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$W^{2,\infty}$ REGULARIZING EFFECT IN A NONLINEAR, DEGENERATE PARABOLIC EQUATION IN ONE SPACE DIMENSION

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ABSTRACT. In this paper we provide and analyze a nonlinear *degenerate* parabolic equation in one space dimension with the following smoothing property: If the initial data is only uniformly continuous, at positive times, the solution has bounded second derivatives (it belongs to $W^{2,\infty}$). We call this surprising phenomenon a $W^{2,\infty}$ regularizing effect. So far, such phenomena have only been observed in *uniformly* parabolic equations.

1. INTRODUCTION

In this paper we are going to study the following initial value problem:

$$\begin{aligned} &u_t + u_x^2 + \max(0, -u_{xx}) = 0 \quad \text{in} \quad (0,T)\times\mathbb{R} := Q_T,\\ &u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}. \end{aligned} \tag{1.1}$$

This is an example of a problem that is nonlinear and *degenerate* parabolic, but still has "smooth" solutions at positive times, even if the initial data is not "smooth". Note that the nonlinearity is nonsmooth. We will show below that if the initial values are only uniformly continuous, then the solution $u(t,\cdot)$ belongs to the Sobolev space $W^{2,\infty}(\mathbb{R})$ for any $t \in (0,T]$. We call this a $W^{2,\infty}$ regularizing effect. Actually the solution u belongs to the parabolic Sobolev space $W^{2,\infty}([\varepsilon,T]\times\mathbb{R})$ for all $\varepsilon \in (0,T)$, and this implies that the equation is satisfied almost everywhere.

This is a surprising phenomenon, which to the best of the author's knowledge, has only been observed before in equations that are *uniformly* parabolic. In general, degenerate parabolic equations do not have smooth solutions, not even when the initial data is smooth. However, regularizing effects have been studied in the subclasses of uniformly parabolic equations [6, 14, 17, 18, 19], and first-order Hamilton-Jacobi equations [15, 16, 7, 4, 2]. In the first case (for convex equations), the solutions typically become continuously differentiable, twice in x and once in t. In the second case (for strictly convex Hamiltonians), the solutions typically become Lipschitz continuous and x-semiconcave.

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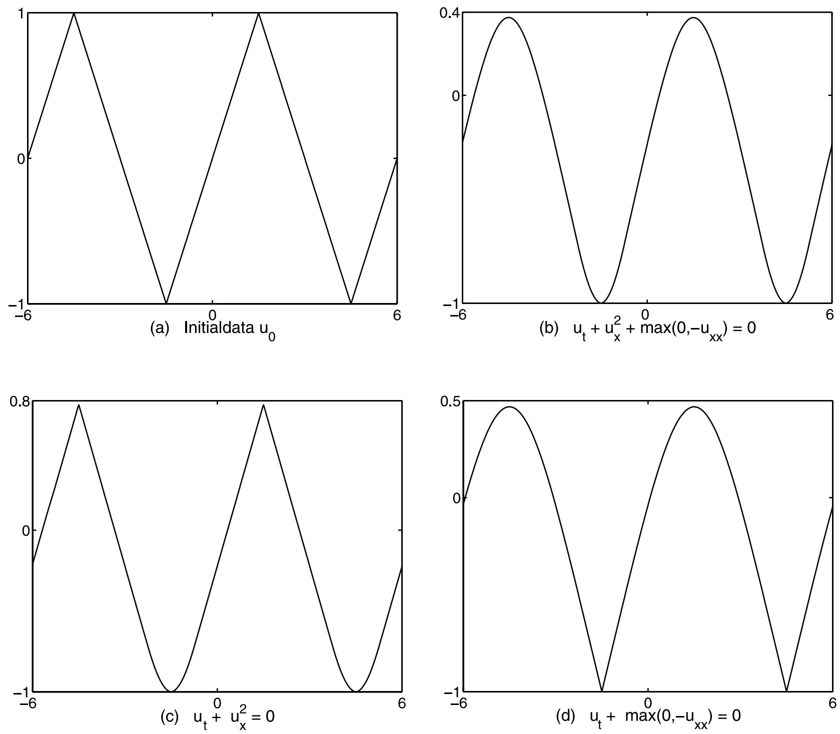

FIGURE 1. Solution of $u_t + f(u_x, u_{xx}) = 0$, $u|_{t=0} = u_0$ at $t = 0.5$: (b) $f = u_x^2 + \max(0, -u_{xx})$, (c) $f = u_x^2$, (d) $f = \max(0, -u_{xx})$.

