Distributed Function and Time Delay Estimation using Nonparametric Techniques

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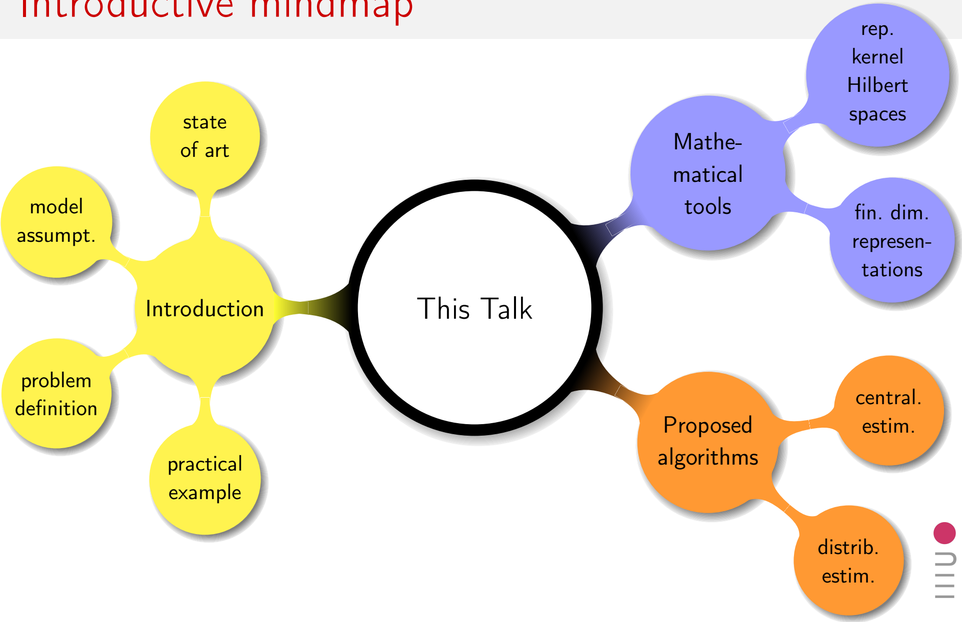
DEPARTMENT OF INFORMATION ENGINEERING
UNIVERSITY OF PADOVA

entirely written in LaTeX2ε using Beamer and TikZ

Introductive mindmap



introduction

mathematical tools

algorithms

Practical example

windfarm:

- windwheels subject to approximatively the same wind force
- wind arrives with unknown delays

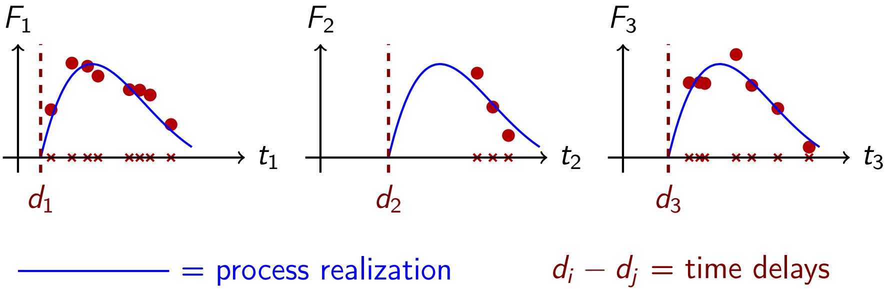
Problem definition

we want to distributely estimate:

- the process realization (e.g. wind force vs. time)
- the time delays

Modeling assumptions

- all sensors measure the same process (simplification)
- measurements noises are independent
- measurements are not synchronized
- there are limits on the amount of exchangeable information
- there are limits on the sizes of the estimates representations



State of the art

Function estimation

- distributed non-parametric regression already proposed
- we add:
  - unknown time-delays complication

Time delay estimation

- usually formulated with inner-products maximization
- we add:
  - distributed estimation
  - non-parametric framework
  - easy management of non-uniform sampling

