\}\\
&+\Big\{1+\Delta t\big(C_0+\sup_{\vartheta\in\Theta}\sum_{i=1}^N([b_i^{\vartheta+}]_{,1}+[b_i^{\vartheta+}]_{,1})\big)\Big\}[v(t,\cdot)]_{,1}\Bigg\}.
\end{aligned}$$

By interchanging the roles of $v(t+\Delta t,x)$ and $v(t+\Delta t,y)$, we see that the same bound holds for $|v(t+\Delta t,x)-v(t+\Delta t,y)|$ as well. Estimate (3.6) now follows after an application of Lemma 3.2. □

Now we give the existence, uniqueness and regularity results.

**Proposition 3.3.** *Assume (A),* (3.1), (3.2) *hold and* $g_h\in C_b([0,\Delta t)\times\mathbb{R}^N)$*, then there exists a unique* $u_h\in C_b(\overline{Q}_T)$ *solving* (3.3) *&* (1.5)*.*

*Proof.* By (3.5) and the boundedness of the data, any solution of (3.3) & (1.5) is bounded. Since the equation is explicit, existence of a continuous solution follows by induction since the coefficients and initial data are continuous. Uniqueness follows from assuming there exists two solutions, subtracting their corresponding equations (3.4), iterating and thus showing that they have to coincide. □

**Proposition 3.4.** *Assume (A),* (3.1)*,* (3.2) *hold, and* $|g_h|_1$ *bounded independently of* $\Delta t,\Delta x$*. If* $u_h$ *is the solution of the initial value problem* (3.4) *&* (1.5)*, then* $u_h\in\mathcal{C}^1(\overline{Q}_T)$ *and* $|u_h|_1$ *is bounded independently of* $\Delta t,\Delta x$*.*

*Proof.* By (3.5), (3.6), and the fact that $|g_h|_1$ is bounded independently of $\Delta x, \Delta t$, it is clear that $|u_h|_0$ and $[u_h]_{,1}$ are bounded independently of $\Delta x, \Delta t$.

We proceed to the regularity in time. Let $t, t+k \in [n\Delta t, (n+1)\Delta t)$, where $(n+1)\Delta t \leq T$, and $v_h(t,x) = u_h(t+k,x)$. This means that at time $t$, $v_h(t,x)$ is the solution of (3.4) with initial values $\bar{g}_h(t-n\Delta t, x) = g_h(t-n\Delta t+k, x)$, and coefficients $a^\vartheta$, $\bar{b}^\vartheta(t,x) = b^\vartheta(t+k,x)$, $\bar{c}^\vartheta(t,x) = c^\vartheta(t+k,x)$, $\bar{d}^\vartheta(t,x) = d^\vartheta(t+k,x)$, and $\bar{f}(t,x) = f(t+k,x)$. So by the continuous dependence result (3.7), we have

$$
\begin{aligned}
&|u_h(t,\cdot) - u_h(t+k,\cdot)|_0 \\
&\leq |f(\cdot,\cdot) - f(\cdot+k,\cdot)|_0 + C\Big(|g_h(t-n\Delta t,\cdot) - g_h(t-n\Delta t+k,\cdot)|_0 \\
&+ \sup_{\vartheta\in\Theta}\Big[|b^\vartheta(\cdot,\cdot) - b^\vartheta(\cdot+k,\cdot)|_0 + |c^\vartheta(\cdot,\cdot) - c^\vartheta(\cdot+k,\cdot)|_0 + |d^\vartheta(\cdot,\cdot) - d^\vartheta(\cdot+k,\cdot)|_0\Big]\Big).
\end{aligned}
$$

Assume for the moment that coefficients and initial data are Lipschitz in $t$, then

(3.8)
$$
|u_h(t,\cdot) - u_h(t+k,\cdot)|_0 \leq Ck \sup_{\vartheta\in\Theta}\Big\{|\partial_t f|_0 + |\partial_t g_h|_0 + |\partial_t b^\vartheta|_0 + |\partial_t c^\vartheta|_0 + |\partial_t d^\vartheta|_0\Big\}.
$$

This bound holds for arbitrary $t, t+k \in [0,T]$ (with the same Lipschitz constant), because if $t \in [(m-1)\Delta t, m\Delta t), k \in [(l-1)\Delta t, l\Delta t)$ for $m, l \in \mathbb{N}$ then

$$
\begin{aligned}
|u_h(t,x) - u_h(t+k,x)| &\leq |u_h(t,x) - u_h(t_m,x)| + |u_h(t_{m+l-1},x) - u_h(t+k,x)| \\
&\quad + \sum_{i=1}^{l-1} |u_h(t_{m+i-1},x) - u_h(t_{m+i},x)|,
\end{aligned}
$$

and we get the conclusion by using (3.8) on each subinterval and adding up.

The coefficients and initial data are only Hölder 1/2 in time, so by extending them appropriately and $t$-mollifying them, we obtain $t$-Lipschitz functions. Let $b^{\vartheta,\varepsilon}, c^{\vartheta,\varepsilon}, d^{\vartheta,\varepsilon}$, $f^\varepsilon$, and $g_h^\varepsilon$ be these smoothed functions, and let $u_h^\varepsilon$ denote the solution of the problem with these (smoothed) coefficients and initial data. By the continuous dependence result (3.7) and the $t$-Hölder regularity of the coefficients and initial data, $|u_h(t,x) - u_h^\varepsilon(t,x)| \leq C\varepsilon^{1/2}$ in $\overline{Q}_T$. Furthermore, by the properties of mollifiers $|\partial_t b^{\vartheta,\varepsilon}|_0, |\partial_t c_t^{\vartheta,\varepsilon}|_0, |\partial_t d_t^{\vartheta,\varepsilon}|_0, |\partial_t f|_0, |\partial_t g_h^\varepsilon|_0 \leq C\varepsilon^{-1/2}$. We can now conclude that

$$
\begin{aligned}
|u_h(t,\cdot) - u_h(t+k,\cdot)|_0 &\leq |u_h(t,\cdot) - u_h^\varepsilon(t,\cdot)|_0 + |u_h^\varepsilon(t,\cdot) - u_h^\varepsilon(t+k,\cdot)|_0 \\
&\quad + |u_h^\varepsilon(t+k,\cdot) - u_h(t+k,\cdot)|_0 \\
&\leq C\varepsilon^{1/2} + C\varepsilon^{-1/2}k \leq Ck^{1/2}.
\end{aligned}
$$

Here we have chosen $\varepsilon = k$. □

We are now in a position to check Assumption 2.5.

**Proposition 3.5.** *If (A),* (3.1), (3.2) *hold and* $|g_h|_1$ *bounded independently of* $\Delta t, \Delta x$*, then Assumption 2.5 is satisfied.*

*Proof.* Existence, uniqueness, boundedness, and regularity follow from Propositions 3.3 and 3.4, since (1.8) can be considered as a special case of (3.3) by introducing the new control parameter $(\vartheta, s, e)$, the new control space $\Theta \times (0,\varepsilon^2) \times B(0,\varepsilon)$, and via a rescaling in time, the new domain $\widetilde{Q}_T^\varepsilon$. $|u_h(t,x) - u_h^\varepsilon(t,x)| \leq C\varepsilon$ in $\overline{Q}_T$ follows after appropriate applications of (3.7). □

By Propositions 3.1 and 3.5, and Theorem 2.6, we have the following result about the rate of convergence for the scheme (3.3):

**Proposition 3.6.** *Assume (A),* (3.1)*,* (3.2) *hold and* $|g_h|_1$ *is bounded independently of* $\Delta t, \Delta x$*. If* $u$ *is the solution of* (1.1) *and* (1.2)*, and* $u_h$ *the solution of* (3.3) *and* (1.5) *then*

$$|u - u_h|_0 \leq C\big( \sup_{[0,\Delta t)\times\mathbb{R}^N} |u - g_h| + \Delta x^{1/2}\big).$$

Here we have also used the CFL condition $\Delta t \leq C\Delta x^2$, see (3.2). This result is optimal since it is what you get for first order equations (with no obstacle), see [24]. It is better than the corresponding results obtained in Barles & Jakobsen [1] and Krylov [16, 17]. They get the rate 1/3, or 1/27 in [17] where also $a$ is allowed to vary in $t, x$.