Even though our equation is neither a first-order Hamilton-Jacobi equation, nor a uniformly parabolic one, "it has regularizing effects of both types". To see what we mean, remove the second-order term from the equation. It then becomes a first-order HJ equation,

$$u_t + u_x^2 = 0 \quad \text{in} \quad Q_T, \tag{1.2}$$

which has Lipschitz continuous and x-semiconcave solutions for positive time. On the other hand, removing the quadratic term from (1.1) leads to the degenerate parabolic equation,

$$u_t = \min(0, u_{xx}) \quad \text{in} \quad Q_T, \tag{1.3}$$

which has x-semiconvex solutions for positive time. It turns out that adding both terms to the equation preserves both regularizing effects! That is, the solution of the full equation becomes both x-semiconvex and x-semiconcave, which means it belongs to $W^{2,\infty}(\mathbb{R})$ for any fixed positive time.

These regularizing effects are illustrated in Figure 1. Here a Lipschitz continuous initial data (a) is evolved according to equations (1.1) – (1.3), see (b) – (d). It is easy to see that the solutions plotted in (b), (c), and (d) are $W^{2,\infty}$, semiconcave, and semiconvex respectively.

The rest of the paper is organized as follows: In Section 2 the main result is given along with a corollary and some comments. The proof of the main result is given in

Section 3, and finally there is an appendix containing the proofs of auxiliary results which are either technical or standard.

2. The main result

We start by remarking that equation (1.1) is a convex Hamilton-Jacobi-Bellman equation, as can be seen more clearly when it is written in the following form:

$$u_t + \max_{\substack{\alpha\in\{0,1\}\\ \beta\in\mathbb{R}}} \{-\alpha u_{xx} + 2\beta u_x - \beta^2\} = 0 \quad \text{in} \quad Q_T. \tag{2.1}$$

It is well known that such equations need not have smooth solutions, not even when the initial values are smooth. The correct notion of weak solutions for these equations, is the notion of viscosity solutions. We refer to the User's Guide [5] for a detailed presentation, and to the book by Fleming & Soner [8] for more information about Hamilton-Jacobi-Bellman equations and connections to the viscosity solutions theory.

Let us introduce some notation. For functions $f : \overline{Q}_T \to \mathbb{R}$ and numbers $\delta \in (0, 1]$ we define the following (semi)norms:

$$\begin{aligned} |f|_0 &= \operatorname*{ess\,sup}_{(t,x)\in\overline{Q}_T} |f(t,x)|, \\ [f]_\delta &= \operatorname*{ess\,sup}_{\substack{(t,x),(s,y)\in\overline{Q}_T\\ x\neq y, t\neq s}} \frac{|f(t,x)-f(s,y)|}{|x-y|^\delta + |t-s|^{\delta/2}}, \\ |f|_\delta &= |f|_0 + [f]_\delta. \end{aligned}$$

Let BUC denote spaces of bounded uniformly continuous functions, and $\mathcal{C}^{0,\delta}$ spaces of functions with finite $|\cdot|_\delta$ norm. The last space contains bounded functions that are δ-Hölder continuous in x and $\delta/2$-Hölder continuous in t. Furthermore, let $W^{2,\infty}(K)$ denote the space of functions $f : K \to \mathbb{R}$ for $K \subset \overline{Q}_T$, with finite norm $|f|_0 + |f_x|_0 + |f_t|_0 + |f_{xx}|_0$, where derivatives are understood in the sense of distributions. Finally, by C we denote constants independent of t, x and any smoothness parameters like ε, ϵ (see below).

Now we proceed to give an existence, uniqueness, and regularity result for solutions of (1.1).

Proposition 2.1. *If $u_0 \in \mathrm{BUC}(\mathbb{R})$, then there exists a unique viscosity solution $u \in \mathrm{BUC}(\overline{Q}_T)$ of (1.1). Furthermore if $u_0 \in \mathcal{C}^{0,\delta}(\mathbb{R})$ for some $\delta \in (0, 1]$, then $u \in \mathcal{C}^{0,\delta}(\overline{Q}_T)$.*

This is a standard result holding for a large class of equations, and the regularity here is the best we can hope for in the general case (without regularizing effects). In fact the result is *optimal* w.r.t. global regularity or regularity up to the boundary. The (global) result is optimal even for the linear heat equation. See e.g. [12] for general regularity results of this type for nonlinear equations. For completeness, we provide a proof of this result in the Appendix. We just remark that the first thing to do is to prove a so-called strong comparison principle for (1.1) [5, 3]. Existence of a bounded solution then follows by Perron's method [10, 5]. Uniqueness and regularity are then (for this problem) rather simple consequences of the comparison principle.

We now give the main result, saying that problem (1.1) has a $W^{2,\infty}$ regularizing effect. In other words, for $t > 0$ the solution becomes $W^{2,\infty}$. So for $t > 0$ it is more regular than the initial data and hence more regular than suggested by Proposition 2.1.

