CONTINUOUS DEPENDENCE ESTIMATES FOR NONLINEAR FRACTIONAL CONVECTION-DIFFUSION EQUATIONS

NATHAËL ALIBAUD, SIMONE CIFANI, AND ESPEN R. JAKOBSEN

Abstract. We develop a general framework for finding error estimates for convection-diffusion equations with nonlocal, nonlinear, and possibly degenerate diffusion terms. The equations are nonlocal because they involve fractional diffusion operators that are generators of pure jump Lévy processes (e.g. the fractional Laplacian). As an application, we derive continuous dependence estimates on the nonlinearities and on the Lévy measure of the diffusion term. Estimates of the rates of convergence for general nonlinear nonlocal vanishing viscosity approximations of scalar conservation laws then follow as a corollary. Our results both cover, and extend to new equations, a large part of the known error estimates in the literature.

1. Introduction

This paper is concerned with the following Cauchy problem:

$$(1.1)\qquad \begin{cases} \partial_t u(x,t) + \operatorname{div}\left(f(u)\right)(x,t) = \mathcal{L}^\mu[A(u(\cdot,t))](x) & \text{in} \quad Q_T := \mathbb{R}^d \times (0,T), \\ u(x,0) = u_0(x), & \text{in} \quad \mathbb{R}^d, \end{cases}$$

where u is the scalar unknown function, div denotes the divergence with respect to (w.r.t.) x, and the operator $\mathcal{L}^\mu$ is defined for all $\phi \in C_c^\infty(\mathbb{R}^d)$ by

$$(1.2)\qquad \mathcal{L}^\mu[\phi](x) := \int_{\mathbb{R}^d\setminus\{0\}} \phi(x+z) - \phi(x) - z \cdot D\phi(x)\mathbf{1}_{|z|\leq 1} \, \mathrm{d}\mu(z),$$

where $D\phi$ denotes the gradient of ϕ w.r.t. x and $\mathbf{1}_{|z|\leq 1} = 1$ for $|z| \leq 1$ and $= 0$ otherwise. Throughout the paper, the data (f, A, u_0, μ) is assumed to satisfy the following assumptions:

$$(1.3)\qquad f = (f_1, \dots, f_d) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \text{ with } f(0) = 0,$$

$$(1.4)\qquad A \in W^{1,\infty}(\mathbb{R}) \text{ is nondecreasing with } A(0) = 0,$$

$$(1.5)\qquad u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d),$$

and

(1.6) μ is a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} |z|^2 \wedge 1 \, \mathrm{d}\mu(z) < \infty,$$

where we use the notation $a \wedge b = \min\{a, b\}$. The measure μ is a Lévy measure.

Remark 1.1.

(1) Subtracting constants to f and A if necessary, there is no loss of generality in assuming that $f(0) = 0$ and $A(0) = 0$.

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(2) Our results also hold for locally Lipschitz-continuous nonlinearities f and A since solutions will be bounded; see Remark 2.3 for more details.
(3) Assumption (1.6) and Taylor expansion reveals that $\mathcal{L}^\mu[\phi]$ is well-defined for e.g. bounded C^2 functions ϕ:

$$|\mathcal{L}^\mu[\phi](x)| \leq \max_{|z|\leq 1} |D^2\phi(x+z)| \int_{0<|z|\leq 1} \frac{1}{2}|z|^2 \mathrm{d}\mu(z) + 2\|\phi\|_{L^\infty} \int_{|z|>1} \mathrm{d}\mu(z)$$

where $D^2\phi$ is the Hessian of ϕ. If in addition $D^2\phi$ is bounded on $\mathbb{R}^d$, then so is $\mathcal{L}^\mu[\phi]$.

Under (1.6), $\mathcal{L}^\mu$ is the generator of a pure jump Lévy process, and reversely, any pure jump Lévy process has a generator of like $\mathcal{L}^\mu$ (see e.g. [6, 54]). This class of diffusion processes contains e.g. the α-stable process whose generator is the fractional Laplacian $-(-\triangle)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$. It can be defined for all $\phi \in C_c^\infty(\mathbb{R}^d)$ via the Fourier transform as

$$(-\triangle)^{\frac{\alpha}{2}}\phi = \mathcal{F}^{-1}\left(|\cdot|^\alpha \mathcal{F}\phi\right),$$

or in the form (1.2) with the following Lévy measure (see e.g. [6, 32]):

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}z}{|z|^{d+\alpha}} \text{ (up to a positive multiplicative constant).} \tag{1.7}$$

Many other Lévy processes/operators of practical interest can be found in e.g. [6, 24]. Under assumption (1.4), $\mathcal{L}^\mu[A(\cdot)]$ is an example of a nonlinear nonlocal diffusion operator. For recent studies of this and similar type of operators, we refer the reader to [8, 9, 15, 19, 27] and the references therein.

Equation (1.1) appears in many different contexts such as overdriven gas detonations [22], mathematical finance [24], flow in porous media [27], radiation hydrodynamics [51, 52], and anomalous diffusion in semiconductor growth [57]. Equations of the form (1.1) constitute a large class of nonlinear degenerate parabolic integro-differential equations (integro-PDEs). Let us give some representative examples. When $A=0$ or $\mu=0$, (1.1) is the well-known scalar conservation law (see e.g. [25] and references therein):

$$\partial_t u + \mathrm{div} f(u) = 0. \tag{1.8}$$

When $A(u)=u$, (1.1) is the so-called Lévy/fractal/fractional conservation law:

$$\partial_t u + \mathrm{div} f(u) = \mathcal{L}^\mu[u]. \tag{1.9}$$

Equation (1.9) has been extensively studied since the nineties [1, 2, 3, 4, 5, 7, 10, 11, 12, 16, 17, 20, 21, 26, 28, 29, 30, 31, 32, 35, 37, 38, 39, 40, 41, 44, 48, 49, 50]. When A is nonlinear, (1.1) can be seen as a generalization of the following classical convection-diffusion equation (possibly degenerate):

$$\partial_t u + \mathrm{div} f(u) = \triangle A(u); \tag{1.10}$$

see e.g. [13, 14, 18, 23] for precise references on (1.10). The case of nonlinear and nonlocal diffusions has been studied in [27] in the setting of nonlocal porous media equations, and in [19] where a general L^1-theory for (1.1) is developed along with connections to Hamilton-Jacobi-Bellman equations of stochastic control theory. Other interesting examples concern the class of nonsingular Lévy measures satisfying $\int_{\mathbb{R}^d\setminus\{0\}} \mathrm{d}\mu(z) < +\infty$. In that case, $\mathcal{L}^\mu$ is a convolution operator and (1.1) can be seen as a generalization of Rosenau's models [42, 43, 47, 48, 55, 56] and nonlinear radiation hydrodynamics models [51] of the respective forms

$$\partial_t u + \mathrm{div} f(u) = g_\mu * u - u, \tag{1.11}$$

$$\partial_t u + \mathrm{div} f(u) = g_\mu * A(u) - A(u), \tag{1.12}$$

where $*$ denotes the convolution product w.r.t. x and $g_\mu \in L^1(\mathbb{R}^d)$ is nonnegative with $\int_{\mathbb{R}^d} g_\mu(z)\mathrm{d}z = 1$.

Most of the results on such nonlocal convection-diffusion equations concern Equation (1.9) whose diffusion is linear. It is known that shocks can occur in finite time [4, 28, 42, 43, 44, 48, 55], that weak solutions can be nonunique [2], and that the Cauchy problem is well-posed with the notion of entropy solutions in the sense of Kruzhkov [1, 41, 47, 55]. Results on nonlinear nonlocal diffusions can be found in [51] where entropy solutions of (1.12) are studied. Very recently, the entropy solution theory has been extended in [19] to cover the full problem (1.1) for general singular Lévy measures and nonlinear A.

The purpose of the present paper is to develop an abstract framework for finding error estimates for entropy solutions of (1.1). As applications, we focus in this paper on continuous dependence estimates and convergence rates for vanishing viscosity approximations. We refer the reader to [13, 18, 23, 46] and the references therein for similar analysis on (1.10) and related local equations. As far as nonlocal equations are concerned, continuous dependence estimates for fully nonlinear integro-PDEs have already been derived in [36] in the context of viscosity solutions of Bellman-Isaacs equations; see also [32, 34, 36] for error estimates on nonlocal vanishing viscosity approximations. To the best of our knowledge, there are only a few results for nonlocal conservation laws. All the results we have found concern Equations (1.9), (1.11) and (1.12) for which we refer the reader to [1, 29, 32, 41, 47, 55]. A large part of these error estimates concern convergence rates for vanishing viscosity approximations. The only result we have found on continuous dependence estimates appears in [41]; it concerns Equation (1.9) in the case of self-adjoint Lévy operators. To finish with the bibliography, let us also refer the reader to [20, 21, 26, 30, 51] for the related topic of error estimates for numerical approximations.

Our main result is stated in Lemma 3.1, and it compares the entropy solution u of (1.1) with a general function v. Our main application consists in comparing u with the entropy solution v of

$$\begin{cases} \partial_t v + \mathrm{div} g(v) = \mathcal{L}^\nu[B(v)], \\ v(x,0) = v_0, \end{cases} \tag{1.13}$$

where the data set (g, B, v_0, ν) is assumed to satisfy (1.3)–(1.6). We obtain explicit continuous dependence estimates on the data stated in Theorems 3.3–3.4. Let us recall that when $B = 0$ or $\nu = 0$, (1.13) is the pure scalar conservation law in (1.8). Equation (1.1) can thus be seen as a nonlinear nonlocal vanishing viscosity approximation of (1.8) if A or μ vanishes. The rate of convergence is then obtained as a consequence of Theorems 3.3–3.4, see Theorem 3.7.

It is natural to compare Theorems 3.3–3.4 and Theorem 3.7 with the known error estimates for Equations (1.9), (1.11) and (1.12). One can see that a quite important part of them are particular cases of our general results. We discuss this point in Section 3 by giving precise examples. Let us mention that we also give a simple example of Hamilton-Jacobi equations suggesting that Theorems 3.3–3.4 are in some sense the "conservation laws' versions" of the results in [36]; see Example 3.2.

To finish, let us mention that in the case of fractional Laplacians of order $\alpha \geq 1$, Theorems 3.3–3.4 can be improved by taking advantage of the homogeneity of the measures in (1.7). In order not to make this paper too long, this special case (including $\alpha < 1$) is investigated in a second paper [3].

The rest of this paper is organized as follows. In Section 2 we list the notation used throughout the paper; we also recall the notion of entropy solution to (1.1). In Section 3, we state and discuss our main results. Sections 4–5 are devoted to the proofs of our main results; Section 4 states some preliminary results on the nonlocal operator. For the reader's convenience, some more or less known technical results have been gathered in the Appendices.

2. Preliminaries

In this section we explain most of the notation used in the paper, and we give the definition of entropy solutions of (1.1) along with a well-posedness result. For the definitions of measures and BV-spaces, we refer to the books [33, 53].

2.1. **Notation.**

2.1.1. *Vectors, sets and functions.* Throughout the paper $d \in \mathbb{N}$ is a fixed dimension, $T > 0$ a fixed time, and $(x, t) = (x_1, \dots, x_d, t) \in Q_T := \mathbb{R}^d \times (0, T)$ is the generic space-time variable. For all $a, b \in \mathbb{R}$ we let $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, $a^+ := a \vee 0$, and $a^- := (-a) \vee 0$. For all $m \in \mathbb{N}$, we let $\cdot$ and $|\cdot|$ denote the Euclidean inner product and norm of $\mathbb{R}^m$, while for a matrix $A \in \mathbb{R}^{m\times m}$, we use the norm $|A| = \max\{A\, w : w \in \mathbb{R}^m,\ |w| \leq 1\}$. We let $-E := \{-w \in \mathbb{R}^m : w \in E\}$, and denote the characteristic function of the set E by $\mathbf{1}_E$.

By C^∞ and C_c^∞ we denote the spaces of infinitely differentiable functions and infinitely differentiable functions with compact support. Moreover, for $p \in [1, +\infty]$, L^p, $W^{k,p}$, L^1_{loc}, and $\mathcal{D}'$ denote the Lebesgue and Sobolev spaces, the locally integrable functions, and the Schwartz distributions respectively. Sometime we indicate range and domain of the functions, e.g. $C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

The support of $u \in \mathcal{D}'$ is denoted by $\operatorname{supp} u$. The restriction of u to a set U is denoted by $u_{|_U}$. By $u * v$ we mean the convolution of two functions $u = u(x, t)$ and $v = v(x, t)$ w.r.t. to the space variable x. We let $\partial_t u$, $D_x u$, and $D_x^2 u$ denote the partial derivative in time, the spatial gradient, and the spatial Hessian matrix of u respectively. If there is no confusion, we write D instead of D_x. The derivative of a one variable function u is written u'. The same notation is also used for distributional derivatives.

2.1.2. *Radon measures.* Let μ be a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$, i.e. a measure $\mu : \mathcal{B}_{\mathbb{R}^d\setminus\{0\}} \to [0, +\infty]$ which is finite on compact sets and where $\mathcal{B}_{\mathbb{R}^d\setminus\{0\}}$ is the Borel σ-algebra of $\mathbb{R}^d \setminus \{0\}$. As usual, μ is extended to a complete measure to the (smallest) μ-completion of $\mathcal{B}_{\mathbb{R}^d\setminus\{0\}}$:

$$\begin{aligned}\overline{\mathcal{B}}^{\mu}_{\mathbb{R}^d\setminus\{0\}} := \{E \subseteq \mathbb{R}^d \setminus \{0\} \text{ s.t. there is } B_i \in \mathcal{B}_{\mathbb{R}^d\setminus\{0\}}\ (i = 1, 2)\\ \text{with } B_1 \subseteq E \subseteq B_2 \text{ and } \mu(B_2 \setminus B_1) = 0\}.\end{aligned}$$

We say that $u : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is μ-measurable (μ-integrable) if for each real interval I, $u^{-1}(I) \in \overline{\mathcal{B}}^{\mu}_{\mathbb{R}^d\setminus\{0\}}$ (if in addition it is integrable w.r.t. μ). For the Lebesgue measure we simply use the terminologies *measurable, integrable, almost everywhere (a.e.), etc.*

Throughout the paper the Lebesgue measure of $\mathbb{R}^m$ is denoted by $\mathrm{d}w$ if w denotes the generic variable of $\mathbb{R}^m$. Its tensor product with μ is denoted by $\mathrm{d}\mu(z)\,\mathrm{d}w$; note that this is a well-defined nonnegative Radon measure on $\overline{\mathcal{B}}^{\mathrm{d}\mu(z)\,\mathrm{d}w}_{(\mathbb{R}^d\setminus\{0\})\times\mathbb{R}^m}$, since μ is σ-finite.

Given another nonnegative Radon measure ν on $\mathbb{R}^d \setminus \{0\}$, the total variation of $\mu - \nu$ is denoted by $|\mu - \nu|$. The positive and negative parts $\frac{|\mu-\nu|\pm(\mu-\nu)}{2}$ are denoted by $(\mu - \nu)^\pm$ respectively.

