Some structure theorems for $RO(G)$-graded cohomology

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$RO(G)$-graded cohomology

- CW-complex $X \rightsquigarrow$ cohomology $H^*(X)$
- Suspension $\Sigma^m X = S^m \wedge X = (S^m \times X)/(S^m \vee X)$
- Suspension isomorphism $\tilde{H}^n(X) \cong \tilde{H}^{n+m}(\Sigma^m X)$

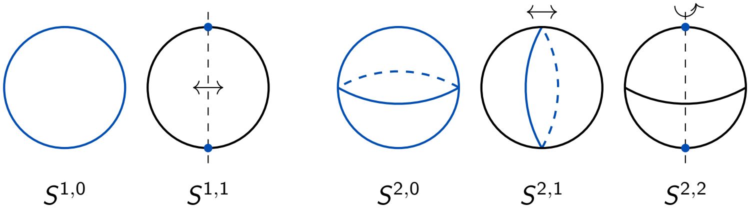
G - finite group

- G-CW complex
- V - real representation of G
- $S^V = \widehat{V}$ one-point compactification
- Suspension $\Sigma^V X = S^V \wedge X$
- Suspension isomorphism $\tilde{H}^{\alpha}_G(X) \cong \tilde{H}^{\alpha+V}_G(\Sigma^V X)$

$RO(C_2)$-graded cohomology

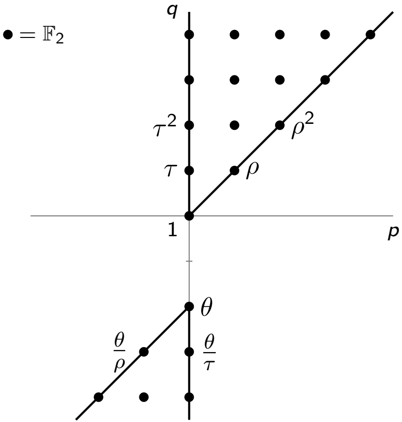
$$G = C_2$$

- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{triv})^{p-q} \oplus (\mathbb{R}_{sgn})^q$
- Representation spheres $S^V = S^{p,q}$

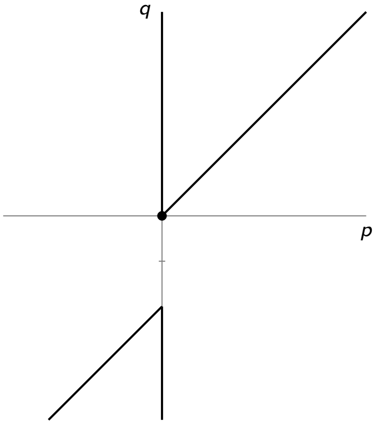


- Coefficients in the constant Mackey functor: $\underline{\mathbb{F}_2}$
- Write $H^{\alpha}_G(X; \underline{\mathbb{F}_2}) = H^{p,q}(X; \underline{\mathbb{F}_2}) = H^{p,q}(X)$

Cohomology of a point



$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$

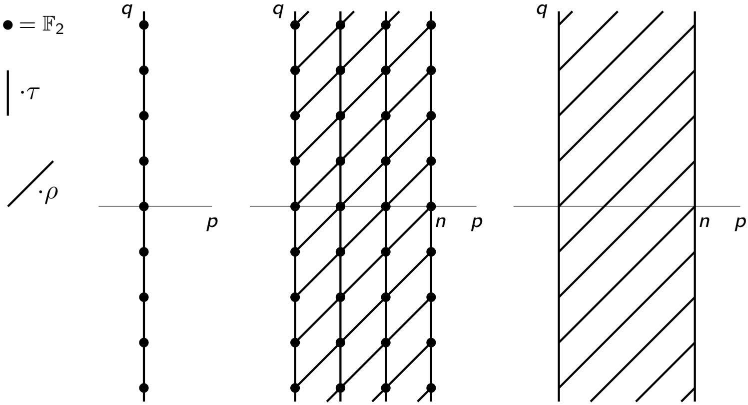


$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$

$$\tilde{H}^{*,*}(S^{p,q}) \cong \Sigma^{p,q}\mathbb{M}_2$$

Examples

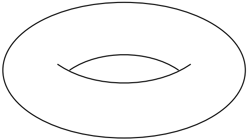
For any X, $H^{*,*}(X)$ is an $\mathbb{M}_2$-module via $X \to pt$