The reason for this improvement is that we use a more general consistency condition (C4) which captures the fact that the finite difference method is second order in its approximation of the second order derivatives. In [16, 17, 1] (3.3) is viewed as a pure first order method, and thus giving a reduced rate.

**Remark 3.7.** The pure control case can be obtained by setting $f \equiv \max\{|u|_0, |u_h|_0\}$, which means that neither the obstacle in the equation nor in the scheme will ever be active. In other words, we are back to equation (1.3) and finite difference methods for this equation.

**Remark 3.8.** In order for $|g_h|_1$ to be bounded independently of $\Delta t, \Delta x$, it is sufficient to take $g_h$ to be the linear in time interpolation of $u_0$ and $u_h(h,\cdot)$.

## 4. Application 2: Control-schemes

In this section we consider so-called control schemes for the Bellman equation in the pure control case (1.3). I.e. we have a parabolic equation with no obstacle. Furthermore for the sake of simplicity we only consider $\mathcal{C}^1(\overline{Q}_T)$ coefficients, and hence solutions; i.e. the case $\delta = 1$ in (A).

What we call control-schemes here are schemes based on a discretization of the so-called dynamic programming principle instead of the Bellman equation itself. These schemes correspond to discretizations in time only, and can themselves be considered as (discrete) dynamic programming principles. We refer to Fleming & Soner [11] for an explanation of the dynamic programming principle and its connection to the Bellman equation. Control schemes were introduced for first-order Hamilton-Jacobi equations (in the viscosity solutions setting) by Capuzzo-Dolcetta [5] and for second-order equations (in a classical setting) by Menaldi [22]. A full discretization in time and space was considered in Camilli and Falcone [4]. While the above mentioned paper considered stationary schemes, time-dependent schemes was considered by Falcone & Giorgi [10] for first order equations. Note that our control-schemes corresponds to the above mentioned schemes via the time-change $t \to T - t$. Finally we mention Barles & Jakobsen [1] where the rate of convergence was obtained for stationary schemes.

Borrowing notation from [4], we define the scheme in the following way

$$u_h(t+h,x) = \max_{\vartheta\in\Theta}\Big\{(1 + hc^\vartheta(t,x))\Pi_h^\vartheta u_h(t,x) + hd^\vartheta(t,x)\Big\} \quad \text{in} \quad \overline{Q}_{T-h}, \tag{4.1}$$

where $\Pi_h^\vartheta$ is the operator:

$$\Pi_h^\vartheta\phi(t,x) = \frac{1}{2N}\sum_{m=1}^{N}\Big(\phi(t, x + hb^\vartheta(t,x) + \sqrt{h}\sigma_m^\vartheta(t,x)) + \phi(t, x + hb^\vartheta(t,x) - \sqrt{h}\sigma_m^\vartheta(t,x))\Big),$$

and $\sigma^\vartheta_m$ is the $m$-th row of $\sigma^\vartheta$. Let us now define what we mean by $S$ and $[\cdot]_{t,x}$. First, let $h = \Delta t = \Delta x$ and for any $\phi \in C_b(\overline{Q}_T)$, we set $[\phi]_{t,x}(\cdot) = \phi(t-h, x+\cdot)$ and then

$$\begin{aligned} S(h,t,y,r,[\phi]_{t,x}) = \min_{\vartheta\in\Theta} \Bigg\{ &\frac{r-[\phi]_{t,x}(0)}{h} \\ &- \big(1+hc^\vartheta(t-h,y)\big)\frac{A(h,\vartheta,t-h,y,[\phi]_{t,x})-[\phi]_{t,x}(0)}{h} \\ &- c^\vartheta(t-h,y)[\phi]_{t,x}(0) - d^\vartheta(t-h,y)\Bigg\}, \end{aligned} \tag{4.2}$$

where $A$ is given by

$$\begin{aligned} A(h,\vartheta,t,y,[\phi]_{t,x}) := \frac{1}{2N}\sum_{m=1}^{N} \Big( &[\phi]_{t,x}\big(hb^\vartheta(t,y)+\sqrt{h}\sigma^\vartheta_m(t,y)\big) \\ &+ [\phi]_{t,x}\big(hb^\vartheta(t,y)-\sqrt{h}\sigma^\vartheta_m(t,y)\big)\Big). \end{aligned}$$

It is easy to see that $S$ defines a scheme which is equivalent to (4.1). Let us check that conditions (C1) – (C5) hold.

**Proposition 4.1.** *Assume (A) holds and $h\sup_{\vartheta\in\Theta}|c^\vartheta|_0 \le 1$. Then the scheme* (4.1) *satisfy (C1) – (C5), where for any smooth $\phi$, (C4) takes the form*

$$\begin{aligned} &\Big|\big[\phi_t + g(s,x,\phi,D\phi,D^2\phi)\big]_{s=t-\Delta t} - S(h,t,x,\phi(t,x),[\phi]_{t,x})\Big| \\ &\qquad \le C\left(|D^2\phi|_0 + |D^4\phi|_0 + |\phi_{tt}|_0\right)h. \end{aligned}$$

*Proof.* Condition (C1) holds because $h\sup_{\vartheta\in\Theta}|c^\vartheta|_0 \le 1$, and (C2) holds by the regularity of the data, see (A). (C3) and (C5) obviously hold, and Taylor expansion of $\phi$ yields (C4). □

From the form of the scheme (4.1) it is easy to see that any solution $v$ has to satisfy

$$|v(t,\cdot)|_0 \le |v(t-nh,\cdot)|_0 + nh\sup_{\vartheta\in\Theta}|d^\vartheta|_0, \quad \text{for} \quad t, t-nh \in [0,T], \tag{4.3}$$

when $h\sup_{\vartheta\in\Theta}|c^\vartheta|_0 \le 1$. Let us now prove existence and uniqueness of $C_b(\overline{Q}_T)$ solutions of (4.1) & (1.5):

**Proposition 4.2.** *Assume (A) holds, $g_h \in C_b([0,h)\times\mathbb{R}^N)$, $h\sup_{\vartheta\in\Theta}|c^\vartheta|_0 \le 1$, and $T \ge h > 0$. Then there exists a unique $u_h \in C_b(\overline{Q}_T)$ solving* (4.1) *&* (1.5).

*Proof.* By (4.3) and the boundedness of the data, any solution of (4.1) and (1.5) is bounded. Since the equation is explicit, existence of a continuous solution follows by induction since the coefficients and initial data are continuous. Uniqueness follows from assuming there exists two solutions, subtracting their corresponding equations (4.1), iterating and thus showing that they have to coincide. □

Now to continue we state a continuous dependence on the nonlinearities estimate.