2.1.3. *BV-spaces.* The space $BV(\mathbb{R}^d)$ of functions with bounded variation on $\mathbb{R}^d$ is defined as the space of $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that Du is a (finite variation Radon) measure $Du : \mathcal{B}_{\mathbb{R}^d} \to \mathbb{R}^d$. Let us recall that its total variation $|Du|$ satisfies for all $B \in \mathcal{B}_{\mathbb{R}^d}$,

(2.1)
$$|Du|(B) = \inf_{B \subseteq U \text{ open}} \sup \left\{ \int_{\mathbb{R}^d} u \operatorname{div}\phi \, \mathrm{d}x : \phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d), \ |\phi| \leq 1, \ \operatorname{supp}\phi \subset U \right\}.$$

The BV-semi-norm of u is defined as $|u|_{BV(\mathbb{R}^d)} := |Du|(\mathbb{R}^d) < +\infty$.

2.1.4. *Functions in $L^\infty \cap C(L^1) \cap L^\infty(BV)$.* Given $u \in L^\infty(Q_T) \cap C([0,T]; L^1)$ we denote by $u(t)$ the function $u(\cdot, t)$ if there is no confusion. The $C([0,T]; L^1)$-norm of u is

$$\|u\|_{C([0,T];L^1)} := \max \left\{ \|u(t)\|_{L^1(\mathbb{R}^d)} : t \in [0,T] \right\}.$$

The modulus of continuity (in time) of $u \in C([0,T]; L^1)$ is denoted by

$$\omega_u(\delta) := \max \left\{ \|u(t) - u(s)\|_{L^1(\mathbb{R}^d)} : t, s \in [0,T] \text{ s.t. } |t-s| \leq \delta \right\}, \ \delta > 0.$$

Throughout the paper we say that

$$u \in L^\infty(Q_T) \cap C([0,T]; L^1) \cap L^\infty(0,T; BV)$$

if for all $t \in [0,T]$, $u(t) \in BV(\mathbb{R}^d)$, and if the $L^\infty(0,T;BV)$-semi-norm of u,

$$|u|_{L^\infty(0,T;BV)} := \sup \left\{ |u(t)|_{BV(\mathbb{R}^d)} : t \in [0,T] \right\} < \infty.$$

The $L^1(0,T;BV)$-semi-norm of u is defined as

$$|u|_{L^1(0,T;BV)} := \int_0^T |u(t)|_{BV(\mathbb{R}^d)} \, \mathrm{d}t.$$

Note that it is standard that $t \to |u(t)|_{BV(\mathbb{R}^d)}$ is measurable since it is lower semi-continuous; see e.g. [33].

2.2. **Entropy formulation and well-posedness.** Let us recall the formal computations leading to the entropy formulation of (1.1). First we split $\mathcal{L}^\mu$ into 3 parts:

$$\mathcal{L}^\mu[\phi](x) = \mathcal{L}^\mu_r[\phi](x) + \operatorname{div}\left(b^\mu_r \, \phi\right)(x) + \mathcal{L}^{\mu,r}[\phi](x) \tag{2.2}$$

for $\phi \in C_c^\infty(\mathbb{R}^d)$, $r > 0$, and $x \in \mathbb{R}^d$, where

$$\mathcal{L}^\mu_r[\phi](x) := \int_{0<|z|\leq r} \phi(x+z) - \phi(x) - z \cdot D\phi(x) \, \mathbf{1}_{|z|\leq 1} \, \mathrm{d}\mu(z), \tag{2.3}$$

$$b^\mu_r := -\int_{|z|>r} z \mathbf{1}_{|z|\leq 1} \, \mathrm{d}\mu(z), \tag{2.4}$$

$$\mathcal{L}^{\mu,r}[\phi](x) := \int_{|z|>r} \phi(x+z) - \phi(x) \, \mathrm{d}\mu(z). \tag{2.5}$$

Consider then the Kruzhkov [45] entropies $|\cdot - k|$, $k \in \mathbb{R}$, and entropy fluxes

$$q_f(u,k) := \operatorname{sgn}(u-k)\,(f(u) - f(k)) \in \mathbb{R}^d, \tag{2.6}$$

where we always use the following everywhere representative of the sign function:

$$\operatorname{sgn}(u) := \begin{cases} \pm 1 & \text{if } \pm u > 0, \\ 0 & \text{if } u = 0. \end{cases} \tag{2.7}$$

By (1.4) it is readily seen that for all $u, k \in \mathbb{R}$,

$$\operatorname{sgn}(u-k)\,(A(u) - A(k)) = |A(u) - A(k)|, \tag{2.8}$$

and we formally deduce from (2.2), (2.8), and the nonnegativity of μ that

$$\begin{aligned}&\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu}[A(u)]\\&\leq \mathcal{L}^{\mu}_{r}[|A(u)-A(k)|]+\operatorname{div}(b^{\mu}_{r}\,|A(u)-A(k)|)+\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)].\end{aligned}$$

Let u be a solution of (1.1), and multiply (1.1) by $\operatorname{sgn}(u-k)$. Formal computations then reveals that

$$\begin{aligned}\partial_t|u-k|+\operatorname{div}\,(q_f(u,k)-b^{\mu}_{r}\,|A(u)-A(k)|)&\\ \leq \mathcal{L}^{\mu}_{r}[|A(u)-A(k)|]+\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)].&\end{aligned}$$

The entropy formulation in Definition 2.1 below consists in asking that u satisfies this inequality for all entropy-flux pairs (i.e. for all $k\in\mathbb{R}$) and all $r>0$. Roughly speaking one can give a sense to $\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]$ for bounded discontinuous u thanks to (1.6). But since μ may be singular at $z=0$, see Remark 1.1 (3), the other terms have to be interpreted in the sense of distributions: Multiply by test functions ϕ and integrate by parts to move singular operators onto test functions. For the nonlocal terms this can be done by change of variables: First take $(z,x,t)\to(-z,x,t)$ to see (formally) that

$$\int_{Q_T}\phi\operatorname{div}\,(b^{\mu}_{r}\,|A(u)-A(k)|)\,\mathrm{d}x\mathrm{d}t=\int_{Q_T}D\phi\cdot b^{\mu^*}_{r}\,|A(u)-A(k)|\,\mathrm{d}x\mathrm{d}t,$$

where μ^* is the Lévy measure (i.e. it satisfies (1.6)) defined by

$$\mu^*(B):=\mu(-B)\quad\text{for all}\quad B\in\mathcal{B}_{\mathbb{R}^d\setminus\{0\}}. \tag{2.9}$$

In view of (2.3), we can take $(z,x,t)\to(-z,x+z,t)$ to find that

$$\int_{Q_T}\phi\,\mathcal{L}^{\mu}_{r}[|A(u)-A(k)|]\,\mathrm{d}x\mathrm{d}t=\int_{Q_T}|A(u)-A(k)|\,\mathcal{L}^{\mu^*}_{r}[\phi]\,\mathrm{d}x\mathrm{d}t.$$

This leads to the following definition introduced in [19].

Definition 2.1. (Entropy solutions) *Assume* (1.3)–(1.6). *We say that a function $u\in L^\infty(Q_T)\cap C\left([0,T];L^1\right)$ is an entropy solution of* (1.1) *provided that for all $k\in\mathbb{R}$, all $r>0$, and all nonnegative $\phi\in C^\infty_c(\mathbb{R}^{d+1})$,*

$$\begin{aligned}&\int_{Q_T}|u-k|\,\partial_t\phi+\Big(q_f(u,k)+b^{\mu^*}_{r}\,|A(u)-A(k)|\Big)\cdot D\phi\,\mathrm{d}x\mathrm{d}t\\&\quad+\int_{Q_T}|A(u)-A(k)|\,\mathcal{L}^{\mu^*}_{r}[\phi]+\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\,\mathrm{d}x\,\mathrm{d}t\\&\qquad-\int_{\mathbb{R}^d}|u(x,T)-k|\,\phi(x,T)\,\mathrm{d}x+\int_{\mathbb{R}^d}|u_0(x)-k|\,\phi(x,0)\,\mathrm{d}x\geq 0.\end{aligned} \tag{2.10}$$

Remark 2.1.

(1) Under assumptions (1.3)–(1.6), the entropy inequality (2.10) is well-defined independently of the a.e. representative of u. To see this note that μ^* obviously satisfies (1.6), and hence it easily follows that $\mathcal{L}^{\mu^*}_{r}[\phi]\in C^\infty_c(\mathbb{R}^{d+1})$. Since $\operatorname{sgn}(u-k)$, $q_f(u,k)$, and $A(u)$ belong to L^∞ by (2.7) and (1.3)–(1.4), it is then clear that all terms in (2.10) are well-defined except possibly the $\mathcal{L}^{\mu,r}$-term. Here it may look like we are integrating Lebesgue measurable functions w.r.t. a Radon measure μ. However, the integrand does have the right measurability/integrability by Lemma 4.2. We therefore find that since $A(u)$ belongs to $C([0,T];L^1)$, so does also $\mathcal{L}^{\mu,r}[A(u)]$ and we are done.

(2) In the definition of entropy solutions, it is possible to consider functions u only defined for a.e. $t\in[0,T]$ by taking test functions with compact support in Q_T and adding an explicit initial condition, see e.g. [19].

(3) One can check that classical solutions are entropy solutions, thus justifying the formal computations leading to Definition 2.1. Moreover entropy solution are weak solutions and hence smooth entropy solutions are classical solutions. We refer the reader to [19] for the proofs.

Here is a well-posedness result from [19].

Theorem 2.2. (Well-posedness) *Assume* (1.3)–(1.6). *There exists a unique entropy solution* u *of* (1.1). *This entropy solution belongs to* $L^\infty(Q_T) \cap C\left([0,T];L^1\right) \cap L^\infty(0,T;BV)$ *and*

$$\begin{cases} \|u\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \\ \|u\|_{C([0,T];L^1)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}, \\ |u|_{L^\infty(0,T;BV)} \leq |u_0|_{BV(\mathbb{R}^d)}. \end{cases} \tag{2.11}$$

Moreover, if v *is the entropy solution of* (1.1) *with* $v(0) = v_0$ *for another initial data* v_0 *satisfying* (1.5)*, then*

$$\|u - v\|_{C([0,T];L^1)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}. \tag{2.12}$$

Remark 2.3. By the L^∞-estimate in (2.11), all the results of this paper also holds for locally Lipschitz-continuous nonlinearities (f, A). Simply replace the data (f, A) by $(f, A)\,\mathbf{1}_{[-M,M]}$ with $M := \|u_0\|_{L^\infty(\mathbb{R}^d)}$.

3. Main results

Our first main result is a Kuznetsov type of lemma that measures the distance between the entropy solution u of (1.1) and an arbitrary function v.

Let $\epsilon, \delta > 0$ and $\phi^{\epsilon,\delta} \in C^\infty(Q_T^2)$ be the test function

$$\phi^{\epsilon,\delta}(x,t,y,s) := \theta_\delta(t-s)\,\bar{\theta}_\epsilon(x-y), \tag{3.1}$$

where $\theta_\delta(t) := \frac{1}{\delta}\tilde{\theta}_1\left(\frac{t}{\delta}\right)$ and $\bar{\theta}_\epsilon(x) := \frac{1}{\epsilon^d}\tilde{\theta}_d\left(\frac{x}{\epsilon}\right)$ are, respectively, time and space approximate units with kernel $\tilde{\theta}_n$ with $n = 1$ and $n = d$ satisfying

$$\tilde{\theta}_n \in C_c^\infty(\mathbb{R}^n), \quad \tilde{\theta}_n \geq 0, \quad \operatorname{supp}\tilde{\theta}_n \subseteq \{|x| < 1\}, \quad \text{and} \quad \int_{\mathbb{R}^n} \tilde{\theta}_n(x)\,\mathrm{d}x = 1. \tag{3.2}$$

Also recall that $\omega_u(\delta)$ is the modulus of continuity of $u \in C\left([0,T];L^1\right)$.

Lemma 3.1 (Kuznetsov type Lemma)**.** *Assume* (1.3)–(1.6). *Let* u *be the entropy solution of* (1.1) *and* $v \in L^\infty(Q_T) \cap C\left([0,T];L^1\right)$ *with* $v(0) = v_0$. *Then for all* $r > 0$,

$\epsilon > 0$, *and* $0 < \delta < T$,

(3.3)

$$\begin{aligned}
&\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\
&\le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + \epsilon\, C_{\tilde\theta}\, |u_0|_{BV(\mathbb{R}^d)} + 2\, \omega_u(\delta) \vee \omega_v(\delta) \\
&\quad - \iint_{Q_T^2} |v(x,t) - u(y,s)|\, \partial_t \phi^{\epsilon,\delta}(x,t,y,s)\, \mathrm{d}w \\
&\quad - \iint_{Q_T^2} \Big(q_f(v(x,t),u(y,s)) + b_r^{\mu^*}\, |A(v(x,t)) - A(u(y,s))| \Big) \cdot D_x \phi^{\epsilon,\delta}(x,t,y,s)\, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v(x,t)) - A(u(y,s))|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\, \mathrm{d}w \\
&\quad - \iint_{Q_T^2} \operatorname{sgn}(v(x,t) - u(y,s))\, \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}(x,t,y,s)\, \mathrm{d}w \\
&\quad + \iint_{\mathbb{R}^d \times Q_T} |v(x,T) - u(y,s)|\, \phi^{\epsilon,\delta}(x,T,y,s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \\
&\quad - \iint_{\mathbb{R}^d \times Q_T} |v_0(x) - u(y,s)|\, \phi^{\epsilon,\delta}(x,0,y,s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s
\end{aligned}$$

where $\mathrm{d}w := \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$, *and* $C_{\tilde\theta} := 2\int_{\mathbb{R}^d} |x| \tilde\theta_d(x)\, \mathrm{d}x$.

Remark 3.2.

(1) The error in time only depends on the moduli of continuity of u and v at $t = 0$ and $t = T$. Here we simply take the global-in-time moduli of continuity $\omega_u(\delta)$ and $\omega_v(\delta)$, since this is sufficient in our settings.
(2) When $A = 0$ or $\mu = 0$ this lemma reduces to the well-known Kuznetsov lemma [46] for multidimensional scalar conservation laws.
(3) Notice that the $\mathcal{L}_r^{\mu^*}$-term vanishes when $r \to 0$, see Lemma 4.6.
(4) Lemma 3.1 has many applications. In this paper and in [3] we focus on continuous dependence results and error estimates for the vanishing viscosity method. In a future paper, we will use the lemma to obtain error estimates for numerical approximations of (1.1).

In this paper we apply Lemma 3.1 to compare the entropy solution u of (1.1) with the entropy solution v of (1.13). This is our second main result , and we present it in the two theorems below. The first focuses on the dependence on the nonlinearities (with $\mu = \nu$) and the second one on the Lévy measure (with $A = B$).

Theorem 3.3. (Continuous dependence on the nonlinearities) *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* $(g, B, v_0, \nu = \mu)$ *satisfying* (1.3)–(1.6). *Then for all* $r_2 > r_1 > 0$,

$$\begin{aligned}
\|u - v\|_{C([0,T];L^1)} &\le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)} \sqrt{c_d\, T \int_{0<|z|\le r_1} |z|^2\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})}} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)}\, T \int_{r_1<|z|\le r_2} |z|\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)}\, T \left| \int_{r_1\wedge 1<|z|\le r_1\vee 1} z\, \mathrm{d}\mu(z) \right| \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&\quad + T \int_{|z|\ge r_2} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)}\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})},
\end{aligned} \tag{3.4}$$

where $c_d = \frac{4d^2}{d+1}$.

