LECTURE : BILINEAR FORMS

Today we will focus on extra structure we can put on a vector space that render certain constructions more natural.

Definition. A bilinear form $\langle -,-\rangle$ on V is a function

$$\langle -,-\rangle \colon V \times V \to \mathbb{F}$$

that is bilinear:

$$\langle a\vec{v} + b\vec{u}, \vec{w}\rangle = a\langle \vec{v}, \vec{w}\rangle + b\langle \vec{u}, \vec{w}\rangle \text{ and}$$

$$\langle \vec{v}, a\vec{u} + b\vec{w}\rangle = a\langle \vec{v}, \vec{u}\rangle + b\langle \vec{v}, \vec{w}\rangle.$$

It is important to note that $\langle -,-\rangle$ is a function on $V \times V$ as a set and *not* a linear map on the direct sum $V \oplus V$. This makes this definition a little unsettling: we consider structure on vector spaces that does not stay in the category of vector spaces.

We have a few types of special bilinear forms.

Definition.
1. $\langle -,-\rangle$ is symmetric if $\langle \vec{v}, \vec{w}\rangle = \langle \vec{w}, \vec{v}\rangle$ for all $\vec{v}, \vec{w} \in V$.
2. $\langle -,-\rangle$ is skew-symmetric if $\langle \vec{v}, \vec{w}\rangle = -\langle \vec{w}, \vec{v}\rangle$ for all $\vec{v}, \vec{w} \in V$.
3. $\langle -,-\rangle$ is alternating if $\langle \vec{v}, \vec{v}\rangle = 0$ for all $\vec{v} \in V$.

These are not unrelated concepts, and for the first time, specific properties of the field $\mathbb{F}$ appear.

Proposition. *Alternating always implies skew-symmetric. If char*$(\mathbb{F}) = 2$*, then skew-symmetric is the same as symmetric. If char*$(\mathbb{F}) \neq 2$*, then skew-symmetric implies alternating.*

Proof. If $\langle -,-\rangle$ is alternating, then we know

$$0 = \langle \vec{v} + \vec{w}, \vec{v} + \vec{w}\rangle = \langle \vec{v}, \vec{v}\rangle + \langle \vec{v}, \vec{w}\rangle + \langle \vec{w}, \vec{v}\rangle + \langle \vec{w}, \vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle + \langle \vec{w}, \vec{v}\rangle.$$

Thus $\langle -,-\rangle$ is skew-symmetric. If the characteristic is 2, then $1 = -1$, so symmetric is clearly the same as skew-symmetric. Assume that the characteristic is not 2 and $\langle -,-\rangle$ is skew-symmetric. Then

$$\langle \vec{v}, \vec{v}\rangle = -\langle \vec{v}, \vec{v}\rangle,$$

and so we conclude that $\langle \vec{v}, \vec{v}\rangle = 0$. □

We should think of $\langle -,-\rangle$ as a generalization of the dot product on $\mathbb{R}^n$. This provides a fantastic example.

Example.
1. The dot product on $\mathbb{R}^n$ is a bilinear form.
2. The generalization of the dot product on $\mathbb{F}^n$ is a bilinear form.
3. If $a_1, \dots, a_n$ are elements of $\mathbb{R}$ (or $\mathbb{F}$), then $\langle \vec{v}, \vec{w}\rangle = a_1 v_1 w_1 + \cdots + a_n v_n w_n$ is a bilinear form.

The last part is very helpful, and it helps us tie bilinear forms to matrices.

Proposition. *Choose a basis $\mathcal{B}$ $\vec{v}_1, \dots, \vec{v}_n$ for V, and let $[\vec{v}]_{\mathcal{B}} \in \mathbb{F}^n$ denote the column vector that represents $\vec{v}$ with respect to $\mathcal{B}$. Let B denote the matrix with*

$$B_{i,j} = \langle \vec{v}_i, \vec{v}_j \rangle.$$

Then

$$[\vec{v}]_{\mathcal{B}}^t B [\vec{w}]_{\mathcal{B}} = \langle \vec{v}, \vec{w} \rangle.$$

Proof. This is an exercise in bilinearity. Both sides are clearly bilinear (the left one is so since multiplication distributes over addition). By linearity in the first factor, to know $\langle \vec{v}, \vec{w} \rangle$ (or the column vector form), it suffices to know $\langle \vec{v}_i, \vec{w} \rangle$ for all i. Similarly, by linearity in the second factor, to know $\langle \vec{v}, \vec{w} \rangle$, it suffices to know $\langle \vec{v}, \vec{v}_j \rangle$ for all j. Thus we are reduced to showing

$$[\vec{v}_i]_{\mathcal{B}}^t B [\vec{v}_j]_{\mathcal{B}} = \langle \vec{v}_i, \vec{v}_j \rangle.$$

However, by definition, $[\vec{v}_i]_{\mathcal{B}} = \vec{e}_i$. Matrix multiplication tells us that

$$[\vec{v}_i]_{\mathcal{B}}^t B [\vec{v}_j]_{\mathcal{B}} = B_{i,j} = \langle \vec{v}_i, \vec{v}_j \rangle,$$

as required. □

Thus when we have a basis $\mathcal{B}$, we have a connection between matrices and bilinear forms.