Theorem 2.2 ($W^{2,\infty}$ regularizing effect)**.** *Let u be the viscosity solution of* (1.1)*, and assume $u_0 \in \mathrm{BUC}(\mathbb{R})$. Then $u \in W^{2,\infty}([\varepsilon, T] \times \mathbb{R})$ for all $\varepsilon \in (0, T)$.*

Note that functions in $W^{2,\infty}([\varepsilon, T] \times \mathbb{R})$ are differentiable a.e. twice w.r.t. x and once w.r.t. t. Since viscosity solutions are point-wise solutions at such points of differentiability, the following corollary is immediate.

Corollary 2.3. *If $u_0 \in \mathrm{BUC}(\mathbb{R})$, then the viscosity solution u of equation* (1.1) *satisfies the equation a.e., i.e., it is a so-called strong solution of* (1.1).

We remark that the proof of the main result given below only works in one space-dimension. To see this, consider the $\mathbb{R}^N$ version of (1.1):

$$w_t + |Dw|^2 + \max(0, -\Delta w) = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^N.$$

It turns out that using the techniques of this paper, we can prove that for $t > 0$, w is Lipschitz continuous in t and x and semiconcave in x. However, our proof of semiconvexity leads to the following inequality,

$$\Delta w(t, x) \geq w_t(t, x) \geq -C(t) \quad \text{in the sense of distributions,}$$

where $C(t)$ is a positive function satisfying $\lim_{t \to 0^+} C(t) = \infty$. It is only in $\mathbb{R}^1$ that this inequality implies semiconvexity. In $\mathbb{R}^N$ we get two-sided control only of the Laplacian.

The result and proofs can be generalized to more general equations. An easy generalization is to consider equations of the form

$$w_t + H(w_x) = f(w_{xx}) \quad \text{in} \quad Q_T,$$

where H is strictly convex and f is increasing, Lipschitz continuous, concave, and satisfies $f(X) \leq CX$ for $X \in \mathbb{R}$. If $f = 0$, it is well known that this equation has Lipschitz continuous and x-semiconcave solutions. Other generalizations are also possible; see e.g. Chapter 13 in Lions [15] for how to treat more general first-order terms.

Finally, numerical simulations seem to indicate that the following equation also exhibits a $W^{2,\infty}$ regularizing effect:

$$w_t + \max(w_x^2, -w_{xx}) = 0 \quad \text{in} \quad Q_T.$$

It would be interesting to have a proof in this (more difficult) case too. So far we have not been able to do it.

3. Proof of the main result

We divide the proof into three parts. First we use the regularizing properties of the u_x^2 term to prove that for positive time, the solution is Lipschitz continuous and semiconcave in x (Proposition 3.1). Then we show that the solution is Lipschitz in t for positive time (Proposition 3.4). This is a regularizing effect that occurs in a large class of parabolic equations [13]. Finally, we show that the solution is semiconvex in x (Proposition 3.7), using the previous mentioned result and the regularizing properties of the $\max(0, -u_{xx})$ term. Taken together, these results imply Theorem 2.2.

We start by stating

Proposition 3.1. *If* $u \in \mathrm{BUC}(\overline{Q}_T)$ *is the viscosity solution of* (1.1)*, then*

$$u_{xx}(t,x) \leq \frac{2}{t} \text{ in the sense of distributions, and } \quad [u(t,\cdot)]_1 \leq \left(\frac{|u|_0}{t}\right)^{1/2}.$$

In order to prove this result, we need to consider a regularized version of (1.1). We regularize by mollifying the max function and the initial data, and adding a viscous term to the equation. Let ρ be a positive, smooth function with mass 1 and support in $\{|x| < 1\}$, and define the mollifier ρ_ε as $\rho_\varepsilon(x) := \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. Furthermore, define

$$f^\varepsilon(x) = \int_{\mathbb{R}} \min(0,y)\rho_\varepsilon(x-y)dy \qquad \text{and} \qquad u_0^\varepsilon(x) = \int_{\mathbb{R}} u_0(y)\rho_\varepsilon(x-y)dy.$$

Then the regularized problem in consideration can be written

$$\begin{aligned} &u_t^\varepsilon + (u_x^\varepsilon)^2 = f^\varepsilon(u_{xx}) + \varepsilon u_{xx}^\varepsilon \quad \text{in} \quad Q_T, \\ &u^\varepsilon(0,x) = u_0^\varepsilon(x) \quad \text{in} \quad \mathbb{R}. \end{aligned} \tag{3.1}$$

Note that $f^\varepsilon, u_0^\varepsilon$ are smooth and have bounded derivatives of all orders, and that f^ε is nondecreasing and concave. This new equation is a smooth and uniformly parabolic Hamilton-Jacobi-Bellman equation. The properties of its solutions are given in the next lemma whose proof will be given in the appendix.

