A FINITE ELEMENT LIKE SCHEME FOR INTEGRO-PARTIAL DIFFERENTIAL HAMILTON-JACOBI-BELLMAN EQUATIONS

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Abstract. We construct a finite element like scheme for fully non-linear integro-partial differential equations arising in optimal control of jump-processes. Special cases of these equations include optimal portfolio and option pricing equations in Finance. The schemes are monotone and robust. We prove that they converge in very general situations, including degenerate equations, multiple dimensions, relatively low regularity of the data, and for most (if not all) types of jump-models used in Finance. In all cases we provide (probably optimal) error bounds. These bounds apply when grids are unstructured and integral terms are very singular, two features that are new or highly unusual in this setting.

1. Introduction

In this paper we introduce and analyze finite element (FEM) like schemes for nonlocal Hamilton-Jacobi-Bellman equations (HJB equations) of the form

\[ \sup_{v \in V} \{- \text{tr} [a(x,v)D^2 u] - b(x,v)Du + c(x,v)u - f(x,v) - I^v u(x)\} = 0 \quad \text{in} \quad \mathbb{R}^N, \]

or

\[ \sup_{v \in V} \{- \text{tr} [a(x,v)D^2 u] - b(x,v)Du + c(x,v)u - f(x,v) - J^v u(x)\} = 0 \quad \text{in} \quad \mathbb{R}^N, \]

where

\[ a(x,v) = \frac{1}{2} \sigma(x,v)\sigma^T(x,v), \]

\[ I^v u(x) = \int_E [u(x + \eta(x,v,z)) - u(x)]\nu(dz), \]

\[ J^v u(x) = \int_E [u(x + \eta(x,v,z)) - u(x) - 1_{|z|<1}\eta(x,v,z)Du(x)]\nu(dz), \]

\[ V \text{ is a compact metric space and } E = \mathbb{R}^M \setminus \{0\}. \]

The coefficients \( \sigma, \eta, b, c, f \) are loosely speaking Lipschitz continuous in \( x \), continuous in \( v \), and Borel measurable in \( z \). The precise assumptions will be given in the next section. These equations are the dynamic programming equations for stochastic control problems involving Levy processes, a class of Markov processes with jumps [11]. The two forms of the equation correspond to different intensities \( \nu(dz) \) of small jumps in the corresponding Levy processes. In general these equations are degenerate and fully-nonlinear, and the corresponding solutions are typically only Hölder continuous and have to be understood in the viscosity sense [25, 29, 4].

Equations like (1.1) and (1.2) appear in advanced models of financial markets where price evolution of stocks (and other risky assets) are modeled as (exponential) pure-jump or jump-diffusion processes. Special cases of (1.1) and (1.2) include linear equations used in option pricing problems, obstacle problems used e.g. in pricing of American options, and fully non-linear equations (the full HJB equation) used in optimal portfolio problems. We refer to [19] for the first two cases and
to [25, 8] for the last case. It is well-known that the standard Black-Scholes model (a diffusion model) give poor fit to real markets, at least on smaller time scales. E.g. log-returns distributions of stock prices are leptokurtic and have longer and fatter tails than predicted by the Black-Scholes model, see e.g. [19]. To improve upon these shortcomings, many pure-jump and jump-diffusion models have been suggested in the literature over the years, see e.g. [19, 40] for the most popular models. The empirical fact that Levy processes with discontinuous sample paths tend to better model e.g. stock prices, is one main reason for the renewed interest in stochastic optimal control of jump-diffusion processes.

Except in very simple cases equations (1.1) and (1.2) do not have closed form solutions, so numerical methods are needed to obtain (approximate) solutions. In this paper we construct FEM like schemes. Since the equations are fully nonlinear we have no weak/variational formulation, and the usual FEM approach does not work. If the measure \( \nu \) is finite and the corresponding HJB equation is (1.1), we discretize in two steps. First we obtain a semi-discrete equation (“discrete in time”) as the dynamic programming equation of a discrete time control problem approximating the underlying continuous time control problems of (1.1):

\[
(1.6) \quad u_h(x) = \inf_{v \in \mathcal{V}} \left\{ \int f(x, v) + e^{-h\sigma_m(x,v)} \sum_{m=1}^d \left( u_h(x + h\nu_m(x,v)) + \frac{1}{2d} \left( \int_{E} u_h(x + \eta(x,v,z)) \nu(dz) \right) \right) \right\},
\]

where \( \sigma_m \) is the \( m \)-th column of the matrix \( \sigma \), \( h \) is the discretization step and \( \lambda \) is the mass of the measure. The next step is obtain a fully discrete equation by introducing a regular triangulation and look for continuous piecewise linear functions over the chosen triangulation satisfying the semi-discrete equation at every vertex of the triangulation. When \( \nu \) is not finite we first approximate it by a finite measure \( 1_{r<|z|<R} \nu(dz) \) (truncation), and then approximate the truncated equation following the above approach. To improve the truncation approximation, we also add small diffusion and/or drift terms to the equation.

We prove that these methods converge and derive (probably optimal) error bounds using the framework developed in [30] and ideas from [17]. We also discuss issues like restricting to a bounded domain, truncating long jumps, and approximating integrals by quadrature. In all cases we provide rigorous error bounds for the various approximations. What remains to do in a computer implementation, is the resolution of the non-linearity. There are various ways to do that, e.g. policy iteration, value iteration, artificial time methods (stationary limits), and so on. We will not address this point in this paper.

FEM schemes like the one we have described above are usually called semi-Lagrangian or control schemes in the literature. They are (usually) monotone and first order accurate, and a comprehensive background and references can be found in [24]. Most results in this field concern deterministic control problems and first order HJB equations without integral terms. Semi-Lagrangian schemes for second order HJB equations with no integral terms have been considered in [37, 17, 5, 1]. Moreover in [16] such schemes were derived for HJB equations associated to piecewise deterministic processes (compound Poisson processes with drift). One advantage of this type of schemes is that in general they produce also an approximation of the optimal control law in feedback form and the optimal trajectories (this is a key point in the study of a control problem). This advantage can also be shared by some (monotone, low order) finite difference schemes following the construction of Kushner, and we refer to [34] for a discussion of this point. An other advantage is that these schemes are formulated on general grids/triangulations just like FEMs. By contrast this is very cumbersome to achieve with finite difference methods. A third advantage is that the semi-Lagrangian scheme automatically handles non-diagonally dominant diffusion matrices \( a \). Such matrices appear in applications and are cumbersome to discretize, we refer to [12] for a finite difference (FDM) approach to this problem.

The construction and analysis of numerical schemes for linear integro-partial differential equations arising as pricing equations in financial markets of jump-diffusion type is currently an active field of research, see e.g. [19, 36, 23, 15] and references therein. By contrast, there are few works on
numerical schemes for fully nonlinear degenerate integro-partial differential equations. We mention the discussion about jump-diffusion processes in [34] and the papers [22, 14, 30, 10]. In all cases mainly monotone FDMs are considered and convergence is obtained. In the last two papers the main focus is on convergence rates. The framework of [30] is again based on ideas of Krylov, Barles and Jakobsen [31, 32, 3] in the pure PDE case. We emphasize that the error bounds of this paper apply when grids are unstructured and integral terms are very singular, a feature which is new or highly unusual in this setting.

The rest of this paper is organized as follows. In section 2 we state the assumptions on the data and give well-posedness/regularity results for equations (1.1) and (1.2). We discuss truncation of the Levy measure and reduction to a bounded domain and give error bounds. In section 3 we construct the schemes via the dynamic programming equation (a semi-discretization) and FEM ideas. We prove existence, uniqueness, consistency, and partial error bounds. We also derive a semicontinuous functions. We will use the following norms

\[ |f|_0 = \text{ess sup}_{x \in \mathbb{R}^N} |f(x)|, \quad |f|_1 = |Df|_0, \quad \text{and} \quad |f|_1 = |f|_0 + |f|_1. \]

2. Preliminaries

In this section we state the assumptions for equations (1.1) and (1.2) and give well-posedness results. We discuss reduction to a bounded domain and reduction to bounded measure \( \nu \) with compact support.

2.1. Assumptions, well-posedness, regularity. We will use the following assumptions:

(A1) The set \( V \) is a compact metric space, the coefficients \( \sigma, \eta, b, c, f \) are continuous in \( x \) and \( v \) and Borel measurable in \( z \), and \( \nu \) is a positive Radon measure on \( E := \mathbb{R}^M \setminus \{0\} \).

(A2) There exists \( L_1, L_2, \ell \geq 0 \) such that for any \( v \in V \) and \( z \in E \)

\[
|c(\cdot, v)|_1 + |f(\cdot, v)|_1 \leq L_1, \\
|\sigma(\cdot, v)|_1 + |b(\cdot, v)|_1 \leq L_2 \\
|\eta(\cdot, v, z)|_1 \leq L_2(|z|_1|z|_1 + e^{\|z\|_1}1_{|z|_1>1}).
\]

(A3) There exists \( c_0 > 0 \) such that, for all \( x \in \mathbb{R}^N \) and \( v \in V \),

\[ c(x, v) \geq c_0. \]

(A4) There exists \( L_3 \geq 0 \), such that for \( \ell \) defined in (A2), any \( v \in V \) and \( x \in \mathbb{R}^N \),

\[ |D_x \eta(x, v, \cdot)|_0 \leq L_3 e^{\ell |z|}. \]

For the measure \( \nu \) we will use the following integrability assumptions.

(B1) The measure \( \nu \) satisfies for \( \ell \) defined in (A2) and some \( k \in \{0, 1, 2\} \),

\[
\int_E (|z|^k 1_{|z|<1} + e^{\ell |z|}1_{|z|>1}) \nu(dz) < \infty.
\]

(B2) The measure \( \nu \) has a positive density \( m : E \to [0, \infty) \) such that \( \nu(dz) = m(z)dz \) and for some \( C, \varepsilon > 0 \), and \( n \in \mathbb{N} \)

\[ |D_k^m(z)| \leq C_k e^{-(\ell + \varepsilon)|z|} \quad \text{for} \quad k = 0, 1, \ldots, n, \]

where \( D_k^m \) is the vector of all order \( k \) derivatives of \( m \) and \( \ell \) is defined in (A2).
continuous, i.e., there is a density \( \nu(dz) = m(z)dz \) and for some \( \alpha \in [0, 2) \), \( C_0 \), \( C_1 \), and \( \varepsilon > 0 \),

\[
|D^k m(z)| \leq C_k \left( \frac{1}{|z|^{N+\alpha+k}} |1_{|z|<1} + e^{-(\ell+\varepsilon)|z|} |1_{|z|>1} ^{\varepsilon} \right) \quad \text{for } k = 0, 1,
\]

where \( D^0 m = m \), \( D^1 m = Dm \) the gradient of \( m \), and \( \ell \) is defined in (A2).