**Proposition 4.3.** *Let $T \ge h > 0$ and let $u$ and $\bar u$ be $C_b(\overline{Q}_T)$ sub- and super solutions of* (4.1) *&* (1.5) *with data $(\sigma^\vartheta, b^\vartheta, c^\vartheta, d^\vartheta, g_h)$ and $(\bar\sigma^\vartheta, \bar b^\vartheta, \bar c^\vartheta, \bar d^\vartheta, \bar g_h)$ respectively. Moreover assume (A) holds and that $|g_h|_1$ and $|\bar g_h|_1$ are bounded independently of $h$. Then for $t \in [0,T]$*

$$\begin{aligned} |\big(u(t,\cdot)-\bar u(t,\cdot)\big)^+|_0 \le |g_h - \bar g_h|_0 &+ \sqrt{t}C\sup_{\vartheta\in\Theta}\Big[|\sigma^\vartheta - \bar\sigma^\vartheta|_0 + |b^\vartheta - \bar b^\vartheta|_0\Big] \\ &+ t\sup_{\vartheta\in\Theta}\Big[|u|_0\wedge|\bar u|_0|c^\vartheta - \bar c^\vartheta|_0 + |d^\vartheta - \bar d^\vartheta|_0\Big]. \end{aligned}$$

Here $|u|_0 \wedge |\bar{u}|_0 = \min(|u|_0, |\bar{u}|_0)$. The proof which is quite technical, is given in the appendix. This result will now be used to obtain regularity results for $u_h$.

**Proposition 4.4.** *Assume (A) holds, $|g_h|_1$ bounded independently of $h$, and that $h\sup_{\vartheta\in\Theta}|c^\vartheta|_0 \leq 1$. If $u_h$ is the solution of* (4.1) *&* (1.5)*, then $u_h \in \mathcal{C}^1(\overline{Q}_T)$ and $|u_h|_1$ is bounded independently of $h$.*

*Proof.* First, by (4.3) $|u_h|_0$ is bounded independently of $h$. Then Proposition 4.3 with $u(t,x) = u_h(t,x+h)$ and $\bar{u}(t,x) = u_h(t,x)$ along with (A) and regularity of $g_h$ implies that $[u_h]_{,1}$ is bounded independently of $h$. In a similar way we get bounds on the $t$-regularity by considering $u(t,x) = u_h(t+s,x)$ and $\bar{u}(t,x) = u_h(t,x)$ in Proposition 4.3. □

Now we verify that Assumption 2.5 hold.

**Proposition 4.5.** *If (A) holds and $|g_h|_1$ bounded independently of $h$, then Assumption 2.5 is satisfied.*

*Proof.* Existence, uniqueness, boundedness, and regularity follow from Propositions 4.2 and 4.4, since (1.8) can be considered as a special case of (4.1) by introducing the new control parameter $(\vartheta, s, e)$, the new control space $\Theta \times (0,\varepsilon^2) \times B(0,\varepsilon)$, and via a rescaling in time, the new domain $\widetilde{Q}^\varepsilon_T$. $|u_h(t,x) - u^\varepsilon_h(t,x)| \leq C\varepsilon$ in $\overline{Q}_T$ follows after appropriate applications of Proposition 4.3. □

Now by Propositions 4.1 and 4.5, and Theorem 2.6, we have the following result about the rate of convergence for the scheme (4.1):

**Theorem 4.6.** *Assume (A) holds, $|g_h|_1$ bounded independently of $h$, and $h\sup_{\vartheta\in\Theta}|c^\vartheta|_0 \leq 1$. If $u$ is the solution of* (1.1) *&* (1.2)*, and $u_h$ is the solution of* (4.1) *&* (1.5)*, then*

$$|u - u_h|_0 \leq C\Big(\sup_{[0,h)\times\mathbb{R}^N}|u - g_h|_0 + h^{1/4}\Big).$$

See Remark 3.8 about the condition on $g_h$. This result is in agreement with Barles & Jakobsen [1]. For first order equations, the rate is 1/2 (see Falcone & Giorgi [10]), and the same rate was obtained by Menaldi [22] for second order equations but under stronger regularity assumptions on the solutions. As opposed to Section 3, here we get the same rate of convergence as in [1]. The reason is that in this case the approximation of the second-derivatives is first order accurate only, so (C4) already had the optimal form in [1].

**Remark 4.7.** It is possible to extend the analysis of control schemes to the optimal stopping case, i.e. to include an obstacle. We have not done this because of the extra technicalities. Let us mention some of them. The result corresponding to Proposition 4.3 which is a key result, would be more complicated to prove. One way to prove it would be to adapt the "elliptic" technique of the proof of Theorem A.1. Furthermore, we will no longer get an estimate on the $t$-regularity of solutions from this result. This means that a separate analysis is necessary to obtain such an estimate.

## 5. The linear case

In this section we consider the linear case. The reason for doing this is that as opposed to the non-linear case, we will get results for finite difference schemes without having to assume that the diffusion coefficients are constant! Moreover we get fewer conditions on the schemes, essentially we only need the schemes to be consistent and have a comparison principle. In fact, the results in this Section improve *in the linear case* all previous results of this paper and [1, 16, 17].

Now consider the following linear initial value problem:

$$L(t,x,u_t,u,Du,D^2u)=0 \quad \text{in} \quad Q_T, \tag{5.1}$$

$$u(0,x)=u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{5.2}$$

where $L$ is the following linear function

$$L(t,x,m,r,p,X):=m-\frac{1}{2}\operatorname{tr}[a(t,x)X]-b(t,x)p-c(t,x)r-d(t,x).$$

The idea here is similar to the idea in Section 2. To obtain the upper bound we do exactly as before: We consider the solution $u^\varepsilon$ of

$$\inf_{\substack{|e|\le\varepsilon\\ s\in(0,\varepsilon^2)}} L(t+s,x+e,u^\varepsilon_t(t,x),u^\varepsilon(t,x),Du^\varepsilon(t,x),D^2u^\varepsilon(t,x))=0 \quad \text{in} \quad Q^\varepsilon_T, \tag{5.3}$$

$$u^\varepsilon(-\varepsilon^2,x)=u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{5.4}$$

where the coefficients have been appropriately extended. Regularize $u^\varepsilon$ by mollification, and use concavity of $L$ ($L$ is linear) to prove that the resulting function denoted by $u_\varepsilon$ is a (smooth) subsolution of (5.1) in $Q_T$. Then we plug $u_\varepsilon$ into the scheme, use the consistency condition, comparison properties for the scheme, and estimates on $u_\varepsilon$, to obtain an upper bound on $u-u_h$.

Now to get the lower bound we want to obtain a smooth *subsolution* of (5.1) approximating the true solution and satisfying the necessary bounds. If this is possible, then we get the lower bound by a similar argument as above. Since the equation is linear, such a smooth subsolution can be obtained by mollifying the solution of

$$\sup_{\substack{|e|\le\varepsilon\\ s\in(0,\varepsilon^2)}} L(t+s,x+e,\bar u^\varepsilon_t(t,x),\bar u^\varepsilon(t,x),D\bar u^\varepsilon(t,x),D^2\bar u^\varepsilon(t,x))=0 \quad \text{in} \quad Q^\varepsilon_T, \tag{5.5}$$

$$\bar u^\varepsilon(-\varepsilon^2,x)=u_0(x) \quad \text{in} \quad \mathbb{R}^N. \tag{5.6}$$

This equation now plays the same role as (5.3) did in the proof of the upper bound.

We will use conditions on the data similar to (A):

**(B)** (Conditions on data) $c\le 0$ and $a\equiv\sigma\sigma^T$ for some $N\times P$ matrix-valued function $\sigma$. Moreover there exists $\delta\in(0,1]$ such that $|\sigma|_1,|b|_1,|c|_\delta,|d|_\delta,|u_0|_\delta\le C$.

The next result is essentially a consequence of Theorem 2.1 and Lemma 2.7.