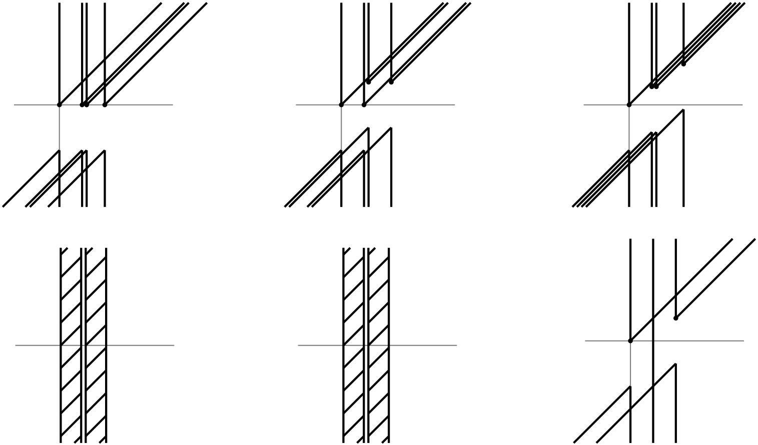


$H^{*,*}(C_2; \underline{\mathbb{F}_2})$ $H^{*,*}(S^n_a; \underline{\mathbb{F}_2})$ $H^{*,*}(S^n_a; \underline{\mathbb{F}_2})$

Torus examples

Cohomologies of C_2-actions on a torus with $\underline{\mathbb{F}_2}$-coefficients

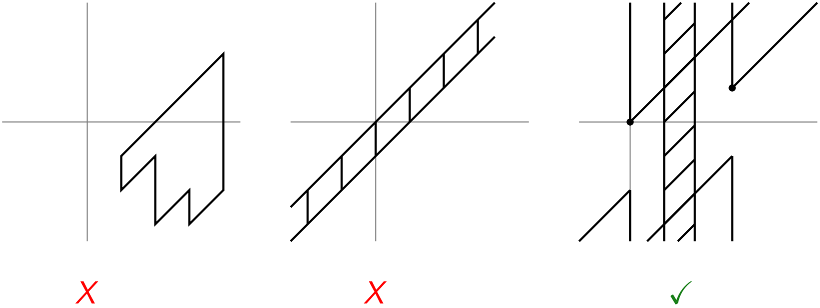




Structure theorem

Theorem (M, 2018)

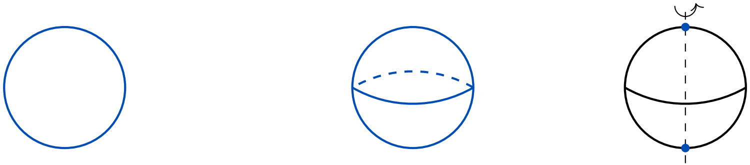
If X is a finite C_2-CW complex then $H^{*,*}(X;\underline{\mathbb{F}_2})$ is a direct sum of shifted copies of $\mathbb{M}_2 = H^{*,*}(pt;\underline{\mathbb{F}_2})$ and $H^{*,*}(S^n_a;\underline{\mathbb{F}_2})$.



$RO(C_3)$-graded cohomology

$$G = C_3$$

- Representations $V = \mathbb{R}^{p,q} = (\mathbb{R}_{triv})^{p-q} \oplus (\mathbb{R}^2_{rot})^{q/2}$
- Representation spheres $S^V = S^{p,q}$

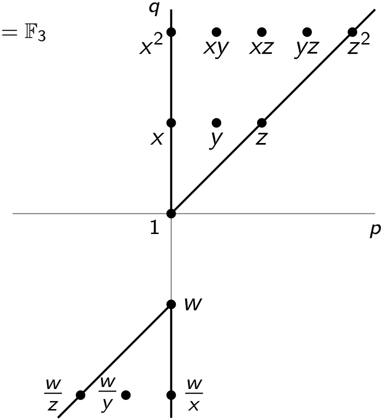


$S^{1,0}$ $S^{2,0}$ $S^{2,2}$

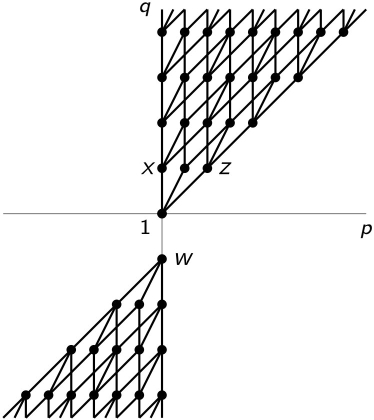
- Coefficients in the constant Mackey functor: $\underline{\mathbb{F}_3}$
- Write $H^{\alpha}_G(X; \underline{\mathbb{F}_3}) = H^{p,q}(X; \underline{\mathbb{F}_3})$ for $q =$ even