Assumptions (A1) – (A3) and (B1)_2 are standard from a stochastic control theory point of view, as they insure existence and uniqueness of strong solutions of the underlying stochastic differential equations, see [26]. Under assumption (B1)_2 or the less general assumption (B1)_1 the measure \( \nu \) may have a (non-integrable) singularity at \( z = 0 \). If (B1)_1 or (B1)_0 holds the HJB equation takes the form (1.1). If (B1)_2 holds but not (B1)_1, then (1.2) gives the correct form of the HJB equation. In this case, the extra term in the integrand of \( \mathcal{J}^\nu \) (compared to \( \mathcal{I}^\nu \)) is needed for the integral to converge: If (B1)_2 holds, then \( \mathcal{J}^\nu u \) converge for all \( C^2 \) functions \( u \) is with polynomial growth at infinity. Finally, (B2) and (B3) prescribe densities. Close to \( z = 0 \), the density of (B3) equals the density of the \( \alpha \)-stable processes related to the fractional Laplacian \( \Delta^{\alpha/2} \). (B3) implies (B1)_2, and if \( \alpha \in [0, 1) \), then (B3) also implies (B1)_1.

**Example 2.1.** When the stock returns are modeled as exponential Levy processes the integral terms equals (1.4) or (1.5) with \( \eta(x,z) = x(e^{\varepsilon z} - 1) \) in the one-dimensional uncontrolled case, see [19]. An important extension of this expression satisfying (A2) and (A4) is \( \eta(x,v,z) = \eta(x,v)\phi(z) \) where \( \eta \) is bounded and Lipschitz and \( |\phi(z)| \leq C(e^{\varepsilon z} - 1) \), \( |D\phi(z)| \leq C e^{\varepsilon z} \).

**Example 2.2.** Assumptions (B2) and (B3) is satisfied by all bounded and unbounded Lévy processes used in the literature to model financial markets, see [19] for a nice overview. In the models of Merton and Kou, the Levy measures have bounded densities (no singularity), in the Merton case [38] given by

\[
\nu(dz) = \frac{\lambda}{\delta\sqrt{2\pi}} e^{-\frac{|z| - \mu^2}{2\delta^2}} dz \quad \text{for constants } \lambda, \delta, \mu.
\]

For these models (B2) holds. The Variance Gamma model has a Levy density with an integrable singularity at \( z = 0 \), and satisfy (B3) with \( \alpha = 0 \). The Normal Inverse Gaussian model [40] has a non-integrable density corresponding to \( \alpha = 1 \) in (B3), and the Levy measure is

\[
\nu(dz) = \frac{C}{|z|} e^{Az} K_1(B|z|) \quad \text{for constants } A, B, C,
\]

where \( K_1 \) is a modified Bessel function of 2nd kind. Finally we mention models using tempered \( \alpha \)-stable processes, e.g. the CGMY model [18]. Here \( \alpha \in (0, 2) \) in (B3) and the Levy measure is give by

\[
\nu(dz) = \frac{c_-}{|z|^{1+\alpha}} e^{-\lambda_-|z|} 1_{z<0} + \frac{c_+}{|z|^{1+\alpha}} e^{-\lambda_+|z|} 1_{z>0}
\]

for constants \( \alpha \in (0, 2), c_-, c_+, \lambda_-, \lambda_+ \geq 0 \).

It is well known that under the above assumptions the solutions to (1.1) and (1.2) need not be smooth, and that the correct concept of (weak) solutions is that of viscosity solutions. For the definition of viscosity solution in this case we refer to [41, 3, 39, 29, 4]. We state without proof a well-posedness and regularity result for (1.2). The proof of this result is standard, and we refer to [3, 39, 28, 29] for the proofs of similar results.

**Theorem 2.1.** Assume (A1) – (A3) and (B1)_2 hold.

(i) There exists a unique viscosity solution \( u \in C_b(\mathbb{R}^N) \) of equation (1.2) which is Hölder continuous, i.e., there is a \( \delta \in (0, 1] \) such that

\[
|u(x) - u(y)| \leq C|x - y|^\delta \quad \text{for all } x, y \in \mathbb{R}^N.
\]

(ii) There exists a constant \( c_1 > 0 \) depending only on \( \sup_x |\sigma(x,v)|_1 \), \( \sup_x |b(x,v)|_1 \), and \( \sup_v \int_{\mathbb{R}^N} |\eta(x,v,z)|_\pi^2 \nu(dz) \) such that if \( c_0 \geq c_1 \), then the viscosity solution \( u \) of (1.2) is Lipschitz continuous \( (\delta = 1 \text{ above}) \).
(iii) Let \( u, v \in USC_0(\mathbb{R}^N) \). If \( u \) and \( v \) are respectively viscosity sub- and supersolutions of (1.2), then \( u \leq v \) in \( \mathbb{R}^N \).

**Remark 2.2.** Note that this result also holds for (1.1) under assumptions (A1) – (A3) and either one of (B1)_1 or (B1)_0. Simply write this equation in the form (1.2) with \( b \) replaced by \( \tilde{b} = b - \int_E \eta \nu(dz) \).

### 2.2. Reduction to bounded measure with compact support.

Consider Levy measures \( \nu \) which are not bounded nor have compact support. We assume that (B3) hold and consider two cases: (i) \( \alpha \in [0, 1) \) and (ii) \( \alpha \in [1, 2) \). In the first case we truncate the Levy and compensate by adding a drift and a diffusion term. Let us introduce the “two-scales” truncated Levy measure

\[
\nu_{r,R}(dz) := 1_{r < |z| < R} \nu(dz),
\]

where \( r \in (0, 1) \) and \( R > 1 \). Clearly, \( \nu_{r,R}(dz) \) is a bounded and compactly supported measure satisfying assumption (B1)_0. When \( \alpha \in (0, 1) \) assumption (B1)_1 holds and the HJB equation is (1.1). This equation is approximated by

\[
\sup_{v \in V} \left\{ - \text{tr} [\tilde{a}(x, v) D^2 u] - \tilde{b}(x, v) D u + c(x, v) u - f(x, v) - \tilde{I}^u u(x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^N,
\]

where

\[
\tilde{a}(x, v) = a(x, v) + \frac{1}{2} \int_{0 < |z| < r} \eta(x, v, z) \eta(x, v, z)^T \nu(dz),
\]

\[
\tilde{b}(x, v) = b(x, v) + \int_{0 < |z| < r} \eta(x, v, z) \nu(dz),
\]

\[
\tilde{I}^u u(x) = \int_E [\phi(x + \eta(x, v, z)) - \phi(x)] \nu_{r,R}(dz).
\]

This gives an approximation of \( \mathcal{I}^\nu \phi \) which is “third order” near \( z = 0 \) since

\[
|\mathcal{I}^\nu \phi + (\tilde{b} - b) D \phi + \text{tr} [(\tilde{a} - a) D^2 \phi] - \mathcal{J}^\nu \phi| \\
\leq K |D^3 \phi|_0 \left| \int_{0 < |z| < r} |\eta(\cdot, \cdot, z)|^3 \nu(dz) \right|_0 + K |\phi|_0 \int_{|z| > R} \nu(dz).
\]

In case (ii) we truncate the Levy measure and compensate by adding a diffusion term. When \( \alpha \in [1, 2) \) assumption (B1)_1 is not satisfied and the HJB equation is (1.2). We approximate this equation by

\[
\sup_{v \in V} \left\{ - \text{tr} [\bar{a}(x, v) D^2 u] - \bar{b}(x, v) D u + c(x, v) u - f(x, v) - \bar{I}^u u(x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^N,
\]

where \( \bar{a} \) and \( \bar{I}^u \) are defined in (2.3) and (2.5), and

\[
\bar{b}(x, v) = b(x, v) + \int_E 1_{|z| < 1} \eta(x, v, z) \nu_{r,R}(dz).
\]

Again we obtain an approximation of the integral term (\( \mathcal{J}^\nu \phi \) this time) which is “third order” near \( z = 0 \) since

\[
|\mathcal{I}^\nu \phi + (\bar{b} - b) D \phi + \text{tr} [(\bar{a} - a) D^2 \phi] - \mathcal{J}^\nu \phi| \\
\leq K |D^3 \phi|_0 \left| \int_{0 < |z| < r} |\eta(\cdot, \cdot, z)|^3 \nu(dz) \right|_0 + K |\phi|_0 \int_{|z| > R} \nu(dz).
\]

Equations (2.2) and (2.6) are almost, but not quite, of the same form as (1.1). In [30] it is proved that Theorem 2.1 still holds for solutions of (2.2) and (2.6). The following result gives error bounds for the approximations (2.2), (2.6) of equations (1.1), (1.2).

**Lemma 2.3.** Assume (A1) – (A3) and (B3) hold and let \( u, \bar{u}, v, \bar{v} \in C^{0,1}(\mathbb{R}^N) \) solve (1.1), (2.2), (1.2), (2.6) respectively. Then for \( r > 0 \) small enough and \( R \) large enough we have

\[
|u - \bar{u}|_0 + |v - \bar{v}|_0 \leq C_1 (r^{1-\alpha/3} + e^{-tR}),
\]

for some constant \( C_1 \) independent of \( r \) and \( R \).
The result about \( v, \bar{v} \) is proved in [30], while the result about \( u, \bar{u} \) is new but the proof is almost identical to the proof of the \( v, \bar{v} \) result. We omit it.

**Remark 2.4.** The compensating drift and diffusion terms are added to improve the convergence of the approximation (of the small jumps). The error is of order \( r^{1-\alpha/2} \) without the compensating diffusion (see [30]), while it is of order \( r^{1-\alpha} \) without compensating drift and diffusion in the case \( \alpha \in [0,1) \).

**Remark 2.5.** In probabilistic terms, the explanation for the improved convergence is that the small jumps of a Levy process can be approximated by a Brownian motion, we refer to e.g. [2, 19] for details. Moreover using probabilistic methods (Berry-Essen type estimates), it is possible to prove convergence of order \( r \) in some cases [2]. We refer to [20] where such estimates are made explicit for linear parabolic equations with constant coefficients.

**Remark 2.6.** When \( \alpha = 0 \) no improvement is obtained by adding drift and diffusion terms to the equation. If \( \nu \) has a bounded density, no truncation is needed close to \( z = 0 \).