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Nonparametric approach

motivations: functional structure of f could be not easily managed by parametric structures

hypotheses: f is a zero-mean Gaussian process with

$$\operatorname{cov}\left(f(t_m),\ f(t_n)^T\right) = K(t_m, t_n)$$

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$$\mathrm{cov}\left(f\left(t_m\right),\ f\left(t_n\right)^T\right) = K\left(t_m, t_n\right)$$

our approach: use $K\left(\cdot,\cdot\right)$ to construct a reproducing kernel Hilbert space $\mathcal{H}_K$

goal: use $K\left(\cdot,\cdot\right)$ + input locations + measurements to construct $\widehat{f} \in \mathcal{H}_K$

(RKHS := reproducing kernel Hilbert space)

RKHS based learning

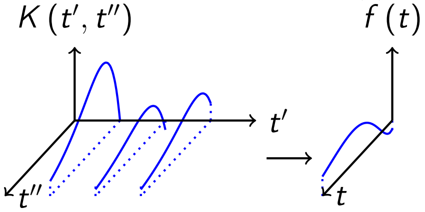
Further hypotheses:

- $K(\cdot,\cdot)$ is a Mercer Kernel
- input domain is compact

Result: MMSE estimator $\widehat{f} = \operatorname{cov}(f,\mathbf{y})\operatorname{var}(\mathbf{y})^{-1}\mathbf{y}$ is:

$$\widehat{f}(\cdot) = \sum_{m=1}^{M} c_m K(t_m,\cdot)$$

$$\mathbf{c} = (K_{\mathbf{t}} + \gamma I_M)^{-1}\mathbf{y} \qquad (1)$$



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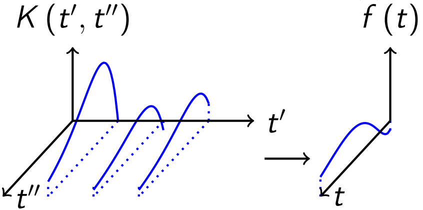

too expensive solution for high M $\Rightarrow$ must approximate solution using other ways of expressing f

Towards approximated RKHS learning

Theorem (with the previous hypotheses:)

- $K(\cdot,\cdot)$ *defines:* $(L_K f)(t_m) := \int_{\mathcal{X}} K(t_m, t') f(t') dt'$
- L_K *has (numerable) eigenvalues and eigenfunctions:* $\phi_k(\cdot) = \lambda_k (L_K \phi_k)(\cdot) \qquad k = 1, 2, \ldots$
- $\{\phi_k(\cdot)\}$ *is an orthonormal basis for the deterministic space*

$$\mathcal{H}_K = \left\{ g \in \mathcal{L}_2 \;\; s.t. \;\; g = \sum_{k=1}^{\infty} a_k \phi_k \;\; with \;\; \sum_{k=1}^{\infty} \frac{a_k \cdot a_k}{\lambda_k} < +\infty \right\} \tag{2}$$

- *MMSE estimate* $\widehat{f}$ *of process realization* f *belongs to this space*

Approximated RKHS learning

a Principal Component Analisys reminding approach

new goal: want to estimate only the first E coefficients ($E \ll M$):

$$\widehat{f}(\cdot) = \sum_{m=1}^{M} c_m K(t_m, \cdot) \qquad \rightarrow \qquad \widehat{f}(\cdot) = \sum_{k=1}^{E} a_k \phi_k(\cdot) \tag{3}$$

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new notation: $y_m = \sum_{k=1}^{+\infty} a_k \phi_k(t_m) + \nu_m \quad \rightarrow \quad y_m = C_m \mathbf{a} + e_m + \nu_m$

$$C_m := \begin{bmatrix} \phi_1(t_m) & \dots & \phi_E(t_m) \end{bmatrix} \quad \mathbf{a} := \begin{bmatrix} a_m \\ \vdots \\ a_E \end{bmatrix} \quad e_m := \sum_{k=E+1}^{+\infty} a_k \phi_k(t_1) \tag{4}$$

use a finite number of eigenfunctions $\Rightarrow$ introduce correlated noise e_m

Approximated learning - Bayesian approach

Prior on eigenfunctions weights $\mathbf{a}$: (depends on the eigenvalues of L_K!)

$$\Sigma_{\mathbf{a}} := \operatorname{diag}\left(\lambda_1, \ldots, \lambda_E\right) \tag{5}$$