Lemma 3.2. *There exists a unique classical solution* u^ε *to the initial value problem* (3.1) *belonging to* $C^\infty(Q_T)$*, the space of infinitely differentiable functions.*

Proof of Proposition 3.1. Let u^ε be the solution of the regularized problem (3.1) provided by Lemma 3.2. We claim that

$$u_{xx}^\varepsilon \leq \frac{1}{2t} \quad \text{in} \quad Q_T. \tag{3.2}$$

To see this, let $w = tu_{xx}^\varepsilon$ and differentiate twice w.r.t. x in (3.1) (ok by Lemma 3.2) to obtain

$$\begin{aligned} w_t - \frac{1}{t}w + \frac{2}{t}w^2 + 2u_x^\varepsilon w_x &= (\varepsilon + (f^\varepsilon)')w_{xx} + \frac{1}{t}(f^\varepsilon)'' w_x^2 \\ &\leq (\varepsilon + (f^\varepsilon)')w_{xx}. \end{aligned}$$

The inequality follows since $(f^\varepsilon)'' \leq 0$ by concavity of f^ε. At an interior maximum point $(\bar{t},\bar{x})$ of w, this inequality reduces to

$$2(w(\bar{t},\bar{x}))^2 - w(\bar{t},\bar{x}) \leq 0,$$

implying $\sup_{\bar{Q}_T} w \leq \frac{1}{2}$. If the maximum of w is attained at $t = 0$, then $\sup_{\bar{Q}_T} w = 0$. The general case can always be reduced to one of these two cases, hence (3.2) holds. It can be proved that u^ε converges uniformly to u as $\varepsilon \to 0$ (see e.g. [12] for the case of $u_0 \in \mathcal{C}^{0,\delta}(\mathbb{R})$). So it immediately follows that u satisfies the bound (3.2) in the sense of distributions.

Since $u \in \mathrm{BUC}(\bar{Q}_T)$ satisfies $u_{xx} \leq C_0$ in the sense of distributions (with $C_0 = (2t)^{-1}$), a standard interpolation inequality (see e.g. [15, p. 240]) yields

$$[u(t,\cdot)]_1 \leq (2|u|_0 C_0)^{1/2} \quad \text{for all} \quad t \in (0,T].$$

This completes the proof. $\square$

Remark 3.3. For a similar proof for first-order Hamilton-Jacobi equations, see Lions ([15], Chapter 13).

Let us proceed to

Proposition 3.4. *If* $u \in \mathrm{BUC}(\overline{Q}_T)$ *is the viscosity solution of* (1.1), *then* $|u_t(t,\cdot)|_0 \leq C(t) < \infty$ *for any* $t \in (0,T]$.

Remark 3.5. It follows from the proof below that $C(t) \to \infty$ as $t \to 0^+$, and that $C(t)$ can be chosen to be a decreasing function.

Proof. The result would follow from Katsoulakis [13] if the equation and initial data were uniformly Lipschitz continuous in all arguments. This can be achieved by considering the restriction of u to the interval $[\epsilon, T]$ for any fixed $\epsilon > 0$. If v denotes this function, then v satisfies

$$\begin{aligned} &v_t + (v_x)^2 + \max(0, -v_{xx}) = 0 \quad \text{in} \quad (\epsilon, T) \times \mathbb{R}, \\ &v(\epsilon, x) = u(\epsilon, x) \quad \text{in} \quad \mathbb{R}. \end{aligned} \tag{3.3}$$

Obviously v is globally x-Lipschitz and $|v_x|_0 \leq C/\epsilon^{1/2}$ since $|u_x(t,\cdot)|_0 \leq C/t^{1/2}$ by Proposition 3.1. Keeping the form (2.1) of equation (1.1) in mind, we may therefore rewrite equation (3.3) in the following way:

$$v_t + \max_{\substack{\alpha \in \{0,1\} \\ |\beta| \leq |v_x|_0}} \{-\alpha v_{xx} + 2\beta v_x - \beta^2\} = 0 \quad \text{in} \quad Q_T. \tag{3.4}$$