Theorem 3.4. (Continuous dependence on the Lévy measure) *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* $(g, B = A, v_0, \nu)$ *satisfying* (1.3)–(1.6). *Then for all* $r_2 > r_1 > 0$,

$$\begin{aligned} \|u - v\|_{C([0,T];L^1)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty} \\ &+ |u_0|_{BV(\mathbb{R}^d)} \sqrt{c_d\, T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{0<|z|\leq r_1} |z|^2 \,\mathrm{d}|\mu - \nu|(z)} \\ &+ |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{r_1<|z|\leq r_2} |z| \,\mathrm{d}|\mu - \nu|(z) \\ &+ |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \left| \int_{r_1\wedge 1<|z|\leq r_1\vee 1} z \,\mathrm{d}(\mu - \nu)(z) \right| \\ &+ T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{|z|\geq r_2} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}|\mu - \nu|(z), \end{aligned} \tag{3.5}$$

where $c_d = \frac{4d^2}{d+1}$.

¿From Theorems 3.3 and 3.4 we can easily find a general continuous dependence estimate when both A and μ are different from B and ν, respectively. E.g. we can take an intermediate solution w of $w_t + \operatorname{div} f(w) = \mathcal{L}^\mu[B(w)]$ and $w(0) = u_0$, and use the triangle inequality. Using this idea we can show that the following estimates always have to hold:

Corollary 3.5. *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* (g, B, v_0, ν) *satisfying* (1.3)–(1.6). *Then*

$$\begin{aligned} \|u - v\|_{C([0,T];L^1)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty} \\ &+ C\,(T^{\frac{1}{2}} \vee T) \left(\sqrt{\|A' - B'\|_{L^\infty(\mathbb{R})}} + \sqrt{\int_{\mathbb{R}^d\setminus\{0\}} |z|^2 \wedge 1 \,\mathrm{d}|\mu - \nu|(z)} \right) \end{aligned} \tag{3.6}$$

where C only depends on d and the data. Moreover, if in addition

$$\int_{\mathbb{R}^d\setminus\{0\}} |z| \wedge 1 \,\mathrm{d}\mu(z) + \int_{\mathbb{R}^d\setminus\{0\}} |z| \wedge 1 \,\mathrm{d}\nu(z) < \infty,$$

then we have the better estimate

$$\begin{aligned} \|u - v\|_{C([0,T];L^1)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty} \\ &+ CT \left(\|A' - B'\|_{L^\infty(\mathbb{R})} + \int_{\mathbb{R}^d\setminus\{0\}} |z| \wedge 1 \,\mathrm{d}|\mu - \nu|(z) \right), \end{aligned} \tag{3.7}$$

where C only depends on d and the data.

Outline of proof. To prove (3.6), we use Theorems 3.3 and 3.4 with $r_1 = 1 = r_2$ and the triangle inequality. We also use estimates like $|a - b| \leq \sqrt{|a| + |b|}\sqrt{|a - b|}$, $|\mu - \nu| \leq |\mu| + |\nu|$ etc. To prove (3.7) we take $r_1 = 0$ and $r_2 = 1$. □

Remark 3.6.

(1) All these estimates hold for arbitrary Lévy measures μ, ν and even for strongly degenerate diffusions where A, B may vanish on large sets. They are consistent (at least for the $|\mu - \nu|$ term) with general results for nonlocal Hamilton-Jacobi-Bellman equations in [36]. When μ, ν have the special form (1.7) (with possibly different α's), then it is possible to use the extra symmetry and homogeneity properties to obtain better estimates, see [3].

(2) The optimal choice of the r_1, r_2 depends on the behavior of the Lévy measures at zero and infinity, see the discussion above and at the end of this section for more details.
(3) In the special case of symmetric Lévy measures ($\mu = \mu^*$), the terms corresponding to the cutting $r_1 \wedge 1 < |z| \leq r_1 \vee 1$ disappear.

Let us now consider the nonlocal vanishing viscosity problem

$$\begin{cases} \partial_t u^\epsilon + \operatorname{div} f(u^\epsilon) = \epsilon\, \mathcal{L}^\mu[A(u^\epsilon)], \\ u^\epsilon(0) = u_0, \end{cases} \tag{3.8}$$

i.e. problem (1.8) with a perturbation term $\epsilon\, \mathcal{L}^\mu[A(u^\epsilon)]$. When $\epsilon > 0$ tend to zero, u^ϵ is expected to converge toward the solution u of (1.8). As an immediate application of Theorem 3.3 or 3.4, we have the following result:

Theorem 3.7 (Vanishing viscosity)**.** *Assume* (1.3)–(1.6)*. Let u and u^ϵ be the entropy solutions of* (1.8) *and* (3.8) *respectively. Then*

$$\begin{aligned} \|u - u^\epsilon\|_{C([0,T];L^1)} \leq C \min_{r_2 > r_1 > 0} \Bigg\{ & T^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \sqrt{\int_{0<|z|\leq r_1} |z|^2 \, \mathrm{d}\mu(z)} \\ & + T\epsilon \left[\int_{r_1 < |z| \leq r_2} |z| \, \mathrm{d}\mu(z) + \left| \int_{r_1 \wedge 1 < |z| \leq r_1 \vee 1} z \, \mathrm{d}\mu(z) \right| + \int_{|z| \geq r_2} \mathrm{d}\mu(z) \right] \Bigg\}, \end{aligned} \tag{3.9}$$

where C only depends on d, $\|u_0\|_{L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)}$, $\|A'\|_{L^\infty(\mathbb{R})}$.

Outline of proof. Note that u can be seen as the entropy solution of (1.1) with $A = 0$ and μ as Lévy measure. Hence we can estimate $\|u - u^\epsilon\|_{C([0,T];L^1)}$ from Theorem 3.3. The error coming from the difference of the derivatives of the nonlinearities is equal to $\epsilon\, \|A'\|_{L^\infty(\mathbb{R})}$. Inequality (3.9) then follows from (3.4). □

Corollary 3.8. *Assume* (1.3)–(1.6)*. Let u and u^ϵ be the entropy solutions of* (1.8) *and* (3.8) *respectively. Then*

$$\|u - u^\epsilon\|_{C([0,T];L^1)} \leq C\, (T^{\frac{1}{2}} \vee T)\, \epsilon^{\frac{1}{2}},$$

where C only depends on d and the data. Moreover, if in addition

$$\int_{\mathbb{R}^d \setminus \{0\}} |z| \wedge 1 \, \mathrm{d}\mu(z) < \infty,$$

then we have the better estimate

$$\|u - u^\epsilon\|_{C([0,T];L^1)} \leq C T \epsilon,$$

where C depends on d and the data.

This corollary follows immediately from Theorem 3.7 or Corollary 3.5.

Remark 3.9.

(1) Our estimates are just as good or better than the standard $\mathcal{O}(\epsilon^{\frac{1}{2}})$ estimate for the classical vanishing viscosity method ((1.10) with $A(u) = \epsilon\, u$).
(2) Our estimates hold for arbitrary Lévy measures μ and even for strongly degenerate diffusions where A may vanish on a large set! This is consistent with general results for nonlocal Hamilton-Jacobi-Bellman equations [36].
(3) If the solutions are smoother, it is possible to obtain better estimates. Also if μ is as in (1.7), the additional symmetry and homogeneity can be used to obtain better estimates which can be proven to be optimal. See Example 3.3 below.
(4) Corollary 3.8 contains less information than Theorem 3.7 and is not strong enough to get optimal results in all cases, e.g. in Example 3.3 with $\alpha \geq 1$.

(5) The error estimates above trivially also holds for the more general vanishing viscosity equation

$$\begin{cases} \partial_t u^\epsilon + \mathrm{div} f(u^\epsilon) = \mathcal{L}^\nu[B(u^\epsilon)] + \epsilon\, \mathcal{L}^\mu[A(u^\epsilon)], \\ u^\epsilon(0) = u_0. \end{cases}$$

Further discussion. We now make a more precise comparison of the results above with known estimates from the literature. We begin with continuous dependence estimates and finish with convergence rates for vanishing viscosity approximations.

Let u and v denote the entropy solutions of (1.1) and (1.13), respectively. To simplify, we take the same data sets $(f, A, u_0) = (g, B, v_0)$ and we only allow the Lévy measures μ and ν to be different. We also let C denote a constant only depending on T, d and the data.

Example 3.1. Let us consider Equation (1.9), i.e. $A(u) = u$. Let us also consider the class of Lévy operators satisfying

$$\begin{cases} \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 \wedge |z| \,\mathrm{d}\mu(z) < +\infty, \\ \mu = \mu^*. \end{cases}$$

For such kind of equations, the following continuous dependence estimate on the Lévy measure has been established in [41]:

$$\|u - v\|_{C([0,T];L^1)} \leq C \int_{0<|z|\leq 1} |z|^2 \,\mathrm{d}|\mu - \nu|(z) + C \int_{|z|>1} |z| \,\mathrm{d}|\mu - \nu|(z).$$

This estimate follows from Theorem 3.4 by taking $r_1 = 1$ and $r_2 = +\infty$ in (3.5).

Example 3.2. Consider the following one-dimensional Hamilton-Jacobi-Bellman equation

$$U_t + f(U_x) = \mathcal{L}^\mu[U]$$

with initial data $U_0(x) := \int_{-\infty}^x u_0(y)\,\mathrm{d}y$. This particular equation is related to the nonlocal conservation law (1.8), its solution $U(x,t) = \int_{-\infty}^x u(y,t)\,\mathrm{d}y$ where u solves (1.8), see [19]. It is also an example of an integro-PDE for which the general theory of [36] applies, and this theory allows us to establish the following continuous dependence estimate on the Lévy measure:

$$\sup_{\mathbb{R}\times[0,T]} |U - V| \leq C \sqrt{\int_{\mathbb{R}\setminus\{0\}} |z|^2 \wedge 1 \,\mathrm{d}|\mu - \nu|(z)},$$

where $V(x,t) := \int_{-\infty}^x v(y,t)\,\mathrm{d}y$. (This result is a version of Theorem 4.1 (in [36]) which follows from Theorem 3.1 by setting $p_0, \ldots, p_4, p_s = 0$ and $\rho = |z| \wedge 1$ in (A0)). Since

$$\sup_{\mathbb{R}\times[0,T]} |U - V| \leq \|u - v\|_{C([0,T];L^1)},$$

this estimate also follows from (3.6) in Corollary 3.5 when $(A, f, u_0) = (B, g, v_0)$.

Let us now compare Theorem 3.7 with known convergence rates. We keep the same notation for u, u^ϵ and $\mathcal{O}(\cdot)$ as in Theorem 3.7.

Example 3.3. Let us consider the case where $A(u^\epsilon) = u^\epsilon$ and $\mathcal{L}^\mu = -(-\triangle)^{\frac{\alpha}{2}}$, $\alpha \in (0,2)$. Then the following optimal rates have been derived in [1, 29]:

$$\|u - u^\epsilon\|_{C([0,T];L^1)} = \begin{cases} \mathcal{O}\left(\epsilon^{\frac{1}{\alpha}}\right) & \text{if } \alpha > 1, \\ \mathcal{O}\left(\epsilon\,|\ln \epsilon|\right) & \text{if } \alpha = 1, \\ \mathcal{O}\left(\epsilon\right) & \text{if } \alpha < 1. \end{cases} \tag{3.10}$$

Let us explain how these results can be deduced from (3.9). First we use (1.7) to explicitly compute the integrals in (3.9) and obtain

$$\|u-u^\epsilon\|_{C([0,T];L^1)}=\mathcal{O}\left(\min_{r_2>r_1>0}\left\{\sqrt{\epsilon\,\frac{r_1^{2-\alpha}}{2-\alpha}}+\epsilon\int_{r_1}^{r_2}\frac{\mathrm{d}\tau}{\tau^\alpha}+\epsilon\, r_2^{-\alpha}\right\}\right).$$

We then deduce (3.10) by taking $r_1=\epsilon^{\frac{1}{\alpha}}$ and $r_2=+\infty$ if $\alpha>1$, $r_1=\epsilon$ and $r_2=1$ if $\alpha=1$, and $r_1=0$ and $r_2=1$ if $\alpha<1$.

Example 3.4. Let us consider the class of Lévy operators where $\mathrm{d}\mu(z)=g_\mu(z)\,\mathrm{d}z$ for $0\le g_\mu\in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}g_\mu(z)\,\mathrm{d}z=1$. This corresponds to problem (1.11)–(1.12) since we may write

$$\mathcal{L}^\mu[u^\epsilon]=g_\mu*u^\epsilon-u^\epsilon\quad(*\text{ is the convolution in space}).$$

The following optimal rate of convergence has been derived in [47, 55]:

$$\|u-u^\epsilon\|_{C([0,T];L^1)}=\mathcal{O}(\epsilon).$$

This result also follows from (3.9) by taking $r_1=r_2=0$.

4. Auxiliary results concerning $\mathcal{L}^\mu$

Before proving our main results in the next section, we state and prove some results concerning $\mathcal{L}^\mu$ that will be needed.

Lemma 4.1. *Assume* (1.6) *and* $r>0$. *Then for all* $\phi\in C_c^\infty(\mathbb{R}^d)$,

$$\|\mathcal{L}^\mu_r[\phi]\|_{L^1(\mathbb{R}^d)}\le\int_{0<|z|\le r}|z|^2\,\mathrm{d}\mu(z)\,\|\phi\|_{W^{2,1}(\mathbb{R}^d)}.$$

Proof. Recall that μ is σ-finite as nonnegative Radon measure, hence the product measure of μ and the Lebesgue measure is a well-defined nonnegative Radon measure for which Fubini's theorem applies. Using in addition Taylor's formula with integral remainder, we find that

$$\begin{aligned}
&\|\mathcal{L}^\mu_r[\phi]\|_{L^1(\mathbb{R}^d)}\\
&=\int_{\mathbb{R}^d}\left|\int_{0<|z|\le r}\phi(x+z)-\phi(x)-z\cdot D\phi(x)\,\mathbf{1}_{|z|\le1}\,\mathrm{d}\mu(z)\right|\mathrm{d}x,\\
&\le\int_{0<|z|\le r\wedge1}\int_0^1|z|^2\,(1-\tau)\int_{\mathbb{R}^d}|D^2\phi(x+\tau z)|\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}\mu(z)\\
&\quad+\int_{1<|z|\le r\vee1}\int_0^1|z|\int_{\mathbb{R}^d}|D\phi(x+\tau z)|\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}\mu(z),\\
&\le\int_{0<|z|\le r\wedge1}|z|^2\,\mathrm{d}\mu(z)\,\|D^2\phi\|_{L^1(\mathbb{R}^d,\mathbb{R}^{d\times d})}\\
&\quad+\int_{1<|z|\le r\vee1}|z|\,\mathrm{d}\mu(z)\,\|D\phi\|_{L^1(\mathbb{R}^d,\mathbb{R}^d)}\quad(\text{by the change of variable }x\to x+\tau\,z),\\
&\le\int_{0<|z|\le r}|z|^2\,\mathrm{d}\mu(z)\,\|\phi\|_{W^{2,1}(\mathbb{R}^d)}.
\end{aligned}$$

The proof is complete. □

Lemma 4.2. *Assume* (1.6), *and let* $r>0$ *and* $u\in L^1(\mathbb{R}^d)$. *Then* $\mathcal{L}^{\mu,r}[u]$ *is a well-defined function in* $L^1(\mathbb{R}^d)$. *Moreover,*

(i) *for a.e.* $x\in\mathbb{R}^d$, $\mathcal{L}^{\mu,r}[u](x)=\int_{|z|>r}u(x+z)-u(x)\,\mathrm{d}\mu(z)$,

(ii) $\|\mathcal{L}^{\mu,r}[u]\|_{L^1(\mathbb{R}^d)}\le2\|u\|_{L^1(\mathbb{R}^d)}\int_{|z|>r}\mathrm{d}\mu(z)$,

(iii) *if* $u \in L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ *and* $r_2 > r_1 > 0$,

$$\|\mathcal{L}^{\mu,r_1}[u]\|_{L^1(\mathbb{R}^d)} \leq |u|_{BV(\mathbb{R}^d)} \int_{r_1<|z|\leq r_2} |z| \,\mathrm{d}\mu(z) + \int_{|z|>r_2} \|u(\cdot+z)-u\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}\mu(z).$$

Remark 4.3. The precise statement of (i) is the following: For each a.e. representative of u and for a.e. $x \in \mathbb{R}^d$, $z \to u(x+z)-u(x)$ is μ-integrable and $x \to \int_{|z|>r} u(x+z)-u(x)\,\mathrm{d}\mu(z)$ is an a.e. representative of $\mathcal{L}^{\mu,r}[u]$. This is not trivial since it is not immediately clear that $z \to u(x+z)-u(x)$ is μ-measurable when u is Lebesgue measurable but not Borel measurable.