### 2.3. Reduction to a bounded domain

Reduction to a bounded domain is key step in order to implement a numerical method. Following ideas of [17], we restrict equations (1.1) and (1.2) to bounded domains by truncating the coefficients outside some large ball,

\[
B_{\frac{1}{\mu}} = \{ x \in \mathbb{R}^n : |x| < \frac{1}{\mu} \} \text{ for } \mu > 0.
\]

Let \( \xi_\mu \in C_\infty^c(\mathbb{R}^N) \) be a cut-off function satisfying \( 0 \leq \xi_\mu \leq 1 \) and \( \xi_\mu(x) = 1 \) for \( x \in B_{\frac{1}{\mu}} \) and define

\[
\sigma_\mu = \xi_\mu(x)\sigma(x,a), \quad b_\mu = \xi_\mu(x)b(x,a), \quad \eta_\mu(x,v,z) = \xi_\mu(x)\eta(x,v,z).
\]

Since \( b_\mu, \sigma_\mu \) and \( \eta_\mu \) satisfies the same assumptions of \( \sigma, b \) and \( \eta \) for any \( \mu \), there exists a unique viscosity solution \( u_\mu \) of the equation

\[
(2.8) \quad \sup_{x \in \mathbb{R}^N} \left\{ -\text{tr} \left[ a_\mu(x,v)D^2u \right] - b_\mu(x,v)Du + c(x,v)u - f(x,v) - \mathcal{F}_\mu(x) \right\} = 0 \quad \text{in } \mathbb{R}^N
\]

where \( a_\mu(x,v) = \frac{1}{2} \sigma_\mu(x,v)\sigma_\mu^T(x,v) \) and \( \mathcal{F}_\mu(\cdot) \) is defined as in (1.5) with \( \eta_\mu \) in place of \( \eta \). Since the coefficients \( b_\mu, \sigma_\mu \) and \( \eta_\mu \) are zero outside of \( \text{supp}(\xi_\mu) \), the solution \( u_\mu \) (2.8) is given by

\[
u \quad u_\mu(x) = \min_{x \in \mathbb{R}^N} f(x,v) \quad \text{for any } x \notin \text{supp}(\xi_\mu).
\]

The next result give a crude bound on the error that this cut-off procedure introduces.

**Lemma 2.7.** Assume (A1), (A2), (B1) hold, that (1.2) and (2.8) satisfy the dynamic programming principle, and that \( u, u_\mu \) solve (1.2), (2.8) respectively. Then there exists a constant \( C \) such that

\[
|u(x) - u_\mu(x)| \leq C\mu^2(1 + |x|^2) \text{ in } B_{\frac{1}{\mu}}.
\]

We skip the proof since it is similar to the proof of Proposition 3.1 in [17] when we are equipped with the moment estimates (3.2) and (3.4) of [39]. The dynamic programming principle has recently been extended to the current setting in [11].

**Remark 2.8.** As in Remark 2.2 we immediately get an analogous result for (1.1) under assumption (B1)_1.

**Remark 2.9.** If the equation is uniformly elliptic or has a non-degenerate singular integral term, then you expect estimates decaying exponentially as \( \mu \to \infty \). We refer to [20, 36] for such result in a linear one-dimensional setting.

### 3. Construction of the scheme

In subsections 3.1 and 3.2 we always assume that (A1) - (A3) and (B1)_0 hold: The Levy measure \( \nu \) is bounded and the HJB equation has the form (1.1). We can always reduce to this case by a truncation. In subsection 3.3 we discuss the general case.
3.1. **Semi-discretization.** We introduce a control problem for which (1.1) is the corresponding dynamic programming or HJB equation. We start by defining the controlled dynamics. Since the measure $\nu$ is bounded, we can normalize $\nu$ and obtain a probability measure $\mu$ as follows

$$\mu(dz) = \frac{1}{X} \nu(dz)$$

where

$$\lambda = \int E \nu(dz)$$

Now we consider a Markov process $X_t^v$ evolving according the SDE

$$dX_t = (X_{t-}, v_{t-}) dt + \sigma(X_{t-}, v_{t-}) dW_t + \int_{|z|>0} \eta(X_{t-}, v_{t-}, z) \mu(dz, dt),$$

where $\mu$ is a Poisson measure corresponding to a compound Poisson process with jump intensity $\lambda$ and jump distribution $\mu$. The control $v_t$ belongs to $\mathcal{V}$, the set of all progressively measurable processes with values in $V$. Since we assume that $\nu$ and hence $\mu$ are bounded, on any finite time interval $X_t$ will only jump finitely many times with probability one. Between two jump times $T_i$ and $T_{i+1}$ the process diffuses according to the SDE

$$dY_t = (Y_{t-}, v_{t-}) dt + \sigma(Y_{t-}, v_{t-}) dW_t.$$ 

For $v_t \equiv v \in V$ and $a(x, v) = \frac{1}{2} \sigma(x, v) \sigma^T(x, v)$, the infinitesimal generator of the process $X_t$ is

$$L^v \psi(x) = \lim_{t \to 0} \frac{E_x[\psi(X_t)] - \psi(x)}{t} = tr [a(x, v) D^2 \psi] + b(x, v) D \psi + I^v \psi(x)$$

for $\psi \in C^2(\mathbb{R}^N)$. On the paths of the process $X_t$ we define the discounted cost functional

$$J(x, v_t) = E_x \left[ \int_0^\infty f(X_t, v_t) e^{-\int_0^t c(X_s, v_s) ds} dt \right],$$

and we consider the corresponding value function

$$u(x) = \inf_{v_t \in \mathcal{V}} J(x, v_t).$$

In [11] it is proved that $u$ is the unique viscosity solution of equation (1.1).

Following the approach of [16, 17] we construct an approximation scheme for the equation (1.1) by discretizing the associated control problem. We fix a discretization step $h > 0$ and consider two stochastic processes $N_n$ and $Z_n$, $n \in \mathbb{N}$, taking values in $\mathbb{N}$ and in $\mathbb{R}^N$ and representing the $n$-th jump time and the corresponding $z$-jump (size and direction) of the Poisson measure $\bar{\mu}$. We set $N_0 = 0$ and $Z_0 = 0$ and assume that $N_n$ has independent $h\lambda$-exponentially distributed increments, i.e. the probability distribution of $N_n$ is given by

$$P[N_{n+1} - N_n \geq j | N_0, N_1, \ldots, N_n] = e^{-h\lambda j}, \quad n = 0, 1, 2, \ldots,$$

while the $Z_n$, $n \in \mathbb{N}$, are i.i.d. random variables with probability density $\mu$. Now we define a discrete time stochastic process $X_n$ approximating the continuous time process $X_t$.

\[
\begin{align*}
X_0 &= x, \\
X_n &= X_{n-1} + hb(X_{n-1}, v_{n-1}) + \sqrt{h} \sum_{m=1}^d \sigma_m(X_{n-1}, v_{n-1}) \xi_m^{n-1}, \\
X_{N_{n+1}} &= X_{N_{n+1}-1} + \eta(X_{N_{n+1}-1}, v_{n_{N_{n+1}-1}}, Z_i),
\end{align*}
\]

for $i = 1, 2, 3, \ldots$, where $\sigma_m$ denote the $m$-th column of $\sigma$ and $\xi_m^{n}$, $m = 1, \ldots, d$ are random variables taking values in $\{-1, 0, 1\}$ such that

$$P[\xi_i = \pm 1] = \frac{1}{2d} \quad \text{and} \quad P[\xi_i \neq 0 \cap \{\xi_j \neq 0\}] = 0, \quad i \neq j.$$ 

The discrete control $\{v_n\}$ is a random variable with values in $V$ which is measurable with respect to the $\sigma$-algebra generated by $X_1, \ldots, X_n$. 
Between jumps the process evolves like a random walk approximating the SDE (3.3), and when the process jumps there is no diffusion/random walk. The generator of the discrete process is

\[ L_h^\psi(x) = \frac{E_x[\psi(X_1)] - \psi(x)}{h} = e^{-h\lambda}L_h^\psi(x) + \frac{1 - e^{-h\lambda}}{h}T_h^\psi(x), \]

for \( \psi \in C^0(\mathbb{R}^N) \) and where

\[ L_h^\psi = \frac{1}{2dh} \sum_{m=1}^d \left[ \psi(x + hb(x, v) + \sqrt{h}\sigma_m(x, v)) + \psi(x + hb(x, v) - \sqrt{h}\sigma_m(x, v)) - 2\psi(x) \right], \]

\[ T_h^\psi = \frac{1}{\lambda} T^\psi, \]

with \( T^\psi \) as in (1.4). Observe that at this level the space variable is not discretized, therefore the discrete process has the same jump distribution as the continuous process.

On the paths of the discrete process we define the cost functional

\[ J_h(x, \{v_n\}) = E_x[\sum_{n=0}^{\infty} he^{-h\sum_{i=0}^{n-1} c(X_i, v_i)} f(X_n, v_n)], \]

(with the convention \( \sum_{i=0}^{-1} = 0 \), and the corresponding value function for the discrete control problem

\[ u_h(x) = \inf_{\{v_n\}} J_h(x, \{v_n\}). \]

Now it is easy to see, at least formally [9], that the following dynamic programming principle holds:

\[ u_h(x) = \inf_{\{v_n\}} E_x \left[ \sum_{n=0}^{P} he^{-h\sum_{i=0}^{n-1} c(X_i, v_i)} f(X_n, v_n) + e^{-h\sum_{i=0}^{P} c(X_i, v_i)} u_h(X_{P+1}) \right], \]

for any \( P \in \mathbb{N} \). Taking \( P = 0 \) in the above equation and noting that \( X_0 = x \), we get

\[ u_h(x) = \inf_{v \in V} E_x[h f(X_0, v) + e^{-hc(x,v)} u_h(X_1)] \]

\[ = \inf_{v \in V} \left\{ hf(x, v) + e^{-hc(x,v)} \left[ \frac{e^{-h\lambda}}{2d} \sum_{m=1}^d \left( u_h(x + hb(x, v) + \sqrt{h}\sigma_m(x, v)) + \frac{1 - e^{-h\lambda}}{\lambda} \int_E u_h(x + \eta(x, v, z))\nu(dz) \right) \right] \right\}. \]

Rearranging the terms in the previous equation and dividing by \( he^{-hc(x,v)} \) we get

\[ \sup_{v \in V} \left\{ -L_h^\psi u_h(x) + \frac{e^{hc(x,v)}}{h} \frac{1}{u_h(x) - e^{hc(x,v)} f(x, v)} \right\} = 0 \quad \text{in} \quad \mathbb{R}^N, \]

where \( L_h^\psi(x) \) is as in (3.4). We will talk about sub- and supersolutions of this equation, meaning that (3.8) holds as an inequality with \( \leq \) and \( \geq \) respectively. For the scheme (3.8) we have the following easy properties:

**Proposition 3.1.** Assume (A1) - (A3) and (B1). \( (ii) \) If \( u_h \) and \( v_h \) are bounded sub- and supersolutions of (3.8), then \( u_h \leq v_h \) in \( \mathbb{R}^N \). \( (ii) \) Any solution \( u_h \) of (3.8) is bounded and satisfies for all \( h > 0 \),

\[ |u_h|_0 \leq \sup_v |f(., v)|_0 \frac{c_0}{c_0}, \quad \text{where } c_0 \text{ is as in (A3).} \]

\( (iii) \) There exists a unique bounded continuous function \( u_h \) solving (3.8).