Here the controls α, β take values in compact sets. So the coefficients in the above equation are bounded. This again implies that the function

$$F(p, X) = \max_{\substack{\alpha \in \{0,1\} \\ |\beta| \leq |u_x|_0}} \{-\alpha X + 2\beta p - \beta^2\}$$

is Lipschitz continuous in both arguments. Since $v_t + F(v_x, v_{xx}) = 0$ in the viscosity sense, and $v(\epsilon, \cdot)$ is x-Lipschitz continuous, the conditions of Theorem 4.2 (or Theorem 6.1) in [13] are satisfied. We may therefore conclude that $|(t-\epsilon)v_t|_0 \leq C_\epsilon$ for some constant C_ϵ depending on ϵ, and $t > \epsilon$. Since $\epsilon > 0$ was arbitrary, and $u = v$ for $t \in [\epsilon, T]$, the proof is complete. □

Remark 3.6. Theorem 6.1 in Katsoulakis [13] essentially says that the solution u of the equation

$$\begin{aligned} &u_t + f(t, x, u, Du, D^2u) = 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^N, \\ &u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \end{aligned}$$

satisfies $|tu_t| \leq C$ in the sense of distributions if the following conditions hold:

(i) u_0 is bounded and Lipschitz continuous.

(ii) f is continuous in all variables.

(iii) $f(t,x,r,p,X) \leq f(t,x,r,p,Y) \quad \text{for} \quad Y \leq X$.

(iv) $|f(t,x,r,p,X) - f(t,y,r,p,X)| \leq C|x-y|(1+|r|+|p|)$.

(v) $|f(t,x,r,p,X) - f(t,x,s,q,Y)| \leq C(|r-s| + |p-q| + |X-Y|)$.

The proof of this result relies on rewriting the equation as an Isaacs equation and studying the associated stochastic differential game. It turns out that the upper (or lower) value function of this game satisfies the Isaacs equation in the viscosity sense

[13, 9]. Moreover this value function has by its definition a stochastic (Feynman-Kac like) representation formula. The result is then proved by working directly with this stochastic formula.

The proof of Theorem 2.2 is completed by giving

Proposition 3.7. *If $u \in \mathrm{BUC}(\overline{Q}_T)$ is the viscosity solution of* (1.1)*, then*

$$u_{xx}(t,x) \geq -C(t) \quad \textit{in the sense of distributions,}$$

where $C(t)$ is defined in Proposition 3.4.

Proof. Since u is the viscosity solution of (1.1), it is not difficult to see that the following inequalities hold in the viscosity sense:

$$u_{xx} \geq \min(0, u_{xx}) = u_t + u_x^2 \geq -C(t) \quad \text{in} \quad \overline{Q}_T,$$

where the last inequality follows from Proposition 3.4. From Lemma 1, p. 268 in [1], this inequality implies that $u(t,x) + \frac{1}{2}C(t)x^2$ is x-convex, which again is equivalent to the distributional inequality in the proposition. $\square$

Appendix A. Proofs of secondary results

Proof of Proposition 2.1. This result is mainly a consequence of the two lemmas below. But first we need some notation: For functions $\phi : \overline{Q}_T \to \mathbb{R}$, $G[\phi] := \phi_t + \phi_x^2 + \max(0, -\phi_{xx})$ in the viscosity sense, and BUSC denotes the space of bounded upper semicontinuous functions.

Lemma A.1 ("Strong" comparison principle)**.** *Let $u, -v \in \mathrm{BUSC}(\overline{Q}_T)$ satisfy $G[u] \leq 0$, $G[v] \geq 0$, $u(0,x) \leq v(0,x)$, and $u(0,x), v(0,x) \in \mathrm{BUC}(\mathbb{R})$. Then $u \leq v$ in $[0,T) \times \mathbb{R}$.*

Proof. Define

$$\begin{aligned} \phi(t,x,y) &= \alpha|x-y|^2 + \varepsilon(|x|^2 + |y|^2) + \frac{\varepsilon}{T-t}, \\ \psi(t,x,y) &= u(t,x) - v(t,y) - \phi(t,x,y), \\ \sigma &= \sup_{t,x,y} \psi(t,x,y) - \sup_{x,y} \psi(0,x,y)^+, \end{aligned}$$

where the suprema are taken over the set $[0,T) \times \mathbb{R} \times \mathbb{R}$. We will derive an upper bound on σ; so we may assume that $\sigma > 0$. By classical arguments there is a maximum point $(t_0, x_0, y_0) \in [0,T) \times \mathbb{R} \times \mathbb{R}$ of $\psi(t,x,y) - \sigma t$. Furthermore, $t_0 > 0$ since otherwise $\sigma \leq 0$. Using the maximum principle for semicontinuous functions (Theorem 8.3 in [5]), and subtracting the inequalities that follow from the definitions of viscosity solutions, we obtain

$$\begin{aligned} \phi_t(t_0,x_0,y_0) + \sigma \leq& (\phi_y(t_0,x_0,y_0))^2 - (\phi_x(t_0,x_0,y_0))^2 \\ &+ \max(0,-Y) - \max(0,-X), \end{aligned}$$

where $X, Y \in \mathbb{R}$ satisfies $X \leq Y + k\varepsilon$ for some number $k > 0$. Simplify this expression to get

$$\frac{\varepsilon}{(T-t_0)^2} + \sigma \leq 2\varepsilon(x_0+y_0)\alpha(x_0-y_0) + \varepsilon^2(y_0^2 - x_0^2) + k\varepsilon \leq \varepsilon\, C\left(1 + \alpha^{1/2}\right).$$