Proof. We start by proving that

$$(z,x) \in \left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{R}^d \to (u(x+z)-u(x))\,\mathbf{1}_{|z|>r} \text{ is } \mathrm{d}\mu(z)\,\mathrm{d}x\text{-measurable} \tag{4.1}$$

(where we still denote by u an a.e. representative of u). It suffices to prove the measurability of $v(z,x) := u(x+z)$, since all the other terms are obviously measurable. Let us first assume that u is a simple function. We want to show that for each real interval I, $v^{-1}(I)$ is an element of the (smallest) $\mathrm{d}\mu(z)\,\mathrm{d}x$-completion $\overline{\mathcal{B}}^{\mathrm{d}\mu(z)\,\mathrm{d}x}_{(\mathbb{R}^d\setminus\{0\})\times\mathbb{R}^d}$ of $\mathcal{B}_{(\mathbb{R}^d\setminus\{0\})\times\mathbb{R}^d}$, see the definition in Section 2.1.2. It suffices to prove it for $u = \mathbf{1}_E$ where $E \in \overline{\mathcal{B}}^{\mathrm{d}x}_{\mathbb{R}^d}$ is a Lebesgue measurable set, i.e. when there are $B_1, B_2 \in \mathcal{B}_{\mathbb{R}^d}$ s.t. $B_1 \subseteq E \subseteq B_2$ and $\int_{B_2\setminus B_1} \mathrm{d}x = 0$. Now let v_i denote the function $(z,x) \to \mathbf{1}_{B_i}(x+z)$ $(i=1,2)$. We have to distinguish between four cases according to whether I contains the values 0 and/or 1. Since the proof is very similar in all cases, we do it for only one case, $0 \in I$ and $1 \notin I$. In that case $v^{-1}(I) = \{(z,x) \text{ s.t. } z \neq 0 \text{ and } x+z \notin E\}$ and $v_i^{-1}(I) = \{(z,x) \text{ s.t. } z \neq 0 \text{ and } x+z \notin B_i\}$ $(i=1,2)$. This implies that $v_2^{-1}(I) \subseteq v^{-1}(I) \subseteq v_1^{-1}(I)$, where $v_i^{-1}(I) \in \mathcal{B}_{(\mathbb{R}^d\setminus\{0\})\times\mathbb{R}^d}$ $(i=1,2)$. Moreover, by the monotone convergence theorem and Fubini,

$$\begin{aligned}\iint_{v_1^{-1}(I)\setminus v_2^{-1}(I)} \mathrm{d}\mu(z)\,\mathrm{d}x &= \sup_{n\geq 1} \int_{\mathbb{R}^d} \int_{|z|>\frac{1}{n}} \mathbf{1}_{B_2\setminus B_1}(x+z)\,\mathrm{d}\mu(z)\,\mathrm{d}x \\ &= \sup\left\{\int_{|z|>\frac{1}{n}} \mathrm{d}\mu(z) \int_{B_2\setminus B_1} \mathrm{d}x : n \geq 1\right\} = 0.\end{aligned}$$

This shows that $v^{-1}(I) \in \overline{\mathcal{B}}^{\mathrm{d}\mu(z)\,\mathrm{d}x}_{(\mathbb{R}^d\setminus\{0\})\times\mathbb{R}^d}$ and hence v is measurable when u is a simple function. Measurability in the general case then follows since any Lebesgue measurable function u is the pointwise limit of simple functions.

Now simple computations show that

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_{|z|>r} |u(x+z)-u(x)|\,\mathrm{d}\mu(z)\,\mathrm{d}x \\ &= \int_{|z|>r} \|u(\cdot+z)-u\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\mu(z) \leq 2 \underbrace{\int_{|z|>r} \mathrm{d}\mu(z)}_{<+\infty \text{ by (1.6)}} \|u\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

By Fubini's theorem, the function $z \to (u(x+z)-u(x))\,\mathbf{1}_{|z|>r}$ is μ-integrable for a.e. $x \in \mathbb{R}^d$, and $x \to \mathcal{L}^{\mu,r}[u](x) := \int_{|z|>r} u(x+z)-u(x)\,\mathrm{d}\mu(z)$ is well-defined in $L^1(\mathbb{R}^d)$. This completes the proofs of (i) and (ii). When u is in addition BV, $\|u(\cdot+z)-u\|_{L^1(\mathbb{R}^d)} \leq |u|_{BV(\mathbb{R}^d)}\,|z|$ – see e.g. Lemma A.2, and (iii) follows easily from the computations above. □

In the next result, we establish a Kato type inequality for $\mathcal{L}^{\mu,r}[A(u)]$.

Lemma 4.4. *Assume* (1.4) *and* (1.6)*. Then for all* $u \in L^1(\mathbb{R}^d)$*,* $k \in \mathbb{R}$*,* $r > 0$*, and* $\phi \in C_c^\infty(\mathbb{R}^d)$*,*

$$\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\, \mathcal{L}^{\mu,r}[A(u)]\, \phi \, \mathrm{d}x \le \int_{\mathbb{R}^d} |A(u)-A(k)|\, \mathcal{L}^{\mu^*,r}[\phi]\, \mathrm{d}x.$$

Proof. Notice first that $A(u)$ is L^1 by (1.4), and hence $\mathcal{L}^{\mu,r}[A(u)]$ is well-defined in L^1 by Lemma 4.2. Easy computations then reveal that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\, \mathcal{L}^{\mu,r}[A(u)]\, \phi \, \mathrm{d}x,\\
&= \int_{\mathbb{R}^d} \int_{|z|>r} \operatorname{sgn}(u(x)-k) \Big(A(u(x+z)) - A(u(x)) \Big) \phi(x) \, \mathrm{d}\mu(z)\, \mathrm{d}x\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 4.2},\\
&= \int_{\mathbb{R}^d} \int_{|z|>r} \operatorname{sgn}(u(x)-k)\\
&\qquad\qquad \Big\{ (A(u(x+z)) - A(k)) - (A(u(x)) - A(k)) \Big\} \phi(x)\, \mathrm{d}\mu(z)\, \mathrm{d}x,\\
&\le \int_{\mathbb{R}^d} \int_{|z|>r} \Big(|A(u(x+z)) - A(k)| - |A(u(x)) - A(k)| \Big) \phi(x)\, \mathrm{d}\mu(z)\, \mathrm{d}x \quad \text{by (2.8)},\\
&= \underbrace{\int_{\mathbb{R}^d} \int_{|z|>r} |A(u(x+z)) - A(k)|\, \phi(x)\, \mathrm{d}\mu(z)\, \mathrm{d}x}_{=:I}\\
&\qquad\qquad - \underbrace{\int_{\mathbb{R}^d} \int_{|z|>r} |A(u(x)) - A(k)|\, \phi(x)\, \mathrm{d}\mu(z)\, \mathrm{d}x}_{=:J}.
\end{aligned}$$

Notice that the integrands above are $\mathrm{d}\mu(z)\,\mathrm{d}x$–integrable by similar arguments as in the proof of Lemma 4.2.

By the respective changes of variable $(z,x) \to (-z, x+z)$ and $(z,x) \to (-z,x)$, we find that

$$\begin{aligned}
I &= \int_{\mathbb{R}^d} \int_{|z|>r} \phi(x+z)\, |A(u(x)) - A(k)|\, \mathrm{d}\mu^*(z)\, \mathrm{d}x,\\
J &= \int_{\mathbb{R}^d} \int_{|z|>r} \phi(x)\, |A(u(x)) - A(k)|\, \mathrm{d}\mu^*(z)\, \mathrm{d}x
\end{aligned}$$

Here measure μ^* in (2.9) appear because of the relabelling of z. This measure has the same properties as μ. Hence we can conclude that

$$\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\, \mathcal{L}^{\mu,r}[A(u)]\, \phi\, \mathrm{d}x \le I - J = \int_{\mathbb{R}^d} |A(u)-A(k)|\, \mathcal{L}^{\mu^*,r}[\phi]\, \mathrm{d}x,$$

and the proposition follows. □

The next lemma is a consequence of the Kato inequality, and it plays a key role in the doubling of variables arguments throughout this paper and in the uniqueness proof of [1, 19].

Lemma 4.5. *Assume* (1.4) *and* (1.6)*, and let* $u, v \in L^\infty(Q_T) \cap C([0,T]; L^1)$*,* $0 \le \psi \in L^1(\mathbb{R}^d \times (0,T)^2)$*, and* $r > 0$*. Then*

$$\begin{aligned}
&\iint_{Q_T^2} \operatorname{sgn}(u(y,s) - v(x,t))\\
&\qquad \cdot \Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y) - \mathcal{L}^{\mu,r}[A(v(\cdot,t))](x) \Big) \psi(x-y,t,s)\, \mathrm{d}w \le 0.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\operatorname{sgn}(u(y,s)-v(x,t))\Big(A(u(y+z,s))-A(u(y,s))\Big)\\
&\qquad-\operatorname{sgn}(u(y,s)-v(x,t))\Big(A(v(x+z,t))-A(v(x,t))\Big)\\
&=\operatorname{sgn}(u(y,s)-v(x,t))\\
&\qquad\cdot\Big\{\Big(A(u(y+z,s))-A(v(x+z,t))\Big)-\Big(A(u(y,s))-A(v(x,t))\Big)\Big\}\\
&\leq|A(u(y+z,s))-A(v(x+z,t))|-|A(u(y,s))-A(v(x,t))|
\end{aligned}$$

where these functions are both defined. By Lemma 4.2 (i) and an integration w.r.t. $\mathbf{1}_{|z|>r}\,\mathrm{d}\mu(z)$, we find that for all $(t,s)\in(0,T)^2$ and a.e. $(x,y)\in\mathbb{R}^{2d}$,

$$\begin{aligned}
&\operatorname{sgn}(u-v)\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\\
&\leq\int_{|z|>r}|A(u(y+z,s))-A(v(x+z,t))|-|A(u(y,s))-A(v(x,t))|\,\mathrm{d}\mu(z).
\end{aligned}$$

After another integration, this time w.r.t. $\psi(x-y,t,s)\,\mathrm{d}w$, we then get that

$$\begin{aligned}
&\iint_{Q_T^2}\operatorname{sgn}(u-v)\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\,\psi\,\mathrm{d}w\\
&\leq\iint_{Q_T^2}\int_{|z|>r}|A(u(y+z,s))-A(v(x+z,t))|\;\psi(x-y,t,s)\,d\mu(z)\,\mathrm{d}w\\
&\quad-\iint_{Q_T^2}\int_{|z|>r}|A(u(y,s))-A(v(x,t))|\;\psi(x-y,t,s)\,d\mu(z)\,\mathrm{d}w,\\
&=:I+J.
\end{aligned}$$

Note that the $\mathrm{d}\mu(z)\,\mathrm{d}w$-measurability of the integrands above follow from arguments like the ones used in the proof of (4.1). Integrability of I and J then follows since $\|A(u)\|_{C([0,T];L^1)}\leq\|A'\|_{L^\infty}\|u\|_{C([0,T];L^1)}$ (A is Lipschitz-continuous and 0 at 0) and by Fubini (note the convolution integrals in x and y)

$$I,J\leq\Big(\|A(u)\|_{C([0,T];L^1)}+\|A(v)\|_{C([0,T];L^1)}\Big)\|\psi\|_{L^1(\mathbb{R}^d\times(0,T)^2)}\int_{|z|>r}d\mu(x).$$

To conclude, we change variables, $(z,x,t,y,s)\to(-z,x+z,t,y+z,s)$ in I and $(z,x,t,y,s)\to(-z,x,t,y,s)$ in J, to find that

$$\begin{aligned}
I&=\iint_{Q_T}\int_{|z|>r}|A(u(y,s))-A(v(x,t))|\;\psi(x+z-(y+z),t,s)\,\mathrm{d}\mu^*(z)\,\mathrm{d}w,\\
J&=-\iint_{Q_T}\int_{|z|>r}|A(u(y,s))-A(v(x,t))|\;\psi(x-y,t,s)\,\mathrm{d}\mu^*(z)\,\mathrm{d}w.
\end{aligned}$$

It follows that $I+J=0$ and the proof is complete. □

Lemma 4.6. *Under the assumptions of Lemma 3.1,*

$$I=\iint_{Q_T^2}|A(v(x,t))-A(u(y,s))|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w\leq C_\epsilon\int_{0<|z|\leq r}|z|^2\,\mathrm{d}\mu(z),$$

where $C_\epsilon>0$ does not depend on $r>0$.