\( (iv) \) For \( 0 < h < 1 \) and \( \phi \in C^4(\mathbb{R}^N) \) satisfying \( |\phi|_0 + \cdots + |D^4\phi|_0 < \infty \),

\[ |L^\psi(x) - L_h^\psi(x)| \leq C_1 h(|D^2\phi|_0 + |D^3\phi|_0 + |D^4\phi|_0) + C_2 h\lambda(1 + |\int_E \eta| |D\phi|_0 + |D^2\phi|_0), \]

where the constants \( C_1 \) and \( C_2 \) only depend on \( \sup_v |\sigma|_0, \sup_v |b|_0 \).
Remark 3.2. For the truncation of an unbounded measure to converge as \( h \to 0 \), we need \( \lambda \to \infty \) as \( h \to 0 \), while the scheme (3.8) converges only if both \( h \to 0 \) and \( \lambda h \to 0 \) by (iv). The last condition means that the small jumps must be resolved in the grid.

Proof. We work with the scheme in the equivalent form (3.7). Note that the scheme is monotone, it has positive coefficients. If \( u \) and \( v \) be sub- and supersolutions of (3.8), an easy and standard computation using (3.7) and assumption (A3) shows that

\[
u_h(x) - v_h(x) \leq e^{-ch}\|u_h - v_h\|_0,
\]

and hence \((1 - e^{-ch})(u_h - v_h)\|_0 \leq 0\) which proves (i). A similar computation shows (ii) after noting that \( \frac{he^{hc}}{e^{hc} - 1} \leq 1 \) for \( hc \geq 0 \). Next denote the right hand side of (3.7) by \( T_hu_h \), and note that \( T_h \) is contraction in the \( \| \cdot \|_0 \) norm,

\[
T_hu_h - T_hv_h \leq e^{-hca}\|u_h - v_h\|.\]

Existence and uniqueness of a continuous bounded solution to (3.8) follows from Banach’s fixed point theorem and this proves (iii). Finally, to prove (iv), note that since \( T_h^k = \lambda^{-1}T^v \) we may write

\[
L^v\psi - L_h^v\psi = (L^v - L_h^v)\psi + (T^v - T_h^v)\psi - (1 - e^{-h\lambda})(1 - e^{-h\lambda})L^v\psi,
\]

where \( L^v\psi = \text{tr} [a(x,v)D^2\psi(x)] + b(x,v)D\psi(x) \). By Taylor expansion, e.g.,

\[
|(L^v - L_h^v)\psi| \leq h|b|_0|D^2\psi|_0 + (h^2|b|_0^2 + h|b|_0|\sigma|_0)|D^3\psi|_0 + (h^3|b|_0^3 + h^2|b|_0|\sigma|_0^2 + h|\sigma|_0)|D^4\psi|_0.
\]

and the estimates \(|1 - e^{-x}|, |1 - x^{-1}(1 - e^{-x})| \leq |x|\) the result follows.

Next we give an optimal Lipschitz regularity and continuous dependence on coefficients result for the scheme (3.8).

Proposition 3.3. Let \( u_h \) and \( \tilde{u}_h \) be solutions of (3.8) corresponding to the data \( \sigma, b, c, f, \eta, \nu \) and \( \tilde{\sigma}, \tilde{b}, \tilde{c}, \tilde{f}, \tilde{\eta}, \tilde{\nu} \), respectively, assume both sets of coefficients satisfy (A1) - (A3) and (B1)\(_0\), and that \( h\lambda \leq C_0 \) and \( h \in (0,1) \). Then there exist constants \( c_1, L, K \geq 0 \) (only depending on the data and \( C_0 \)) such that if \( c_0 \geq c_1 \) (\( c_0 \) as in (A3)), then for all \( h > 0, x, y \in \mathbb{R}^N \),

\[
|u_h(x) - \tilde{u}_h(y)| \leq L|x - y| + K \sup_{v \in V} \left[ |f - \tilde{f}|_0 + |c - \tilde{c}|_0 \right]
\]

\[
+ |b - \tilde{b}|_0 + |\sigma - \tilde{\sigma}|_0 + \left( \int_E |\eta(z) - \tilde{\eta}(z)|^2\nu(dz) \right)^{1/2}_0.
\]

If the coefficients are equal and \( u_h = \tilde{u}_h \), then this is a Lipschitz regularity result, while if \( x = y \) then this is a continuous dependence on the coefficients result. This result one of the main contributions of this paper, and it will play a key role in the next section where error bounds are derived. It extends similar results of [5] to equations with integral terms. The proof is rather long and technical and we have put the main bulk in the appendix. In the pure diffusion case, the current proof simplifies considerably the arguments of [5].

Proof. The result follows from Theorem A.1 in the Appendix after writing

\[
b = \left[ b - \frac{1 - e^{-h\lambda}}{\lambda e^{-h\lambda}} \int_E \eta \nu \right] + \frac{1 - e^{-h\lambda}}{\lambda e^{-h\lambda}} \int_E \eta \nu,
\]

and noting that by the Cauchy-Schwartz inequality,

\[
\left| b - \tilde{b} - \frac{1 - e^{-h\lambda}}{\lambda e^{-h\lambda}} \int_E |\eta - \tilde{\eta}| \nu \right| \leq |b - \tilde{b}|_0 + (e^{h\lambda} - 1)\lambda^{-1/2} \int_E |\eta - \tilde{\eta}|^2 \nu \right|_0^{1/2}.
\]

\[
\square
\]

Remark 3.4. When \( c_0 < c_1 \) the solution to (3.8) is only Hölder continuous. We will not discuss this case here, but refer instead to [5] for how to obtain results in this case.
3.2. The fully-discrete scheme. In this section we introduce a FEM like discretization of (3.8) yielding a fully discrete scheme. For a nice introduction to FEMs we refer to [13]. For \( k > 0 \) let \( T^k = \{ S^k_j \}_{j \in \mathbb{N}} \) be a non-degenerate triangulation of \( \mathbb{R}^N \), i.e. a collection of \( N \)-simplices \( S^k_j \) such that

\[
\bigcup_{j \in \mathbb{N}} S^k_j = \mathbb{R}^N, \quad \sup_{j \in \mathbb{N}} (\text{diam } S^k_j) \leq k, \quad pk \leq \sup_{j \in \mathbb{N}} (\text{diam } B_{S^k_j}),
\]

where \( \rho \in (0,1) \), \( \text{diam} \) denotes the diameter of the set, and \( B_{S^k_j} \) is the greatest ball contained in \( S^k_j \). We denote by \( X^k = \{ x_i \}_{i \in \mathbb{N}} \) the corresponding set of the vertices, and introduce the space of continuous piecewise linear functions on \( T^k \),

\[
W^k = \{ w \in C(\mathbb{R}^N) : Dw(x) \text{ is constant in } S^k_j \}.
\]

Every element \( w \) in \( W^k \) can be expressed as

\[
w(x) = \sum_{i \in \mathbb{N}} \beta_i(x) w(x_i),
\]

for basis functions (the so-called tent functions) \( \beta_i \in W^k \) satisfying \( \beta_i(x_j) = \delta_{ij} \) for \( i, j \in \mathbb{N} \). It immediately follows that \( 0 < \beta_i(x) \leq 1 \), \( \sum_{i \in \mathbb{N}} \beta_i(x) = 1 \), \( \beta_i \) has compact support, and at any \( x \in \mathbb{R}^N \) at most \( N + 1 \) \( \beta_i \)'s are non-zero. The family \( \{ \beta_i \}_j \) is a partition of unity. On any simplex \( S^k_i \), the \( \beta_i \)'s are called the barycentric coordinates of \( S^k_i \).

The fully discrete scheme can then be formulated as follows: Find the function \( u \in W^k \) that satisfies (3.8) at every vertex \( x_i \in X^k \), or equivalently,

\[
(3.9) \quad u(x_i) = \inf_{v \in V} \left\{ e^{-h(x_i,v)} \left[ e^{-\lambda h} \sum_j M_{ij}(v) u(x_j) + (1 - e^{-\lambda h}) \sum_j P_{ij}(v) u(x_j) \right] + hf(x_i,v) \right\},
\]

for every \( x_i \in X^k \). Here the matrices \( M(v) \) and \( P(v) \) are given by

\[
(3.10) \quad M(v) = \sum_{m=1}^d \frac{1}{2d} (M^+_m(v) + M^-_m(v))
\]

for \( M^\pm_{m,ij}(v) = \beta_j(x_i + hb(x_i,v) \pm \sqrt{h} \sigma_m(x_i,v)) \), and

\[
(3.11) \quad P_{ij}(v) = \frac{1}{h} \int_E \beta_j(x_i + \eta(x_i,v,z)) \nu(dz).
\]

Note that \( M \) is a stochastic matrix, and for any \( m \), only \( N + 1 \) entries of any row of \( M^+_m \) are non-zero. The matrix \( P \) is non-zero only if the vertex \( x_j \) belong to a simplex which has nonempty intersection with the set \( x_i + \eta(x_i,v,\text{supp}(\nu)) \) for all \( v \in V \).