Proposition. *A choice of basis $\mathcal{B}$ of V provides a $1-1$ correspondence between bilinear forms and the associated matrices.*

What happens then if we change basis? Let $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_n\}$ be another basis, and let $P = {}_{\mathcal{C}}P_{\mathcal{B}}$ be the change-of-basis matrix. Then we know

$$[\vec{v}]_{\mathcal{C}} = {}_{\mathcal{C}}P_{\mathcal{B}} [\vec{v}]_{\mathcal{B}}.$$

Let C be the matrix associated to $\langle -, - \rangle$ in the $\mathcal{C}$-basis, and similarly for B. Then we have

$$[\vec{v}]_{\mathcal{C}}^t C [\vec{w}]_{\mathcal{C}} = \langle \vec{v}, \vec{w} \rangle = [\vec{v}]_{\mathcal{B}}^t B [\vec{w}]_{\mathcal{B}}.$$

Combining these two equations, we see that for all $\vec{v}$ and $\vec{w}$, we have

$$[\vec{v}]_{\mathcal{C}}^t C [\vec{w}]_{\mathcal{C}} = \big(P[\vec{v}]_{\mathcal{B}}\big)^t C \big(P[\vec{w}]_{\mathcal{B}}\big) = [\vec{v}]_{\mathcal{B}}^t \big(P^t C P\big) [\vec{w}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}} B [\vec{w}_{\mathcal{B}}.$$

We've therefore shown the following.

Proposition. *If P is the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$, and if B (resp C) represents $\langle -, - \rangle$ in the $\mathcal{B}$-basis (resp $\mathcal{C}$-basis), then*

$$P^t C P = B.$$

Thus we'll say that two matrices are transpose equivalent if there is an invertible matrix P such that $P^t C P = B$. This is not standard.

Row vectors showed up in another context: dual spaces. This is one of the most powerful aspects of bilinear forms: they provide a *natural* way to connect a vector space and its dual.

Theorem. *Let $\vec{v} \in V$, and let $R_{\vec{v}}$ denote $\langle \vec{v}, - \rangle$. Then*

1. *For all $\vec{v}$, $R_{\vec{v}}$ is a linear transformation $V \to \mathbb{F}$.*
2. *The assignment $R \colon V \to V^*$ given by $\vec{v} \mapsto R_{\vec{v}}$ is linear.*

Proof. Both of these are given by linearity in their respective factors. The bilinear form $\langle -,-\rangle$ is linear in the second factor, so for all $\vec{v}$, $\langle \vec{v},-\rangle$ is a linear map. Thus we have the first part, and we learn that R is a function from V to V^*.

For the second, we need to know that for all $\vec{u}\in V$,

$$R_{a\vec{v}+b\vec{w}}(\vec{u}) = aR_{\vec{v}}(\vec{u}) + bR_{\vec{w}}(\vec{u}).$$

This will show that $V\to V^*$ is a linear map. So we check:

$$R_{a\vec{v}+b\vec{w}}(\vec{u}) = \langle a\vec{v}+b\vec{w},\vec{u}\rangle = a\langle \vec{v},\vec{u}\rangle + b\langle \vec{w},\vec{u}\rangle = aR_{\vec{v}}(\vec{u}) + bR_{\vec{w}}(\vec{u}),$$

since $\langle -,-\rangle$ is linear in the first factor. □

To fully understand the map R, we have to determine $\ker(R)$ and the image of R.

Definition. The radical of V, $Rad(V)$ is defined by

$$Rad(V) = \{\vec{v}\in V \mid \forall \vec{w}\in V,\quad \langle \vec{v},\vec{w}\rangle = 0\}.$$

Proposition. *We have*

$$\ker(R) = Rad(V).$$

Proof. This is immediate. The zero functional is the one that assigns the value 0 to all $\vec{w}\in V$. So $R(\vec{v})=0$ iff for all $\vec{w}\in V$, $\langle \vec{v},\vec{w}\rangle = 0$. This is the same thing as $\vec{v}\in Rad(V)$. □

To make this more useful, we tether this to the matrix form. If $\vec{v}\in V$ is in the radical, then we have

$$[\vec{v}]_{\mathcal{B}}^t B[\vec{w}]_{\mathcal{B}} = 0$$

for all $\vec{w}\in V$. This is the same condition as $[\vec{v}]_{\mathcal{B}}^t B = \vec{0}^t$, or $[\vec{v}]_{\mathcal{B}}$ is in the null-space of B^t.

Definition. A bilinear form $\langle -,-\rangle$ is singular if B is singular.

A bilinear form $\langle -,-\rangle$ is non-singular if B is invertible.

Corollary (Reisz Representation Theorem)**.** *If $\langle -,-\rangle$ is non-singular, then*

$$R\colon V\to V^*$$

is an injection. If V is finite dimensional, then R is an isomorphism.

Thus in the finite dimensional case, for any $f\in V^*$, we have a vector $\vec{v}_f\in V$ such that $f(\vec{w}) = \langle \vec{v}_f,\vec{w}\rangle$ for all $\vec{w}\in V$. This is the Reisz vector for f.

Now if S is a subspace of V, then S inherits a bilinear form by restriction. It can be the case that S is singular even if V is not. If V is non-singular and S is a singular subspace, then the Reisz vector for $f\in S^*$ need not be a vector in S. In fact, it could be just a vector in V. S is non-singular exactly when the Reisz vector is back in S.