**Lemma 5.1.** *Assume (B). Then for any $0\le\varepsilon\le 1$ there exist $\mathcal{C}^\delta$-functions $u,u^\varepsilon,\bar u^\varepsilon$ which are the unique viscosity solutions of* (5.1) *&* (5.2), (5.3) *&* (5.4) *and* (5.5) *&* (5.6) *respectively, and satisfy*

$$|u^\varepsilon|_\delta+|\bar u^\varepsilon|_\delta\le C \quad \text{and} \quad |u^\varepsilon(t,x)-u(t,x)|+|\bar u^\varepsilon(t,x)-u(t,x)|\le C\varepsilon^\delta \quad \text{in} \quad \overline{Q}_T.$$

For the schemes, we now only need conditions (C1), (C2), and (C4) to be satisfied, where (C4) takes the following form:

**(C4)** (Consistency) There exists integers $n,m,k_i,\bar k>0$, $i=1,2,\dots,n$ such that for every smooth $\phi$, $\Delta t,\Delta x\ge 0$, and $(t,x)\in\overline{Q}_T$:

$$\begin{aligned}&\big|L(t,x,\phi_t,\phi,D\phi,D^2\phi)-S(h,t,x,\phi(t,x),[\phi]_{t,x})\big|\\ &\qquad\le C\left(\sum_{i=1}^n|D^i\phi|_0\Delta x^{k_i}+|(\partial_t)^m\phi|_0\Delta t^{\bar k}\right).\end{aligned}$$

In particular we do not need the restrictive Assumption 2.5, and this is the reason we obtain stronger results.

Now we state the main result in this section which says that the solution of the scheme (1.4) converges to the viscosity solution of (5.1) with given *a priori* error estimate.

**Theorem 5.2** (The Rate of Convergence). *Let (B), (C1), (C2), and (C4) hold, let $u$ be the viscosity solution of* (5.1) *&* (5.2)*, and let $u_h$ be the solution of the scheme* (1.4) *&* (1.5)*. Then if $\Delta t, \Delta x \geq 0$ are sufficiently small*

$$|u-u_h|_0 \leq C\left(\sup_{[0,\Delta t)\times\mathbb{R}^N}|u-g_h| + \Delta x^{\gamma_x} + \Delta t^{\gamma_t}\right),$$

*where*

$$\gamma_x := \min_{i=1,\dots,n}\left\{\frac{k_i}{i}\right\} \quad \textit{and} \quad \gamma_t := \frac{\bar{k}}{m}.$$

*Proof.* We only give the proof of the upper bound of $u-u_h$. The proof of the lower bound is similar, and it is in fact a special case of the proof already given for Theorem 2.6.

1. Consider (5.5) whose properties are given by Lemma 5.1. The following inequality hold in the viscosity sense for every $s \in (0,\varepsilon^2)$ and $|e| \leq \varepsilon$

$$L(t+s, x+e, \bar{u}^\varepsilon_t(t,x), \bar{u}^\varepsilon(t,x), D\bar{u}^\varepsilon(t,x) D^2\bar{u}^\varepsilon(t,x)) \leq 0 \quad \text{in} \quad Q^\varepsilon_T,$$

which implies implies that $\bar{u}^\varepsilon(t-s,x-e)$ is a subsolution of (5.1) in $Q_T$.

2. We regularize $\bar{u}^\varepsilon$ and define $\bar{u}_\varepsilon := \bar{u}^\varepsilon * \rho_\varepsilon$ where $\rho_\varepsilon$ is defined in (2.1). By an argument like the one leading to Lemma 2.8, $\bar{u}_\varepsilon$ satisfy $L[\bar{u}_\varepsilon] \leq 0$ in $Q_T$ in the viscosity sense.

3. By properties of mollifiers, $\bar{u}_\varepsilon \in C^\infty(Q_T)$ with $|(\partial_t)^m \bar{u}_\varepsilon|_0 \leq C(\varepsilon^2)^{\delta/2-m}$ and $|D^i\bar{u}_\varepsilon|_0 \leq C\varepsilon^{\delta-i}$. By consistency (C4) and since $L[\bar{u}_\varepsilon] \leq 0$ in $Q_T$ by 2, we deduce that

$$S(h,t,y,\bar{u}_\varepsilon(t,y),[\bar{u}_\varepsilon]_{t,y}) \leq C\left(\sum_{i=1}^n \varepsilon^{\delta-i}\Delta x^{k_i} + \varepsilon^{\delta-2m}\Delta t^{\bar{k}}\right).$$

4. By comparison Lemma 2.4 (needs (C1) and (C2)), we see that in $\overline{Q}_T$

$$\bar{u}_\varepsilon - u_h \leq \sup_{[0,\Delta t]\times\mathbb{R}^N}|\bar{u}_\varepsilon - u_h| + C\left(\sum_{i=1}^n \varepsilon^{\delta-i}\Delta x^{k_i} + \varepsilon^{\delta-2m}\Delta t^{\bar{k}}\right).$$

5. The properties of mollifiers and the uniform boundedness in $\mathcal{C}^\delta(\overline{Q}^\varepsilon_T)$ of $\{\bar{u}^\varepsilon\}_\varepsilon$ imply $|\bar{u}^\varepsilon(t,x) - \bar{u}_\varepsilon(t,x)| \leq C\varepsilon^\delta$ in $\overline{Q}_T$. Moreover from Lemma 5.1 it follows that $|u(t,x) - \bar{u}^\varepsilon(t,x)| \leq C\varepsilon^\delta$ in $\overline{Q}_T$, so we can conclude that $|u-\bar{u}_\varepsilon|_0 \leq C\varepsilon^\delta$.

6. Combining 4 and 5 and choosing $\varepsilon$ as in the proof of Theorem 2.6 then leads to

$$u-u_h \leq \sup_{[0,\Delta t]\times\mathbb{R}^N}|u-u_h| + C\left(\Delta t^{\gamma_t} + \Delta x^{\gamma_x}\right) \quad \text{in} \quad \overline{Q}_T.$$

This completes the proof of Theorem 5.2. □

Now we consider the class of monotone finite difference schemes defined in Section 3 allowing the diffusion coefficients to vary in $t,x$. The stability conditions corresponding to (3.1) and (3.2) then becomes:

$$a_{ii}(t,x) - \sum_{j\neq i} |a_{ij}(t,x)| \geq 0, \quad i = 1,\dots,N, \quad \text{in} \quad \overline{Q}_T, \tag{5.7}$$

$$\frac{\Delta t}{\Delta x^2} \sum_{i=1}^{N} \Big\{ a_{ii}(t,x) - \sum_{j\neq i} |a_{ij}(t,x)| + \Delta x |b_i(t,x)| \Big\} - \Delta t c(t,x) \leq 1 \quad \text{in} \quad \overline{Q}_T. \tag{5.8}$$

We proceed to the definition of the scheme. For each $x$, $t$, $r$, $p_i^{\pm}$, $A_{ii}$, $A_{ij}^{\pm}$, $i,j = 1,\dots,N$, let

$$\begin{aligned}\widetilde{L}(t,x,r,p_i^{\pm},A_{ii},A_{ij}^{\pm}) = \sum_{i=1}^{N} \Big[ &- \frac{a_{ii}(t,x)}{2} A_{ii} + \sum_{j\neq i} \Big( - \frac{a_{ij}^{\vartheta+}(t,x)}{2} A_{ij} + \frac{a_{ij}^{\vartheta-}(t,x)}{2} A_{ij} \Big) \\ &- b_i^{\vartheta+}(t,x) p_i^+ + b_i^{\vartheta-}(t,x) p_i^- \Big] - c(t,x) r - d(t,x).\end{aligned}$$

