Some structure theorems for $RO(G)$-graded cohomology

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$RO(G)$-graded cohomology

G - finite group

- G-CW complex: attach orbit cells $G/K \times D^n$, for $K \leq G$
- Bredon cohomology $H^*_G(-)$
- coefficient system $H^*_G(G/K) \longrightarrow H^*_G(G/J)$

V - real representation of G

$S^V = \widehat{V}$ one-point compactification

$$\Sigma^V X = S^V \wedge X$$

- $RO(G)$ = Grothendieck group of finite-dimensional real orthogonal representations

$RO(G)$-graded cohomology

Theorem (Lewis, May, McClure, 1981)

The ordinary $\mathbb{Z}$-graded theory $H^*_G(-; M)$ with coefficients in a coefficient system M extends to an $RO(G)$-graded theory iff M extends to a Mackey functor.

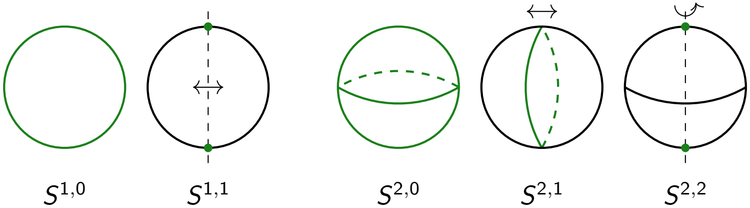
$$H^*_G(G/K) \overset{\dashleftarrow}{\longrightarrow} H^*_G(G/J)$$

- For $\alpha \in RO(G)$ any virtual representation and M a Mackey functor, get $H^\alpha_G(-; M)$
- Suspension isomorphism $\tilde{H}^\alpha_G(X; M) \cong \tilde{H}^{\alpha+V}_G(\Sigma^V X; M)$

$RO(C_2)$-graded cohomology

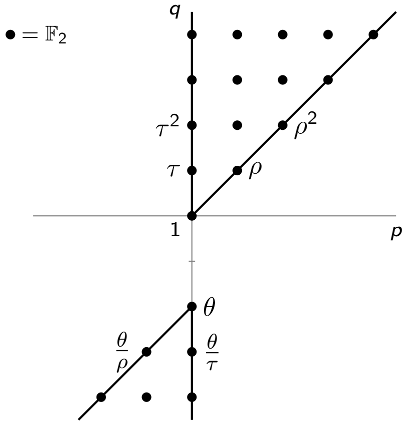
$$G = C_2$$

- Two orbits: $pt = C_2/C_2$ and $C_2 = C_2/e$
- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{triv})^{p-q} \oplus (\mathbb{R}_{sgn})^q$
- Representation spheres $S^V = S^{p,q}$

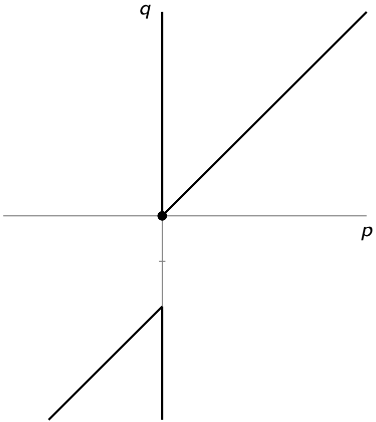


- Coefficients in the constant Mackey functor: $\underline{\mathbb{F}_2}$
- Write $H^V_G(X;\underline{\mathbb{F}_2}) = H^{p,q}(X;\underline{\mathbb{F}_2}) = H^{p,q}(X)$