As a final step we also discretize \( P_{ij} \) by (monotone) quadrature

\[
(3.12) \quad Q_{\Delta z}[\phi] := \sum_{j \in \mathbb{N}} \phi(z_j) \omega_j \quad \text{where } z_j \in E, \quad \omega_j \geq 0,
\]

satisfying the error bound

\[
(3.13) \quad E_{\Delta z}[\phi] := \int_E \phi(z)dz - Q_{\Delta z}[\phi] \leq K ||D\phi||_{L^1} \Delta z \quad \text{for } \phi \in W^{1,1}(\mathbb{R}^n).
\]

Here \( \Delta z > 0 \) is the discretization parameter. An example is the compound midpoint rule where \( Q_{\Delta z}[\phi] = \sum_j \phi(z_j) \Delta z M \) and \( \{ z_j \}_j \) is a renumeration of the uniform \( z \)-grid \( (\Delta z) M \) \( \cap E \). All sensible tensor product quadratures satisfy the (first order) error bound, and the bound is optimal if the integrand \( \phi \) is not more regular. The monotonicity assumption \( \omega_j \geq 0 \) is satisfied for compound Gauss and Newton-Cotes types of quadratures in any space dimension and when the order is less than 9 in the Newton-Cotes case. We refer to [30] for examples and a wider discussion of these issues. If we assuming that \( v \) has a density \( m \) satisfying (B2), we get the following final scheme: Find \( u \in W^k \) such that

\[
(3.14) \quad u(x_i) = \inf_{v \in V} \left\{ e^{-h(x_i,v)} \left[ e^{-\lambda h} \sum_j M_{ij}(v) u(x_j) + (1 - e^{-\lambda h}) \sum_j P_{ij}(v) u(x_j) \right] + hf(x_i,v) \right\},
\]
where $M_{ij}$ is defined in (3.10) and

$$P_{ij}(v) = \frac{1}{\lambda} Q_{\Delta z}[\beta_j(x_i + \eta(x_i, v, \cdot))m(\cdot)].$$

We have the following existence, uniqueness, and partial convergence result for (3.14).

**Theorem 3.5.** Assume (A1) – (A4), (B2), (3.12), (3.13) hold, $h\lambda \leq \bar{C}_0$, and $h \in (0, 1]$. Then there exists a unique bounded solution $u_{hk\Delta z} \in W^k$ to (3.14). Furthermore, if the solution $u_h$ of (3.8) belongs to $C^{0,1}(\mathbb{R}^N)$, then

$$|u_h - u_{hk\Delta z}| \leq \frac{|u_h|_1}{1 - e^{-\lambda h}} (2k + C\Delta z).$$

**Proof.** Existence and uniqueness follow from a fixed point argument like in the proof of Proposition 3.1. To prove the error bound, note that since $u_{hk\Delta z}(x) = \sum_{j \in \mathbb{N}} \beta_j(x)u_{hk\Delta z}(x_j)$, $eta_j \geq 0$, and $\sum \beta_j = 1$,

$$|u_h(x) - u_{hk\Delta z}(x)| \leq \sum_{j \in \mathbb{N}} \left( \beta_j(x)|u_h(x) - u_h(x_j)| + \beta_j(x)|u_h(x_j) - u_{hk\Delta z}(x_j)| \right) \leq |u_h|_1k + \sum_{j \in \mathbb{N}} \beta_j(x)|u_h(x_j) - u_{hk\Delta z}(x_j)|.$$

The last term can be estimated by using (3.8) and (3.14). Easy computations show that

$$|u_h(x_j) - u_{hk\Delta z}(x_j)| \leq e^{-h\lambda}e^{-\lambda h}|u_{hk\Delta z} - u_h|_0 + e^{-h\lambda}(1 - e^{-\lambda h}) \left( \sum_j P_{ij}|u_h(x_j) - u_{hk\Delta z}(x_j)| \right)$$

where $P_{ij} = \frac{1}{\lambda} Q_{\Delta z}[1] = 1$ by (3.13) since $\sum \beta_j = 1$. Furthermore,

$$A = \left| \frac{1}{\lambda} \int_E I_k u_h(x_i + \eta(x_i, v, z))m(z)dz - Q_{\Delta z}\left[ I_k u_h(x_i + \eta(x_i, v, \cdot))m(\cdot) \right] \right|$$

$$\leq \frac{\Delta z}{\lambda} \|D\left[ I_k u_h(x_i + \eta(x_i, v, z))m(z) \right]\|_{L_1} \leq \frac{\Delta z}{\lambda} \left( |u_h|_1 L_\beta C_0 + |u_h|_0 C_1 \right) \int_E e^{-\lambda z}dz,$$

$$B = \left| \frac{1}{\lambda} \int_E (u_h - I_k u_h)(x_i + \eta(x_i, v, z))\nu(dz) \right| \leq k|u_h|_1 \int_E \nu(dz) \leq k|u_h|_1,$$

where $I_k \phi(x) = \sum_i \beta(x_i)\phi(x_i)$ is piecewise linear interpolation of $\phi$ on $X^k$ and we have used (B2), (A4), and (3.13). Combining these estimates and (3.15), using the properties of $\beta_i(x)$, and remembering that $h \in (0, 1]$ then gives the result.

**Remark 3.6.** Since $h \in (0, 1]$, $|u_h - u_{hk\Delta z}| \leq \frac{C|u_h|_1}{\lambda^2} \frac{k + \Delta z}{h}$, which is consistent with the estimates obtained in [16] in a different setting.

**Remark 3.7.** If we use the cut-off procedure explained in Section 2.3 (a way of reducing to a bounded domain), then equation (3.14) gives a finite system of equations. In view of the integral term the new effective domain then becomes $B_\mu + \text{supp}(\nu)$ (see Section 2.3).

**3.3. Schemes for unbounded measures $\nu$.** We consider general unbounded Levy measures $\nu$ under assumption (B3). There are two different cases: (i) $\alpha \in (0, 1)$ with HJB equation (1.1), and (ii) $\alpha \in [1, 2)$ with HJB equation (1.2). To derive our schemes, we first reduce to a bounded Levy measure $\nu_{r, R}$ as explained in Section 2. The result are the approximate HJB equations (2.2) and (2.6). These equations are then approximated by a slightly modified version of the semi-Lagrangian scheme (3.8) (or equivalently (3.7)) defined in Section 3.1.
We propose the following semi-Lagrangian scheme in case (i)

\begin{equation}
\label{eq:semi-Lagrangian}
v_h(x) = \inf_{v \in V} \left\{ hf(x,v) + e^{-hc(x,v)} \left[ \frac{-h \lambda_{r,R}}{4d} \sum_{m=1}^{d} (v_h(x + h\bar{b}(x,v) + \sqrt{h}\bar{\sigma}_{+,m}(x,v))
+ v_h(x + h\bar{b}(x,v) + \sqrt{h}\bar{\sigma}_{-,m}(x,v)) + v_h(x + h\bar{b}(x,v) - \sqrt{h}\bar{\sigma}_{+,m}(x,v))
+ v_h(x + h\bar{b}(x,v) - \sqrt{h}\bar{\sigma}_{-,m}(x,v)) \right) + \frac{1 - e^{-h \lambda_{r,R}}}{\lambda_{r,R}} \int_{E} w_h(x + \eta(x,v,z)) \nu_{r,R}(dz) \right\},
\end{equation}

and in case (ii)

\begin{equation}
\label{eq:semi-Lagrangian2}
w_h(x) = \inf_{v \in V} \left\{ hf(x,v) + e^{-hc(x,v)} \left[ \frac{-h \lambda_{r,R}}{4d} \sum_{m=1}^{d} (w_h(x + h\bar{b}(x,v) + \sqrt{h}\bar{\sigma}_{+,m}(x,v))
+ w_h(x + h\bar{b}(x,v) + \sqrt{h}\bar{\sigma}_{-,m}(x,v)) + w_h(x + h\bar{b}(x,v) - \sqrt{h}\bar{\sigma}_{+,m}(x,v))
+ w_h(x + h\bar{b}(x,v) - \sqrt{h}\bar{\sigma}_{-,m}(x,v)) \right) + \frac{1 - e^{-h \lambda_{r,R}}}{\lambda_{r,R}} \int_{E} w_h(x + \eta(x,v,z)) \nu_{r,R}(dz) \right\},
\end{equation}

where \( \lambda_{r,R} := \int_{E} \nu_{r,R}(dz) \), \( \bar{b}, \bar{b}, \nu_{r,R} \) are defined in Section 2, and \( \bar{\sigma}_{\pm,m} \) is \( m \)-th column of

\begin{equation}
\bar{\sigma}_{\pm}(x,v) = \sigma(x,v) \pm \sqrt{\int_{0<|z|<r} \eta(x,v,z)\eta(x,v,z)T \nu(dz)},
\end{equation}

where the square root denotes the matrix square root.

**Remark 3.8.** The additional terms in (3.16) and (3.17) compared with (3.8), enable us to use \( \sigma \pm \sqrt{\int \eta T \nu} \) instead of \( \sqrt{\sigma \sigma^T + \int \eta \eta^T \nu} \) as diffusion matrix. The consistency relation Proposition 3.1 (iv) still holds, and if \( \eta(x,v,z) = \eta_1(x,v)\eta_2(z) \), then the square root in (3.18) equals \( C \eta_1(x,v) \) where \( C \) is the precomputable constant matrix \( \sqrt{\int_{0<|z|<r} \eta_2(z)\eta_2^T(z) \nu(dz)} \).

**Remark 3.9.** These schemes are similar to (3.8), and can be derived in a similar way. The conclusions of Propositions 3.1 and 3.3 (when \( b \)-terms are replaced by \( \bar{b} \)-terms as defined in Section 2) still hold for solutions of (3.16) and (3.17). We refer to [30] for the technical modifications needed to handle the integral term in the diffusion coefficients.

**Remark 3.10.** Previously bounded quantities may blow up as \( r \to 0 \). Indeed by (B3) and (A2) we have for \( r \in (0,1) \),

\begin{equation}
\lambda_{r,R} \leq \frac{K}{\alpha r^s} \text{ for } \alpha \in (0,2), \quad \left| \int_{E} \eta \nu_{r,R}(dz) \right| \leq \left\{ \begin{array}{ll}
\frac{L_2 K}{(\alpha-1)r^{\alpha s}} & \text{for } \alpha \in (1,2) \\
\frac{L_2 K}{r^{s}} & \text{for } \alpha \in (0,1).
\end{array} \right.
\end{equation}

4. Convergence estimates for the discrete-time problem

In this section we prove a priori error bounds for the convergence of solutions \( u_h \) of the semidiscrete scheme (3.8) to the unique viscosity solution \( u \) of (1.1). We consider two cases: (i) The measure \( \nu \) is bounded and (B1) holds and (ii) the measure \( \nu \) is the truncation of an unbounded measure satisfying (B3).

In view of the equi-boundedness, equi-continuity and consistency results of Propositions 3.1 and 3.3, \( u_h \) converge locally uniformly to \( u \) by the Arzela-Ascoli Theorem, stability and uniqueness results for viscosity solutions (see e.g. [4]), and the consistency result in Proposition 3.1. It is also possible to obtain convergence without equi-continuity (i.e. under weaker assumptions on the coefficients) using so-called half relaxed limits [7]. Such results are given in [14] for some non-local equations, but these results does not cover the HJB equations we consider here.