Proof. Easy computations show that

$$\begin{aligned}
&\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\\
&=\theta_\delta(t-s)\int_{0<|z|\le r}\bar\theta_\epsilon(x-y-z)-\bar\theta_\epsilon(x-y)+z\cdot D\bar\theta_\epsilon(x-y)\,\mathbf{1}_{|z|\le 1}\,\mathrm{d}\mu^*(z)\\
&=\theta_\delta(t-s)\int_{0<|z|\le r}\bar\theta_\epsilon(x-y+z)-\bar\theta_\epsilon(x-y)-z\cdot D\bar\theta_\epsilon(x-y)\,\mathbf{1}_{|z|\le 1}\,\mathrm{d}\mu(z)\\
&=\theta_\delta(t-s)\,\mathcal{L}_r^{\mu}[\bar\theta_\epsilon](x-y),
\end{aligned}$$

and by Fubini (there are again convolution integrals in I!),

$$\begin{aligned}
I &\le \iint_{Q_T^2}|A(u(y,s))-A(v(x,t))|\,\theta_\delta(t-s)\left|\mathcal{L}_r^{\mu}[\bar\theta_\epsilon](x-y)\right|\,\mathrm{d}w\\
&\le \Big(\|A(u)\|_{L^1(Q_T)}+\|A(v)\|_{L^1(Q_T)}\Big)\|\theta_\delta\,\mathcal{L}_r^{\mu}[\bar\theta_\epsilon]\|_{L^1(\mathbb{R}^{d+1})}\\
&\le T\|A'\|_{L^\infty}\Big(\|u\|_{C([0,T];L^1)}+\|v\|_{C([0,T];L^1)}\Big)\|\theta_\delta\,\mathcal{L}_r^{\mu}[\bar\theta_\epsilon]\|_{L^1(\mathbb{R}^{d+1})}.
\end{aligned}$$

By classical properties of approximate units (Lemma A.1 in the appendix) and Lemma 4.1,

$$\begin{aligned}
\|\theta_\delta\,\mathcal{L}_r^{\mu}[\bar\theta_\epsilon]\|_{L^1(\mathbb{R}^{d+1})} &= \underbrace{\|\theta_\delta\|_{L^1(\mathbb{R})}}_{=1}\,\|\mathcal{L}_r^{\mu}[\bar\theta_\epsilon]\|_{L^1(\mathbb{R}^{d})}\\
&\le \frac{1}{2}\|\bar\theta_\epsilon\|_{W^{2,1}(\mathbb{R}^d)}\int_{0<|z|\le r}|z|^2\,\mathrm{d}\mu(z),
\end{aligned}$$

and the proof is complete since $\bar\theta_\epsilon\in C_c^\infty(\mathbb{R}^d)$ in (3.1) does not depend on $r>0$. □

5. Proofs of the main results

The proofs of this section use the so-called doubling of variables technique of Kruzhkov [45] (see also [1, 19] for nonlocal equations) along with ideas from [46]. It consists in considering u as a function of the new variables (y,s) and using the approximate units $\phi^{\epsilon,\delta}$ in (3.1) as test functions. For brevity, we do not specify the variables of $u=u(y,s)$, $v=v(x,t)$ and $\phi^{\epsilon,\delta}=\phi^{\epsilon,\delta}(x,t,y,s)$ when the context is clear. Moreover, the Lebesgue measure $\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$ is denoted by $\mathrm{d}w$.

5.1. **Proof of Lemma 3.1.** Let $(x,t)\in Q_T$ be fixed and $u=u(y,s)$, $k=v(x,t)$, and $\phi(y,s):=\phi^{\epsilon,\delta}(x,t,y,s)$. The entropy inequality for u (see (2.10)) then takes the form

$$\begin{aligned}
&\int_{Q_T}|u-v|\,\partial_s\phi^{\epsilon,\delta}+\Big(q_f(u,v)+|A(u)-A(v)|\,b_r^{\mu^*}\Big)\cdot D_y\phi^{\epsilon,\delta}\,\mathrm{d}y\,\mathrm{d}s\\
&+\int_{Q_T}|A(u)-A(v)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}y\,\mathrm{d}s\\
&+\int_{Q_T}\mathrm{sgn}\,(u-v)\,\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}y\,\mathrm{d}s\\
&-\int_{\mathbb{R}^d}|u(y,T)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}y\\
&+\int_{\mathbb{R}^d}|u_0(y)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,0)\,\mathrm{d}y\ge 0.
\end{aligned}$$

We integrate this inequality w.r.t. $(x,t)\in Q_T$, noting that q_f in (2.6) is symmetric, and that $\partial_s\phi^{\epsilon,\delta}=-\partial_s\phi^{\epsilon,\delta}$ and $D_y\phi^{\epsilon,\delta}=-D_x\phi^{\epsilon,\delta}$ by (3.1). Consequently we find that

$$
\begin{aligned}
&I_1+\cdots+I_5\\
&:=\iint_{Q_T^2}|u-v|\underbrace{\partial_s\phi^{\epsilon,\delta}}_{=-\partial_t\phi^{\epsilon,\delta}}+\Big(\underbrace{q_f(u,v)}_{=q_f(v,u)}+|A(u)-A(v)|\,b_r^{\mu^*}\Big)\cdot\underbrace{D_y\phi^{\epsilon,\delta}}_{=-D_x\phi^{\epsilon,\delta}}\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}|A(u)-A(v)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}\operatorname{sgn}(u-v)\,\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&\quad-\iint_{Q_T\times\mathbb{R}^d}|u(y,T)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\\
&\quad+\iint_{Q_T\times\mathbb{R}^d}|u_0(y)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\geq 0.
\end{aligned}
\tag{5.1}
$$

Note that the terms in the inequality above are well-defined since they are all essentially of the form of convolution integrals of L^1-functions. See Lemmas 4.1 and 4.2 and the discussions in the proofs of Lemmas 4.5 and 4.6 for more details.

A classical computation from [46] reveals that

$$
\begin{aligned}
&I_4+I_5\underbrace{-\iint_{\mathbb{R}^d\times Q_T}|u(y,s)-v(x,T)|\,\phi^{\epsilon,\delta}(x,T,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s}_{=:I_4^v}\\
&\quad\underbrace{+\iint_{\mathbb{R}^d\times Q_T}|u(y,s)-v_0(x)|\,\phi^{\epsilon,\delta}(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s}_{=:I_5^v}\\
&\leq-\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)}+\|u_0-v_0\|_{L^1(\mathbb{R}^d)}\\
&\quad+\epsilon\,C_{\tilde\theta}\,|u_0|_{BV(\mathbb{R}^d)}+2\omega_u(\delta)\vee\omega_v(\delta),
\end{aligned}
\tag{5.2}
$$

where $C_{\tilde\theta}$ is as in Lemma 3.1. For the readers convenience we sketch the proof in Section B.1 in the appendix. Lemma 3.1 now follows from (5.1) and the above estimates on I_4 and I_5.

5.2. **Proof of Theorem 3.3.** We adapt ideas from [36] to the current entropy solution setting by considering the region where $A'\geq B'$ and its complementary. Let $E_\pm$ be sets satisfying

$$
\begin{cases}
E_\pm\in\mathcal{B}_\mathbb{R};\\
\cup_\pm E^\pm=\mathbb{R}\text{ and }\cap_\pm E_\pm=\emptyset;\\
\mathbb{R}\setminus\operatorname{supp}(A'-B')^\mp\subseteq E_\pm.
\end{cases}
\tag{5.3}
$$

For all $u\in\mathbb{R}$, we define

$$
\begin{aligned}
A_\pm(u)&:=\int_0^u A'(\tau)\,\mathbf{1}_{E_\pm}(\tau)\,\mathrm{d}\tau,\\
B_\pm(u)&:=\int_0^u B'(\tau)\,\mathbf{1}_{E_\pm}(\tau)\,\mathrm{d}\tau,\\
C_\pm(u)&:=\pm(A_\pm(u)-B_\pm(u)).
\end{aligned}
\tag{5.4}
$$

We will need the following two technical lemmas.

Lemma 5.1. *Under the assumptions of Theorem 3.3,*

(i) $A=A_++A_-$ *and* $B=B_++B_-$*;*
(ii) $A_\pm,B_\pm,C_\pm$ *satisfy* (1.4)*;*

(iii) $\sum_{\pm} |C_{\pm}(u)|_{L^1(0,T;BV)} \leq \|A' - B'\|_{L^\infty(\mathbb{R})} |u|_{L^1(0,T;BV)}$;
(iv) *for all* $z \in \mathbb{R}^d \setminus \{0\}$,

$$\sum_{\pm} \|C_{\pm}(u(\cdot + z, \cdot)) - C_{\pm}(u)\|_{L^1(Q_T)} \leq \|A' - B'\|_{L^\infty(\mathbb{R})} \|u(\cdot + z, \cdot) - u\|_{L^1(Q_T)}.$$

We prove these Lemmas after the proof of Theorem 3.3.

Proof of Theorem 3.3. Let us divide the proof into several steps.

1. We argue as in the beginning of the proof of Lemma 3.1 changing the roles of u and v. We fix (y,s) and take $k = u(y,s)$ and $\phi^{\epsilon,\delta} = \phi^{\epsilon,\delta}(x,t,y,s)$ in the entropy inequality for $v = v(x,t)$ to find that

$$\begin{aligned}
&\iint_{Q_T^2} |v-u|\,\partial_t \phi^{\epsilon,\delta} + \Big(q_g(v,u) + |B(v) - B(u)|\, b_r^{\mu^*}\Big) \cdot D_x \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&+ \iint_{Q_T^2} |B(v) - B(u)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \,\mathrm{d}w \\
&+ \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{\mu,r}[B(v(\cdot,t))](x)\, \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&- \iint_{\mathbb{R}^d \times Q_T} |v(x,T) - u(y,s)|\, \phi^{\epsilon,\delta}(x,T,y,s) \,\mathrm{d}x \;\mathrm{d}y \,\mathrm{d}s \\
&+ \iint_{\mathbb{R}^d \times Q_T} |v_0(x) - u(y,s)|\, \phi^{\epsilon,\delta}(x,0,y,s) \,\mathrm{d}x \;\mathrm{d}y \,\mathrm{d}s \geq 0.
\end{aligned}$$

Then we add this inequality and inequality (3.3) in Lemma 3.1,

$$\begin{aligned}
&\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + \epsilon\, C_{\tilde{\theta}}\, |u_0|_{BV(\mathbb{R}^d)} + 2\,\omega_u(\delta) \vee \omega_v(\delta) \\
&\quad + \underbrace{\iint_{Q_T^2} (q_g - q_f)(v,u) \cdot D_x \phi^{\epsilon,\delta} \,\mathrm{d}w}_{=:I_1} \\
&\quad + \underbrace{\iint_{Q_T^2} |B(v) - B(u)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(\cdot,y,t,s)](x) \,\mathrm{d}w}_{=:I_2} \\
&\quad + \underbrace{\iint_{Q_T^2} |A(v)) - A(u)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) \,\mathrm{d}w}_{=:I_2'} \\
&\quad + \underbrace{\iint_{Q_T^2} \Big(|B(v) - B(u)| - |A(v) - A(u)|\Big)\, b_r^{\mu^*} \cdot D_x \phi^{\epsilon,\delta} \,\mathrm{d}w}_{=:I_3} \\
&\quad + \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\mu,r}[B(v(\cdot,t))](x) - \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\Big)\, \phi^{\epsilon,\delta} \,\mathrm{d}w,}_{=:I_4}
\end{aligned} \tag{5.5}$$

where $r, \epsilon > 0$, $0 < \delta < T$, and $C_{\tilde{\theta}} > 0$ only depends on the kernel $\tilde{\theta}_d$ from (3.2).

2. It is standard to estimate I_1 (cf. e.g. [25, 46]), and $I_2 + I_2'$ can be estimated by Lemma 4.6,

$$I_1 \leq |u_0|_{BV(\mathbb{R}^d)}\, T \operatorname{ess-sup}_{\mathbb{R}} |f' - g'|, \tag{5.6}$$

$$I_2 + I_2' \leq C_\epsilon \int_{0<|z|\leq r} |z|^2 \,\mathrm{d}\mu(z), \tag{5.7}$$

where C_ϵ does not depend on $r > 0$. For the sake of completeness, the proof of (5.6) is given in Appendix B Section B.3. Now we focus on I_3 and I_4.

3. *Cutting w.r.t.* $E_\pm$. We split I_3 and I_4 into four new terms using the sets $E_\pm$, see (5.3)–(5.4). By Lemma 5.1 (i), I_4 can be written as

$$I_4 = \sum_\pm \iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\mu,r}[B_\pm(v(\cdot,t))](x) - \mathcal{L}^{\mu,r}[A_\pm(u(\cdot,s))](y) \Big)\, \phi^{\epsilon,\delta} \,\mathrm{d}w.$$

By Lemma 5.1 (ii), we can apply twice Lemma 4.5 with B_+ and A_- instead of A, followed by the definitions of $C_\pm$, see (5.4), to show that

$$\begin{aligned}
I_4 &\leq \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{\mu,r}\Big[B_+(u(\cdot,s)) - A_+(u(\cdot,s))\Big](y)\, \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{\mu,r}\Big[B_-(v(\cdot,t)) - A_-(v(\cdot,t))\Big](x)\, \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&= \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu,r}\big[C_+(u(\cdot,s))\big](y)\, \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{\mu,r}\big[C_-(v(\cdot,t))\big](x)\, \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&=: I_4^+ + I_4^- .
\end{aligned} \tag{5.8}$$

Note that it is crucial to have u in the first term and v in the second – otherwise we will not be able to apply the Kato inequality later on!

We now consider I_3. By (2.8), Lemma 5.1 (i)–(ii), the formula $D_x\phi^{\epsilon,\delta} = -D_y\phi^{\epsilon,\delta}$, and the definitions $D_+ = D_y$ and $D_- = D_x$, it follows that

$$\begin{aligned}
&\Big(|B(v) - B(u)| - |A(v) - A(u)|\Big) D_x\phi^{\epsilon,\delta} \\
&= \operatorname{sgn}(u-v) \Big\{ (A(u) - B(u)) - (A(v) - B(v)) \Big\} D_y\phi^{\epsilon,\delta} \\
&= \sum_\pm \operatorname{sgn}(u-v) \Big\{ \pm (A_\pm(u) - B_\pm(u)) \mp (A_\pm(v) - B_\pm(v)) \Big\} D_\pm\phi^{\epsilon,\delta} \\
&= \sum_\pm |C_\pm(u) - C_\pm(v)|\, D_\pm\phi^{\epsilon,\delta}.
\end{aligned}$$

We can then rewrite I_3 as

$$I_3 = \sum_\pm \underbrace{\iint_{Q_T} |C_\pm(u) - C_\pm(v)|\, b_r^{\mu^*} \cdot D_\pm\phi^{\epsilon,\delta} \,\mathrm{d}w}_{=:I_3^\pm} . \tag{5.9}$$

4. *Cutting w.r.t.* z. We decompose $\mathcal{L}^{\mu,r}$ into two new terms using a new cutting parameter $r_1 > r$. Let $\mu = \mu_1 + \mu_{|_{|z|>r_1}}$ for

$$\mu_1 := \mu_{|_{0<|z|\leq r_1}},$$

and note that by (2.5), $\mathcal{L}^{\mu,r} = \mathcal{L}^{\mu_1,r} + \mathcal{L}^{\mu,r_1}$. Then

$$
\begin{aligned}
I_4^+ = {} & \underbrace{\iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu_1,r}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w}_{=:I_5^+} \\
& + \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu,r_1}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w.
\end{aligned} \tag{5.10}
$$

Since C_+ satisfies (1.4) by Lemma 5.1 (ii) and μ_1 clearly satisfies (1.6), we can apply the Kato type inequality in Lemma 4.4 (with $k = v(x,t)$ and $A = C_+$) to show that

$$
\begin{aligned}
I_5^+ &= \int_{Q_T}\int_{Q_T} \operatorname{sgn}(u(y,s)-v(x,t))\, \mathcal{L}^{\mu_1,r}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}y\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}t \\
&\leq \int_{Q_T}\int_{Q_T} |C_+(u(y,s)) - C_+(v(x,t))|\, \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\, \mathrm{d}y\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}
$$

Adding I_3^+ in the form (5.9) then gives

$$
I_3^+ + I_5^+ \leq \iint_{Q_T^2} |C_+(u) - C_+(v)| \left(b_r^{\mu^*} \cdot D_y\phi^{\epsilon,\delta} + \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) \right) \mathrm{d}w. \tag{5.11}
$$