Cohomology of a point



$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$

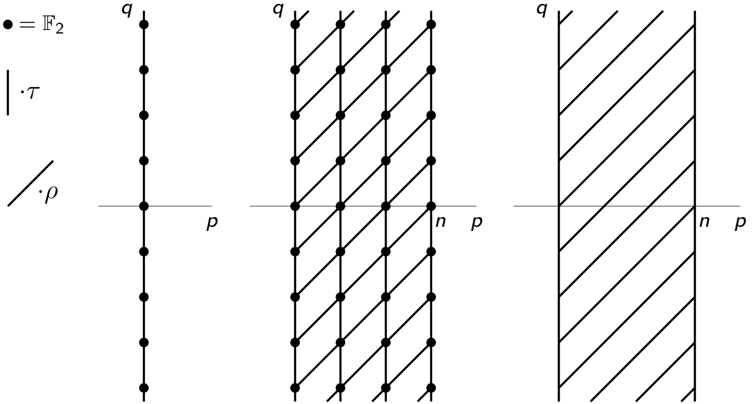


$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$

$$\tilde{H}^{*,*}(S^{p,q}) \cong \Sigma^{p,q}\mathbb{M}_2$$

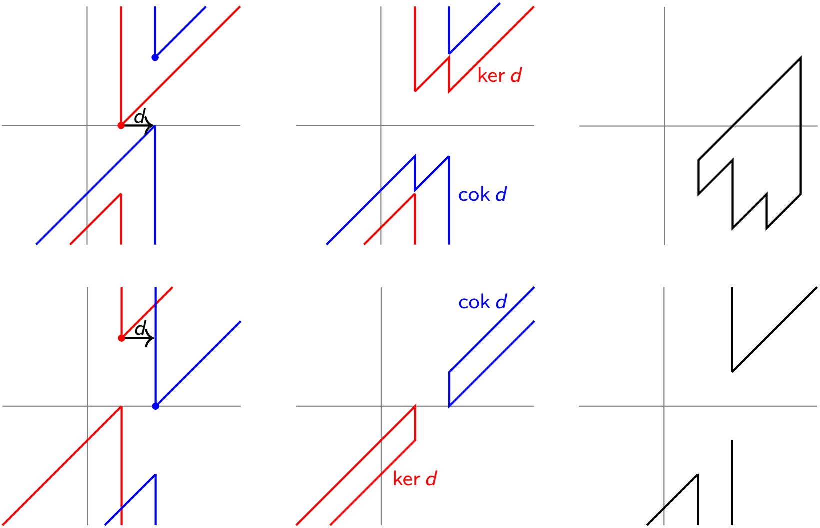
Examples

For any X, $H^{*,*}(X; \underline{\mathbb{F}_2})$ is an $\mathbb{M}_2$-module via $X \to pt$



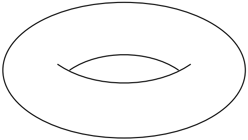
$H^{*,*}(C_2; \underline{\mathbb{F}_2})$	$H^{*,*}(S^n_a; \underline{\mathbb{F}_2})$	$H^{*,*}(S^n_a; \underline{\mathbb{F}_2})$
$\mathbb{F}_2[\tau, \tau^{-1}]$	$\mathbb{F}_2[\tau, \tau^{-1}, \rho]/(\rho^{n+1})$	$\mathbb{F}_2[\tau, \tau^{-1}, \rho]/(\rho^{n+1})$

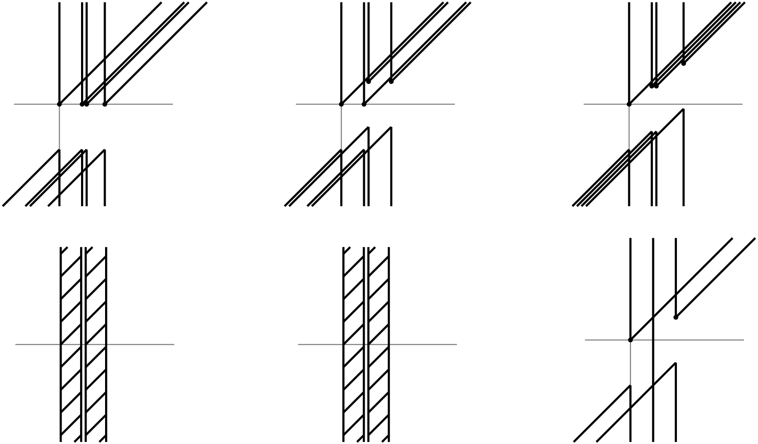
Some $\mathbb{M}_2$-modules



Torus examples

Cohomologies of C_2-actions on a torus with $\underline{\mathbb{F}_2}$-coefficients

