Math 32B - Fall 2019
Practice Exam 2

Full Name: ______________________________

UID: ______________________________

Circle the name of your TA and the day of your discussion:

Steven Gagniere   Jason Snyder   Ryan Wilkinson

Tuesday   Thursday

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Simplify your answers as much as possible.
- Include units with your answer where applicable.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Let $\mathbf{F}(x,y,z) = \langle e^y, xe^y, (z+1)e^z \rangle$ and let C be the curve parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ for $0 \le t \le 1$.

(a) Show that the vector field $\mathbf{F}$ is conservative.

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & xe^y & (z+1)e^z \end{vmatrix} = \langle 0-0, -(0-0), e^y - e^y \rangle = \vec{0}$$

$\vec{F}$ is defined on $\mathbb{R}^3$ and curl $\vec{F} = \vec{0}$ so $\vec{F}$ is conservative

(b) Find a potential function for $\mathbf{F}$.

want f s.t. $\nabla f = \vec{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$\frac{\partial f}{\partial x} = e^y$ so $f = xe^y + ??$

$\frac{\partial f}{\partial y} = xe^y$ ✓ $\quad \frac{\partial f}{\partial z} = (z+1)e^z$ X so $f = \int (z+1)e^z dz = ze^z + ??$

$\boxed{f(x,y,z) = xe^y + ze^z}$ $\quad \nabla f = \langle e^y, xe^y, ze^z + e^z \rangle$ ✓

(c) Use parts (a) and (b) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

By the Fundamental Thm for Line Integrals

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1,1,1) - f(0,0,0) = (e+e) - (0+0)$$

$$= \boxed{2e}$$

(d) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \mathbf{F}$? $\boxed{\text{No}}$

div(curl $\vec{G}$) = 0 so if there is such a $\vec{G}$, div($\vec{F}$) = 0

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle e^y, xe^y, (z+1)e^z \rangle$$

$$= 0 + xe^y + e^z + (z+1)e^z$$

$$= xe^y + (z+2)e^z \neq 0$$

So no such $\vec{G}$ exists.

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2. (10 points) Evaluate the line integral $\int_C (x^2+y^2+z^2)\, ds$ where C is the helix parameterized by $x = t$, $y = \cos 2t$, $z = \sin 2t$ for $0 \le t \le 2\pi$.

arc length

$\vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle, \quad 0 \le t \le 2\pi$ $\qquad f(x,y,z) = x^2+y^2+z^2$

$\vec{r}\,'(t) = \langle 1, -2\sin(2t), 2\cos(2t) \rangle$

$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$

$\int_C x^2+y^2+z^2\,ds = \int_0^{2\pi} (t^2 + \underbrace{\cos^2(2t)+\sin^2(2t)}_{1}) \sqrt{1+\underbrace{4\sin^2(2t)+4\cos^2(2t)}_{4}}\,dt$

$= \int_0^{2\pi} (t^2+1)\sqrt{5}\,dt = \sqrt{5}\int_0^{2\pi} t^2+1\,dt$

$= \sqrt{5}\left[\frac{t^3}{3}+t\right]_0^{2\pi} = \boxed{\sqrt{5}\left(\frac{8\pi^3}{3}+2\pi\right)}$

3. (10 points) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle 2y + z, x - 3z, x + y \rangle$ and C is the line segment from $(1, 0, 2)$ to $(2, 3, -1)$.

$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y+z & x-3z & x+y \end{vmatrix} = \langle 1+3, -(1-1), 1-2 \rangle \ne \vec{0}$

so $\vec{F}$ not conservative

evaluate directly

$\vec{r}(t) = \langle 1+t, 3t, 2-3t \rangle, \quad 0 \le t \le 1$

$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$= \int_0^1 \langle 2(3t)+2-3t,\ 1+t-3(2-3t),\ 1+t+3t \rangle \cdot \langle 1, 3, -3 \rangle\,dt$

$= \int_0^1 \langle 3t+2,\ 10t-5,\ 4t+1 \rangle \cdot \langle 1, 3, -3 \rangle\,dt$

$= \int_0^1 3t+2+3(10t-5)-3(4t+1)\,dt = \int_0^1 21t-16\,dt$

$= \left[\frac{21t^2}{2} - 16t\right]_0^1 = \frac{21}{2} - 16 = \boxed{\frac{-11}{2}}$

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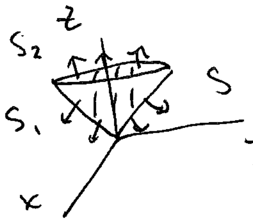
4. (20 points) The velocity field of a fluid is given by $\mathbf{F}(x, y, z) = \langle x, y, z^4 \rangle$. Find the flux of the fluid across the closed surface given by $z^2 = x^2 + y^2$ for $0 \le z \le 1$ and $x^2 + y^2 \le 1$ at $z = 1$ with positive orientation.

cone disk

outward normal

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

S_1 = cone : $\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$ for $D: u^2+v^2 \le 1$

$$\vec{r}_u = \langle 1, 0, \frac{u}{\sqrt{u^2+v^2}} \rangle$$

$$\vec{r}_v = \langle 0, 1, \frac{v}{\sqrt{u^2+v^2}} \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = \langle -\frac{u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \rangle$$

upward, not downward

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = -\iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$$= -\iint_D \langle u, v, (u^2+v^2)^2 \rangle \cdot \langle \frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \rangle\, dA$$

$$= -\iint_D \frac{-u^2}{\sqrt{u^2+v^2}} + \frac{-v^2}{\sqrt{u^2+v^2}} + (u^2+v^2)^2\, dA = -\iint_D -\frac{u^2+v^2}{\sqrt{u^2+v^2}} + (u^2+v^2)^2\, dA$$

