TRACES OF VANISHING HÖLDER SPACES

KAUSHIK MOHANTA, CARLOS MUDARRA, AND TUOMAS OIKARI

ABSTRACT. For an arbitrary subset $E \subset \mathbb{R}^n$ and the scales $\Gamma = small$, large, far, we introduce and study the Γ -vanishing subspaces $\dot{\mathrm{VC}}^{\omega}_{\Gamma}(E) \subset \dot{C}^{0,\omega}(E)$ of the Hölder spaces. We find a single bounded linear extension operator that maps $\dot{\mathrm{VC}}^{\omega}_{\Gamma}(E) \to \dot{\mathrm{VC}}^{\omega}_{\Gamma}(\mathbb{R}^n)$ and works separately for each of the three vanishing scales. In fact, this follows as the zeroth order special case of a more general problem involving higher order derivatives. As a corollary, we obtain complete characterizations of approximability of Hölder functions $\dot{C}^{0,\omega}(E)$ by Lipschitz and boundedly supported functions, for an arbitrary subset $E \subset \mathbb{R}^n$.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. The vanishing Hölder spaces $\dot{\mathrm{VC}}^{\omega}_{\Gamma}(X,Y)$, for $\Gamma \in \{\text{small}, \text{large}, \text{far}\}$, for normed spaces X, Y and for many moduli of continuity ω , were recently defined and studied by the two last named authors [14]. In particular, these vanishing Hölder spaces were shown to often provide a complete description of those Hölder functions approximable by Lipschitz or smooth or boundedly supported functions. In this article we continue to study the vanishing spaces $\dot{\mathrm{VC}}^{\omega}_{\Gamma}(X,Y)$, when $X = \mathbb{R}^n$ but Y is allowed to be an arbitrary normed space.

We are interested in understanding when a given vanishing scale restricted to a proper subset $E \subset \mathbb{R}^n$, admits a bounded linear extension operator $L : \dot{\mathrm{VC}}^{\omega}_{\Gamma}(E,Y) \to \dot{\mathrm{VC}}^{\omega}_{\Gamma}(\mathbb{R}^n,Y)$ to the whole ambient space. Our answer is that for a completely arbitrary subset $E \subset \mathbb{R}^n$, a single linear bounded extension operator exists and works simultaneously for all the scales $\Gamma \in \{\text{small}, \text{large}, \text{far}\}$. We use the classical Whitney extension operator.

We do not only consider the possibility of extending the function itself, but also its *putative* derivatives. In particular, we provide a full description of exactly when a jet $\mathcal{A} \in \dot{J}^{m,\omega}(E,Y)$ is obtained by restricting a function $F \in \dot{C}_{\Gamma}^{m,\omega}(\mathbb{R}^n,Y)$. This amounts to showing that the Whitney extension operator maps the vanishing jet spaces $\dot{J}_{\Gamma}^{m,\omega}(E,Y)$ to $\dot{C}_{\Gamma}^{m,\omega}(\mathbb{R}^n,Y)$, i.e. preserves separately each of the vanishing scales. See Section 1.2 for the precise definitions and statements of these results.

One perspective to our results is as follows. The scale of Hölder spaces $\dot{C}^{0,\omega}$ can be seen to measure the smoothness of a function relative to any fixed modulus $\omega(t) > 0$, and when $\omega(t) = 0$ the space BMO of bounded mean oscillations is often substituted as the zeroth order smoothness measure. As opposed to real-valued Hölder functions $\dot{C}^{0,\omega}(E)$, with $E \subset \mathbb{R}^n$ arbitrary, being extendable to the whole ambient space $\dot{C}^{0,\omega}(\mathbb{R}^n)$, for example by the classical infimal convolution formula

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 $x \mapsto \inf_{y \in E} \{f(y) + M\omega(|x - y|)\}$, this is far from being true for functions in BMO. Indeed, given a connected open set (a domain) $\Omega \subset \mathbb{R}^n$, a classical theorem of Jones [12] tells us that a bounded linear extension operator $L : BMO(\Omega) \to BMO(\mathbb{R}^n)$ exists if and only if Ω is uniform. With BMO, geometry of the domain appears naturally. A recent result of Dalia and Gafni [4, Theorem 3.] (see also [5]) extends Jone's result by providing a single bounded extension operator on BMO(Ω) and also on several of its subspaces determined by approximability by nice functions, or by having appropriate vanishing mean oscillations. Our results are in the same spirit as Butaev's and Gafni's, but opposed to Jones'; we consider vanishing subspaces, but get rid of all geometry by reintroducing an order of smoothness $\omega(t) > 0$.