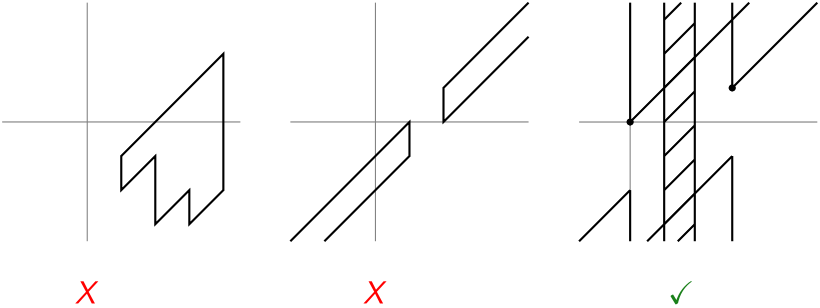




Structure theorem

Theorem (M, 2018)

If X is a finite C_2-CW complex then $H^{*,*}(X;\underline{\mathbb{F}_2})$ is a direct sum of shifted copies of $\mathbb{M}_2 = H^{*,*}(pt;\underline{\mathbb{F}_2})$ and $H^{*,*}(S^n_a;\underline{\mathbb{F}_2})$.



Structure theorem

Theorem (M, 2018)

Let X be a finite C_2-CW complex. There is a decomposition of $H^{*,*}(X;\underline{\mathbb{F}_2})$ as a module over $\mathbb{M}_2 = H^{*,*}(pt;\underline{\mathbb{F}_2})$

$$H^{*,*}(X;\underline{\mathbb{F}_2}) \cong (\oplus_i \Sigma^{p_i,q_i}\mathbb{M}_2) \oplus (\oplus_j \Sigma^{r_j,0} H^{*,*}(S_a^{n_j}))$$

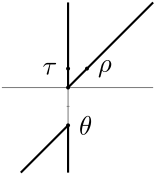
where $\mathbb{R}^{p_i,q_i}$ and $\mathbb{R}^{r_j,0}$ are elements of $RO(C_2)$ corresponding to actual representations.

Corollary

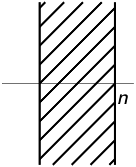
Let X be a finite C_2-CW spectrum. There is a weak equivalence of genuine C_2-spectra

$$X_+ \wedge H\underline{\mathbb{F}_2} \simeq \bigvee_i \left(S^{p_i,q_i} \wedge H\underline{\mathbb{F}_2}\right) \vee \bigvee_j \left(S^{r_j,0} \wedge {S_a^{n_j}}_+ \wedge H\underline{\mathbb{F}_2}\right)$$

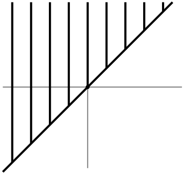
Ingredients for the proof



$\mathbb{M}_2 = H^{*,*}(pt)$



$H^{*,*}(S^n_a)$



$\rho^{-1}\mathbb{M}_2$

- If $x \in H^{*,*}(X)$ and $\theta x \neq 0$ then $\mathbb{M}_2\langle x \rangle \hookrightarrow H^{*,*}(X)$.
- $\mathbb{M}_2$ is self-injective
- $$0 \to \oplus_i \Sigma^{p_i,q_i}\mathbb{M}_2 \to H^{*,*}(X) \to Q \to 0$$
- For a finite C_2-CW complex
$$\rho^{-1}H^{*,*}(X) \cong H^*_{sing}(X^{C_2}; \mathbb{F}_2) \otimes \rho^{-1}\mathbb{M}_2$$
- $\langle \tau, \theta, \rho \rangle = 1$

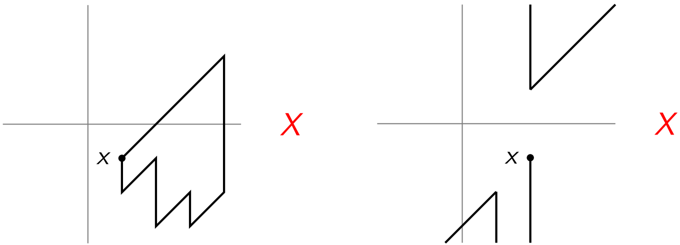
Toda bracket

Use $\langle \tau, \theta, \rho \rangle = 1$ to exclude many $\mathbb{M}_2$-modules.