$u = r\cos\theta$
$u = r\sin\theta$

$$= -\int_0^{2\pi} \int_0^1 \left(\frac{-r^2}{r} + (r^2)^2 \right) r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r^2 - r^5\, dr\, d\theta$$

$$= [\theta]_0^{2\pi} \left(\frac{r^3}{3} - \frac{r^6}{6} \right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{6} \right) = \frac{\pi}{3}$$

S_2 = disk $\quad \vec{n} = \langle 0, 0, 1 \rangle \quad$ for $D: x^2+y^2 \le 1$

$z = 1$ on all of S_2

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot \vec{n}\, dS = \iint_{S_2} \langle x, y, z^4 \rangle \cdot \langle 0, 0, 1 \rangle\, dS = \iint_{S_2} z^4\, dS$$

$$= \iint_{S_2} 1\, dS = A(D) = \pi \quad . \quad \text{So } \iint_S \vec{F} \cdot d\vec{S} = \frac{\pi}{3} + \pi = \boxed{\frac{4\pi}{3}}$$

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5. (20 points) Let S be a portion of the helicoid parameterized by

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle \qquad \text{for} \quad 0 \le u \le 1, \quad 0 \le v \le \pi.$$

D

(a) Compute $\iint_S 2y\, dS$.

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,dA$$

$$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$$
$$\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \langle \sin v, -\cos v, u\cos^2 v + u\sin^2 v\rangle$$

$$= \langle \sin v, -\cos v, u\rangle$$

$$\iint_S 2y\,dS = \iint_D 2u\sin v\sqrt{\underbrace{\cancel{\sin^2 v + \cos^2 v}}_{1} + u^2}\,dA$$

$$= \int_0^\pi \sin v\,dv \int_0^1 2u\sqrt{1+u^2}\,du = \left(-\cos v\right]_0^\pi \left(\tfrac{2}{3}(1+u^2)^{3/2}\right]_0^1$$

$$= (1+1)\left(\tfrac{2}{3}(2^{3/2}) - \tfrac{2}{3}\right) = \boxed{\tfrac{4}{3}\left(\sqrt{8}-1\right)}$$

(b) Let $\mathbf{F}(x,y,z) = \langle x, y, z\rangle$ and compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

$$= \iint_D \langle u\cos v, u\sin v, v\rangle \cdot \langle \sin v, -\cos v, u\rangle\,dA$$

$$= \iint_D \cancel{u\cos v\sin v} - \cancel{u\cos v\sin v} + uv\,dA = \iint_D uv\,dA$$

$$= \int_0^\pi v\,dv\int_0^1 u\,du = \left[\tfrac{v^2}{2}\right]_0^\pi \left[\tfrac{u^2}{2}\right]_0^1 = \tfrac{\pi^2}{2}\cdot\tfrac{1}{2} = \boxed{\tfrac{\pi^2}{4}}$$

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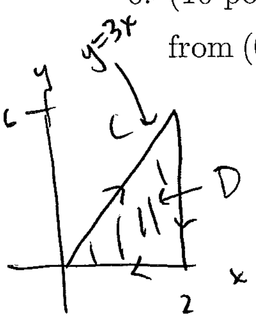
$\qquad\qquad\qquad\qquad F_1 \qquad F_2$

6. (10 points) Let $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x\rangle$ and let C be the path along the triangle from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ and back to $(0, 0)$. Use Green's Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.



C is negatively oriented so

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r} \underset{\text{Greens}}{=} -\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

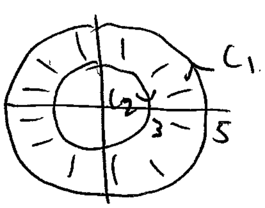
$$= -\iint_D 2x + \cancel{2y\cos x} - \cancel{2y\cos x}\, dA$$

$$= -\iint_D 2x\, dA = -\int_0^2 \int_0^{3x} 2x\, dy\, dx$$

$$= -\int_0^2 \left[2xy\right]_0^{3x} dx = -\int_0^2 6x^2\, dx = -\left[2x^3\right]_0^2$$

$$= \boxed{-16}$$

7. (10 points) Use Green's Theorem to find the area of the annulus $\mathcal{R}$ bounded by two circles centered at the origin, one with radius 3 and the other with radius 5. (You should be able to check your answer easily by computing the area another way).



$$C = C_1 \cup C_2$$

use $A = \iint_D 1\, dA = \overbrace{\int_C x\, dy = -\int_C y\, dx = \frac{1}{2}\int_C x\, dy - y\, dx}^{\text{Green's Thm}}$

$A = \pi R^2 - \pi r^2$
$= 25\pi - 9\pi$
$= 16\pi$

$C_1 : \vec{r}(t) = \langle 5\cos t, 5\sin t\rangle \quad 0 \le t \le 2\pi$

$C_2 : \vec{r}(t) = \langle 3\cos t, -3\sin t\rangle \quad 0 \le t \le 2\pi$ — clockwise

$$A = \iint_D 1\, dA = \int_C x\, dy = \int_{C_1} x\, dy + \int_{C_2} x\, dy$$

$dy = y' dt$

$$= \int_0^{2\pi} 5\cos t \cdot 5\cos t\, dt + \int_0^{2\pi} 3\cos t\,(-3\cos t)\, dt$$

$$= \int_0^{2\pi} 25\cos^2 t - 9\cos^2 t\, dt = \int_0^{2\pi} 16\cos^2 dt$$

$$= \int_0^{2\pi} 16 \cdot \frac{1}{2}(\cos(2t) + 1)\, dt = 8\left[\frac{1}{2}\sin(2t) + t\right]_0^{2\pi} = \boxed{16\pi}$$ ✓

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