Decomposing C_2-equivariant spectra

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$RO(G)$-graded cohomology

G - finite group

- G-CW complex: attach orbit cells $G/K \times D^n$, for $K \leq G$
- Bredon cohomology $H^*_G(-)$
- Coefficient sytem $H^*_G(G/K) \longrightarrow H^*_G(G/J)$

V - real representation of G

$S^V = \widehat{V}$ one-point compactification

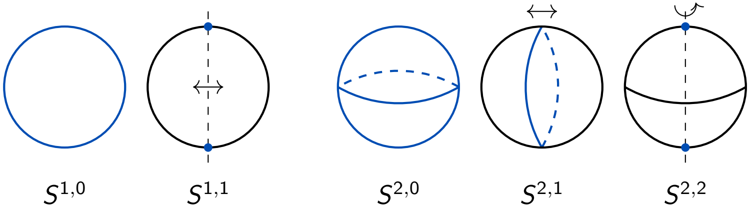
$$\Sigma^V X = S^V \wedge X$$

- For $\alpha \in RO(G)$ any virtual representation and M a Mackey functor, get $H^\alpha_G(-; M)$
- Suspension isomorphism $\tilde{H}^\alpha_G(X; M) \cong \tilde{H}^{\alpha+V}_G(\Sigma^V X; M)$

$RO(C_2)$-graded cohomology

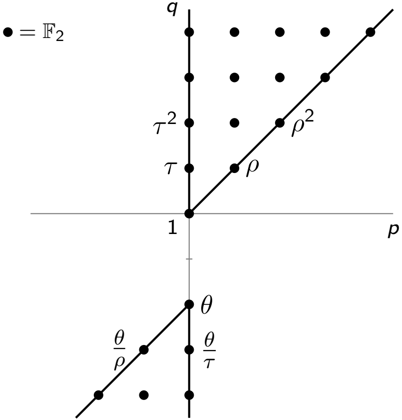
$$G = C_2$$

- Two orbits: $pt = C_2/C_2$ and $C_2 = C_2/e$
- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{triv})^{p-q} \oplus (\mathbb{R}_{sgn})^q$
- Representation spheres $S^V = S^{p,q}$

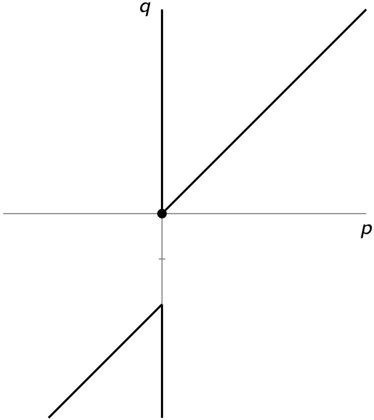


- Write $H^{\alpha}_G(X; \underline{\mathbb{F}_2}) = H^{p,q}(X; \underline{\mathbb{F}_2}) = H^{p,q}(X)$
- Represented by the Eilenberg–MacLane spectrum $H\underline{\mathbb{F}_2}$

Cohomology of a point



$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$

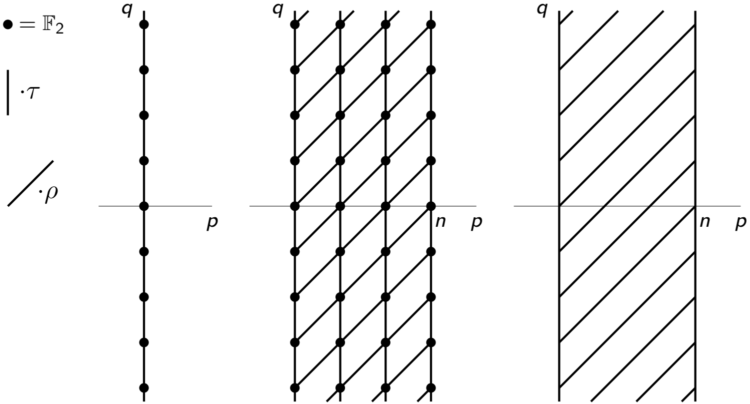


$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$

$$\tilde{H}^{*,*}(S^{p,q}) \cong \Sigma^{p,q}\mathbb{M}_2$$

Examples

For any X, $H^{*,*}(X;\underline{\mathbb{F}_2})$ is an $\mathbb{M}_2$-module via $X \to pt$