Bayesian approach: find the best linear estimator:

$$\mathbf{y} = C\mathbf{a} + \mathbf{e} + \nu \quad \rightarrow \quad \widehat{\mathbf{a}} = \operatorname{cov}\left(\mathbf{a}, \mathbf{y}\right) \operatorname{var}\left(\mathbf{y}\right)^{-1} \mathbf{y} \tag{6}$$

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Numerical solution:

$$\widehat{\mathbf{a}} = \Sigma_{\mathbf{a}} C^T \left(C\Sigma_{\mathbf{a}} C^T + \Sigma_{\mathbf{e}} + \sigma^2 I_M\right)^{-1}\mathbf{y} \tag{7}$$

approximation noise: $\Sigma_{\mathbf{e}} := \operatorname{var}\left(\mathbf{e}\right)$ can be small even for small E

computations load: $O\left(M^3\right)$ operations

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Centralized joint function and TD estimation

proposed solution: cost-function based regularization:

$$\mathcal{L} := -\ln P\left(t_{1,1}, y_{1,1}, \ldots, t_{S,M}, y_{S,M} \mid \tau_1, \ldots, \tau_S, a_1, \ldots, a_E\right) + \gamma \sum_{k=1}^{E} \frac{a_k^2}{\lambda_k} \tag{8}$$

$\Rightarrow$ proposed minimization uses a 2-steps gradient descent:

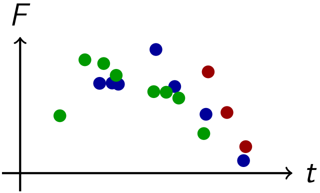
1. keep delays τ_i fixed and update the weights a_k
2. keep the weights a_k fixed and update the delays τ_i

Caveat: initialization will strongly affects results!

Gradient descents steps

Weights a_k update: (τ_i fixed)

1. join all the shifted data sets



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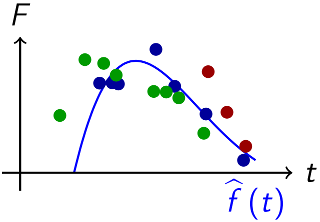

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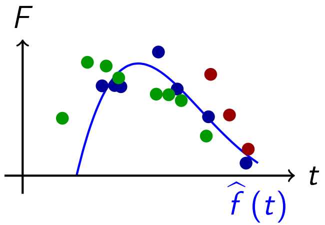
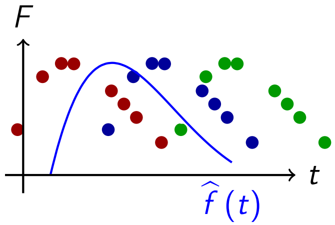


Time delays τ_i update: (a_k fixed)

1. shift optimally each data set



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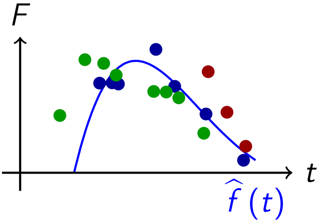

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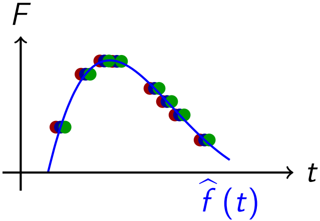

Distributed function estimation with known delays

1. assume to know the delays between the various functions
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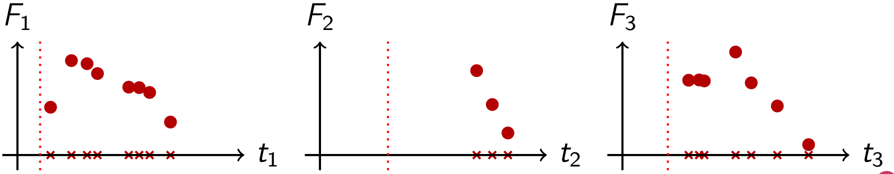
3. compute (locally) the eigenfunctions weights a_i^k
4. make average consensus on the weights a_i^k
5. shift back the eigenfunctions (locally)



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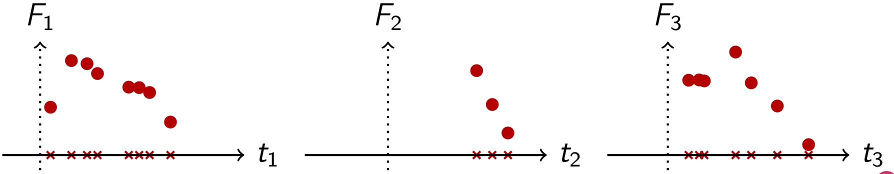