Cohomology of a point

$\bullet = \mathbb{F}_3$



$\mathbb{M}_3 = H^{*,*}(pt; \underline{\mathbb{F}_3})$

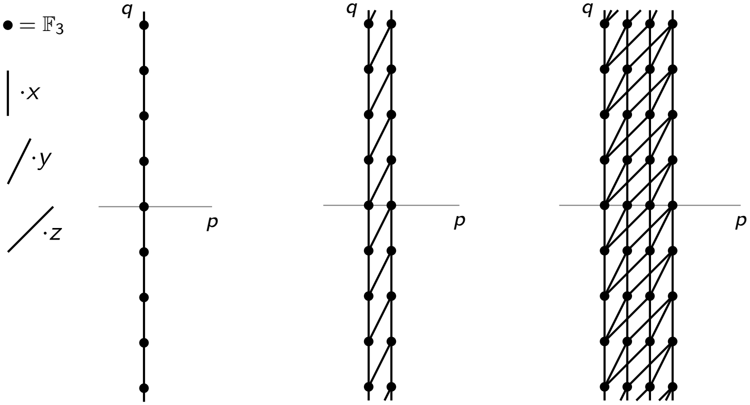


$\mathbb{M}_3 = H^{*,*}(pt; \underline{\mathbb{F}_3})$

$$\tilde{H}^{*,*}(S^{p,q}) \cong \Sigma^{p,q}\mathbb{M}_3$$

Examples

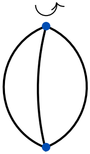
For any X, $H^{*,*}(X;\underline{\mathbb{F}_3})$ is an $\mathbb{M}_3$-module via $X \to pt$

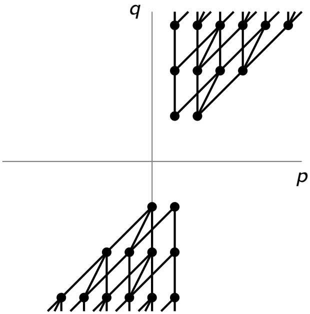


$H^{*,*}(C_3;\underline{\mathbb{F}_3})$ $\qquad$ $H^{*,*}(S^1_{free};\underline{\mathbb{F}_3})$ $\qquad$ $H^{*,*}(S^3_{free};\underline{\mathbb{F}_3})$

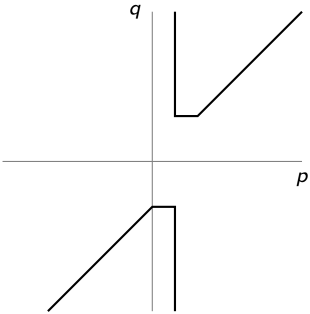
Egg-beater

Cofiber sequence $C_{3+} \to S^{0,0} \to EB$





$\widetilde{H}^{*,*}(EB; \underline{\mathbb{F}_3})$



$\widetilde{H}^{*,*}(EB; \underline{\mathbb{F}_3})$

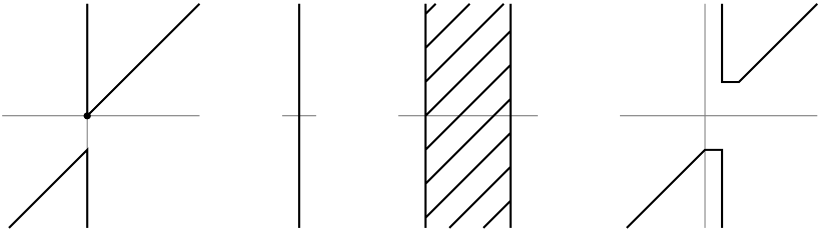
For $G = C_2$ this cofiber sequence is $C_{2+} \to S^{0,0} \to S^{1,1}$

Structure theorem

"Theorem" (M, in progress)

If X is a finite C_3-CW complex then $H^{*,*}(X;\underline{\mathbb{F}_3})$ is a direct sum of shifted copies of:

$$\mathbb{M}_3 = H^{*,*}(pt), \quad H^{*,*}(C_3), \quad H^{*,*}(S^{2n+1}_{free}), \quad \text{and } \widetilde{H}^{*,*}(EB).$$



Thank you!