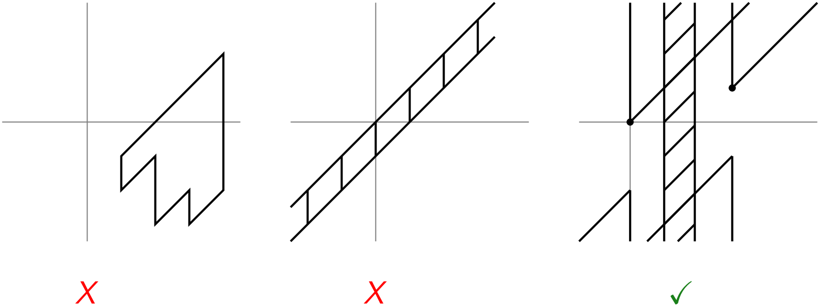
$H^{*,*}(C_2)$	$H^{*,*}(S^n_a)$	$\mathbb{A}_n = H^{*,*}(S^n_a)$
$\mathbb{F}_2[\tau, \tau^{-1}]$	$\mathbb{F}_2[\tau, \tau^{-1}, \rho]/(\rho^{n+1})$	$\mathbb{F}_2[\tau, \tau^{-1}, \rho]/(\rho^{n+1})$

Structure theorem

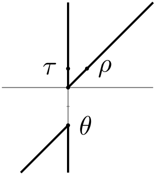
Theorem (M, 2018)

If X is a finite C_2-CW complex then $H^{*,*}(X)$ is a direct sum of shifted copies of $\mathbb{M}_2 = H^{*,*}(pt)$ and $\mathbb{A}_n = H^{*,*}(S^n_a)$.

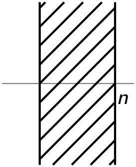
$$H^{*,*}(X; \underline{\mathbb{F}_2}) \cong (\oplus_i \Sigma^{p_i,q_i}\mathbb{M}_2) \oplus (\oplus_j \Sigma^{r_j,0}\mathbb{A}_{n_j})$$



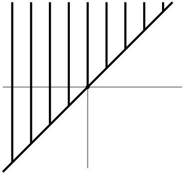
Ingredients for the proof



$\mathbb{M}_2 = H^{*,*}(pt)$



$\mathbb{A}_n = H^{*,*}(S^n_a)$



$\rho^{-1}\mathbb{M}_2$

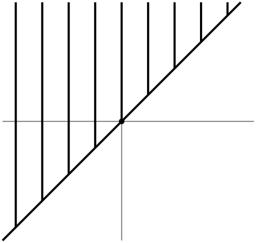
- If $x \in H^{*,*}(X)$ and $\theta x \neq 0$ then $\mathbb{M}_2\langle x\rangle \hookrightarrow H^{*,*}(X)$.
- $\mathbb{M}_2$ is self-injective
- $$0 \to \oplus_i \Sigma^{p_i,q_i}\mathbb{M}_2 \to H^{*,*}(X) \to Q \to 0$$
- For a finite C_2-CW complex
$$\rho^{-1}H^{*,*}(X) \cong H^*_{sing}(X^{C_2};\mathbb{F}_2) \otimes \rho^{-1}\mathbb{M}_2$$
- $\langle \tau, \theta, \rho \rangle = 1$

Localization

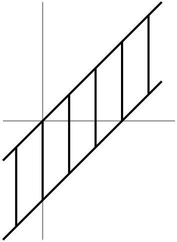
Recall: For a finite C_2-CW complex

$$\rho^{-1}H^{*,*}(X) \cong H^*_{sing}(X^{C_2}; \mathbb{F}_2) \otimes \rho^{-1}\mathbb{M}_2$$

Implies any class that survives ρ-localization is not τ-torsion.



