1. Let $\mathcal{S}_1$ be the surface $x^2 + y^2 + z^2 = 4$ for $z \geq 0$ and let $\mathcal{S}_2$ be the surface $z = 4 - x^2 - y^2$ for $z \geq 0$, with both surfaces oriented upward. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_{\mathcal{S}_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

2. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle x, y, xyz \rangle$ and let $\mathcal{S}$ be the part of the plane $2x + y + z = 2$ that lies in the first octant oriented upward.

   (a) Explicitly compute $\displaystyle\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

   (b) Verify that Stokes' theorem holds by explicitly computing $\displaystyle\int_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{r}$.

3. Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$ that is closed and bounded (e.g. a sphere) and let $\mathbf{F}$ be a vector field defined on all of $\mathbb{R}^3$ with continuous partial derivatives. Find $\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

4. In this problem you will show that Stokes' Theorem implies Green's Theorem. Let $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle$ be a vector field in the plane with continuous partial derivatives and define $\mathbf{G}$, a vector field on $\mathbb{R}^3$ by

$$\mathbf{G}(x, y, z) = \langle F_1(x, y), F_2(x, y), 0 \rangle .$$

Suppose $\mathcal{S}$ is a flat surface in the xy-plane with upward orientation and $\mathcal{C}$ is the boundary of $\mathcal{S}$ with positive orientation.

(a) Show that

$$\int_{\mathcal{C}} F_1(x, y)\, dx + F_2(x, y)\, dy = \int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r}.$$

(b) Show using Stokes' Theorem that

$$\int_{\mathcal{C}} F_1(x, y)\, dx + F_2(x, y)\, dy = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{G} \cdot d\mathbf{S}.$$

(c) Show the equation in part (b) is exactly Green's Theorem by explicitly writing the right-hand side.

5. Let $\mathcal{W}$ be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$. Let $\mathcal{S} = \partial\mathcal{W}$ be the surface formed by the boundary (i.e. the cylinder together with two planes). Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$.

   (a) Compute $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

   (b) Compute $\displaystyle\iiint_{\mathcal{W}} \operatorname{div} \mathbf{F} \, dV$.

   (c) Compare your answers in parts (a) and (b). What do you notice?