The last inequality follows since the inequality $\psi(t_0,x_0,y_0) \geq \psi(t_0,0,0)$, and boundedness of u and v, yields the bound $\alpha|x_0-y_0|^2 + \varepsilon(|x_0|^2 + |y_0|^2) \leq C$. Since

$u_0 \in \mathrm{BUC}(\mathbb{R})$ it is easy to get the estimate $\sup_{x,y} \psi(0,x,y)^+ \le \omega\left(\alpha^{-1}+\varepsilon\right)$ for some modulus ω. By the definition of σ, for any $(t,x) \in [0,T) \times \mathbb{R}$ we now have

$$u(t,x) - v(t,x) \le \sigma + \sup_{x,y} \psi(0,x,y)^+ + 2\varepsilon|x|^2 + \frac{\varepsilon}{T-t}.$$

By the above estimates we can conclude the proof by first sending ε to 0, and then sending α to ∞. □

We give a standard and immediate corollary.

Corollary A.2. *Assume $u_0 \in \mathrm{BUC}(\mathbb{R})$, and let u be a viscosity solution of* (1.1). *Then u is unique. Furthermore, if $h \in \mathbb{R}$ and $t \in [0,T]$, then*

$$\sup_{x\in\mathbb{R}} |u(t,x) - u(t,x+h)| \le \sup_{x\in\mathbb{R}} |u(0,x) - u(0,x+h)|.$$

We see that the x-regularity of the solution u is the same as the regularity of the initial function u_0. So if u_0 belongs to $\mathrm{BUC}(\mathbb{R})$ or $\mathcal{C}^{0,\delta}(\mathbb{R})$ for $\delta \in (0,1]$, then $u(t,\cdot)$ (if it exists) belongs to $\mathrm{BUC}(\mathbb{R})$ or $\mathcal{C}^{0,\delta}(\mathbb{R})$ for $t \in (0,T]$. Regularity in t can be obtained from the comparison principle if one knows the t-regularity at $t=0$. This again can be obtained by a barrier argument. See [11] for the details in the Hölder continuous case.

Lemma A.3. *If $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$, then* (1.1) *has a solution $u \in \mathrm{BUC}(\overline{Q}_T)$ satisfying $|u|_0 \le |u_0|_0$.*

Outline of proof. Here, we need to consider discontinuous viscosity solutions and initial values in the viscosity sense, meaning that a subsolution v and a supersolution $\bar{v}$ have to satisfy in the viscosity sense:

$$\begin{aligned} &G[v^*] \le 0 \quad \text{and} \quad G[\bar{v}_*] \ge 0 \quad \text{in} \quad Q_T, \\ &\min\left(G[v^*], v^* - u_0^*\right) \le 0 \quad \text{in} \quad \{0\}\times\mathbb{R}, \\ &\max\left(G[\bar{v}_*], \bar{v}_* - u_{0*}\right) \ge 0 \quad \text{in} \quad \{0\}\times\mathbb{R}. \end{aligned}$$

Here $v^*(v_*)$ denotes the upper (lower) semicontinuous envelopes of the possibly discontinuous function v.

To continue, we note that the constants $|u_0|_0$ and $-|u_0|_0$ are smooth sub- and supersolutions of the initial value problem in the viscosity sense. Then by Perron's method [10, 5], the following function is a discontinuous viscosity solution of this problem:

$$u(t,x) = \sup\left\{w(t,x) \ : \ -|u_0|_0 \le w \le |u_0|_0 \ \text{ and } \ G[w^*] \le 0 \text{ in } Q_T\right\}.$$

From Lemma 4.7 in [3] it follows that $u(0,x) \le u_0^*(x)$ and $u(0,x) \ge u_{0*}(x)$. Since $u_0 = u_0^* = u_{0*}$ (u_0 is continuous), this means that $u(0,x) = u_0(x) \in \mathrm{BUC}(\mathbb{R})$. By the "strong" comparison principle (Lemma A.1) it follows that $u^* \le u_*$. Since the opposite inequality is always true, $u = u^* = u_*$, implying that u is continuous. It is then immediate from Corollary A.2 that u is uniformly continuous in x, and uniform continuity in time follows as explained below Corollary A.2. □