The scheme can then be defined as follows:

$$\begin{aligned}&\frac{u_h(t+\Delta t,x) - u_h(t,x)}{\Delta t} \\ &+ \widetilde{L}(t,x,u_h(t,x),\Delta_{x_i}^{\pm} u_h(t,x),\Delta_{x_i}^2 u_h(t,x),\Delta_{x_i x_j}^{\pm} u_h(t,x)) = 0,\end{aligned} \tag{5.9}$$

for any $(t,x) \in \{t_1,t_2,\dots,t_{N_t}\} \times \Delta x \mathbb{Z}^N$. See Section 3 for the definitions of the differencing operators. We also refer to Section 3 to see how $S$ can be defined in this case. We just mention that (5.9) corresponds to $S$ being evaluated at time $t + \Delta t$, and that $[u_h]_{t+\Delta t,x}$ in this case contains all the relevant $u_h$-values at time $t$. Monotonicity in this case is a consequence of the fact that if (5.7) and (5.8) hold, then all coefficients in (5.9) of $u_h(t,\cdot)$-terms are negative. It is not difficult to see that the following result now holds:

**Lemma 5.3.** *If (B),* (5.7)*,* (5.8) *hold, then the scheme* (5.9) *satisfy (C1) and (C2). Furthermore (C4) holds such that for smooth* $\phi$

$$\begin{aligned}\Big| \big[ L(s,x,\phi_t,\phi,D\phi,D^2\phi) \big]_{s=t-\Delta t} &- S(h,t,x,\phi(t,x),[\phi]_{t,x}) \Big| \\ &\leq C \left( |D^2\phi|_0 \Delta x + |D^4\phi|_0 \Delta x^2 + |\phi_{tt}|_0 \Delta t \right).\end{aligned}$$

It follows from this result that $\gamma_x = \delta/2$ and $\gamma_t = \delta/4$ where $\gamma_x, \gamma_t$ were defined in Theorem 5.2. By Lemma 5.3 and Theorem 5.2 we get the following result giving the rate of convergence for the scheme (5.9):

**Proposition 5.4.** *Assume (B),* (5.7)*,* (5.8) *hold. If $u$ is the solution of* (5.1) *&* (5.2)*, and $u_h$ the solution of* (5.9) *&* (1.5) *then*

$$|u - u_h|_0 \leq C \Big( \sup_{[0,\Delta t)\times\mathbb{R}^N} |u_0 - g_h| + \Delta x^{\delta/2} \Big).$$

If the data is Lipschitz continuous ($\delta = 1$), then the rate is $1/2$. The result gives the same rate of convergence as we obtained in Section 3, but this result is stronger in the linear case since $a$ now can vary in $t, x$. Of course, this result also improves the results of [1, 16, 17] in the linear case: In [1, 16] they were not able to handle the case of variable $a$ and essentially got the rate $1/3$, while in [17] the convergence rate obtained for variable $a$ (in the non-linear problem) was quite low: $1/27$.

## Appendix A. Proof of Lemma 2.7

The key ingredient in the proof of Lemma 2.7 is the continuous dependence result given below. This result is an extension to the case of parabolic variational inequalities of results in [15], and the extra technicalities are due to the obstacle.

**Theorem A.1.** *Let $u, \bar{u} \in C(\overline{Q}_T)$ be solutions of (1.1) with coefficients $\{a^\vartheta,\ b^\vartheta,\ c^\vartheta,\ d^\vartheta, f\}$ and $\{\bar{a}^\vartheta,\ \bar{b}^\vartheta,\ \bar{c}^\vartheta,\ \bar{d}^\vartheta,\ \bar{f}\}$ respectively. Assume (A) holds for both sets of coefficients with constants $M, \bar{M}$ and common $\delta$. Then there is a constant $\bar{C}$ depending only on $M$, $\bar{M}$, $\delta$, and $T$ such that*

$$|u-\bar{u}|_0 \le \bar{C}\Big(|u(0,\cdot)-\bar{u}(0,\cdot)|_0 + \sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta-\bar{\sigma}^\vartheta|_0^\delta + |b^\vartheta-\bar{b}^\vartheta|_0^\delta\big\} + \sup_{\vartheta\in\Theta}\big\{|c^\vartheta-\bar{c}^\vartheta|_0 + |d^\vartheta-\bar{d}^\vartheta|_0\big\} + |f-\bar{f}|_0\Big).$$

*Proof.* First we may assume that $c^\vartheta, \bar{c}^\vartheta \le -1$, i.e. that the coefficients multiplying the $u$ and $\bar{u}$ terms in the parabolic parts are bigger than 1. If this was not true, simply consider $v = e^{-Ct}u$ and $\bar{v} = e^{-Ct}\bar{u}$ where $C = 1 + \sup_\Theta \max\{|c^\vartheta|_0, |\bar{c}^\vartheta|_0\}$. The new functions satisfy

$$\min\big\{v_t + Cv + e^{-Ct}g(t,x,e^{Ct}v, e^{Ct}Dv, e^{Ct}D^2v), v - e^{-Ct}f\big\} = 0,$$
$$\min\big\{\bar{v}_t + C\bar{v} + e^{-Ct}g(t,x,e^{Ct}\bar{v}, e^{Ct}D\bar{v}, e^{Ct}D^2\bar{v}), \bar{v} - e^{-Ct}f\big\} = 0.$$

Here it is easy to see that the coefficients multiplying the $v$ and $\bar{v}$ terms in parabolic parts are bigger than 1. So if the result now holds, we get a bound on $|v-\bar{v}|_0$. A back-substitution then yields the result for $|u-\bar{u}|_0$ (but with a different constant $\bar{C}$).

Now to continue we define $\phi(t,x,y) := e^{\lambda t}\alpha|x-y|^2 + \varepsilon(|x|^2+|y|^2)$, and $\psi(t,x,y) := u(t,x) - \bar{u}(t,y) - \phi(t,x,y)$ in $[0,T]\times\mathbb{R}^N\times\mathbb{R}^N$. Then we set $m^0_{\alpha,\varepsilon} := \sup_{\mathbb{R}^N\times\mathbb{R}^N}\psi(0,x,y)^+$ and $m_{\alpha,\varepsilon} := \sup_{[0,T]\times\mathbb{R}^N\times\mathbb{R}^N}\psi(t,x,y) - m^0_{\alpha,\varepsilon}$. By classical arguments, there exists $t_0 \in [0,T]$ and $x_0, y_0 \in \mathbb{R}^N$ such that $m_{\alpha,\varepsilon} + m^0_{\alpha,\varepsilon} = \psi(t_0,x_0,y_0)$. Note that $t_0, x_0, y_0$ depends on $\alpha$ and $\varepsilon$.

We assume that $m_{\alpha,\varepsilon} > 0$ and derive a (positive) upper bound for this quantity. Of course this upper bound still holds if $m_{\alpha,\varepsilon} \le 0$. Note that this assumption implies that $t_0 > 0$, because if $t_0 = 0$, then $m_{\alpha,\varepsilon} = \sup_{\mathbb{R}^N\times\mathbb{R}^N}\psi(0,x,y) - m^0_{\alpha,\varepsilon} \le 0$.