Now easy computations show that

$$
D_y\phi^{\epsilon,\delta} = -\theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y), \quad \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) = \theta_\delta(t-s)\, \mathcal{L}^{\mu_1,r}[\bar{\theta}_\epsilon](x-y).
$$

Hence by adding and subtracting $z \cdot D\bar{\theta}_\epsilon(x-y)$, we get that

$$
\begin{aligned}
& b_r^{\mu^*} \cdot D_y\phi^{\epsilon,\delta} + \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) \\
&= \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y+z) - \bar{\theta}_\epsilon(x-y) - z\cdot D\bar{\theta}_\epsilon(x-y)\, \mathrm{d}\mu(z) \\
&\quad + \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \cdot \underbrace{\left(-b_r^{\mu^*} + \int_{r<|z|\leq r_1} z\, \mathrm{d}\mu(z) \right)}_{=\operatorname{sgn}(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\, \mathrm{d}\mu(z)},
\end{aligned} \tag{5.12}
$$

where the last equality comes from (2.4) and the change of variable $z \to -z$. We insert (5.12) into (5.11) and combine the resulting inequality with (5.10),

$$
\begin{aligned}
& I_3^+ + I_4^+ \leq \\
& \iint_{Q_T^2} |C_+(u) - C_+(v)| \\
& \qquad \cdot \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y+z) - \bar{\theta}_\epsilon(x-y) - z\cdot D\bar{\theta}_\epsilon(x-y)\, \mathrm{d}\mu(z)\, \mathrm{d}w \\
& + \iint_{Q_T^2} |C_+(u) - C_+(v)| \\
& \qquad \cdot \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \cdot \operatorname{sgn}(r_1-1) \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\, \mathrm{d}\mu(z)\, \mathrm{d}w \\
& + \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu,r_1}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \\
& =: J_1^+ + J_2^+ + J_3^+.
\end{aligned} \tag{5.13}
$$

Similar arguments show that we can bound $I_3^- + I_4^-$ (see (5.8)–(5.9)) as follows,

$$
\begin{aligned}
& I_3^- + I_4^- \leq \\
& \iint_{Q_T^2} |C_-(v) - C_-(u)| \\
& \quad \cdot \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y-z) - \bar{\theta}_\epsilon(x-y) + z\cdot D\bar{\theta}_\epsilon(x-y)\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
& - \iint_{Q_T^2} |C_-(v) - C_-(u)| \\
& \quad \cdot \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y)\cdot \operatorname{sgn}(r_1-1) \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
& + \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r_1}[C_-(v(\cdot,t))](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w}_{\leq \iint_{Q_T^2} \operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r_1}[C_-(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w \text{ by Lemma 4.5}} \\
& =: J_1^- + J_2^- + J_3^- .
\end{aligned}
\tag{5.14}
$$

5. *$L^1\cap BV$-regularity.* It remains to estimate $J_i^\pm$ for $i = 1,\ldots,3$ in (5.13)–(5.14). For $J_1^\pm$ and $J_2^\pm$, we use Fubini and integrate by parts to take advantage of the BV-regularity of the entropy solution u. After some computations given in Appendix B (see Lemma B.1), we find that

$$
\begin{aligned}
|J_1^\pm| &\leq \frac{1}{2\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z)\, |C_\pm(u)|_{L^1(0,T;BV)}, \\
|J_2^\pm| &\leq \left| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\right| |C_\pm(u)|_{L^1(0,T;BV)},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\sum_\pm (J_1^\pm + J_2^\pm) \leq{}& \frac{1}{2\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z) \sum_\pm |C_\pm(u)|_{L^1(0,T;BV)} \\
&+ \left| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\right| \sum_\pm |C_\pm(u)|_{L^1(0,T;BV)}.
\end{aligned}
$$

By Lemma 5.1 (iii) and a priori estimates for u, cf. (2.11), we see that

$$
\begin{aligned}
\sum_\pm (J_1^\pm + J_2^\pm) \leq{}& \frac{1}{2\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x \underbrace{|u|_{L^1(0,T;BV)}}_{\leq |u_0|_{BV(\mathbb{R}^d)}T} \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z)\, \|A'-B'\|_{L^\infty(\mathbb{R})} \\
&+ \underbrace{|u|_{L^1(0,T;BV)}}_{\leq |u_0|_{BV(\mathbb{R}^d)}T} \left| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\right| \|A'-B'\|_{L^\infty(\mathbb{R})}.
\end{aligned}
\tag{5.15}
$$

Let us now estimate J_3^+ in (5.13). Easy computations (see the proofs of Lemmas 4.5–4.6) show that

$$
J_3^+ \leq \|\theta_\delta\,\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^{d+1})}\, \|\mathcal{L}^{\mu,r_1}[C_+(u)]\|_{L^1(Q_T)}.
$$

By Lemma A.1, $\|\theta_\delta\,\bar\theta_\epsilon\|_{L^1(\mathbb{R}^{d+1})} = \|\theta_\delta\|_{L^1(\mathbb{R})}\,\|\bar\theta_\epsilon\|_{L^1(\mathbb{R}^d)} = 1$, and then we can use Lemma 4.2 (iii) to find that

$$\begin{aligned} J_3^+ \leq \int_0^T \Bigg(&\int_{r_1<|z|\leq r_2} |C_+(u(\cdot,s))|_{BV(\mathbb{R}^d)}\,|z|\,\mathrm{d}\mu(z) \\ &+ \int_{|z|>r_2} \|C_+(u(\cdot+z,s)) - C_+(u(\cdot,s))\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\mu(z)\Bigg)\,\mathrm{d}s \end{aligned}$$

for all $r_2 > r_1$. Since $C_+(u) \in L^\infty \cap C([0,T];L^1) \cap L^\infty(0,T;BV)$, $(z,s) \to \|C_+(u(\cdot+z,s))-C_+(u(\cdot,s))\|_{L^1(\mathbb{R}^d)}$ and $s \to |C_+(u(\cdot,s))|_{BV(\mathbb{R}^d)}$ are continuous and lower semi-continuous functions respectively and hence Borel measurable. They are thus $\mathrm{d}\mu(z)\,\mathrm{d}s$-measurable, and we may change the order of the integration to find

$$\begin{aligned} J_3^+ \leq &\int_{r_1<|z|\leq r_2} |z|\,\mathrm{d}\mu(z)\,|C_+(u)|_{L^1(0,T;BV)} \\ &+ \int_{|z|>r_2} \|C_+(u(\cdot+z,\cdot)) - C_+(u)\|_{L^1(Q_T)}\,\mathrm{d}\mu(z). \end{aligned}$$

We get a similar estimates for J_3^- and find by Lemma 5.1 (iii)–(iv) and (2.11) that

$$\begin{aligned} \sum_\pm J_3^\pm \leq &\int_{r_1<|z|\leq r_2} |z|\,\mathrm{d}\mu(z)\,\sum_\pm |C_\pm(u)|_{L^1(0,T;BV)} \\ &+ \int_{|z|>r_2} \sum_\pm \|C_\pm(u(\cdot+z,\cdot)) - C_\pm(u)\|_{L^1(Q_T)}\,\mathrm{d}\mu(z), \\ \leq &\, |u_0|_{BV(\mathbb{R}^d)}\,T \int_{r_1<|z|\leq r_2} |z|\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)} \\ &+ \int_{|z|>r_2} \underbrace{\|u(\cdot+z,\cdot)) - u\|_{L^1(Q_T)}}_{\leq T\,\|u_0(\cdot+z)-u_0\|_{L^1(\mathbb{R}^d)}}\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}. \end{aligned} \tag{5.16}$$

The last inequality (under the bracket) comes from (2.12) applied to the solution $u(\cdot+z,\cdot)$ of (1.1) with initial data $u_0(\cdot+z)$.

6. *Conclusion.* By (5.8)–(5.9) and (5.13)–(5.14), $I_3+I_4 \leq \sum_\pm \sum_{i=1}^3 J_i^\pm$. Therefore we may estimate (5.5) by (5.6)–(5.7) and (5.15)–(5.16). For all $r_2 > r_1 > r > 0$, $\epsilon > 0$, and $T > \delta > 0$, we find that

$$\begin{aligned} &\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\ &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\,T\,\text{ess-sup}_{\mathbb{R}}\,|f'-g'| \\ &\quad + \epsilon\,C_{\tilde\theta}\,|u_0|_{BV(\mathbb{R}^d)} + 2\,\omega_u(\delta)\vee\omega_v(\delta) + C_\epsilon \int_{0<|z|\leq r} |z|^2\,\mathrm{d}\mu(z) \\ &\quad + \frac{1}{2\epsilon}\int_{\mathbb{R}^d} |D\tilde\theta_d|\mathrm{d}x\,|u_0|_{BV(\mathbb{R}^d)}\,T \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + |u_0|_{BV(\mathbb{R}^d)}\,T \left|\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\right|\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + |u_0|_{BV(\mathbb{R}^d)}\,T \int_{r_1<|z|\leq r_2} |z|\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + T\int_{|z|>r_2} \|u_0(\cdot+z) - u_0\|_{L^1(Q_T)}\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}, \end{aligned} \tag{5.17}$$

where $C_\epsilon > 0$ does not depend on $r > 0$.

To finish, we first pass to the limit as $r \to 0$ in (5.17). By the dominated convergence theorem, the result is equivalent to setting $r = 0$ in each term, and in particular the term $C_\epsilon \int_{0<|z|\le r} |z|^2 \,\mathrm{d}\mu(z)$ vanishes. Secondly, we pass to the limit as $\delta \to 0$ to get rid of the term $2\,\omega_u(\delta) \vee \omega_v(\delta)$. Finally, we optimize the remaining terms w.r.t. $\epsilon > 0$ by using the formula $\min_{\epsilon>0}\left(\epsilon\, a + \frac{b}{\epsilon}\right) = 2\sqrt{ab}$ (for $a, b \ge 0$). This gives us the following continuous dependence estimate: F or all $r_2 > r_1 > 0$,

$$\begin{aligned}
\|u - v\|_{C([0,T];L^1)} \le{}& \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T \operatorname{ess-sup}_{\mathbb{R}} |g' - f'| \\
&+ 2\sqrt{\frac{1}{2}\, C_{\tilde{\theta}} \int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x\, |u_0|^2_{BV(\mathbb{R}^d)}\, T \int_{0<|z|\le r_1} |z|^2 \,\mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})}} \\
&+ |u_0|_{BV(\mathbb{R}^d)}\, T \int_{r_1<|z|\le r_2} |z| \,\mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&+ |u_0|_{BV(\mathbb{R}^d)}\, T \left| \int_{r_1 \wedge 1<|z|\le r_1 \vee 1} z \,\mathrm{d}\mu(z) \right| \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&+ T \int_{|z|\ge r_2} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})},
\end{aligned} \tag{5.18}$$

where $\tilde{\theta}_d$ is an arbitrary approximate unit (3.2) and $C_{\tilde{\theta}} = 2\int_{\mathbb{R}^d} |x|\, \tilde{\theta}_d(x) \,\mathrm{d}x$ by Lemma 3.1.

Let $\tilde{\theta}_d = \theta_n$ where $\{\theta_n\}_{n\in\mathbb{N}}$ is a sequence of kernels s.t. θ_n satisfies (3.2), $\theta_n \to {\omega_d}^{-1}\mathbf{1}_{|\cdot|<1}$ in L^1, and $\int_{\mathbb{R}^d} |D\theta_n| \,\mathrm{d}x \to {\omega_d}^{-1}|\mathbf{1}_{|\cdot|<1}|_{BV(\mathbb{R}^d)}$. Here ω_d is the volume of the unit ball in $\mathbb{R}^d$. Note that the BV-semi-norm of the indicator function of the unit ball is equaled to the surface area of the unit sphere, i.e. $|\mathbf{1}_{|\cdot|<1}|_{BV(\mathbb{R}^d)} = d\omega_d$. Moreover, we have

$$\int_{\mathbb{R}^d} |x||\theta_n(x)| \,\mathrm{d}x \to \frac{1}{\omega_d} \int_{|x|<1} |x| \,\mathrm{d}x = \frac{d}{d+1}.$$

The proof of (3.4) is then complete after passing to the limit as $n \to +\infty$ in (5.18). □

Let us now prove Lemma 5.1.

Proof of Lemma 5.1. The proofs of (i)–(ii) are easy and left to the reader. Let us prove (iii)–(iv). We differentiating (5.4), $C'_\pm = \pm(A'_\pm - B'_\pm) = \pm(A' - B')\,\mathbf{1}_{E_\pm}$ a.e., and use (5.3) to see that

$$C'_\pm = (A' - B')^\pm \quad \text{a.e.}. \tag{5.19}$$

Since $u(t) \in L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ for fixed $t \in (0,T)$, there is $\{\phi_n\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$ s.t. $\phi_n \to u(t)$ in L^1 and $\int_{\mathbb{R}^d} |D\phi_n| \,\mathrm{d}x \to |u(t)|_{BV(\mathbb{R}^d)}$, see e.g. Theorem 2 in Section 5.2 of [33]. Since $C_\pm$ satisfy (1.4), it follows that $C_\pm(\phi_n) \to C_\pm(u(t))$ in L^1. Moreover, by lower semi-continuity of the BV-semi-norm,

$$\sum_\pm |C_\pm(u(t))|_{BV(\mathbb{R}^d)} \le \sum_\pm \liminf_{n\to+\infty} \int_{\mathbb{R}^d} |DC_\pm(\phi_n)| \,\mathrm{d}x \le \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \sum_\pm |DC_\pm(\phi_n)| \,\mathrm{d}x.$$

Since ϕ_n is smooth and $C_\pm$ is Lipschitz-continuous, we can use the chain rule and (5.19) to show that

$$\sum_\pm |DC_\pm(\phi_n)| = |A' - B'|(\phi_n)\ |D\phi_n| \le \|A' - B'\|_{L^\infty(\mathbb{R})}\, |D\phi_n| \quad \text{a.e.}.$$

In the limit as $n \to +\infty$, one thus gets

$$\begin{aligned}
&\sum_{\pm} |C_{\pm}(u(t))|_{BV(\mathbb{R}^d)} \\
&\leq \|A' - B'\|_{L^\infty(\mathbb{R})} \lim_{n\to+\infty} \int_{\mathbb{R}^d} |D\phi_n| \,\mathrm{d}x = \|A' - B'\|_{L^\infty(\mathbb{R})} \, |u(t)|_{BV(\mathbb{R}^d)}.
\end{aligned}$$

An integration w.r.t. $t \in (0,T)$ now completes the proof of (iii).