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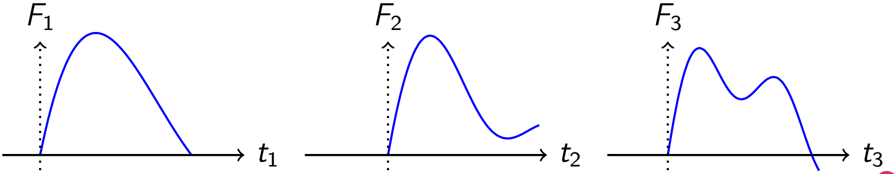

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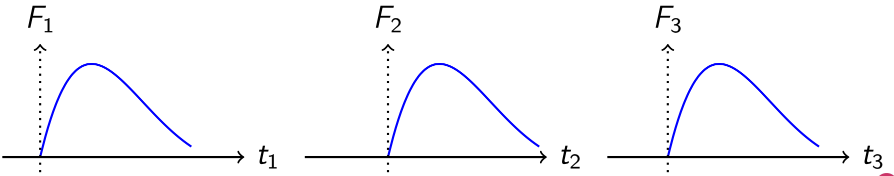

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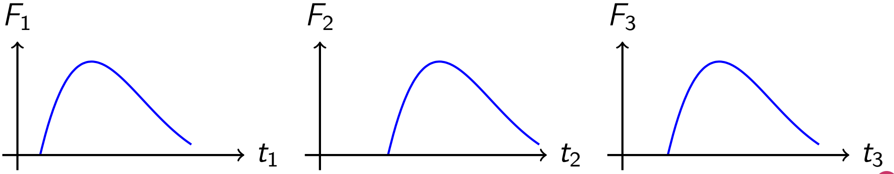

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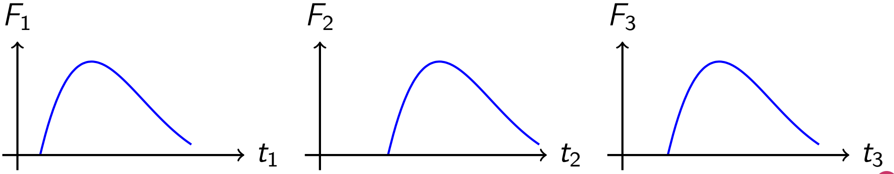

result in general not equivalent to centralized estimate!

Distributed joint function and TDE estimation

Hypothesis: data sets are conditionally independent given the delays and the eigenfunctions weights:

$$P\left(t_{1,1}, y_{1,1}, \ldots, t_{S,M}, y_{S,M} \,\middle|\, \tau_1, \ldots, \tau_S, a_1, \ldots, a_E\right) = \\ = \prod_{i=1}^{S} P\left(t_{i,1}, y_{i,1}, \ldots, t_{i,M_i}, y_{i,M_i} \,\middle|\, \tau_i, a_1, \ldots, a_E\right) \tag{9}$$

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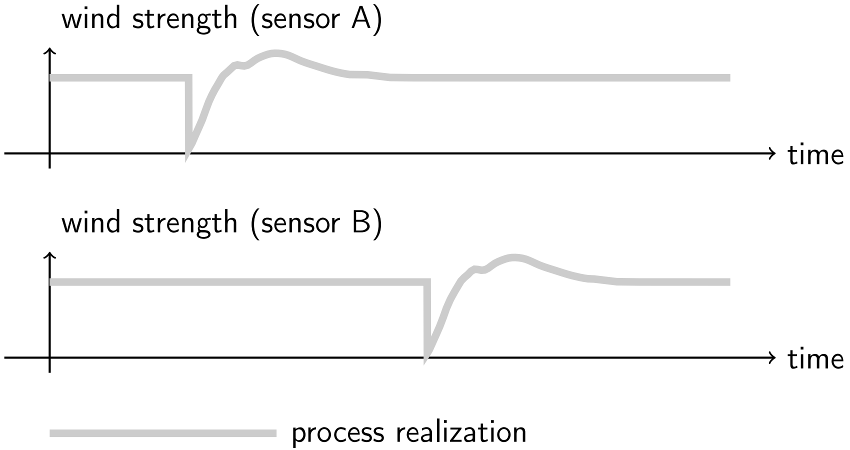
Proposed solution: based on [Schizas et al., Consensus in ad hoc WSNs with noisy links, 2008]