This completes the proof of Proposition 2.1. □

Proof of Lemma 3.2. The result follows from the two lemmas below. In the following, let $C^{1,\alpha}(Q_T)$ and $C^{2,\alpha}(Q_T)$ denote the space of functions $f : \overline{Q}_T \to \mathbb{R}$ with finite norms $|f|_0 + |f_x|_\alpha$ and $|f|_0 + |f_x|_0 + |f_{xx}|_\alpha + |f_t|_\alpha$, respectively. Furthermore, let $C^{1,\alpha}_{\text{loc}}(Q_T)$ and $C^{2,\alpha}_{\text{loc}}(Q_T)$ be their local versions (the norms being finite on compact sets).

Lemma A.4. *The initial value problem* (3.1) *has a unique solution* $u^\varepsilon \in C^{2,\beta}_{\text{loc}}(Q_T)$, *for some* $\beta \in (0,1)$.

Proof. 1. By viscosity methods there exists a unique viscosity solution $u^\varepsilon \in \mathcal{C}^{0,1}(\overline{Q}_T)$. The proof is similar to the proof of Proposition 2.1.

2. It immediately follows that u^ε also is the (unique) viscosity solution of the following truncated equation:

$$u^\varepsilon_t + H_R(u^\varepsilon_x) = f^\varepsilon(u_{xx}) + \varepsilon u^\varepsilon_{xx} \quad \text{in} \quad Q_T,$$

where $H_R(t)$ is t^2 for $|t| < R$ and R otherwise for some fixed $R > |u^\varepsilon_x|_0$. Alternatively, and in view of the global Lipschitz continuity of f^ε, we may write

$$u^\varepsilon_t + \max_{\substack{\alpha\in[0,1]\\ \beta\in[-R,R]}} \{-F^\varepsilon_1(\alpha)u_{xx} - F^\varepsilon_2(\alpha) + 2\beta u^\varepsilon_x - \beta^2\} = \varepsilon u^\varepsilon_{xx} \quad \text{in} \quad Q_T,$$

for some appropriately defined functions F^ε_1 and F^ε_2, bounded on $[0,1]$. Now we have achieved a Hamilton-Jacobi-Bellman equation with bounded coefficients.

3. For the equation in 2, consider the Cauchy-Dirichlet problem on cylinders for the form $[0,T) \times B(x,r)$ where $x \in \mathbb{R}$ and $r > 0$. At the parabolic boundary, we require that the solution equals the function u^ε given by 1. We are now in a situation where the $C^{1,\alpha}$-theory of [6] applies. All the requirements of Theorem 9.3 of [6] are satisfied for the problem on the cylinder, and this implies the existence of a unique viscosity solution $\bar u \in C^{1,\bar\alpha}_{\text{loc}}$ for all $\bar\alpha \in (0,1)$. Uniqueness implies that $\bar u = u^\varepsilon$ on the cylinder, and since x, r was arbitrary, $u^\varepsilon \in C^{1,\bar\alpha}_{\text{loc}}(Q_T)$.

4. By 3, u^ε actually solves the following equation:

$$u_t = f^\varepsilon(u_{xx}) + \varepsilon u_{xx} + g(t,x) \quad \text{in} \quad Q_T,$$

where $g(t,x) = -(u^\varepsilon_x(t,x))^2$ is a $\mathcal{C}^{0,\bar\alpha}_{\text{loc}}(Q_T)$ function. An argument similar to the one in 3, using this time Evans-Krylov $C^{2,\alpha}$-theory (see Theorem 14.10 in [14]), yields $u^\varepsilon \in C^{2,\beta}_{\text{loc}}(Q_T)$ for some $\beta \in (0,1)$. $\square$

Lemma A.5. *If* $u^\varepsilon \in C^{2,\beta}_{\text{loc}}(Q_T)$ *for some* $\beta \in (0,1)$, *then* u^ε *belongs to* $C^\infty(Q_T)$.

Proof. Here we will use difference quotients to obtain higher regularity of the solution. Let $h \in \mathbb{R}$ be nonzero, and define

$$w(t,x) = \frac{u^\varepsilon(t,x+h) - u^\varepsilon(t,x)}{h}.$$

Simple calculations show that for $t > 0$, w satisfies

$$\begin{aligned} &w_t + \left(\int_0^1 2\big(su^\varepsilon_x(t,x+h) + (1-s)u^\varepsilon_x(t,x)\big)ds\right) w_x \\ &\quad = \left(\int_0^1 f^{\varepsilon\prime}\big(su^\varepsilon_{xx}(t,x+h) + (1-s)u^\varepsilon_{xx}(t,x)\big)ds\right) w_{xx} + \varepsilon w_{xx}. \end{aligned} \tag{A.1}$$