We now prove (iv). Since $C_{\pm}$ is Lipschitz-continuous, it can be written as the integral of its derivative. This implies for a.e. $(x,t) \in Q_T$,

$$\begin{aligned}
&\sum_{\pm} |C_{\pm}(u(x+z,t)) - C_{\pm}(u(x,t))| \\
&= |u(x+z,t) - u(x,t)| \sum_{\pm} \Big| \int_0^1 C'_{\pm}\big((1-\tau)\,u(x,t) + \tau\, u(x+z,t)\big)\,\mathrm{d}\tau \Big| \\
&\leq |u(x+z,t) - u(x,t)| \int_0^1 |A' - B'|\big((1-\tau)\,u(x,t) + \tau\, u(x+z,t)\big)\,\mathrm{d}\tau \quad \text{by (5.19),} \\
&\leq \|A' - B'\|_{L^\infty(\mathbb{R})} \, |u(x+z,t) - u(x,t)|\,.
\end{aligned}$$

The proof of (iv) is complete by integrating w.r.t. $(x,t) \in Q_T$. □

5.3. **Proof of Theorem 3.4.** We argue step by step as in the proof of Theorem 3.3. This time, $E_{\pm}$ are taken such as

$$\begin{cases} E_{\pm} \in \mathcal{B}_{\mathbb{R}}; \\ \cup_{\pm} E^{\pm} = \mathbb{R} \text{ and } \cap_{\pm} E_{\pm} = \emptyset; \\ \mathbb{R}^d \setminus \operatorname{supp}(\mu-\nu)^{\mp} \subseteq E_{\pm}. \end{cases} \tag{5.20}$$

Let $\mu_{\pm}$ and $\nu_{\pm}$ denote the restrictions of μ and ν to $E_{\pm}$. It is clear that

$$\begin{cases} \mu = \sum_{\pm} \mu_{\pm} \quad \text{and} \quad \nu = \sum_{\pm} \nu_{\pm}, \\ \pm(\mu_{\pm} - \nu_{\pm}) = (\mu-\nu)^{\pm}, \\ \mu_{\pm}, \nu_{\pm}, \text{ and } \pm(\mu_{\pm} - \nu_{\pm}) \text{ all satisfy (1.6).} \end{cases} \tag{5.21}$$

Proof of Theorem 3.4.

1. We apply Lemma 3.1 with $A = B$, but different Lévy measures μ and ν, along with the entropy inequality for v to show that for all $r, \epsilon > 0$, $0 < \delta < T$

$$\begin{aligned}
&\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + \epsilon\, C_{\tilde{\theta}}\, |u_0|_{BV(\mathbb{R}^d)} + 2\,\omega_u(\delta) \vee \omega_v(\delta) \\
&\quad + \iint_{Q_T^2} (q_g - q_f)(v,u) \cdot D_x \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v) - A(u)|\, \mathcal{L}_r^{\nu^*}[\phi^{\epsilon,\delta}(\cdot,y,t,s)](x)\,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v)) - A(u)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w \\
&\quad + \underbrace{\iint_{Q_T^2} |A(v) - A(u)| \left(b_r^{\nu^*} - b_r^{\mu^*}\right) \cdot D_x \phi^{\epsilon,\delta} \,\mathrm{d}w}_{=:I_3} \\
&\quad + \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u) \left(\mathcal{L}^{\nu,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\right) \phi^{\epsilon,\delta} \,\mathrm{d}w,}_{=:I_4}
\end{aligned} \tag{5.22}$$

where $C_\epsilon > 0$ does not depend on $r > 0$. Except for I_3 and I_4, the other terms were estimated in the proof of Theorem 3.3.

2. *Cutting w.r.t. $E_\pm$.* We use the notation introduced in (5.20). We apply Lemma 4.5 twice with ν_+ and μ_- instead of μ, along with linearity of $\mathcal{L}^{\mu,r}$ in μ, see (2.3), to see that

$$\begin{aligned}
I_4 &= \sum_{\pm} \iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\nu_\pm,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu_\pm,r}[A(u(\cdot,s))](y) \Big) \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&\leq \iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\nu_+,r}[A(u(\cdot,s))](y) - \mathcal{L}^{\mu_+,r}[A(u(\cdot,s))](y) \Big) \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\nu_-,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu_-,r}[A(v(\cdot,t))](x) \Big) \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&= \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu_+-\nu_+,r}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_--\nu_-),r}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta} \,\mathrm{d}w \\
&=: I_4^+ + I_4^- .
\end{aligned} \tag{5.23}$$

Again, it is crucial to have u in I_4^+ and v in I_4^- in order to use Kato's inequality later on.

Let us now consider I_3. By (2.4) and (2.9), b_r^μ and μ^* are linear w.r.t μ. Easy computations using (5.21) then leads to

$$\Big(b_r^{\nu^*} - b_r^{\mu^*} \Big) \cdot D_x \phi^{\epsilon,\delta} = \sum_{\pm} b_r^{\pm(\mu_\pm-\nu_\pm)^*} \cdot D_\pm \phi^{\epsilon,\delta}$$

where $D_+ = D_y$ and $D_- = D_x$, and hence

$$I_3 = \sum_{\pm} \iint_{Q_T} |A(u) - A(v)|\, b_r^{\pm(\mu_\pm-\nu_\pm)^*} \cdot D_\pm \phi^{\epsilon,\delta} \,\mathrm{d}w =: I_3^+ + I_3^- .$$

3. *Cutting w.r.t. z.* The computations of this step are similar to the ones in the proof of Theorem 3.3. For the reader's convenience, we estimate $I_3^- + I_4^-$, the terms that was left to the reader in the preceding proof.

For any measure $\tilde{\mu}$ we let $\tilde{\mu}_1 = \tilde{\mu}_{|0<|z|\leq r_1}$ and write $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_{||z|>r_1}$ for $r_1 > r$. Then

$$\begin{aligned}
I_4^- \leq\ & \underbrace{\iint_{Q_T} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_--\nu_-)_1,r}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta} \,\mathrm{d}w}_{=:I_5^-} \\
& + \iint_{Q_T} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_--\nu_-),r_1}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta} \,\mathrm{d}w.
\end{aligned}$$

Recall that $-(\mu_- - \nu_-)_1$ is a positive Lévy measure by (5.21), so we can apply Lemma 4.4 with $-(\mu_- - \nu_-)_1$ instead of μ and $k = u(y,s)$ to find that

$$I_5^- \leq \iint_{Q_T^2} |A(v) - A(u)|\, \mathcal{L}^{-(\mu_--\nu_-)_1^*,r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \,\mathrm{d}w$$

and

$$\begin{aligned}
& I_3^- + I_5^- \leq \\
& \iint_{Q_T^2} |A(v) - A(u)| \Big(b_r^{-(\mu_--\nu_-)^*} \cdot D_x \phi^{\epsilon,\delta} + \mathcal{L}^{-(\mu_--\nu_-)_1^*,r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \Big) \,\mathrm{d}w.
\end{aligned}$$

Easy computations then leads to

$$\begin{aligned}
&\mathcal{L}^{-(\mu_--\nu_-)^*_1,r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x)\\
&=\theta_\delta(t-s)\int_{r<|z|\le r_1}\bar\theta_\epsilon(x-y-z)-\bar\theta_\epsilon(x-y)\,\mathrm{d}(\nu_--\mu_-)(z),
\end{aligned}$$

and we can rewrite the nonlocal operator as follows,

$$\begin{aligned}
&b_r^{-(\mu_--\nu_-)^*}\cdot D_x\phi^{\epsilon,\delta}+\mathcal{L}^{-(\mu_--\nu_-)^*_1,r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x)\\
&=\theta_\delta(t-s)\int_{r<|z|\le r_1}\bar\theta_\epsilon(x-y-z)-\bar\theta_\epsilon(x-y)+z\cdot D\bar\theta_\epsilon(x-y)\,\mathrm{d}(\nu_--\mu_-)(z)\\
&\quad-\theta_\delta(t-s)\,D\bar\theta_\epsilon(x-y)\cdot\underbrace{\left(-b_r^{-(\mu_--\nu_-)^*}+\int_{r<|z|\le r_1}z\,\mathrm{d}(\nu_--\mu_-)(z)\right)}_{=\operatorname{sgn}(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1}z\,\mathrm{d}(\nu_--\mu_-)(z)}.
\end{aligned}$$

Compare this expression with (5.12) that appear when I_3^+ and I_4^+ are considered.
We add the different estimates and find that for all $r_1>r$,

$$\begin{aligned}
&I_3^-+I_4^-\\
&\le\iint_{Q_T^2}|A(u)-A(v)|\,\theta_\delta(t-s)\\
&\qquad\cdot\int_{r<|z|\le r_1}\bar\theta_\epsilon(x-y-z)-\bar\theta_\epsilon(x-y)+z\cdot D\bar\theta_\epsilon(x-y)\,\mathrm{d}(\nu_--\mu_-)(z)\,\mathrm{d}w\\
&\quad-\iint_{Q_T^2}|A(u)-A(v)|\,\theta_\delta(t-s)\,D\bar\theta_\epsilon(x-y)\\
&\qquad\cdot\operatorname{sgn}(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1}z\,\mathrm{d}(\nu_--\mu_-)(z)\,\mathrm{d}w\\
&\quad+\underbrace{\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{-(\mu_--\nu_-),r_1}[A(v(\cdot,t))](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w}_{\le\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{-(\mu_--\nu_-),r_1}[A(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\text{ by Lemma 4.5}}\\
&=J_1^-+J_2^-+J_3^-.
\end{aligned}$$

Similar arguments also lead to

$$\begin{aligned}
&I_3^++I_4^+\\
&\le\iint_{Q_T^2}|A(u)-A(v)|\theta_\delta(t-s)\\
&\qquad\cdot\int_{r<|z|\le r_1}\bar\theta_\epsilon(x-y+z)-\bar\theta_\epsilon(x-y)-z\cdot D\bar\theta_\epsilon(x-y)\,\mathrm{d}(\mu_+-\nu_+)(z)\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}|A(u)-A(v)|\,\theta_\delta(t-s)\,D\bar\theta_\epsilon(x-y)\\
&\qquad\cdot\operatorname{sgn}(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1}z\,\mathrm{d}(\mu_+-\nu_+)(z)\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}\operatorname{sgn}(u-v)\,\mathcal{L}^{(\mu_+-\nu_+),r_1}[A(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w,\\
&=:J_1^++\cdots+J_3^+.
\end{aligned}$$

4. *$L^1 \cap BV$-regularity.* We estimate $J_i^\pm$ $(i = 1, \dots, 3)$. By (B.1) of Lemma B.1 and (2.11), it follows that

$$\sum_\pm J_1^\pm \leq \frac{1}{2\epsilon} \int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x \, |u_0|_{BV(\mathbb{R}^d)} \, T \, \|A'\|_{L^\infty(\mathbb{R})} \int_{r<|z|\leq r_1} |z|^2 \, \mathrm{d} \underbrace{\sum_\pm \pm(\mu_\pm - \nu_\pm)}_{=|\mu-\nu| \text{ by (5.21)}}(z).$$

Note now that $\sum_\pm (\mu_\pm - \nu_\pm) = \mu - \nu$, and hence

$$\sum_\pm J_2^\pm = \iint_{Q_T^2} |A(u) - A(v)| \, \theta_\delta(t-s) \, D\bar{\theta}_\epsilon(x-y) \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1 \vee r) < |z| \leq r_1 \vee 1} z \, \mathrm{d}(\mu - \nu)(z) \, \mathrm{d}w.$$

An other application of (B.2) of Lemma B.1 and (2.11), can be used to see that

$$\sum_\pm J_2^\pm \leq |u_0|_{BV(\mathbb{R}^d)} \, T \, \|A'\|_{L^\infty(\mathbb{R})} \left| \int_{r_1 \wedge (1 \vee r) < |z| \leq r_1 \vee 1} z \, \mathrm{d}(\mu - \nu)(z) \right|.$$

Finally, we use again Lemma 4.2 (iii) and (2.12) to show that

$$\sum_\pm J_3^\pm \leq |u_0|_{BV(\mathbb{R}^d)} \, T \, \|A'\|_{L^\infty(\mathbb{R}^d)} \int_{r_1 < |z| \leq r_2} |z| \, \mathrm{d}|\mu - \nu|(z) + T \, \|A'\|_{L^\infty(\mathbb{R})} \int_{|z| \geq r_2} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \, \mathrm{d}|\mu - \nu|(z).$$

5. *Conclusion.* The rest of the proof is the same as for Theorem 3.3; i.e. we use the estimates on $J_i^\pm$ to estimate $I_3 + I_4 \leq \sum_{i=1}^3 \sum_\pm J_i^\pm$ in (5.22) and pass to limit and/or optimizes w.r.t. the parameters $r, \epsilon, \delta > 0$. The proof is complete. □

Appendix A. Properties of approximate units and BV-functions

The results in this section are quite classical, see e.g. [13, 33, 53].

Lemma A.1. *Assume* (1.6) *and let $\phi^{\epsilon,\delta}$ be defined as in Lemma 3.1. Then for all $\epsilon, \delta > 0$*

(i) $\|\theta_\delta\|_{L^1(\mathbb{R})} = \|\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^d)} = 1$,

(ii) $\int_{\mathbb{R}^d} |D\bar{\theta}_\epsilon| \, \mathrm{d}x \leq \frac{1}{\epsilon} \|D\tilde{\theta}_d\|_{L^1(\mathbb{R})}$.

Lemma A.2. *Let $u \in L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. Then for all $h \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |u(x+h) - u(x)| \, \mathrm{d}x \leq |h| \, |u|_{BV(\mathbb{R}^d)}.$$

Lemma A.3. *Let $u \in L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ and $q : \mathbb{R} \to \mathbb{R}^d$ be Lipschitz-continuous. Then for all $\phi \in C_c^\infty(\mathbb{R}^d)$,*

$$\left| \int_{\mathbb{R}^d} q(u) \cdot D\phi \, \mathrm{d}x \right| \leq \operatorname{ess-sup}_{\mathbb{R}} |q'| \int_{\mathbb{R}^d} |\phi| \, \mathrm{d}|Du|(x). \tag{A.1}$$

Moreover, if $\eta : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, then $\eta(u) \in BV(\mathbb{R}^d)$ with

$$|D\eta(u)| \leq \|\eta'\|_{L^\infty(\mathbb{R})} \, |Du|.$$

Let us give a short proof of the last lemma for the reader's convenience. Remember that $|\cdot|$ denotes the total variation measure of a Radon measure (see the definition in (2.1)).