1. construct a constrained optimization problem (i.e impose $a_k^i = a_k^j$)
2. distributely minimize it

Simulations - distributed function estimation



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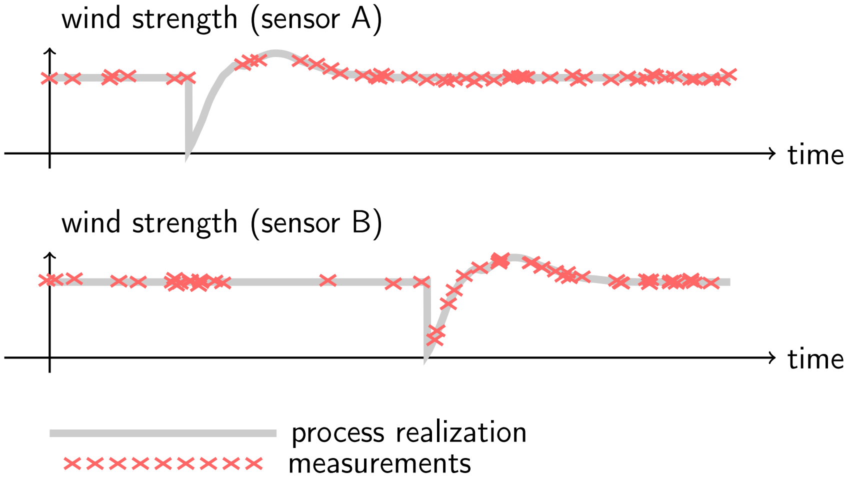

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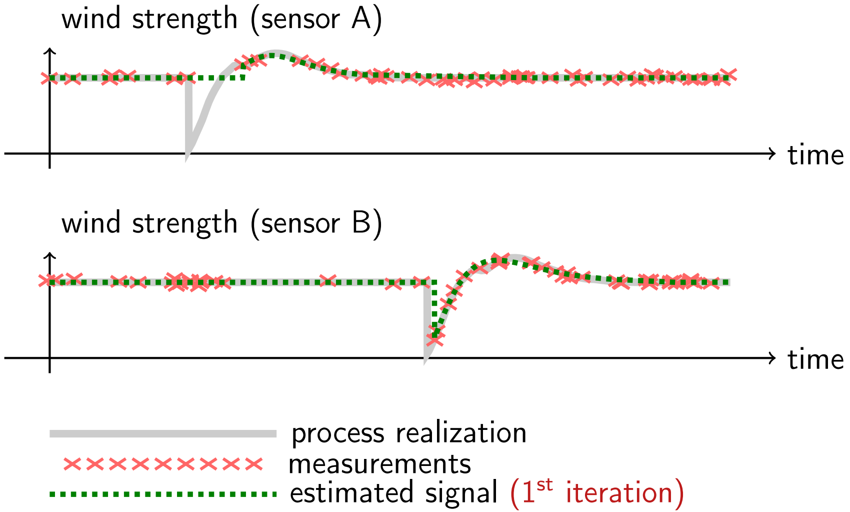