Lemma

If $x \in H^{*,*}(X)$ and $\tau x = 0$ then $x = \rho y$ for some $y \in H^{*,*}(X)$.

Follows from $x = x \cdot \langle \tau, \theta, \rho \rangle = \langle x, \tau, \theta \rangle \cdot \rho$



Use several similar results to show Q is a $\mathbb{F}_2[\tau, \tau^{-1}, \rho]$-module.

Finiteness required

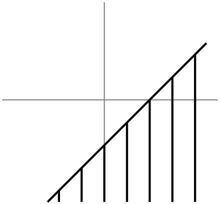
Would want to extend to locally finite C_2-CW complexes:

$$H^{*,*}(S_a^n) \cong \mathbb{F}_2[\tau, \tau^{-1}, \rho]/(\rho^{n+1})$$

$$H^{*,*}(S_a^\infty) \cong \mathbb{F}_2[\tau, \tau^{-1}, \rho]$$

Counterexample

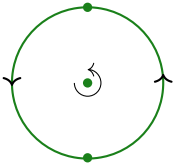
$$S^{\infty,\infty} = \mathrm{colim}(S^{0,0} \xrightarrow{\rho} S^{1,1} \xrightarrow{\rho} S^{2,2} \xrightarrow{\rho} \cdots \to S^{n,n} \to \cdots)$$

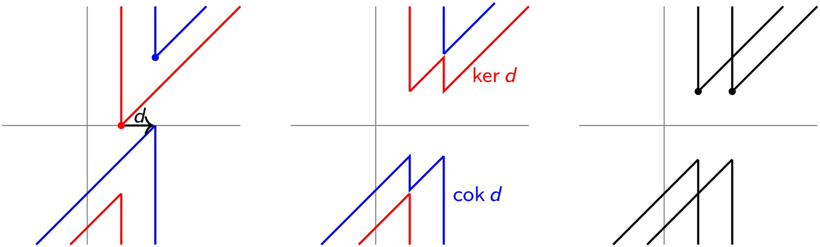


$H^{*,*}(S^{\infty,\infty})$

Application of theorem to $\mathbb{R}P^2_{tw}$

- Consider $\mathbb{R}P^2_{tw}$
- Cofiber sequence $S^{1,0} \hookrightarrow \mathbb{R}P^2_{tw} \to S^{2,2}$
- Long exact sequence in $\tilde{H}^{*,*}(-)$
- Extension problem
  $0 \to \operatorname{cok} d \to \tilde{H}^{*,*}(\mathbb{R}P^2_{tw}) \to \ker d \to 0$





Freeness Theorems

- G-CW complex: attach orbit cells $G/K \times D^n$
- $Rep(G)$-complex: attach representation cells $D(V)$

e.g. Grassmannian $Gr_k(\mathbb{R}^{p,q})$

Theorem (Kronholm, 2010)

If X is a finite $Rep(C_2)$-complex, $H^{*,*}(X)$ is a free $\mathbb{M}_2$-module.

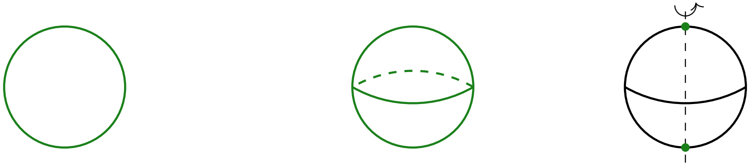
Theorem (Ferland, 1999)

If X is a finite $Rep(C_p)$-complex for p odd and X has only even dimensional cells, then $H^*_G(X)$ is a free $H^*_G(pt)$-module (with coefficients in $\mathcal{A}$ or $\underline{\mathbb{Z}}$).