Now we proceed to obtain a priori estimates for the convergence of \( u_h \) to \( u \). To do this we will make use of an abstract result in [30], which we will describe below. Consider the equation

\begin{equation}
F(x,u,Du,D^2u,u(\cdot)) = 0 \quad x \in \mathbb{R}^N
\end{equation}
where \(u(\cdot)\) represents non-local (integral) terms. Let \(h > 0\) be a discretization parameter and consider an approximation scheme for (4.1) written in abstract form as

\[
S(h, x, u_h(x), [u_h]_x) = 0 \quad x \in \mathbb{R}^N,
\]

where \([u_h]_x\) represents a function defined at \(x\) via all the possible value of \(u_h\). We need the following set of assumptions.

(C1) **(Monotonicity)** There exists \(\tau_0 > 0\) such that for any \(h > 0\), \(x \in \mathbb{R}^N\), \(\zeta \in \mathbb{R}\), \(\tau > 0\) and bounded functions \(u, v\) such that \(u \leq v\) in \(\mathbb{R}^N\), then

\[
S(h, x, \zeta + \tau, [u + \tau]_x) \geq S(h, x, \zeta, [v]_x) + \bar{c}_0 \tau
\]

(C2) **(Regularity)** For any \(h > 0\) and any continuous, bounded function \(\phi\), the function

\[
x \mapsto S(h, x, \phi(x), [\phi]_x)
\]

is bounded and continuous on \(\mathbb{R}^N\) and the function

\[
\zeta \mapsto S(h, x, \zeta, [\phi]_x)
\]

is uniformly continuous for bounded \(\zeta\), uniformly in \(x\).

(C3) **(Consistency)** There exists a function \(E(\tilde{K}, h, \epsilon)\) such that for any sequence \(\{\phi_\epsilon\}_\epsilon\) of smooth functions satisfying

\[
|D^\beta \phi_\epsilon(x)| \leq \tilde{K} \epsilon^{1-|\beta|} \quad \text{in} \ \mathbb{R}^N, \quad \text{for any} \ \beta \in \mathbb{N}^N,
\]

where \(|\beta| = \sum_{i=1}^N \beta_i\), the following inequality holds:

\[
|S(h, x, \phi_\epsilon(x), [\phi_\epsilon]_x) - F(x, \phi, D\phi_\epsilon, D^2\phi_\epsilon)| \leq E(\tilde{K}, h, \epsilon) \quad \text{in} \ \mathbb{R}^N.
\]

(C4) **(Convexity)** Let \((\rho_\epsilon)_{\epsilon > 0}\) be a family of mollifiers (smooth, positive functions with mass 1 and support in \(\{|x| < \epsilon\}\)). For any Lipschitz-continuous function \(\phi\), there exists a constant \(C\) such that for any \(x\) and \(h\)

\[
\int_{\mathbb{R}^N} S(h, x, \phi(x - \epsilon), [\phi(\cdot - \epsilon)]_x) \rho(\epsilon) d\epsilon \geq S(h, x, (\phi \ast \rho_\epsilon)(x), [(\phi \ast \rho_\epsilon)]_x) - C\epsilon
\]

(C5) **(Commutation with translations)** For any \(h > 0\) small enough, \(0 < \epsilon < 1\), \(y \in \mathbb{R}^N\), \(\zeta \in \mathbb{R}\), continuous bounded function \(\phi\) and \(|\epsilon| \leq y\), we have

\[
S(h, y, \zeta, [\phi]_{y - \epsilon}) = S(h, y, \zeta, [\phi(\cdot - \epsilon)]_y).
\]

(D) For \(h\) small enough and \(\epsilon \in [0, 1)\), there is a unique solution \(u_h^0\) of the scheme

\[
(4.3) \quad \max_{|\epsilon| \leq \epsilon} S(h, x + \epsilon, u_h^0(x), [u_h^0]_x) = 0 \quad \text{in} \ \mathbb{R}^N,
\]

where \(u_h := u_h^0\) also solve (4.2), and a constant \(C\) independent of \(h, \epsilon\) such that

\[
|u_h^0|_1 \leq C \quad \text{and} \quad |u_h^0 - u_h|_0 \leq C\epsilon.
\]

We remark that we are using a more general consistency relation here than in [30], and that this extra generality will be needed when we consider unbounded measures \(\nu\). The next result is a restatement of Theorem 3.4 in [30] in view of the new consistency relation (C3).

**Theorem 4.1.** Assume \((A1) - (A3), (B1)_2, (C1) - (C5), \text{and} (D)\) hold, and let \(u_0\) and \(u_h\) be solutions of respectively (4.1) and (4.2) satisfying \(K := |u_0| \vee |u_h| < \infty\). Then there exists a constant \(C\) depending only on \(L_1, L_2, c_0\) from \((A2)\) and \((A3)\) such that

\[
|u - u_h| \leq C \min_{\epsilon > 0} \left( \epsilon + E_1(\tilde{K}, h, \epsilon) \right) \quad \text{in} \ \mathbb{R}^N.
\]

**Remark 4.2.** To prove (D) for the scheme (3.8), we will need to assume also \((B1)_0, h\lambda \leq \bar{c}_0, h \leq 1,\) and \(c_0 \geq c_1\) for both \(c_1\) defined in Theorem 2.1 and Proposition 3.3. Under these assumptions we also have \(K := |u| \vee |u_h| < \infty\).
We will apply this abstract result to derive error bounds for the scheme (3.8). We rewrite the scheme in the form (4.2) with
\[
S(h, x, r, [\psi]_x) =
\]
\[
\text{(4.4)} \sup_{v \in \mathcal{V}} \left\{ -\frac{e^{-\lambda h}}{2dh} \sum_{m=1}^{d} \left[ [\psi]_x(hb(x, v) + \sqrt{m} \sigma_m(x, v)) - 2r + [\psi]_x(hb(x, v) - \sqrt{m} \sigma_m(x, v)) \right] \right. \\
- \frac{1 - e^{-\lambda h}}{\lambda h} \int_{E} [\psi(x + \eta(x, v, z)) - r] + \frac{\epsilon h c(x, v) - 1}{h} - \epsilon h c(x, v) f(x, v) \right\},
\]
and \([\psi]_x(z) = \psi(x + z)\).

**Lemma 4.3.** Assume (A1) – (A3) and (B1), \(h \lambda \leq \bar{C}_0, h \leq 1\). Then the scheme (3.8) (and equivalently (4.4)) satisfies assumptions (C1)–(C5) with
\[
E(\tilde{K}, h, \varepsilon) = C_1 h (\tilde{K} \varepsilon^{-1} + \tilde{K} \varepsilon^{-2} + \tilde{K} \varepsilon^{-3}) + C_2 h \lambda ((1 + |\int_{E} \eta \nu|) \tilde{K} + \tilde{K} \varepsilon^{-1})
\]
where the constant \(C_1\) and \(C_2\) only depend on sup \(\sigma\), sup \(b\).

If in addition \(c_0 \geq c_1\) for both \(c_1\)'s in Theorem 2.1 and Proposition 3.3, then assumption (D) also holds with constants \(C\) only depending on the data and \(\bar{C}_0\).

**Proof.** It is straightforward to verify (C1) with \(\tau_0 = c_0\) where \(c_0\) is defined in (A3). (C2) follows from the assumptions on the coefficients, while (C3) follows from Proposition 3.1 (iv). By a straightforward computation, it follows that (C4) holds with \(C = 0\). We refer to [30] for similar computations. Finally, (C5) holds since \([\phi']_{x-e} = [\phi' - e]_x\) for any continuous function \(\phi\).

To prove (D), observe that (4.3) can be rewritten in the form (4.4) (by defining a new control \(\theta = (v, e)\)). The coefficients of this new equation still satisfy (A1) – (A3). Therefore (D) follows after an application of Propositions 3.1 and 3.3. \(\square\)

Now we are in a position to state the error bounds. First we consider the bounded case, i.e. (B1) holds. In this case the equation is (1.1) which is approximated by the scheme (3.8). Note that the integral operator has not yet been discretized.

**Theorem 4.4** (Bounded measure). Assume (A1) – (A3), (B1), \(h \leq 1\), and \(c_0 \geq c_1\) for both \(c_1\)'s in Theorem 2.1 and Proposition 3.3. Let \(u\) be the solution of (1.1) and \(u_h\) be the solution of (3.8) (or equivalently (4.4)).

(a) (General IPDEs) Then \(|u - u_h|_0 \leq Ch^{1/4}\).

(b) (1st order IPDEs) If in addition \(\sigma \equiv 0\), then \(|u - u_h|_0 \leq Ch^{1/2}\).

In both cases the constant \(C\) depends only on the coefficients, \(c_1\), and \(\lambda\).

**Proof.** Part (a) is an easy consequence of Theorem 4.1 and Lemma 4.3. Part (b) follows in a similar manner after noting that the consistency relation corresponding to Proposition 3.1 (iv) now becomes
\[
|L^* \psi(x) - L^*_h \psi(x)| \leq C_1 h |D^2 \phi|_0 + C_2 h \lambda ((1 + |\int_{E} \eta \nu|) |D \phi|_0 + |D^2 \phi|_0).
\]
\(\square\)

**Remark 4.5.** The convergence rate obtained in Theorem 4.4 is the same as in the pure PDE case, see [27, 5]. In the first order case and when the Levy measure is bounded, convergence rate 1/2 for semi-Lagrangian schemes like (3.8) have previously been obtained in [16]. However the integral term in [16] has a different form compared to the one we consider here.

**Remark 4.6.** The scheme (3.8) uses a first order accurate approximation of 2nd derivatives as can be seen from the consistency relation Proposition 3.1 (iv). This leads to lower rates of convergence than for some monotone FDMs that use 2nd order accurate approximations of 2nd derivatives, see [33, 30, 10]. There the rate is 1/2, while for a more general class of FDMs the rate is at least 1/5, see [6] for the pure PDE case.
We proceed to the case of general unbounded Levy measure \( \nu \) under assumption (B3). There are two different cases: (i) \( \alpha \in [0, 1) \) with HJB equation (1.1), and (ii) \( \alpha \in [1, 2) \) with HJB equation (1.2). In case (i), \( \alpha \in (0, 1) \), as a consequence of Lemma 4.3, Theorems 4.1 and 2.3 we have the following convergence result.

**Theorem 4.7** (Unbounded measure I). Assume (A1) – (A3), (B3) with \( \alpha \in (0, 1) \), \( h \leq 1 \), and \( c_0 \geq c_1 \) for both \( c_1 \)’s in Theorem 2.1 and Proposition 3.3. Let \( u \) be the solution of (1.1) and \( u_h \) be the solution of (3.16).