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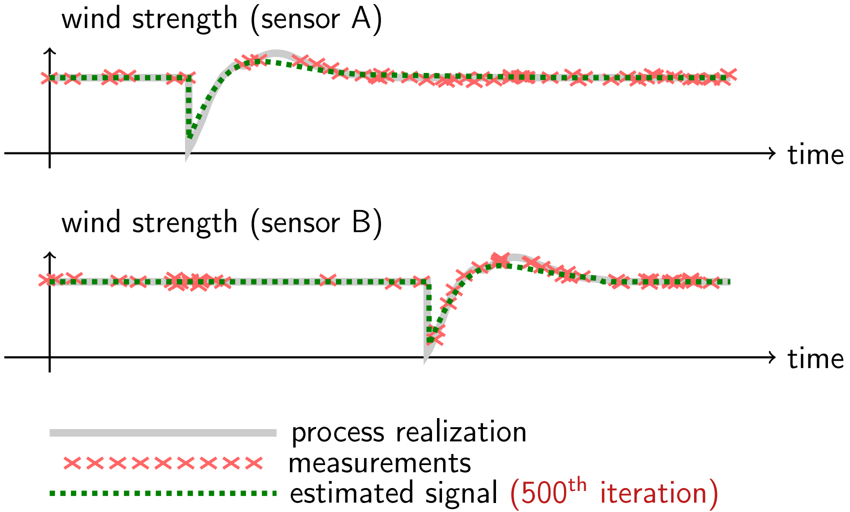
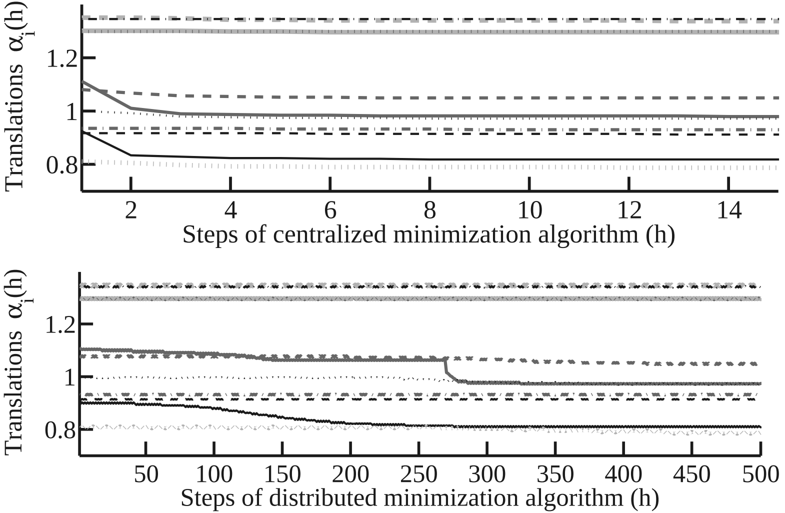


Simulations - convergence speed comparisons



Conclusions and future work

Qualities of the proposed algorithms

- good accuracy with compact representations

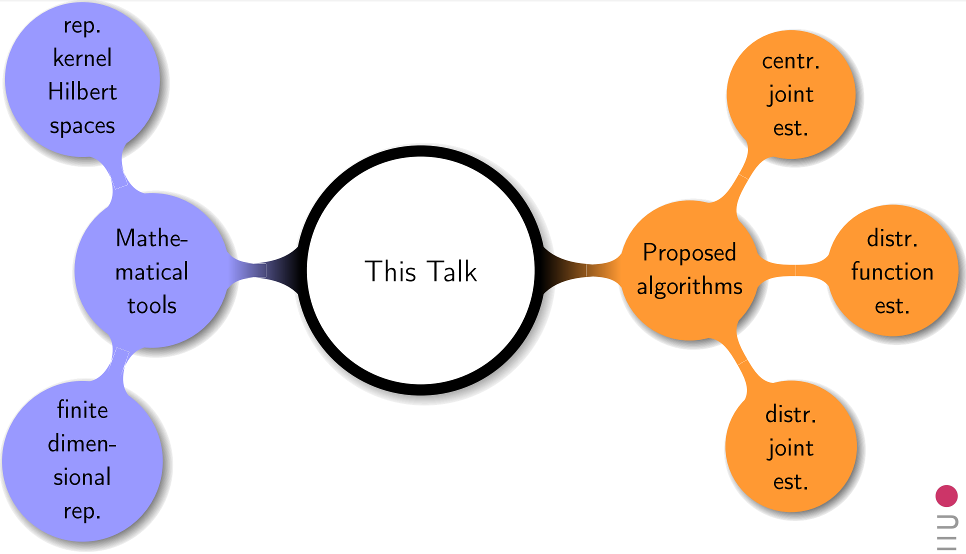
Drawbacks of the distributed algorithms

- convergence can be extremely slow
- results strongly affected by initialization

Future works

- develop hierarchical approaches
- characterize time delay estimators (biasness, variance)

Conclusive mindmap



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Integral operator associated to a Mercer Kernel