By the (parabolic) maximum principle for semicontinuous functions, Theorem 8.3 in [6], there are $a, b \in \mathbb{R}$ and $X, Y \in \mathcal{S}^N$ such that $(a, D_x\phi(t_0,x_0,y_0), X) \in \bar{\mathcal{P}}^{2,+}u(t_0,x_0)$ and $(b, -D_y\phi(t_0,x_0,y_0), Y) \in \bar{\mathcal{P}}^{2,-}\bar{u}(t_0,y_0)$. Moreover, $a - b = \phi_t(t_0,x_0,y_0)$ and the following inequality holds for some constant $k > 0$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le ke^{\lambda t}\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + k\varepsilon\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{A.1}$$

Subtracting the viscosity solutions' inequalities we obtain after using the definitions of viscosity sub- and supersolutions, and using the inequality $\inf(\cdots) - \inf(\cdots) \ge \inf(\cdots - \cdots)$ twice we get

$$\begin{aligned} 0 \ge \min\Big\{&\lambda e^{\lambda t}\alpha|x_0-y_0|^2 + \inf_{\vartheta\in\Theta}\Big\{-\frac{1}{2}\operatorname{tr}[a^\vartheta(t_0,x_0)X - \bar{a}^\vartheta(t_0,y_0)Y] \\ &\quad - b^\vartheta(t_0,x_0)\big(2e^{\lambda t}\alpha(x_0-y_0) + 2\varepsilon x_0\big) \\ &\quad + \bar{b}^\vartheta(t_0,y_0)\big(2e^{\lambda t}\alpha(x_0-y_0) - 2\varepsilon y_0\big) \\ &\quad - c^\vartheta(t_0,x_0)u(t_0,x_0) + \bar{c}^\vartheta(t_0,y_0)\bar{u}(t_0,y_0) \\ &\quad - d^\vartheta(t_0,x_0) + \bar{d}^\vartheta(t_0,y_0)\Big\}, \\ &u(t_0,x_0) - \bar{u}(t_0,y_0) - f(t_0,x_0) + \bar{f}(t_0,y_0)\Big\}. \end{aligned} \tag{A.2}$$

So either the first term in the minimum is less than or equal to zero, or so is the second term. If it is the second term, then because of the regularity of $f, \bar{f}$

$$m_{\alpha,\varepsilon} \le u(t_0,x_0) - \bar{u}(t_0,y_0) \le |f-\bar{f}|_0 + C|x_0-y_0|^\delta. \tag{A.3}$$

Now we assume that the first term in the minimum in (A.2) is less than or equal to zero. By the computations given in Ishii and Lions [13, p. 35], the inequality (A.1), and the inequality $(s+t)^2 \leq 2(s^2+t^2)$ for $s,t \in \mathbb{R}$, we get

$$-\operatorname{tr}[a^\vartheta(t_0,x_0)X - \bar{a}^\vartheta(t_0,y_0)Y] \geq -2ke^{\lambda t}\alpha\Big\{|\sigma^\vartheta(t_0,x_0) - \bar{\sigma}^\vartheta(t_0,x_0)|^2 + |\bar{\sigma}^\vartheta(t_0,x_0) - \bar{\sigma}^\vartheta(t_0,y_0)|^2\Big\} - k\varepsilon\Big\{|\sigma^\vartheta(t_0,x_0)|^2 + |\bar{\sigma}^\vartheta(t_0,y_0)|^2\Big\}.$$

Furthermore the following estimates hold

$$\begin{aligned}
&-\big(b^\vartheta(t_0,x_0) - \bar{b}^\vartheta(t_0,y_0)\big)(x_0-y_0)\\
&\quad \geq -2|b^\vartheta(t_0,x_0) - \bar{b}^\vartheta(t_0,x_0)|^2 - 2|x_0-y_0|^2 - |\bar{b}^\vartheta(t_0,x_0) - \bar{b}^\vartheta(t_0,y_0)||x_0-y_0|,\\
&-c^\vartheta(t_0,x_0)u(t_0,x_0) + \bar{c}^\vartheta(t_0,y_0)\bar{u}(t_0,y_0)\\
&\quad \geq -|u(t_0,x_0)||c^\vartheta(t_0,x_0) - \bar{c}^\vartheta(t_0,x_0)| - |\bar{u}(t_0,y_0)||\bar{c}^\vartheta(t_0,x_0) - \bar{c}^\vartheta(t_0,y_0)| + m_{\alpha,\varepsilon}.
\end{aligned}$$

In the second estimate we used that $u(t_0,x_0) = \bar{u}(t_0,y_0) + \phi(t_0,x_0,y_0) + m_{\alpha,\varepsilon} \geq \bar{u}(t_0,y_0) + m_{\alpha,\varepsilon}$ and the fact that $c^\vartheta, \bar{c}^\vartheta \leq -1$. So in particular $-\bar{c}^\vartheta(t_0,x_0)(u(t_0,x_0) - \bar{u}(t_0,y_0)) \geq u(t_0,x_0) - \bar{u}(t_0,y_0) \geq m_{\alpha,\varepsilon}$! Inserting all these estimates into the parabolic part of (A.2) and using (A) yield

$$\begin{aligned}
&\lambda e^{\lambda t}\alpha|x_0-y_0|^2 + m_{\alpha,\varepsilon} \leq\\
&2ke^{\lambda t}\alpha \sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + |b^\vartheta - \bar{b}^\vartheta|_0^2\big\} + \sup_{\vartheta\in\Theta}\big\{|u|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |d^\vartheta - \bar{d}^\vartheta|_0\big\}\\
&+ k_1 e^{\lambda t}\alpha|x_0-y_0|^2 + k_2|x_0-y_0|^\delta + \varepsilon\,\mathrm{Const}\,(1 + |x_0|^2 + |y_0|^2)
\end{aligned} \tag{A.4}$$

where $k_1 = \sup_{\vartheta\in\Theta}\{k[\bar{\sigma}^\vartheta]_1^2 + 4 + 2[\bar{b}^\vartheta]_1\}$ and $k_2 = \sup_{\vartheta\in\Theta}\{|\bar{u}|_0[\bar{c}^\vartheta]_\delta + [\bar{d}^\vartheta]_\delta\}$.

Compare (A.3) and (A.4). They give upper bounds on $m_{\alpha,\varepsilon}$ in the two situations that can occur. We add these bounds to get a bound holding in all situations. Furthermore, we choose $\lambda = k_1 + 1$, collect all $\alpha|x_0-y_0|^2$ and $|x_0-y_0|^\delta$ terms, and maximize with respect to $r = |x_0-y_0|$ noting that

$$\max_{r\geq 0}(r^\delta - \alpha r^2) = C\alpha^{-1/(2-\delta)}.$$

The result is the following estimate

$$\begin{aligned}
m_{\alpha,\varepsilon} \leq |f-\bar{f}|_0 &+ \sup_{\vartheta\in\Theta}\big\{|u|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |d^\vartheta - \bar{d}^\vartheta|_0\big\} + \omega_\alpha(\varepsilon)\\
&+ 2ke^{\lambda t}\alpha\sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + |b^\vartheta - \bar{b}^\vartheta|_0^2\big\} + C\alpha^{-1/(2-\delta)},
\end{aligned} \tag{A.5}$$

where $\omega_\alpha$ is a modulus for fixed $\alpha$ (i.e. $\omega_\alpha : [0,\infty) \to [0,\infty)$, continuous, non-decreasing, $\omega_\alpha(0) = 0$) such that $\varepsilon(|x_0|^2 + |y_0|^2) = \omega_\alpha(\varepsilon)$. This last fact is a consequence of $\lim_{\varepsilon\to 0}\varepsilon(|x_0|^2 + |y_0|^2) = 0$ for fixed $\alpha$, see e.g. Lemma 2.3 in [14].