$\rho^{-1}\mathbb{M}_2$



X

Lifting to spectra

Theorem (M. 2019)

Let Y be a based finite C_2-CW spectrum. Then

$$Y \wedge H\underline{\mathbb{F}_2} \simeq \left(\bigvee_i S^{p_i,q_i} \wedge H\underline{\mathbb{F}_2} \right) \vee \left(\bigvee_j S^{r_j,0} \wedge S^{n_j}_{a}{}_+ \wedge H\underline{\mathbb{F}_2} \right).$$

Theorem (Dugger–Hazel–M. in progress)

Let Z be a finite $H\underline{\mathbb{F}_2}$-module. Then

$$Z \simeq \left(\bigvee_i S^{p_i,q_i} \wedge H\underline{\mathbb{F}_2} \right) \vee \left(\bigvee_j S^{r_j,0} \wedge S^{n_j}_{a}{}_+ \wedge H\underline{\mathbb{F}_2} \right)$$

$$\vee \left(\bigvee_k S^{a_k,b_k} \wedge \mathrm{cof}(\tau^{n_k}) \right).$$

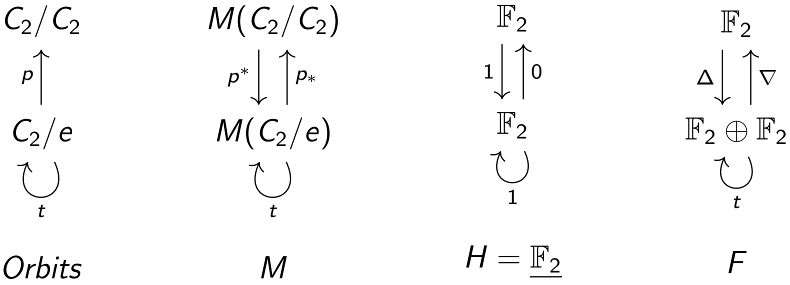
Proof Idea

Theorem (Schwede–Shipley 2003)

There is a Quillen equivalence

$$H\underline{\mathbb{F}_2} - \mathrm{Mod} \simeq Ch(\,\underline{\mathbb{F}_2}\,).$$

Mackey functors



Decomposing chain complexes

Any $C_\bullet \in Ch(\underline{\mathbb{F}_2})$ with finitely many F and H summands decomposes up to quasi-isomorphism as a direct sum of shifted copies of the following:

$$[F \xrightarrow{1+t} F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \xrightarrow{1+t} F] \qquad S^n_{a+}$$

$$[F \xrightarrow{1+t} F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \longrightarrow H] \qquad S^{p,q}$$

$$[H \longrightarrow F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \xrightarrow{1+t} F] \qquad S^{-p,-q}$$

$$[H \longrightarrow F \xrightarrow{1+t} \cdots \xrightarrow{1+t} F \longrightarrow H] \qquad \mathrm{cof}(\tau^n)$$

Decomposition up to isomorphism if we include

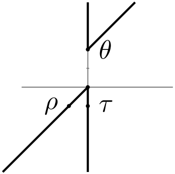
$$[F \xrightarrow{id} F], \qquad [H \xrightarrow{id} H]$$

Classification of finite modules

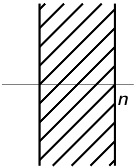
Theorem (Dugger–Hazel–M. in progress)

Let Z be a finite $H\underline{\mathbb{F}_2}$-module. Then

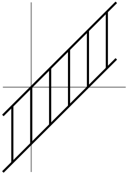
$$Z \simeq \left(\bigvee_i S^{p_i,q_i} \wedge H\underline{\mathbb{F}_2} \right) \vee \left(\bigvee_j S^{r_j,0} \wedge S^{n_j}_{a\,+} \wedge H\underline{\mathbb{F}_2} \right) \vee \left(\bigvee_k S^{a_k,b_k} \wedge \mathrm{cof}(\tau^{n_k}) \right).$$



$\mathbb{M}_2 = H_{*,*}(pt)$



$H_{*,*}(S^n_a)$



$\pi_{*,*}\,\mathrm{cof}(\tau^n)$

Thank you!