Spinning bagels and other symmetries of surfaces

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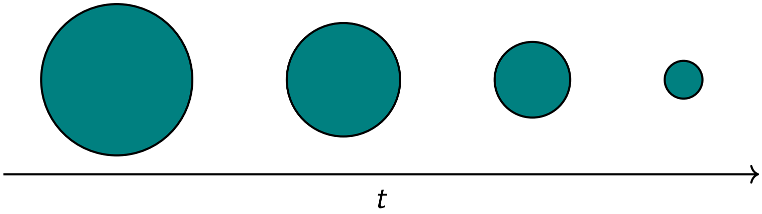
Homotopy theory

Consider shapes and spaces up to continuous deformation

Two spaces X and Y are **homeomorphic** $X \cong Y$ if there is a continuous bijective function $f\colon X \to Y$ with continuous inverse

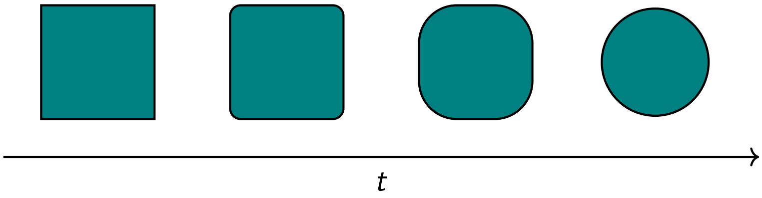
Example The unit disk D^2



In homotopy theory, we are not concerned with size, surface area, or volume

Homotopy theory

Example The unit square $[0,1] \times [0,1]$



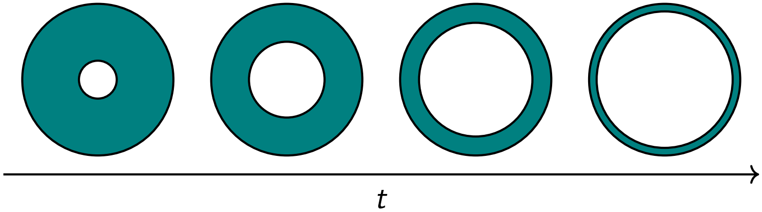
$$[0,1] \times [0,1] \cong D^2$$

Corners can always be rounded

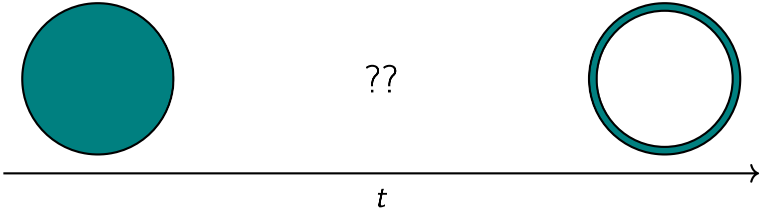
Question Does this mean every shape can be deformed into every other shape?

Homotopy theory

Example An annulus (washer)



A disk is not homeomorphic to an annulus



Invariants

Idea Build computational algebraic tools that are **invariant**

Meaning the algebraic computation for X gives the same result as the computation for Y whenever $X \cong Y$

Example For vector spaces, dimension (the number of elements in a basis) is an invariant

$$\dim \mathbb{R}^2 = 2 \quad \text{and} \quad \dim \mathbb{R}^3 = 3 \qquad \implies \qquad \mathbb{R}^2 \ncong \mathbb{R}^3$$

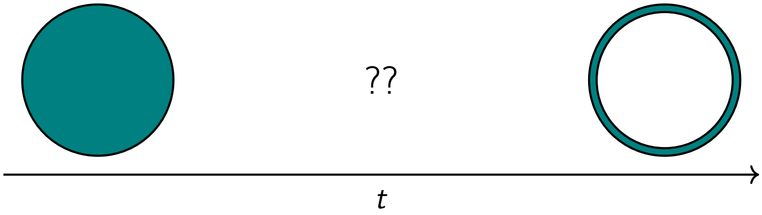
non-isomorphic vector spaces

In fact, dimension is a **complete invariant** for real vector spaces

$$\dim V = \dim W \quad \implies \quad V \cong W$$

Invariants in homotopy

Example Cohomology $H^n(X; \mathbb{R})$ is an invariant of spaces



Proof that $D^2 \not\cong A$:

$$H^n(D^2) \cong \begin{cases} \mathbb{R} & \text{if } n = 0 \\ 0 & \text{else} \end{cases} \qquad H^n(A) \cong \begin{cases} \mathbb{R} & \text{if } n = 0 \\ \mathbb{R} & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

