Math 115A - Spring 2019
Practice Final Exam

Full Name: ______________________________

UID: ______________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	15	
2	10	
3	15	
4	15	
5	10	

Page	Points	Score
6	15	
7	10	
8	10	
Bonus		
Total:	100	

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You may use this page for scratch work. Work found on this page will not be graded unless clearly indicated here and in the exam.

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1. (15 points) Consider the vector space $V = P_2(\mathbb{R})$ with standard basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$\begin{aligned} T : V \to V, &\quad T(f) = f(1) + f(-1)x + f(0)x^2, \\ S : V \to V, &\quad S(ax^2 + bx + c) = cx^2 + bx + a. \end{aligned}$$

(a) Find $[T]_\beta^\beta$ and $[S]_\beta^\beta$. Then show that

$$[TS]_\beta^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Compute $[(TS)^{-1}]_\beta^\beta$.

(c) What is $(TS)^{-1}(x^2 + x + 1)$?

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2. (10 points) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

in $M_{3\times 3}(\mathbb{R})$.

(a) Compute the characteristic polynomial of A. Find all the eigenvalues of A and their algebraic multiplicities.

(b) Is A diagonalizable? If so, find a basis β of eigenvectors for A and write $[T_A]_\beta^\beta$.

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3. (15 points) Consider the vector space $V = \mathbb{R}^4$ with the standard inner product. Let S be

$$S = \{w_1 = (1,0,1,0), w_2 = (1,1,1,1), w_3 = (2,2,0,2)\}.$$

(a) Apply the Gram-Schmidt orthogonalization algorithm to S to compute an orthogonal basis β' of $\text{span}(S)$. You may use that S is linearly independent.

(b) Use your result from part (a) to compute an orthonormal basis β of $\text{span}(S)$.

(c) Let $x = (1,2,3,2) \in \text{span}(S)$. Compute the coordinate vector $[x]_\beta$.

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4. (15 points) Let V be a finite-dimensional vector space over $\mathbb{R}$ with an inner product so that $\langle x, y\rangle \in \mathbb{R}$ for $x, y \in V$.

   (a) Let $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that

   $$\langle x, y\rangle' = \lambda \langle x, y\rangle$$

   for $x, y \in V$ defines an inner product on V.

   (b) The inner product on V defines an induced norm. Show that

   $$\langle x, y\rangle = \frac{1}{2}\left(||x+y||^2 - ||x||^2 - ||y||^2\right)$$

   for all $x, y \in V$. Hence the inner product can be recovered from the norm.

   (c) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V. The *Gram matrix* $G \in M_{n\times n}(\mathbb{R})$ of the inner product $\langle -, -\rangle$ with respect to the basis β is defined by

   $$G_{ij} = \langle v_i, v_j\rangle .$$

   Show that G is invertible.

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5. (10 points) Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let $S, T : V \to V$ be two linear operators.

   (a) Show that $\operatorname{rank}(ST) \leq \min\{\operatorname{rank}(S), \operatorname{rank}(T)\}$.

   (b) Suppose $T^2 = T$. Show that $\ker(T) \cap \operatorname{im}(T) = \{0\}$.

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6. (15 points) True or False: Prove or disprove the following statements.

(a) An upper-triangular matrix is invertible if and only if all of its diagonal entries are nonzero.

(b) If $T : V \to V$ is an invertible linear operator then T is diagonalizable.

(c) If $T : V \to V$ is a diagonalizable linear operator then T is invertible.

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7. (10 points) Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$ and define $\langle -,- \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ via

$$\langle w, z \rangle = \frac{1}{2}\left(w\overline{z} + z\overline{w}\right)$$

for all $w, z \in \mathbb{C}$.

(a) Show that $\langle -,- \rangle$ defined above is an inner product on $\mathbb{C}$.

(b) Let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = \overline{z}$. Show that T is an isometry.

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8. (10 points) True or False: Prove or disprove the following statements.

Let V be a finite-dimensional inner product space over $\mathbb{F} = \mathbb{C}$. Let $T : V \to V$ be a a linear operator and T^* its adjoint.

(a) The linear operator $S = T + T^*$ is diagonalizable.

(b) If T is normal then $||Tv|| = ||T^*v||$ for all $v \in V$.

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