Since $f^{\varepsilon\prime} \ge 0$, this is a linear uniformly elliptic equation in w. Furthermore, on each compact subset of Q_T, the coefficients are as Hölder continuous as are $u^\varepsilon_x, u^\varepsilon_{xx}$ with bounds independent of h. Linear Schauder theory then implies that $w \in C^{2,\beta}_{\text{loc}}(Q_T)$

with bounds independent of h. This in turn implies that $u^\varepsilon_{xxx} \in \mathcal{C}^{0,\beta}_{\text{loc}}(Q_T)$. A similar but simpler argument, using the *linear* equation (A.1), shows that

$$\bar{w}(t,x) = \frac{w(t,x+k) - w(t,x)}{k}$$

belongs to $C^{2,\beta}_{\text{loc}}(Q_T)$ with bounds independent of k (and h), and we can conclude that $u^\varepsilon_{xxxx} \in \mathcal{C}^{0,\beta}_{\text{loc}}(Q_T)$. The regularity of u^ε_{txx} is obtained by differentiating the equation and using the x-regularity of u^ε.

Continuing the above argument shows that u^ε belongs to $C^\infty(Q_T)$. □

This completes the proof of Lemma 3.2. □

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References

1. O. Alvarez, J.-M. Lasry, and P.-L. Lions. *Convex viscosity solutions and state constraints.* J. Math. Pures Appl. (9), 76(3):265–288, 1997. MR 98k:35045
2. M. Arisawa and A. Tourin. *Regularizing effects for a class of first-order Hamilton-Jacobi equations.* Nonlinear Anal. 29(12):1405-1419, 1997. MR 99b:35020
3. G. Barles. *Solutions de visosité des équations de Hamilton-Jacobi.* Springer-Verlag, Berlin, Heidelberg, 1994. MR 2000b:49054
4. G. Barles. *Uniqueness and regularity results for first-order Hamilton-Jacobi equations.* Indiana Univ. Math. J. 39(2):443-466, 1990. MR 92b:35038
5. M. G. Crandall, H. Ishii, and P.-L. Lions. *User's guide to viscosity solutions of second-order partial differential equations.* Bull. Amer. Math. Soc. (N.S.), 27(1):1–67, 1992. MR 92j:35050
6. M. G. Crandall, M. Kocan, and A. Swiech. *L^p-Theory for fully nonlinear uniformly parabolic equations.* Comm. Partial Differential Equations 25(11&12):1997–2053, 2000. MR 2003b:35093
7. L. C. Evans. *Partial Differential Equations.* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. MR 99e:35001
8. W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York, 1993. MR 94e:93004
9. W. H. Fleming and P. E. Souganidis. *On the existence of value functions of two-player, zero-sum stochastic differential games.* Indiana Univ. Math. J. 38(2):293-314, 1989. MR 90e:93089
10. H. Ishii. *Perron's method for Hamilton-Jacobi equations.* Duke Math. J. 55(2):369-384, 1987. MR 89a:35053
11. E. R. Jakobsen. *On the rate of convergence of approximation schemes for Bellman equations associated with optimal stopping time problems.* Math. Models Methods Appl. Sci. (M3AS) 13(5): 613-644, 2003.
12. E. R. Jakobsen and K. H. Karlsen. *Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations.* J. Differential Equations 183:497-525, 2002. MR 2003i:35135
13. M. A. Katsoulakis. *A representation formula and regularizing properties for viscosity solutions of second-order fully nonlinear degenerate parabolic equations.* Nonlinear Anal., 24(2):147–158, 1995. MR 95m:35039
14. G. M. Lieberman. *Second Order Parabolic Differential Equations.* World Scientific, Singapore, 1996. MR 98k:35003
15. P. L. Lions. *Generalized solutions of Hamiton-Jacobi equations.* Pitman, London, 1982. MR 84a:49038
16. P. L. Lions. *Regularizing effects for first-order Hamilton-Jacobi equations.* Applicable Anal. 20(3-4):283-307, 1985. MR 87h:35054
17. L. Wang. *On the regularity theory of fully nonlinear parabolic equations. I.* Comm. Pure Appl. Math. 45(1):27-76, 1992. MR 92m:35126

18. L. Wang. *On the regularity theory of fully nonlinear parabolic equations. II.* Comm. Pure Appl. Math. 45(1):141-178, 1992. MR 92m:35127
19. N. Zhu. $C^{1+\alpha}$ *regularity of viscosity solutions for fully nonlinear parabolic equations.* Nonlinear Anal. 38(8):977-994, 1999. MR 2003b:35100

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