Note Cohomology is not a complete invariant

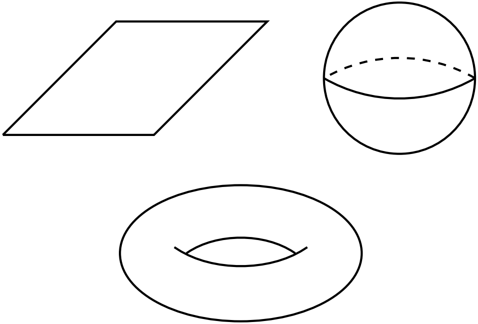
Surfaces

A **surface** is a 2-dimensional manifold, a space that locally looks like the plane $\mathbb{R}^2$

A tiny ant living on a surface has two degrees of freedom at each point

Examples

- A plane
- A sphere
- A torus



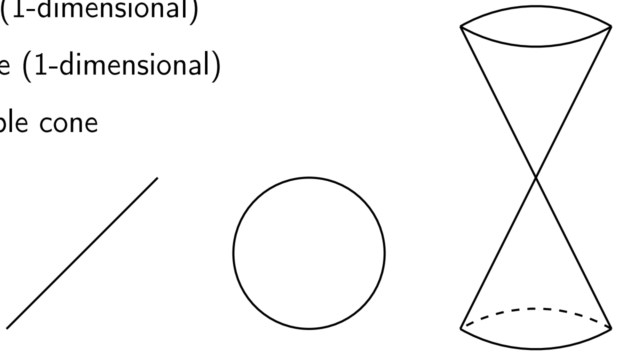
Surfaces

A **surface** is a 2-dimensional manifold, a space that locally looks like the plane $\mathbb{R}^2$