Then the best rate is obtained choosing \( r = h^{\frac{1}{1+\alpha}} \), and in this case

\[
|u - u_h|_0 \leq C(h^{1/4} + h^{\frac{1}{1+\alpha}}) \leq C h^{1/4},
\]

where the constant \( C \) depend only on the coefficients, \( c_1 \), and quantities from (B3)/(B1)2.

**Proof.** When \( \alpha \in (0, 1) \), Lemma 4.3 still holds for the scheme (3.16), and in view of (3.19) we have the following form of \( E \),

\[
E(\tilde{K}, h, r, \varepsilon) = C_1 \tilde{K} h (e^{-1} + \varepsilon^{-2} + \varepsilon^{-3}) + C_2 \tilde{K} h^{-\alpha} (1 + \varepsilon^{-1}),
\]

where the constant \( C_1 \) and \( C_2 \) are independent of \( h, r, \varepsilon \). Let \( u_r \) denote the solution of (2.2). Theorem 4.1 and (“term-wise”) minimization in \( \varepsilon \), lead to the bound

\[
|u_r - u_h|_0 \leq C(h^{1/4} + r^{-\alpha} h + r^{-2} h^{1/2}),
\]

where the constant \( C \) depend only on the coefficients, \( c_1 \), and quantities from (B3)/(B1)2. In view of Theorem 2.3 and the optimal choice of \( r \), \( r = h^{\frac{1}{1+\alpha}} \), the result follows. \( \square \)

**Remark 4.8.** Since \( \frac{1}{4} < \frac{3-\alpha}{3+5\alpha} < \frac{1}{2} \) for \( \alpha \in (0, 1) \), there is no reduction of rate due to truncation of the measure \( \nu \).

The case \( \alpha \in (1, 2) \) is more difficult, since now also \( \int_E \eta \nu_{r, R} \) and hence \( \bar{b} \) in (3.17) blows up as \( r \to 0 \). As a consequence Theorem 4.1 can no longer be used directly. The convergence result is the following:

**Theorem 4.9** (Unbounded measure II). Assume (A1) – (A3), (B3) with \( \alpha \in (1, 2) \), \( h \leq 1 \), and \( c_0 \geq c_1 \) for both \( c_1 \)’s in Theorem 2.1 and Proposition 3.3. Let \( u \) be the solution of (1.2) and \( u_h \) be the solution of (3.17).

Then the best rate is obtained choosing \( r = h^{\frac{1}{3+5\alpha}} \), and in this case

\[
|u - u_h|_0 \leq Ch^{\frac{1}{3+5\alpha}},
\]

where the constant \( C \) depend only on the coefficients, \( c_1 \), and quantities from (B3)/(B1)2.

**Remark 4.10.** This result is consistent with Theorem 4.7 since \( \lim_{\alpha \to 1^-} \frac{3-\alpha}{3+5\alpha} = \frac{1}{4} \). For \( \alpha \in (1, 2) \) the rate degrades as \( \alpha \) increases, and \( \lim_{\alpha \to 2^-} \frac{3-\alpha}{3+5\alpha} = \frac{1}{13} \).

**Outline of proof.** From Proposition 3.3 we have uniform Lipschitz continuity, but the continuous dependence estimates will be proportional to \( \int_E |z| \nu_{r, R} \) through the \( \bar{b} \)-term. Because of this we must redo the arguments leading to Theorem 4.1, and the result will be an estimate of the form

\[
|u - u_h| \leq C \min_{\varepsilon > 0} \left( \varepsilon \int_E |z| \nu_{r, R} + E_1(\tilde{K}, h, r, \varepsilon) \right) \quad \text{in } \mathbb{R}^N,
\]

when \( u_r \) solve (2.6). We omit the details, since the argument is same as the one used to prove Theorem 4.1 in [30]. Because of the blow up in \( \int_E \eta \nu_{r, R} \), we also need a much more precise consistency relation than given in Proposition 3.1 (iv) tracking all \( \bar{b} \) dependence. From the proof of Proposition 3.1, it is easy to see that it will have the following form

\[
E(\tilde{K}, h, \varepsilon) = C_1 \tilde{K} \left[ \bar{b}_{10}^2 + (h^2 \bar{b}_{10}^2 + h \bar{b}_{10}) \varepsilon^{-2} + (h + h^3 \bar{b}_{10}^2 + h^2 \bar{b}_{10}^2 \varepsilon^{-3} \right]
\]

\[+ C_2 \tilde{K} \lambda_h \left[ \bar{b}_{10}^2 + \varepsilon^{-1} \right] + C_3 \tilde{K} \lambda_h \left[ \int_E \eta \nu_{r, R} \right], \]
where the constants $C_1, C_2, C_3$ only depend on $\sup_{\nu} |\tilde{\sigma}_0|$. In view of (3.19), $\lambda = Cr^{-\alpha}$ and $|\tilde{b}_0 + \int_E |z|\nu_{e,R}| \leq C(1 + r^{\alpha-1})$. The rest of the proof is a long computation consisting of choosing optimal $\varepsilon$ and $r$ as in the proof of Theorem 4.7. We omit the details only remarking that the worst term in $E$ turns out to be the $he^{-\lambda t}$-term.}

**Appendix A. Lipschitz regularity and continuous dependence**

In this section we obtain a combined Lipschitz regularity and continuous dependence result for solutions to the equation

\[
(A.1) \quad u_h(x) = \inf_{v \in \mathcal{V}} \left\{ \bar{h} f(x,v) + e^{-hc(x,v)} \left[ \frac{e^{-h\lambda}}{2d} \sum_{m=1}^{d} \left( u_h(x + h\bar{b}(x,v) + \sqrt{h}\sigma_m(x,v)) + u_h(x + h\bar{b}(x,v) - \sqrt{h}\sigma_m(x,v)) + \frac{1 - e^{-h\lambda}}{\lambda} \int_E u_h(x + \eta(x,v,z))\nu(dz) \right) \right] \right\},
\]

where $\lambda = \nu(E)$ and

\[
b = b + \frac{1 - e^{-h\lambda}}{\lambda e^{-h\lambda}} \int_E \eta \nu.
\]

**Theorem A.1.** Let $u_h$ and $\tilde{u}_h$ be solutions of (A.1) corresponding to the data $\sigma, \tilde{b}, c, f, \eta, \nu$ and $\tilde{\sigma}, \tilde{b}, \tilde{c}, \tilde{f}, \tilde{\eta}, \nu$ respectively, and assume both sets of coefficients satisfy (A1) – (A3) and (B1), and that $h \lambda \leq C$ and $h \leq 1$. Then there exist constants $c_1, L, K \geq 0$ (only depending on the data and $C$) such that if $0 \leq c_1$, then for all $h > 0, x, y \in \mathbb{R}^N$,

\[
|u_h(x) - \tilde{u}_h(y)| \leq L|x - y| + K \sup_{v \in \mathcal{V}} \left[ |f - \tilde{f}|_0 + |c - \tilde{c}|_0 + |b - \tilde{b}|_0 + |\sigma - \tilde{\sigma}|_0 + \int |\eta(\cdot, z) - \tilde{\eta}(\cdot, z)|^2 \nu(dz)\right]^{1/2}.
\]

**Remark A.2.** The precise dependence of the constants $c_1, L, K$ is given in the proof below.

**Remark A.3.** This result extends the corresponding results of [5] to non-local HJB equations, with (general singular) Levy measures. Moreover, the proof below simplifies the corresponding proofs of Barles and Jakobsen [5] because we do not use the somewhat unnatural “doubling schemes” as in [5]. Instead we give a more direct proof.

**Proof.** We will use doubling of variables techniques similar to those used to prove corresponding results for equation (1.1). We define

\[
\phi(x, y) = m + \alpha M + \frac{L}{2}(\alpha^{-1} + \alpha|x - y|^2) + \varepsilon(|x|^2 + |y|^2),
\]

\[
\psi(x, y) = u_h(x) - \tilde{u}_h(y) - \phi(x, y),
\]

for $\alpha, \varepsilon > 0$, $M, L \geq 0$, and then we let $(\bar{x}, \bar{y})$ be a point such that $\psi(\bar{x}, \bar{y}) = \sup_{x,y} \psi(x, y)$. We will prove that $\psi(\bar{x}, \bar{y}) \leq o(1)$ as $\varepsilon \to 0$ for a suitable constant $L$, and for

\[
(A.2) \quad m = \left( \frac{e^{ch} - 1}{h} \right)^{-1} \sup_{v \in \mathcal{V}} e^{c_0 \nu|\tilde{\sigma}| h} \left[ |f - \tilde{f}|_0 + |u_h|_0 \land |\tilde{u}_h|_0 + |b|_0 \land |\tilde{b}|_0 |c - \tilde{c}|_0 \right],
\]

\[
(A.3) \quad M = K \left( \frac{e^{ch} - 1}{h} \right)^{-1} \sup_{v \in \mathcal{V}} \left[ |\sigma - \tilde{\sigma}|_0^2 + |b - \tilde{b}|_0^2 + \int |\eta(\cdot, z) - \tilde{\eta}(\cdot, z)|^2 \nu(dz)\right].
\]

This implies Theorem A.1 after sending $\varepsilon \to 0$. To see this, fix $x, y$ and note that for any $\alpha > 0$,

\[
u_{e,R}, R \leq C(1 + r^{\alpha-1}).
\]

In this inequality we send $\varepsilon \to 0$ and choose $\alpha^{-1} = |x - y| \lor (2M/L)^{1/2}$ to obtain

\[
u_{e,R} \leq m + 2(LM)^{1/2} + L|x - y|,
\]

and hence Theorem A.1 follows since $x, y$ were arbitrary.
For simplicity we will not be explicit about the form of the $\varepsilon$-terms appearing in the computations below. The role of these terms is only to guarantee that the maximum is attained at a point $(\bar{x}, \bar{y})$, and they vanish in the final estimate when $\varepsilon \to 0$. We refer to the proof of Theorem 3.4 in [5] for details concerning the $\varepsilon$-terms.