1^{th} assumption:	$K(\cdot, \cdot)$ is a Mercer Kernel (continuous, symmetric, positive definite)
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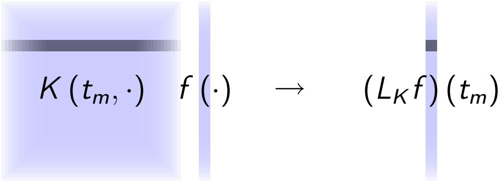
2^{nd} assumption:	input locations domain is compact
1^{th} implication:	$K(\cdot,\cdot)$ defines a compact linear positive definite integral operator: $(L_K f)(t_m) := \int_{\mathcal{X}} K(t_m, t') f(t') dt'$
2^{nd} implication:	L_K has an at most numerable set of eigenfunctions: $\phi_k(\cdot) = \lambda_k (L_K \phi_k)(\cdot) \qquad k = 1, 2, \dots$



Nonparametric approach

assumption: realizations live in a functions space: $f \in \mathcal{H}_K$

goal: search the estimate $\widehat{f}$ inside $\mathcal{H}_K$

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our approach: use Reproducing Kernel Hilbert Spaces

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Workflow:

- assume the existence of
  $K(\cdot,\cdot): \quad \text{Input locations} \times \text{Input locations} \quad \rightarrow \quad \mathbb{R}$
- use $K(\cdot,\cdot)$ to construct $\mathcal{H}_K$
- use $K(\cdot,\cdot)$ + input locations + measurements to construct $\widehat{f}$

RKHS based learning - Bayesian interpretation

Hypotheses:

- f is a zero-mean Gaussian process
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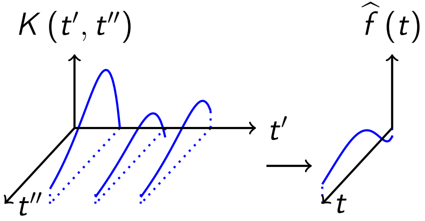

How RKHSs are built

Theorem (with the previous hypotheses:)

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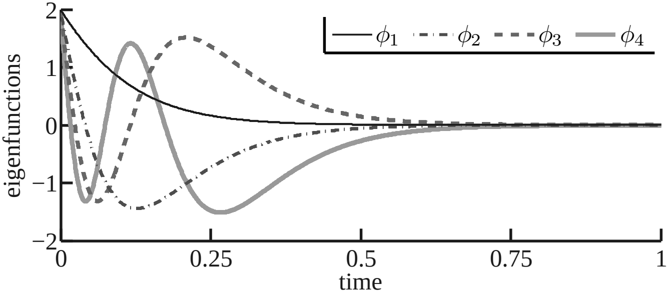
- $f_1 = \sum_{k=1}^{\infty} a_k \phi_k, f_2 = \sum_{k=1}^{\infty} b_k \phi_k \Rightarrow \langle f_1, f_2 \rangle_{\mathcal{H}_K} = \sum_{k=1}^{+\infty} \frac{a_k \cdot b_k}{\lambda_k}$

Example of kernel and eigenfunctions

Kernel for BIBO stable linear time-invariant systems:

$$K(t, t'; \beta) = \begin{cases} \frac{\exp(-2\beta t)}{2}\left(\exp(-\beta t') - \frac{\exp(-\beta t)}{3}\right) & \text{if } t \leq t' \\ \frac{\exp(-2\beta t')}{2}\left(\exp(-\beta t) - \frac{\exp(-\beta t')}{3}\right) & \text{if } t \geq t' \end{cases}$$

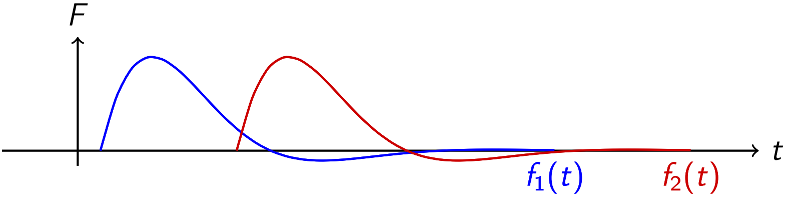


Classic Time Delay Estimation

notation: $f_1, f_2 =$ noisy and fixed delayed versions of the same f

classic TDE: maximization of $\mathcal{L}_2$'s inner product:

$$\tau_{\text{optimal}} = \arg\max_{\tau} \langle f_1(t), f_2(t-\tau) \rangle_{\mathcal{L}_2}$$