$RO(C_3)$-graded cohomology

$$G = C_3$$

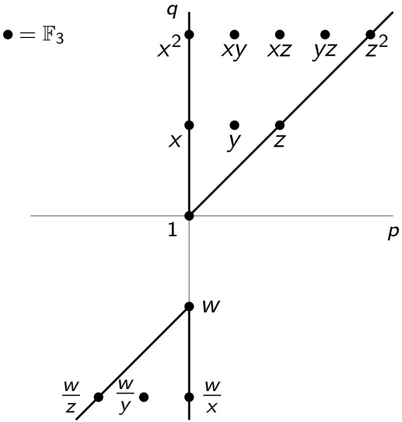
- Two orbits: $pt = C_3/C_3$ and $C_3 = C_3/e$
- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{triv})^{p-q} \oplus (\mathbb{R}^2_{rot})^{q/2}$
- Representation spheres $S^V = S^{p,q}$



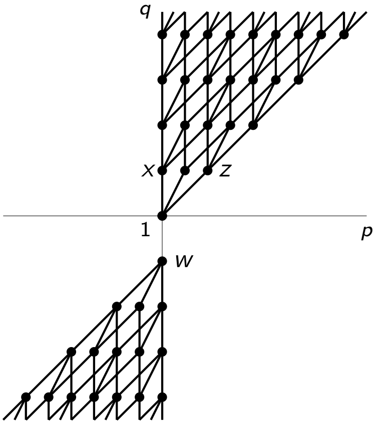
$S^{1,0}$ $S^{2,0}$ $S^{2,2}$

- Coefficients in the constant Mackey functor: $\underline{\mathbb{F}_3}$
- Write $H^V_G(X;\underline{\mathbb{F}_3}) = H^{p,q}(X;\underline{\mathbb{F}_3})$ for $q =$ even

Cohomology of a point



$$\mathbb{M}_3 = H^{*,*}(pt; \underline{\mathbb{F}_3})$$

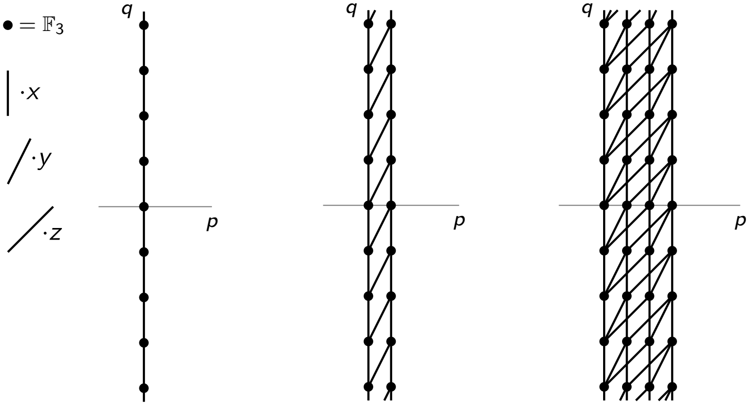


$$\mathbb{M}_3 = H^{*,*}(pt; \underline{\mathbb{F}_3})$$

$$\tilde{H}^{*,*}(S^{p,q}) \cong \Sigma^{p,q}\mathbb{M}_3$$

Examples

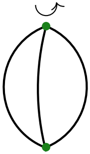
For any X, $H^{*,*}(X;\underline{\mathbb{F}_3})$ is an $\mathbb{M}_3$-module via $X \to pt$

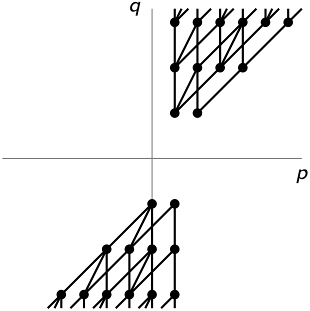


$H^{*,*}(C_3;\underline{\mathbb{F}_3})$ $\quad$ $H^{*,*}(S^1_{free};\underline{\mathbb{F}_3})$ $\quad$ $H^{*,*}(S^3_{free};\underline{\mathbb{F}_3})$