A tiny ant living on a surface has two degrees of freedom at each point

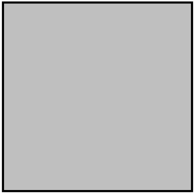
Non-examples

- A line (1-dimensional)
- A circle (1-dimensional)
- A double cone

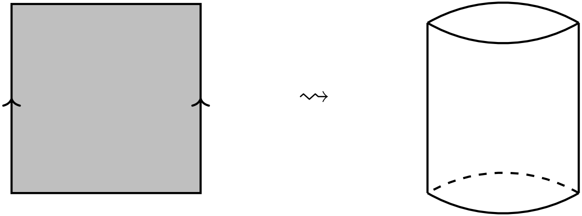


More surfaces

Examples of surfaces from identifying edges of polygons



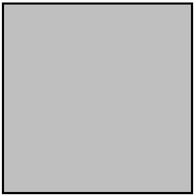
Stretchy/rubbery material



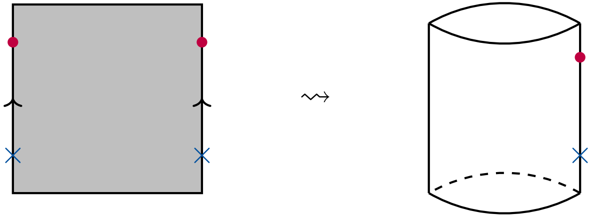
Cylinder

More surfaces

Examples of surfaces from identifying edges of polygons

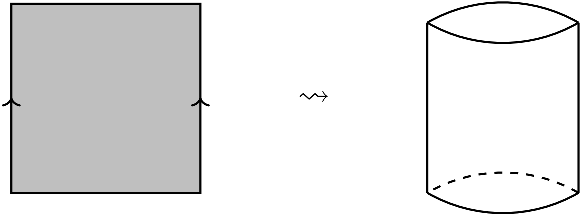


Stretchy/rubbery material

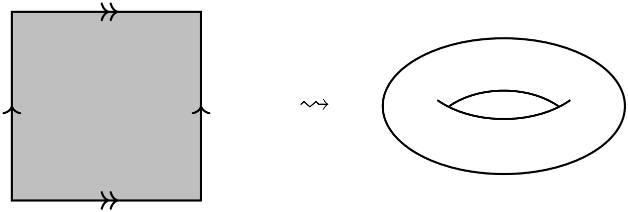


Cylinder

More surfaces

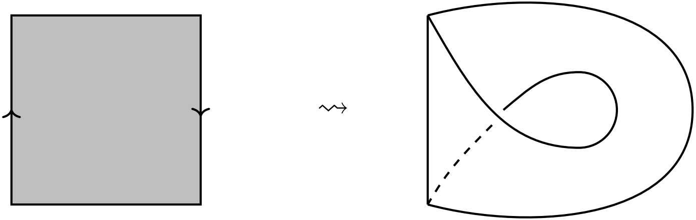


Cylinder

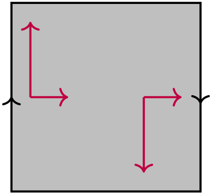


Torus

More surfaces

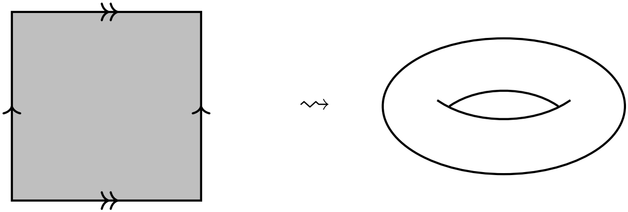


Möbius Band

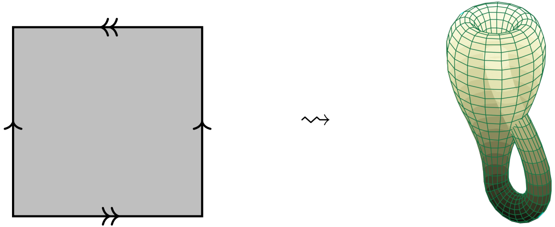


If you walk once around the Möbius band you'll come back flipped! This surface is **non-orientable**

More surfaces

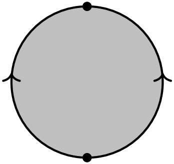


Torus

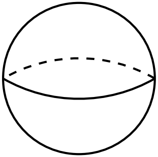


Klein bottle (non-orientable)

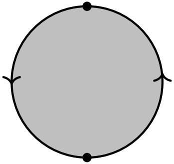
More surfaces



$\rightsquigarrow$



Sphere



$\rightsquigarrow$ $\mathbb{R}P^2$

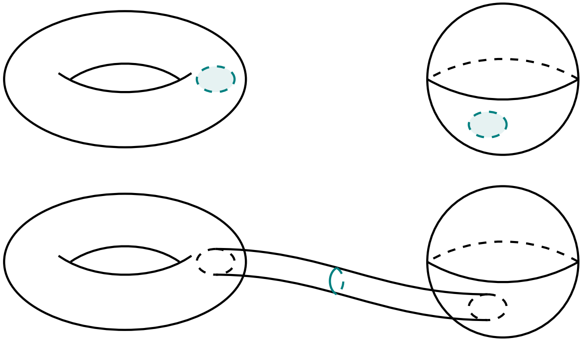
Real projective space (non-orientable)

Building new surfaces by gluing

Given two surfaces S_1 and S_2

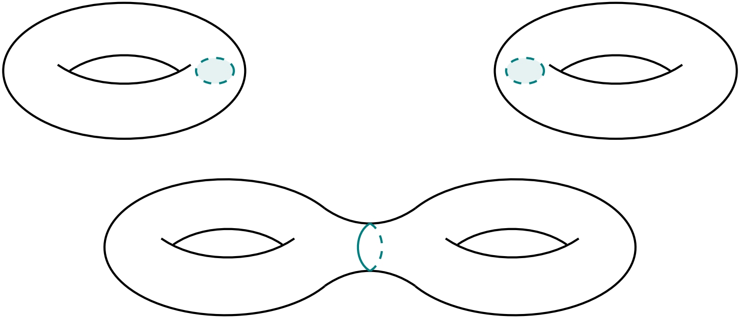
- cut a small disk out of each surface
- sew or glue the edges of the holes together