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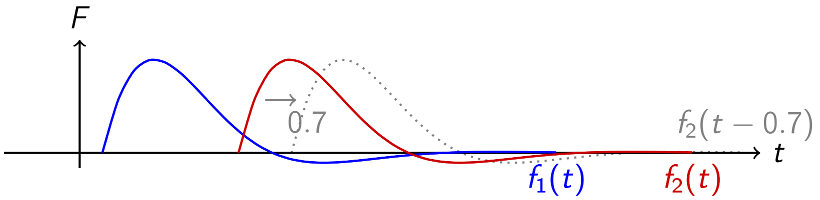

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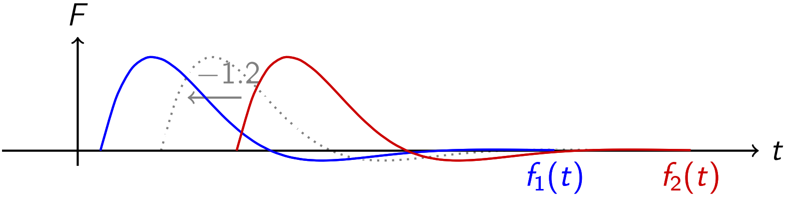

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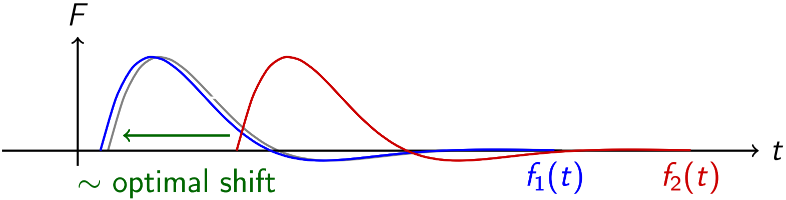

Time Delay Estimation in RKHS framework

RKHS based TDE: maximization of $\mathcal{H}_K$'s inner product:

$$\tau_{\text{optimal}} = \arg\max_{\tau} \langle f_1(t), f_2(t-\tau) \rangle_{\mathcal{H}_K} = \arg\max_{\tau} \sum_{k=1}^{+\infty} \frac{a_k \cdot b_k(\tau)}{\lambda_k}$$

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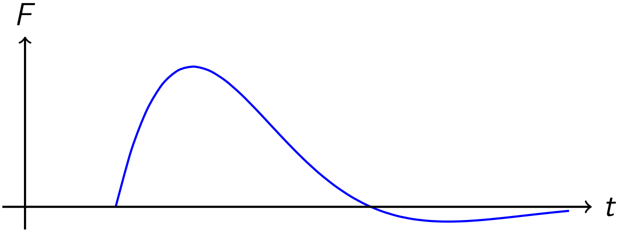
Note: requires $f_1(t)$ and $f_2(t-\tau)$ in the same reference system $\Rightarrow$ for each τ recompute the eigenfunctions weights $b_k(\tau)$



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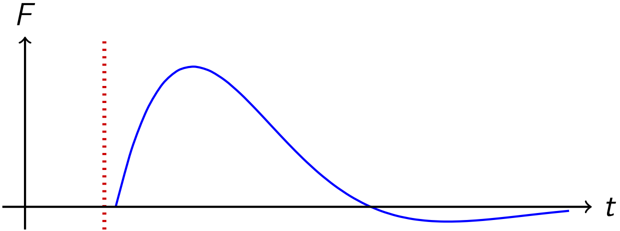

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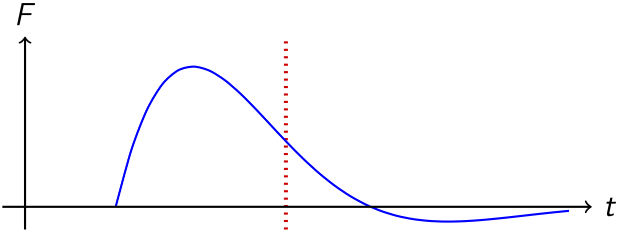

Simulations - temporal behavior of eigenfunctions weights