By the definition of $m_{\alpha,\varepsilon}$, we see that for any $(t,x) \in \overline{Q_T}$

$$u(t,x) - \bar{u}(t,x) - 2\varepsilon|x|^2 \leq m_{\alpha,\varepsilon} + m^0_{\alpha,\varepsilon}. \tag{A.6}$$

Furthermore, Hölder continuity of initial values yields

$$m^0_{\alpha,\varepsilon} \leq |u(0,\cdot) - \bar{u}(0,\cdot)|_0 + C|x_0-y_0|^\delta - \alpha|x_0-y_0|^2 \leq |u(0,\cdot) - \bar{u}(0,\cdot)|_0 + C\alpha^{-1/(2-\delta)},$$

so inserting (A.5) into (A.6) leads to

$$\begin{aligned}
u(t,x) - \bar{u}(t,x) \leq |u(0,\cdot) - \bar{u}(0,\cdot)|_0 &+ \sup_{\vartheta\in\Theta}\big\{|u|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |d^\vartheta - \bar{d}^\vartheta|_0\big\}\\
&+ 2ke^{\lambda t}\alpha\sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + |b^\vartheta - \bar{b}^\vartheta|_0^2\big\} + C\alpha^{-1/(2-\delta)}\\
&+ |f-\bar{f}|_0 + \omega_\alpha(\varepsilon) + 2\varepsilon|x|^2,
\end{aligned}$$

Since this inequality holds for all $\alpha > 0$, we choose the $\alpha$ minimizing the right hand side. Note that this $\alpha$ is finite and non-zero, and note also that for any $k > 0$

$$\min_{\alpha>0}(k\alpha + \alpha^{-1/(2-\delta)}) = Ck^{\delta/2}.$$

Finally we send $\varepsilon$ to 0 and obtain the following estimate

$$u(t,x) - \bar{u}(t,x) \le |u(0,\cdot) - \bar{u}(0,\cdot)|_0 + \sup_{\vartheta\in\Theta}\left\{|u|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |d^\vartheta - \bar{d}^\vartheta|_0\right\}$$
$$C\sup_{\vartheta\in\Theta}\left\{|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + |b^\vartheta - \bar{b}^\vartheta|_0^2\right\}^{\delta/2} + |f - \bar{f}|_0.$$

Now we can conclude since $(s^2+t^2)^{\delta/2} \le |t|^\delta + |s|^\delta$ for any $s,t \in \mathbb{R}$, and since the argument is symmetric in $u$ and $\bar{u}$. □

For a more detailed proof of a similar result, see [15]. Now we give the

*Proof of Lemma 2.7.* Equation (1.6) can be viewed as a special case of equation (1.1) by considering $(\vartheta, s, e)$ as the new control parameter, $\Theta \times (0,\varepsilon^2) \times B(0,\varepsilon)$ as the new space of controls, and, via a scaling in time, $Q_T^\varepsilon$ as the new domain. So existence, uniqueness, and regularity of a bounded viscosity solution follow from Theorem 2.1. The inequality for $|u^\varepsilon(t,x) - u(t,x)|$ follows from the regularity of the data and the continuous dependence result Theorem A.1. □

## Appendix B. The proof of Proposition 4.3

The proof given here is a parabolic version of the elliptic proof given in [1]. It can also be seen as a "discrete" version of arguments given in [15]. We will give a doubling of variables argument mimicking the corresponding PDE proof. In the place of the so-called maximum principle for semicontinuous functions, we introduce a new scheme for a problem in $[0,T] \times \mathbb{R}^{2N}$. This scheme will be related to the original $\overline{Q}_T$ schemes in such a way that $u(t,x) - \bar{u}(t,y)$ will be a subsolution. Moreover it will operate on the test function $|x-y|^2$ in the way we hope for. This new scheme is roughly speaking based on replacing the operator $\Pi_h^\vartheta$ in (4.1) by the operator $\Delta_h^\vartheta$ defined as

$$\Delta_h^\vartheta g(t,x,y) =$$
$$\frac{1}{2N}\sum_{m=1}^{N}\Big[g\big(t, x + hb^\vartheta(t,x) + \sqrt{h}\sigma_m^\vartheta(t,x), y + h\bar{b}^\vartheta(t,y) + \sqrt{h}\bar{\sigma}_m^\vartheta(t,y)\big)$$
$$+ g\big(t, x + hb^\vartheta(t,x) - \sqrt{h}\sigma_m^\vartheta(t,x), y + h\bar{b}^\vartheta(t,y) - \sqrt{h}\bar{\sigma}_m^\vartheta(t,y)\big)\Big],$$

and letting the new scheme act on functions defined on $[0,T] \times \mathbb{R}^{2N}$.

We proceed with the doubling of variable argument. Define

$$E_0^h = \sup_{[0,h]\times\mathbb{R}^{2N}}\Big(u(s,x) - \bar{u}(s,y) - \phi(0,x,y)\Big)^+,$$
$$E = -E_0^h + \sup_{[0,t]\times\mathbb{R}^{2N}}\Big\{u(s,x) - \bar{u}(s,y) - \phi(s,x,y))\Big\},$$
$$\psi(s,x,y) = u(s,x) - \bar{u}(s,y) - \frac{\delta E s}{t} - \phi(s,x,y), \quad 0 < \delta < 1,$$

where

$$\phi(s,x,y) = \alpha(1+\gamma)^{s/h}|x-y|^2 + \varepsilon(|x|^2 + |y|^2),$$

and $\gamma > 0$ will be determined later. The purpose of the following calculations is to establish an upper bound on $E$, so we assume that $E > 0$ (if not, 0 would be an upper bound). By standard arguments there is a point $(s_0, x_0, y_0) \in [0, t] \times \mathbb{R}^{2N}$ such that

$$m := \sup_{[0,t]\times\mathbb{R}^{2N}} \psi = \psi(s_0, x_0, y_0).$$

Note that $s_0 \notin [0, h]$, because then $\sup \psi \le v(s_0, x_0) - \bar{u}(s_0, y_0) - \phi(s_0, x_0, y_0) \le E_0^h$ which contradicts the fact that $\sup \psi \ge E_0^h + (1-\delta)E > E_0^h$.

Now subtract $u(s_0, x_0)$ and $\bar{u}(s_0, y_0)$, use (4.1), the inequality $\inf\{\cdots\} - \inf\{\dots\} \le \sup\{\cdots - \dots\}$, and the fact that $\Delta_h^\vartheta(u(s,x) - \bar{u}(s,y)) = \Pi_h^\vartheta u(s,x) - \bar{\Pi}_h^\vartheta \bar{u}(s,y)$ to get the following inequality:

$$\begin{aligned}
&u(s_0, x_0) - \bar{u}(s_0, y_0) \\
&\le \sup_{\vartheta\in\Theta} \Big\{ (1 - hc^\vartheta(s_0 - h, x_0))\Delta_h^\vartheta(u(s_0 - h, x_0) - \bar{u}(s_0 - h, y_0)) \\
&\qquad + h|\bar{u}|_0([c^\vartheta]_{,1}|x_0 - y_0| + |c^\vartheta - \bar{c}^\vartheta|_0) \\
&\qquad + h([d^\vartheta]_{,1}|x_0 - y_0| + |d^\vartheta - \bar{d}^\vartheta|_0)\Big\}.
\end{aligned} \tag{B.1}$$

Now since $\psi(s_0, x_0, y_0) \ge \psi(s_0 - h, x_0, y_0)$ the following hold:

$$u(s_0, x_0) - \bar{u}(s_0, y_0) = m + \frac{\delta E s_0}{t} + \phi(s_0, x_0, y_0),$$
$$u(s_0 - h, x_0) - \bar{u}(s_0 - h, y_0) \le m + \frac{\delta E(s_0 - h)}{t} + \phi(s_0 - h, x_0, y_0).$$

Furthermore note that $(\Delta_h^\vartheta - 1)(m + \frac{\delta E(s_0-h)}{t}) = 0$, and that after easy computations using (A) we get

$$(\Delta_h^\vartheta - 1)|x - y|^2 \le C(|x - y|^2 + h|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + h^2|b^\vartheta - \bar{b}^\vartheta|_0^2).$$

In a similar way $(\Delta_h^\vartheta - 1)(|x|^2 + |y|^2) \le C(1 + |x|^2 + |y|^2)$. Using the above estimates in (B.1) then yield

$$\begin{aligned}
&m + \frac{\delta E s_0}{t} + \phi(s_0, x_0, y_0) \\
&\le \big(1 + hc^\vartheta(s_0 - h, x_0)\big)\Big(m + \frac{\delta E(s_0 - h)}{t} + \phi(s_0 - h, x_0, y_0)\Big) \\
&\quad + C\Big(h|x_0 - y_0| + \alpha(1+\gamma)^{\frac{s_0-h}{h}}|x_0 - y_0|^2\Big) + \varepsilon C(1 + |x_0|^2 + |y_0|^2) \\
&\quad + h\sup_{\vartheta\in\Theta}\Big\{\alpha C\big(|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + |b^\vartheta - \bar{b}^\vartheta|_0^2\big) + |\bar{u}|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |d^\vartheta - \bar{d}^\vartheta|_0\Big\}.
\end{aligned}$$