This forms a new surface $S_1 \# S_2$ called the **connected sum**



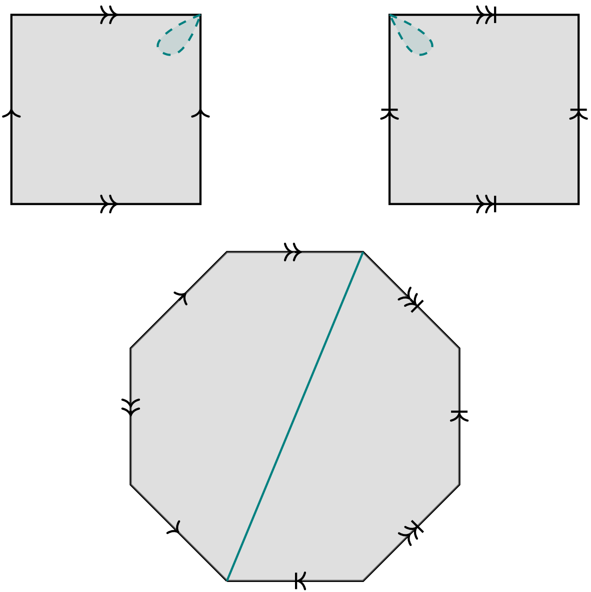
Connected sums

Example The genus two torus is a connected sum



$T_2 = T \# T$

Genus two torus via polygons



Video

Classification of surfaces

Classical result due to work of a number of people

- Early versions due to Möbius (1861) and Jordan (1866)
- More detailed proofs by von Dyck (1888) and Dehn–Heegard (1907)
- Rigorous proof by Brahana (1921)

Theorem

Up to homeomorphism, every compact surface (closed and bounded with no boundary) is

- *a sphere* S^2,
- *a connected sum of tori* $T_g = T \# T \# \cdots \# T$,
- *or a connected sum of real projective spaces* $N_r = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$.

Classification of surfaces

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Proof uses identifications of polygons and decomposes connected sum into fundamental building blocks.

Note The Klein bottle is $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.

Equivariant homotopy

An **involution** is a continuous function $f\colon X \to X$ such that

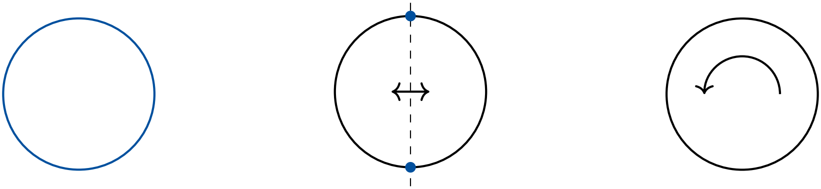
$$f(f(x)) = x \qquad \forall\, x \in X$$

i.e. $f^2 = id$

Also called a $\mathbb{Z}/2$-**action** on X

A point $p \in X$ is **fixed** if $f(p) = p$.

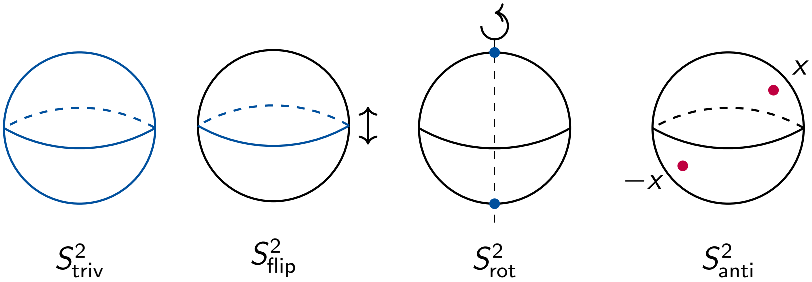
If no points are fixed, the action is **free**



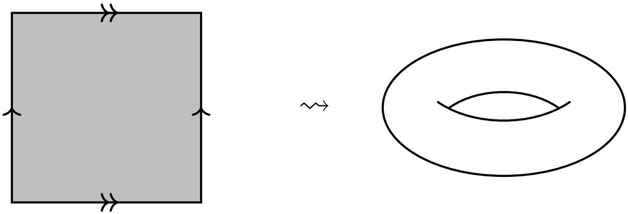
S^1_{triv} $\qquad$ S^1_{flip} $\qquad$ S^1_{rot}

Involutions on surfaces

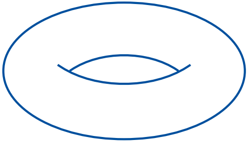
On a sphere



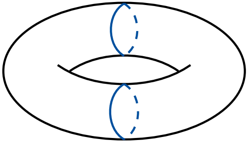
On a torus?