We proceed by contradiction assuming that $\psi(\bar{x}, \bar{y}) > o(1)$ as $\varepsilon \to 0$. Note that by the definition of $\psi$ this implies that $u_h(\bar{x}) - u_h(\bar{y}) > 0$. Furthermore, observe that since $(\bar{x}, \bar{y})$ is a maximum point,

$$
\psi(\bar{x} + b + a, \bar{y} + b + \bar{a}) + \psi(\bar{x} + b - a, \bar{y} + b - \bar{a}) \leq 2\psi(\bar{x}, \bar{y}),
$$

$$
\psi(\bar{x} + \zeta, \bar{y} + \bar{\zeta}) \leq \psi(\bar{x}, \bar{y}),
$$

for every $a, b, \zeta, \bar{a}, \bar{b}, \bar{\zeta} \in \mathbb{R}^N$, and hence by the definition of $\psi$,

$$
I_1 := [u_h(\bar{x} + b + a) - 2u_h(\bar{x}) + u_h(\bar{x} + b - a)]
$$

$$
- [\bar{u}_h(\bar{y} + b + \bar{a}) - 2\bar{u}_h(\bar{y}) + \bar{u}_h(\bar{y} + b - \bar{a})]
$$

$$
\leq \phi(\bar{x} + b + a, \bar{y} + b + \bar{a}) - 2\phi(\bar{x}, \bar{y}) + \phi(\bar{x} + b - a, \bar{y} + b - \bar{a}),
$$

$$
I_2 := [u_h(\bar{x} + \zeta) - u_h(\bar{x}) - [\bar{u}_h(\bar{y} + \zeta) - \bar{u}_h(\bar{y})] \leq \phi(\bar{x} + \zeta, \bar{y} + \bar{\zeta}) - \phi(\bar{x}, \bar{y}).
$$

Finally, by the definition of $\phi$ and the simple identities

$$
|\bar{x} + b \pm a - (\bar{y} + b \pm \bar{a})|^2 = |\bar{x} \pm a - (\bar{y} \pm \bar{a})|^2 + 2(\bar{x} \pm a - (\bar{y} \pm \bar{a})) \cdot (b \pm \bar{b}) + |b - \bar{b}|^2,
$$

$$
|\bar{x} - \bar{y} + (a - \bar{a})|^2 - 2|\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{y} - (a - \bar{a})|^2 = 2|a - \bar{a}|^2,
$$

we are lead to

$$
I_1 \leq 2\frac{L}{2}\alpha|a - \bar{a}|^2 + 2\frac{L}{2}\alpha|b - \bar{b}|^2 + 4\frac{L}{2}\alpha|\bar{x} - \bar{y}, b - \bar{b}| + o(1) \quad \text{as } \varepsilon \to 0,
$$

$$
I_2 \leq 2\frac{L}{2}\alpha|\bar{x} - \bar{y}, \zeta - \bar{\zeta}| + \frac{L}{2}\alpha|\zeta - \bar{\zeta}|^2 + o(1) \quad \text{as } \varepsilon \to 0.
$$

These two inequalities are crucial for the rest of the proof.

Now we divide (A.1) by $h e^{-hc}$ and rewrite it as

$$
0 = \sup_{v \in V} \left\{ \frac{e^{hc} - 1}{h} u_h(x) - e^{hc} f(x, v) \right\}
$$

$$
- \frac{e^{-h\lambda}}{2dh} \sum_{m=1}^d \left( u_h(x + h\tilde{b}(x, v) + \sqrt{h}\sigma_m(x, v)) - 2u_h(x) + u_h(x + h\tilde{b}(x, v) - \sqrt{h}\sigma_m(x, v)) \right)
$$

$$
- \frac{1}{\lambda h} \int_E \left( u_h(x + \eta(x, v, z)) - u_h(x) \right) \nu(dz) \right\}.
$$

Upon subtracting the equations (in this new form) for $\bar{u}_h$ and $u_h$, we get

$$
0 \leq \sup_{v \in V} \left\{ \frac{e^{hc} - 1}{h} \bar{u}_h(\bar{y}) - \frac{e^{hc} - 1}{h} u_h(\bar{x}) \right\}
$$

$$
+ \left[ e^{hc} f(\bar{x}, v) - e^{hc} f(\bar{y}, v) \right] + \frac{e^{-h\lambda}}{2dh} \sum_{m=1}^d I_{1,m} + \frac{1}{\lambda h} \int_E I_2 \nu(dz) \right\},
$$

where $I_{1,m}$ corresponds to $I_1$ above with the choice $a = \sqrt{h}\sigma_m(\bar{x}, v)$, $b = h\tilde{b}(\bar{x}, v)$, $\bar{a} = \sqrt{h}\sigma_m(\bar{y}, v)$, and $\bar{b} = h\tilde{b}(\bar{y}, v)$, and for $I_2$ we have taken $\zeta = \eta(\bar{x}, v, z)$ and $\bar{\zeta} = \eta(\bar{y}, v, z)$. By (A.4) and (A.5)
we then have

\[
0 \leq \sup_{v \in V} \left\{ \frac{e^{ch} - 1}{h} \vec{u}_h(\vec{y}) - \frac{e^{ch} - 1}{h} u_h(\vec{x}) + [e^{ch} f(\vec{x}, v) - e^{ch} \tilde{f}(\vec{y}, v)] + \frac{e^{-\lambda h}}{2} L \alpha \left[ \frac{1}{d} \sum_{m=1}^{d} 2h|\sigma_m(\vec{x}, v) - \sigma_m(\vec{y}, v)|^2 
+ 2h^2 |b(\vec{x}, v) - \tilde{b}(\vec{y}, v)|^2 + 4h(\vec{x} - \vec{y}, b(\vec{x}, v) - \tilde{b}(\vec{y}, v)) \right] \right. \\
\left. + \frac{e^{-\lambda h}}{\lambda h} \frac{L}{2} \alpha \int_E \left[ |\eta(\vec{x}, v, z) - \tilde{\eta}(\vec{y}, v, z)|^2 + 2(\vec{x} - \vec{y}, \eta(\vec{x}, v, z) - \tilde{\eta}(\vec{y}, v, z)) \right] \nu(dz) \right\} + o(1) \quad \text{as } \varepsilon \to 0.
\]

Since \( \bar{b} = b + \frac{1 - e^{-\lambda h}}{\lambda h} \int_E \eta \nu \) and \( \bar{\tilde{b}} \) is defined similarly we see that the \( (\vec{x} - \vec{y}, \eta - \tilde{\eta}) \)-terms cancel in the above inequality. Since \( \nu(E) = \lambda \), Jensen’s inequality implies that

\[
\lambda^2 \left| \int_E (\eta(\vec{x}, v, z) - \tilde{\eta}(\vec{y}, v, z)) \frac{\nu(dz)}{\lambda} \right|^2 \leq \lambda^2 \int_E |\eta(\vec{x}, v, z) - \tilde{\eta}(\vec{y}, v, z)|^2 \frac{\nu(dz)}{\lambda}.
\]

Also note that since \( u_h(\vec{x}) - \vec{u}_h(\vec{y}) > 0 \) and \( c \geq c_0 > 0 \),

\[
\frac{e^{ch} - 1}{h} \vec{u}_h(\vec{y}) - \frac{e^{ch} - 1}{h} u_h(\vec{x}) \leq - \frac{e^{ch} - 1}{h} [u_h(\vec{x}) - \vec{u}_h(\vec{y})] + \frac{e^{ch} - e^{ch}}{h} |u_h(\vec{x})| \wedge |u_h(\vec{y})|.
\]

Therefore after cancellations, Jensen’s inequality, and the inequality \( \frac{1 - e^{-\lambda h}}{\lambda h} \leq he^{\lambda h} \), we get

\[
\frac{e^{ch} - 1}{h} |u_h(\vec{x}) - \vec{u}_h(\vec{y})| \leq \sup_{v \in V} \left\{ \frac{e^{ch} - 1}{h} [f(\vec{x}, v) - \tilde{f}(\vec{y}, v)] + (|u_h| \wedge |\vec{u}_h| + h|f|) |e(\vec{x}, v) - \tilde{c}(\vec{y}, v)| \\
+ \frac{e^{-\lambda h}}{2h} L \alpha \left[ \frac{1}{d} \sum_{m=1}^{d} 2h|\sigma_m(\vec{x}, v) - \sigma_m(\vec{y}, v)|^2 
+ 2h^2 |b(\vec{x}, v) - \tilde{b}(\vec{y}, v)|^2 + 4h(\vec{x} - \vec{y}, b(\vec{x}, v) - \tilde{b}(\vec{y}, v)) \right] \\
+ \frac{1 - e^{-\lambda h}}{\lambda h} \frac{L}{2} \alpha \int_E \left[ 4h^2 \lambda he^{\lambda h} \int_E |\eta(\vec{x}, v, z) - \tilde{\eta}(\vec{y}, v, z)|^2 \nu(dz) \right. \\
\left. + |\eta(\vec{x}, v, z) - \tilde{\eta}(\vec{y}, v, z)|^2 \right] \nu(dz) \right\} + o(1) \quad \text{as } \varepsilon \to 0.
\]
Now since $\lambda h \leq C$ and $h < 1$, it follows from simple computations that
\[
\frac{e^{\alpha h} - 1}{h} \left[ \psi(\bar{x}, \bar{y}) + \phi(\bar{x}, \bar{y}) \right] = \frac{e^{\alpha h} - 1}{h} \left[ u_h(\bar{x}) - \tilde{u}_h(\bar{y}) \right]
\]
\[
\leq \sup_{v \in V} \left\{ e^{c|v|/|\alpha h|} \left[ f - \bar{f}(0) + (|u_h|_0 + b|f|_0 + 1) |c - c_0| \right] \right\}
\]
\[
+ \frac{L}{2} \alpha e^{-\lambda h} e^{\alpha h} \left[ \frac{1}{d} \sum_{m=1}^{d} 4|\sigma_m|_0^2 + 16h|b - \bar{b}|_0^2 \right]
\]
\[
+ \frac{L}{2} \alpha^{5} C e^{C} \left[ \int_{E} |\eta(\cdot, v, z) - \tilde{\eta}(\cdot, v, z)|^2 \nu(dz)|_0 \right]
\]
\[
+ \sup_{v \in V} \left\{ e^{c|v|/|\alpha h|} \left[ L_f + (|u_h|_0 + b|f|_0 + 1) |c - c_0| \right] \cdot L_0 \right\}
\]
\[
+ o(1) \quad \text{as } \varepsilon \to 0.
\]

Let $L, L_0$ be defined as in the inequality above. If $c_0$ is so big that
\[
\frac{e^{\alpha h} - 1}{h} - L_0 > 0,
\]
and we choose $m, M$ as in (A.2) and (A.3) for $K$ big enough, and
\[
L = \frac{\frac{L}{h} - L_0}{e^{\alpha h} - 1 - L_0},
\]
then $\psi(\bar{x}, \bar{y}) \leq o(1)$ as $\varepsilon \to 0$ and the proof is complete. \qed

References


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