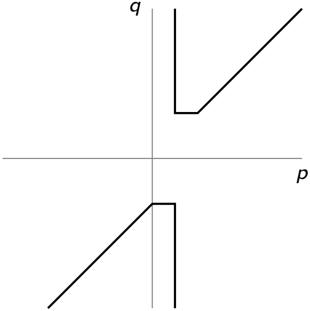
Egg-beater

Cofiber sequence $C_{3+} \to S^{0,0} \to EB$





$\widetilde{H}^{*,*}(EB; \underline{\mathbb{F}_3})$



$\widetilde{H}^{*,*}(EB; \underline{\mathbb{F}_3})$

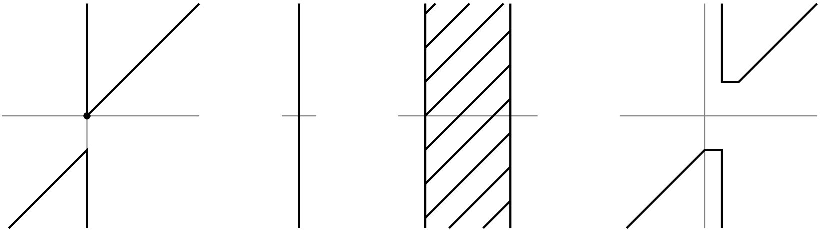
For $G = C_2$ this cofiber sequence is $C_{2+} \to S^{0,0} \to S^{1,1}$

Structure theorem

"Theorem" (M, in progress)

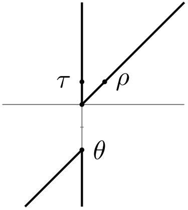
If X is a finite C_3-CW complex then $H^{*,*}(X;\underline{\mathbb{F}_3})$ is a direct sum of shifted copies of:

$$\mathbb{M}_3 = H^{*,*}(pt), \quad H^{*,*}(C_3), \quad H^{*,*}(S^{2n+1}_{free}), \quad \text{and } \widetilde{H}^{*,*}(EB).$$



C_2 structure theorem:

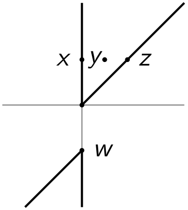
- θ detects $\mathbb{M}_2$
- $\mathbb{M}_2$ is self-injective
- ρ-localization: for finite C_2-CW complexes
  $\rho^{-1}H^{*,*}(X;\underline{\mathbb{F}_2}) \cong H^*_{sing}(X^{C_2};\mathbb{F}_2) \otimes \rho^{-1}\mathbb{M}_2$
- $\langle \tau, \theta, \rho \rangle = 1$



$\mathbb{M}_2$

C_3 structure theorem:

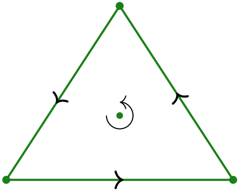
- w detects $\mathbb{M}_3$
- $\mathbb{M}_3$ is self-injective
- z-localization: for finite C_3-CW complexes
  $z^{-1}H^{*,*}(X;\underline{\mathbb{F}_3}) \cong H^*_{sing}(X^{C_3};\mathbb{F}_3) \otimes z^{-1}\mathbb{M}_3$
- $\langle x, \frac{w}{y}, z \rangle = 1$

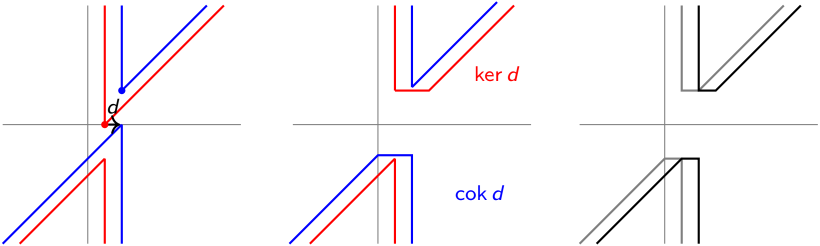


$\mathbb{M}_3$

An analogue of $\mathbb{R}P^2_{tw}$

- Consider Y
- Cofiber sequence $S^{1,0} \hookrightarrow Y \to S^{2,2}$
- Long exact sequence in $\tilde{H}^{*,*}(-)$
- Extension problem
  $0 \to \operatorname{cok} d \to \tilde{H}^{*,*}(Y) \to \ker d \to 0$





Thank you!