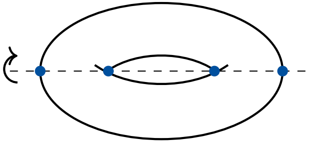
Involutions on a torus



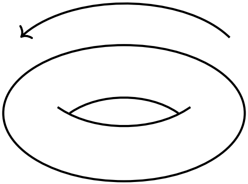
T_{triv}



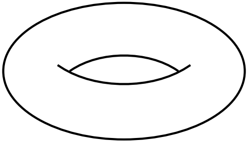
T_{flip}



T_{spit}



T_{rot}



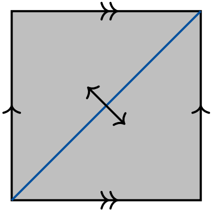
T_{anti}

Classification of involutions on surfaces

Theorem (Dugger 2019)

Up to isomorphism, there are exactly six involutions on a torus.

The last one



T_{swap}

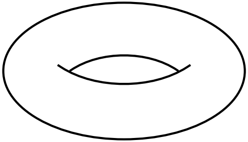
Theorem (Dugger 2019)

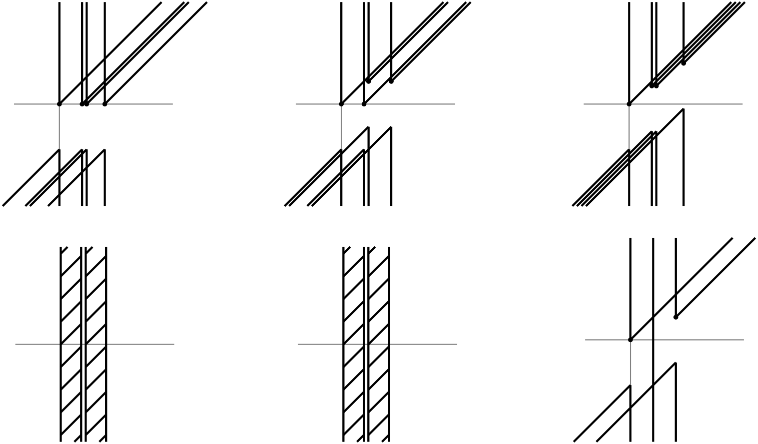
Up to isomorphism, there are exactly $4 + 2g$ *involutions on the genus* g *torus* T_g.

Even more: Dugger completely classified isomorphism classes of involutions on all compact surfaces.

Equivariant cohomology

Equivariant cohomologies of involutions on a torus

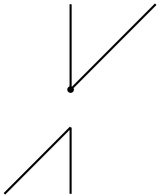


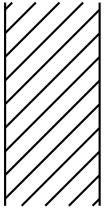


Equivariant cohomology

Theorem (M. 2019)

Equivariant cohomology of an involution involves only two types of pieces.





Theorem (Hazel 2021)

Computation of equivariant cohomology for all involutions on compact surfaces.

Proof uses Dugger's classification of equivariant surfaces and this structure theorem for equivariant cohomology.

Trivolutions

A **trivolution** is a continuous function $f : X \to X$ such that

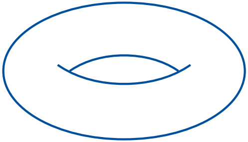
$$f(f(f(x))) = x \qquad \forall\, x \in X$$

i.e. $f^3 = id$

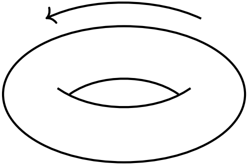
Also called a $\mathbb{Z}/3$-**action** on X

Theorem (Pohland 2023)

Complete classification of trivolutions on surfaces. There are three trivolutions on a torus T, up to isomorphism.



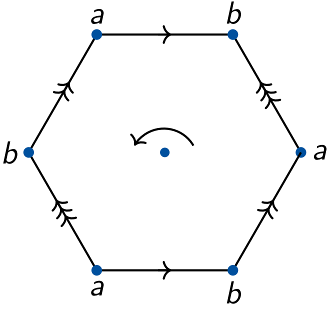
T_{triv}



T_{rot}

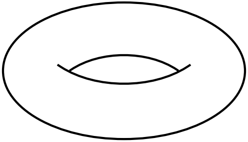
Trivolutions

The last trivolution on a torus



Three fixed points

Classical classification of surfaces $\implies$ this is a torus T



Thank you!