Note that $\phi(s_0, x_0, y_0) - \phi(s_0 - h, x_0, y_0) = \alpha\gamma(1+\gamma)^{\frac{s_0-h}{h}}|x_0 - y_0|^2$, and since $c^\vartheta \le 0$ we have

$$\begin{aligned}
\frac{\delta E}{t} &\le \frac{C - \gamma}{h}\alpha(1+\gamma)^{\frac{s_0-h}{h}}|x_0 - y_0|^2 + C|x_0 - y_0| + \frac{\varepsilon}{h}C(1 + |x_0|^2 + |y_0|^2) \\
&\quad + \sup_{\vartheta\in\Theta}\Big\{\alpha C\big(|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^2 + |b^\vartheta - \bar{b}^\vartheta|_0^2\big) + |\bar{u}|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |d^\vartheta - \bar{d}^\vartheta|_0\Big\}.
\end{aligned} \tag{B.2}$$

Now choose $\gamma$ such that $C - \gamma = -h$. The terms in the above equation involving $|x_0 - y_0| =: r$ then takes the form $Cr - \alpha Cr^2$. Maximizing with respect to $r$ yields

$$\max_{r>0}\{Cr - \alpha Cr^2\} = C\alpha^{-1}.$$

Furthermore by a standard argument (see e.g. [14, Lemma 2.3]) $\varepsilon(|x_0|^2+|y_0|_0^2)\to 0$ as $\varepsilon\to 0$. So for some modulus $\omega$ (B.2) can be reduced to

$$\frac{\delta E}{t}\le \sup_{\vartheta\in\Theta}\Big\{\alpha C\big(|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+|b^\vartheta-\bar b^\vartheta|_0^2\big)+|\bar u|_0|c^\vartheta-\bar c^\vartheta|_0+|d_h^\vartheta-\bar d_h^\vartheta|_0\Big\}$$
$$+C\alpha^{-1}+\frac{1}{h}\omega(\varepsilon).$$

To continue, we also need the estimate

$$E_0^h\le |g_h-\bar g_h|_0+C\alpha^{-1},$$

which is obtained by using $x$-Lipschitz continuity of $g_h$ or $\bar g_h$ and maximizing with respect to $|x_0-y_0|$.

Now by the definition of $E$, for any $x\in\mathbb{R}^N$ and $s\in[0,t]$ we have $u(s,x)-\bar u(s,x)\le E_0^h+E+2\varepsilon|x|^2$. Using the previous estimates we get:

$$u(s,x)-\bar u(s,x)\le |g_h-\bar g_h|_0+(1+\frac{1}{\delta})C\alpha^{-1}+\frac{1}{\delta h}\omega(\varepsilon)+2\varepsilon|x|^2$$
$$+\frac{t}{\delta}\sup_{\vartheta\in\Theta}\Big\{|\bar u|_0|c^\vartheta-\bar c^\vartheta|_0+|d_h^\vartheta-\bar d_h^\vartheta|_0+\alpha C(|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+|b^\vartheta-\bar b^\vartheta|_0^2)\Big\}.$$

Letting $\varepsilon\to 0$, $\delta\to 1$, minimizing with respect to $\alpha$, and finally further approximations, we have the following:

$$|(u(t,\cdot)-\bar u(t,\cdot))^+|_0\le |g_h-\bar g_h|_0+\sqrt{t}C\sup_{\vartheta\in\Theta}\Big\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0+|b^\vartheta-\bar b^\vartheta|_0\Big\}$$
$$+t\sup_{\vartheta\in\Theta}\Big\{|\bar u|_0|c^\vartheta-\bar c^\vartheta|_0+|d^\vartheta-\bar d^\vartheta|_0\Big\}.$$

The proof is complete by noting that we could have interchanged $|\bar u|_0$ with $|u|_0$ (so we get the factor $|u|_0\wedge|\bar u|_0$ in front of the $|c^\vartheta-\bar c^\vartheta|_0$ term).

## References


[1] G. Barles and E. R. Jakobsen. On the convergence rate for approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.* 36(1):33–54, 2002.

[2] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3):271–283, 1991.

[3] M. Brennan and E. Schwartz. The valuation of American put options. *J. Finance*, 32(2):449-462, 1977.

[4] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes. *RAIRO, Modelisation Math. Anal. Numer.*, 29(1):97-122, 1995.

[5] I. Capuzzo-Dolcetta. On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming. *Appl. Math. Optim.* 10:367-377, 1983.

[6] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[7] M. G. Crandall and P.-L. Lions. Two approximations of solutions of Hamilton-Jacobi equations. *Math. Comp.*, 43(167):1–19, 1984.

[8] M. Falcone. A numerical approach to the infinite horizon problem of deterministic control theory. *Appl. Math. Optim.* 15(1):1–13, 1987.

[9] I. Capuzzo-Dolcetta and M. Falcone. Discrete dynamic programming and viscosity solutions of the Bellman equation. *Ann. Inst. H. Poincar Anal. Non Linaire* 6:161–183, 1989.

[10] M. Falcone and T. Giorgi. An approximation scheme for evolutive Hamilton-Jacobi equations. In *Stochastic analysis, control, optimization and applications*, 289–303, Systems Control Found. Appl., Birkhuser, Boston, 1999.

[11] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York, 1993.

[12] R. Glowinski, J. L. Lions, and R. Trmolires. *Numerical Analysis of Variational Inequalities.* North-Holland Publishing Co., Amsterdam-New York, 1981.

[13] H. Ishii and P. L. Lions. Viscosity solutions of Fully Nonlinear Second-Order Elliptic Partial Differential Equations. *J. Differential Equations*, 83:26-78, 1990.

[14] E. R. Jakobsen and K. H. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate elliptic equations. *Electron. J. Diff. Eqns.* Vol. 2002(39):1–10, 2002.

[15] E. R. Jakobsen and K. H. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations. *J. Differential Equations.* To appear.

[16] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations. *St. Petersburg Math. J.*, 9(3):639–650, 1997.

[17] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients. *Probab. Theory Ralat. Fields*, 117:1–16, 2000.

[18] H. J. Kushner and P. Dupuis. *Numerical methods for approximations in stochastic control problems in continuous time.* Springer-Verlag, New York, 1992.

[19] P. Jaillet, D. Lamberton, and B. Lapeyre. Variational inequalities and the pricing of American options. *Acta Appl. Math.* 21(3):263–289, 1990.

[20] P. L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations I: The dynamic programming principle and applications. *Comm. Partial Diff. Equations*, 8:1101-1174, 1983.

[21] P.-L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part II: Viscosity solutions and uniqueness. *Comm. Partial Differential Equations* 8(11):1229–1276, 1983.

[22] J. L. Menaldi Some estimates for finite difference approximations. *SIAM J. Control and Optimization*, Vol. 27, No 3, pp. 579-607, 1989.

[23] H. Pham. Optimal stopping of controlled diffusion processes: A viscosity solution approach. *J. Math. Systems Estim. Control* 8(1):1–27, 1998.

[24] P. E. Souganidis. Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations*, 59(1):1–43, 1985.

[25] P. Wilmott, J. Dewynne, and S. Howison. *Option Pricing: Mathematical Models and Computations.* Oxford Financial Press, Oxford, 1993.

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