Proof. Let us prove (A.1). For all $n \in \mathbb{N}$, define $\phi_n := u * \bar{\theta}_{\frac{1}{n}} \in C^\infty(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$ where $\bar{\theta}_{\frac{1}{n}}$ is the approximate unit in (3.1). Also assume $\tilde{\theta}_d$ in (3.2) is even. By classical results on approximate units, $\phi_n \to u$ in L^1 as $n \to +\infty$. Moreover, for all $x \in \mathbb{R}^d$,

$$
\begin{aligned}
|D\phi_n(x)| &= \left| \int_{\mathbb{R}^d} \bar{\theta}_{\frac{1}{n}}(x-y)\,\mathrm{d}Du(y) \right| \\
&\leq \int_{\mathbb{R}^d} \bar{\theta}_{\frac{1}{n}}(x-y)\,\mathrm{d}|Du(y)| = \int_{\mathbb{R}^d} \bar{\theta}_{\frac{1}{n}}(y-x)\,\mathrm{d}|Du(y)|,
\end{aligned}
$$

since $\bar{\theta}_{\frac{1}{n}}$ is nonnegative and even. It then follows that

$$
\begin{aligned}
I_n :&= \left| \int_{\mathbb{R}^d} q(\phi_n) \cdot D\phi \,\mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}^d} \phi\, q'(\phi_n) \cdot D\phi_n \,\mathrm{d}x \right| \quad \text{by integration by parts and the chain rule,} \\
&\leq \text{ess-sup}_{\mathbb{R}} |q'| \int_{\mathbb{R}^d} |\phi|\, |D\phi_n| \,\mathrm{d}x, \\
&\leq \text{ess-sup}_{\mathbb{R}} |q'| \int_{\mathbb{R}^d} |\phi(x)| \int_{\mathbb{R}^d} \bar{\theta}_{\frac{1}{n}}(y-x)\,\mathrm{d}|Du|(y)\,\mathrm{d}x.
\end{aligned}
$$

Since $|Du|$ is σ-finite as finite Radon measure, the product measure $\mathrm{d}x\,\mathrm{d}|Du|(y)$ is well-defined and Fubini's theorem applies. After changing the order of the integration, we find that

$$
I_n \leq \text{ess-sup}_{\mathbb{R}} |q'| \int_{\mathbb{R}^d} |\phi| * \bar{\theta}_{\frac{1}{n}}(y)\,\mathrm{d}|Du|(y).
$$

Again by classical results on approximate units, $|\phi| * \bar{\theta}_{\frac{1}{n}} \to |\phi|$ uniformly on $\mathbb{R}^d$, and hence

$$
\limsup_{n \to +\infty} I_n \leq \text{ess-sup}_{\mathbb{R}} |q'| \int_{\mathbb{R}^d} |\phi| \,\mathrm{d}|Du|(y).
$$

Finally since $q(\phi_n)$ converges to $q(u)$ in L^1_{loc}, we also have that

$$
\lim_{n \to +\infty} I_n = \left| \int_{\mathbb{R}^d} q(u) \cdot D\phi \,\mathrm{d}x \right|,
$$

and the proof of (A.1) is complete.

To prove the assertion for $\eta(u)$, we repeat the above arguments with η instead of q. The result is that for all $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

$$
\left| \int_{\mathbb{R}^d} \eta(u) \operatorname{div} \phi \,\mathrm{d}x \right| \leq \|\eta'\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^d} |\phi| \,\mathrm{d}|Du|(y),
$$

and we may conclude from Riesz's representation theorem [33, Section 1.8 Theorem 1] that $\eta(u) \in BV(\mathbb{R}^d)$. Moreover, by (2.1) , we see that for all $B \in \mathcal{B}_{\mathbb{R}^d}$

$$
|D\eta(u)|(B) \leq \|\eta'\|_{L^\infty(\mathbb{R})} \inf \{ |Du|(U) \text{ s.t. } U \text{ open, } B \subseteq U \}.
$$

Finally, by classical regularity properties of Radon (Borel) measures [33, 53], the infimum above is equaled to $|Du|(B)$ and the proof is complete. □

Appendix B. Technical results and computations

B.1. **Proof of** (5.2)**.** We start by estimating I_4. Since

$$
-|u(y,T) - v(x,t)| \leq |u(y,T) - u(x,T)| - |u(x,T) - v(x,T)| + |v(x,T) - v(x,t)|,
$$

and $\phi^{\epsilon,\delta}$ is nonnegative,

$$\begin{aligned}
I_4 &= -\iint_{Q_T\times\mathbb{R}^d} |u(y,T)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\
&\le -\iint_{Q_T\times\mathbb{R}^d} |u(x,T)-v(x,T)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\
&\quad + \iint_{Q_T\times\mathbb{R}^d} |u(y,T)-u(x,T)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\
&\quad + \iint_{Q_T\times\mathbb{R}^d} |v(x,T)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\
&=: J_1+J_2+J_3.
\end{aligned}$$

By Lemma A.1,

$$\begin{aligned}
J_1 &= -\int_{\mathbb{R}^d} |u(x,T)-v(x,T)| \int_0^T \theta_\delta(t-T) \underbrace{\int_{\mathbb{R}^d} \bar\theta_\epsilon(x-y)\,\mathrm{d}y}_{=1}\,\mathrm{d}t\,\mathrm{d}x \\
&= -\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)} \int_{-\frac{T}{\delta}}^0 \tilde\theta_1(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

By the change of variables $(x,t,y)\to(x,t,x-y)$,

$$\begin{aligned}
J_2 &= \int_{Q_T} \theta_\delta(t-T)\,\bar\theta_\epsilon(y) \int_{\mathbb{R}^d} |u(x-y,T)-u(x,T)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \\
&\le |u(T)|_{BV(\mathbb{R}^d)} \int_{Q_T} \theta_\delta(t-T)\,|y|\,\bar\theta_\epsilon(y)\,\mathrm{d}y\,\mathrm{d}t \quad \text{by Lemma A.2,} \\
&\le |u_0|_{BV(\mathbb{R}^d)} \int_{-\frac{T}{\delta}}^0 \tilde\theta_1(\tau)\,\mathrm{d}\tau \int_{\mathbb{R}^d} |y|\,\frac{1}{\epsilon^d}\,\tilde\theta_d\Big(\frac{y}{\epsilon}\Big)\,\mathrm{d}y \quad \text{by (2.11),} \\
&= \epsilon \underbrace{\int_{\mathbb{R}^d} |y|\,\tilde\theta_d(y)\,\mathrm{d}y}_{=:\frac{C_{\tilde\theta}}{2}}\; |u_0|_{BV(\mathbb{R}^d)} \int_{-\frac{T}{\delta}}^0 \tilde\theta_1(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
J_3 &= \int_{Q_T} |v(x,t)-v(x,T)|\,\theta_\delta(t-T) \int_{\mathbb{R}^d} \bar\theta_\epsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \\
&\le \int_{T-\delta}^T \theta_\delta(t-T) \int_{\mathbb{R}^d} |v(x,t)-v(x,T)|\,\mathrm{d}x\,\mathrm{d}t \quad \text{since } \operatorname{supp}\theta_\delta\subseteq[-\delta,\delta] \text{ by (3.2),} \\
&\le \omega_v(\delta) \int_{-\frac{T}{\delta}}^0 \tilde\theta_1(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

We conclude that

$$I_4 \le \left(-\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)} + \frac{C_{\tilde\theta}}{2}\,\epsilon\,|u_0|_{BV(\mathbb{R}^d)} + \omega_v(\delta) \right) \int_{-\frac{T}{\delta}}^0 \tilde\theta_1(\tau)\,\mathrm{d}\tau.$$

Starting from

$$-|u(y,s)-v(x,T)| \le |u(y,s)-u(x,s)| - |u(x,T)-v(x,T)| + |u(x,s)-u(x,T)|,$$

similar reasoning leads to

$$I_4^v \le \left(-\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)} + \frac{C_{\tilde\theta}}{2}\,\epsilon|u_0|_{BV(\mathbb{R}^d)} + \omega_u(\delta) \right) \int_0^{+\frac{T}{\delta}} \theta(\tau)\,\mathrm{d}\tau.$$

Hence, $I_4 + I_4^v \leq -\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} + \frac{C_{\tilde{\theta}}}{2}\,\epsilon\,|u_0|_{BV(\mathbb{R}^d)} + \omega_u(\delta) \vee \omega_v(\delta)$ when $\frac{T}{\delta} > 1$ and we finish the proof of (5.2) by estimating $I_5 + I_5^v$ with similar arguments.

B.2. **Estimates of the singular nonlocal parts.**

Lemma B.1. *Assume* (1.4) *and* (1.6). *Let* $u, v \in L^\infty(Q_T) \cap C([0,T]; L^1) \cap L^\infty(0,T; BV)$, $\phi^{\epsilon,\delta}$ *be as in Lemma 3.1, and* $r_1 > r > 0$. *Then*

$$\begin{aligned} &\Bigg| \iint_{Q_T^2} |A(v(x,t)) - A(u(y,s))| \\ &\cdot\; \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y\pm z) - \bar{\theta}_\epsilon(x-y) \mp z \cdot D\bar{\theta}_\epsilon(x-y)\,\mathrm{d}\mu(z)\,\mathrm{d}w \Bigg| \\ &\leq \frac{1}{2\epsilon} \int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z)\,|A(u)|_{L^1(0,T;BV)}. \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} &\Bigg| \iint_{Q_T^2} |A(v(x,t)) - A(u(y,s))| \\ &\cdot\; \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\,\mathrm{d}w \Bigg| \\ &\leq \Bigg| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z) \Bigg|\,|A(u)|_{L^1(0,T;BV)}. \end{aligned} \tag{B.2}$$

Proof. We start by proving (B.1) in the + case. Similar arguments give the proof also in the − case. From Taylor's formula with integral remainder,

$$\bar{\theta}_\epsilon(x-y+z) - \bar{\theta}_\epsilon(x-y) - z \cdot D\bar{\theta}_\epsilon(x-y) = \int_0^1 (1-\tau)\, D^2\bar{\theta}_\epsilon(x-y+\tau\, z)\, z \cdot z\,\mathrm{d}\tau.$$

Let I denote the integral in the left-hand side of (B.1). By Fubini's theorem,

$$\begin{aligned} I = &\iint_{(0,T)^2} \int_{r<|z|\leq r_1} \int_0^1 \theta_\delta(t-s)\,(1-\tau) \\ &\cdot \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |A(v(x,t)) - A(u(y,s))|\, D^2\bar{\theta}_\epsilon(x-y+\tau\, z)\, z\cdot z\,\mathrm{d}y\,\mathrm{d}x}_{=:J}\,\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s. \end{aligned} \tag{B.3}$$

The $\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}w$-integrability of the integrands follows from similar arguments as in the proof of Lemma 4.2. Note that $\eta_k(A(u(\cdot,s))) = |k - A(u(\cdot,s))| \in BV$ with

$$|D\eta_k(A(u(\cdot,s))| \leq |DA(u(\cdot,s))|$$

for any $k \in \mathbb{R}$ by Lemma A.3 and $L^1 \cap BV$ regularity of $A(u(\cdot,s))$. Integration by parts w.r.t. y (for fixed z, x, t, s), then leads to

$$\begin{aligned} |J| &= \left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D\bar{\theta}_\epsilon(x-y+\tau\, z)\cdot z\, z\cdot \mathrm{d}D\eta_{A(v(x,t))}(A(u(\cdot,s)))(y)\,\mathrm{d}x \right|, \\ &\leq |z|^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |D\bar{\theta}_\epsilon(x-y+\tau\, z)|\,\mathrm{d}|DA(u(\cdot,s))|(y)\,\mathrm{d}x. \end{aligned}$$

We change the order of integration (using Fubini) and use Lemma A.1 to see that

$$|J| \leq |z|^2\,|A(u(s))|_{BV(\mathbb{R}^d)} \int_{\mathbb{R}^d} |D\bar{\theta}_\epsilon(x)|\,\mathrm{d}x \leq |z|^2\,|A(u(s))|_{BV(\mathbb{R}^d)} \frac{1}{\epsilon} \int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x,$$

and then from (B.3) that

$$|I| \le \frac{1}{\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x \\ \cdot \iint_{(0,T)^2} \int_{r<|z|\le r_1} \int_0^1 \theta_\delta(t-s)\,(1-\tau)\,|z|^2\,|A(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s.$$

Let us recall that the integrand above is $\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s$-measurable since $s \to |u(s)|_{BV(\mathbb{R}^d)}$ is lower semi-continuous. By Fubini we then integrate first w.r.t. t and use that $\int_0^T \theta_\delta(t-s)\,\mathrm{d}t \le 1$ to see that

$$|I| \le \frac{1}{\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x \int_0^1 (1-\tau)\,\mathrm{d}\tau \int_{r<|z|\le r_1} |z|^2 \mathrm{d}\mu(z) \int_0^T |A(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}s,$$

and the proof of (B.1) is complete.

We prove (B.2) by similar arguments. Define

$$q(v,u) := |v-u|\,\mathrm{sgn}\,(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1} z\,\mathrm{d}\mu(z), \tag{B.4}$$

and note that it is Lipschitz-continuous. Again we denote by I the integral of the left-hand side of (B.2). By Fubini's theorem,

$$I = \iint_{(0,T)^2} \theta_\delta(t-s) \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D\bar{\theta}_\epsilon(x-y)\cdot q(A(v(x,t), A(u(y,s)))\,\mathrm{d}y\,\mathrm{d}x}_{=:J}\,\mathrm{d}t\,\mathrm{d}s. \tag{B.5}$$

For fixed (x,t), $q(A(v(x,t),\cdot)$ is Lipschitz-continuous and we may use (A.1) and integration by parts in y to see that

$$|J| \le \text{ess-sup}_{\mathbb{R}^2} |q_u| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{\theta}_\epsilon(x-y)\,\mathrm{d}|DA(u(\cdot,s))|(y)\,\mathrm{d}x,$$

where $\text{ess-sup}_{\mathbb{R}^2} |q_u|$ denotes the Lipschitz constant of q w.r.t. its second variable u. Changing the order of integration, we find that

$$J \le |A(u(s))|_{BV(\mathbb{R}^d)}\,\text{ess-sup}_{\mathbb{R}^2} |q_u|,$$

and hence by (B.5) and integrating first w.r.t. t, we get that

$$|I| \le \text{ess-sup}_{\mathbb{R}^2} |q_u| \int_0^T |A(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}s. \tag{B.6}$$

The proof of (B.2) is now complete since by (B.4),

$$\text{ess-sup}_{\mathbb{R}^2} |q_u| = \left| \int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1} z\,\mathrm{d}\mu(z)\right|.$$

□

B.3. **Proof of** (5.6)**.** It remains to prove (5.6) from the proof of Theorem 3.3. Recall that u is the entropy solution to (1.1) and that I_1 is defined in (5.5). We want to prove that

$$I := \iint_{Q_T^2} (q_g - q_f)(v(x,t), u(y,s)) \cdot \theta_\delta(t-s)\,D\bar{\theta}_\epsilon(x-y)\,\mathrm{d}w \\ \le |u_0|_{BV(\mathbb{R}^d)}\,T\,\text{ess-sup}_{\mathbb{R}} |f'-g'|.$$

Define

$$q(v,u) := (q_g - q_f)(v,u) = \mathrm{sgn}\,(v-u)\ \{(g(v)-g(u)) - (f(v)-f(u))\}$$

(see Definition 2.6). Since f, g are Lipschitz-continuous, q will be so too, and I will satisfy (B.5) with $A(u) = u$ and new flux q. Arguing word by word as in the preceding proof from (B.5) until (B.6) using also (2.11), we find that

$$|I| \leq \text{ess-sup}_{\mathbb{R}^2} |q_u| \int_0^T |u(s)|_{BV(\mathbb{R}^d)} \, \mathrm{d}s \leq |u_0|_{BV(\mathbb{R}^d)} \, T \, \text{ess-sup}_{\mathbb{R}^2} |q_u|.$$

The proof is complete since $\text{ess-sup}_{\mathbb{R}^2} |q_u| \leq \text{ess-sup}_{\mathbb{R}} |f' - g'|$ by definition.

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(Nathaël Alibaud) UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25 030 Besançon cedex, France and Department of Mathematics, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, Thailand, 90112
E-mail address: nathael.alibaud@ens2m.fr
URL: http://www-math.univ-fcomte.fr/pp_Annu/NALIBAUD/

(Simone Cifani) Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway
E-mail address: simone.cifani@math.ntnu.no
URL: http://www.math.ntnu.no/~cifani/

(Espen R. Jakobsen) Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway
E-mail address: erj@math.ntnu.no
URL: http://www.math.ntnu.no/~erj/