It follows from our results that a single linear and bounded extension operator exists for the intersection $\dot{\mathrm{VC}}^{\omega}$ of the three vanishing scales. In particular, for the Hölder moduli $\alpha(t) := t^{\alpha}$, for $\alpha \in (0, 1)$, [14, Theorem 1.13.] states that $\dot{\mathrm{VC}}^{\alpha}(\mathbb{R}^n) = \mathrm{VMO}^{\alpha}(\mathbb{R}^n)^1$, where the latter fractional vanishing mean oscillation space was defined in Guo et al. [11, Theorem 1.7.] as the space that characterizes the off-diagonal $L^p \to L^q$ (for $1) compactness of commutators of singular integrals. The study of these vanishing Hölder scales is thus also motivated by the needs of singular integrals and harmonic analysis on <math>\mathbb{R}^n$. Moreover, the classes $\dot{\mathrm{VC}}^{0,\alpha}_{\mathrm{small}}$ are also called the *little Hölder spaces*. These are fundamental in the study of Lipschitz algebras in metric spaces; we refer to the monographs by Weaver [17, Chapters 4 and 8], and for an exposition of these little Hölder spaces in the setting of Ahlfors regular sets we refer to D. Mitrea, I. Mitrea, and M. Mitrea [13, Chapter 3].

For the extension of $\dot{C}^{0,\omega}$ mappings between subsets of Hilbert spaces, preserving the $\dot{C}^{0,\omega}$ seminorms, see Grünbaum and Zarantonello [10], and the monograph [2] by Benyamini and Lindenstrauss. Concerning classical (and fundamental) results on $C^{m,\omega}$ extension of jets in \mathbb{R}^n , we refer to the work of Whitney [18] and Glaeser [9]. For Whitney-type extensions of jets generated by Sobolev functions in \mathbb{R}^n , see, for instance, the recent paper of Shvartsman [15]. For infinite dimensional results on $C^{1,\omega}$ extension of jets, see the recent paper [1] by Azagra and the second named author of the present paper. For extension results for functions (instead of jets) of order C^m or $C^{m,\omega}$, or for Sobolev functions, see for instance the papers by Brudnyi and Shvartsman [3], by Fefferman [6,7], by Fefferman, Israel and Luli [8].

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1.2. Basic definitions and main results.

Definition 1.1. Let $\omega : (0, \infty) \to (0, \infty)$ be a modulus of continuity, $E \subset \mathbb{R}^n$ an arbitrary set, $m \in \mathbb{N} \cup \{0\}$, and V a normed space. By an m-jet on E (to V) we simply understand a family of m + 1 functions $\{A_k : E \to \mathcal{L}^k(\mathbb{R}^n, V)\}_{k=0}^m$. For us $\mathcal{L}^k(\mathbb{R}^n, V)$ denotes the vector space of all symmetric k-linear mappings from \mathbb{R}^n to V, and the norm we are using on $\mathcal{L}^k(\mathbb{R}^n, V)$ is the one given by

$$||T|| := \sup\{||T(u_1, \dots, u_k)||_V : u_1, \dots, u_k \in \mathbb{S}^{n-1}\}.$$

¹We remark that notation-wise $VMO^{\alpha} = CMO^{\alpha}$ for Guo et al.

Then we define the trace jet space of $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ to E, denoted by $\dot{J}^{m,\omega}(E, V)$, as the vector space of all *m*-jets $\{A_k\}_{k=0}^m$ on E so that

$$\|\{A_k\}_{k=0}^m\|_{\mathbf{j}^{m,\omega}(E,V)} := \sup\left\{\frac{\|A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^j\|}{\omega(|x-y|)|x-y|^{m-k}} : x, y \in E, \ k = 0, \dots, m\right\} < \infty.$$
⁽¹⁾

It is clear that $\|\cdot\|_{\mathbf{j}^{m,\omega}(E,V)}$ defines a seminorm on $\mathbf{\dot{J}}^{m,\omega}(E,V)$, and it can be made an actual norm if we fix a point $x_0 \in E$ and define

$$|||\{A_k\}_{k=0}^m|||_{\mathbf{J}^{m,\omega}(E,V)} := ||\{A_k\}_{k=0}^m||_{\mathbf{J}^{m,\omega}(E,V)} + \max_{k=0,\dots,m} ||A_k(x_0)||.$$

Moreover, if V is a Banach space, this norm is complete, and $(J^{m,\omega}(E,V), ||| \cdot |||_{J^{m,\omega}(E,V)})$ becomes a Banach space.

Remark 1.2. In the case m = 0, the trace space $\dot{J}^{0,\omega}(E, V)$ coincides with the homogeneous Hölder space $\dot{C}^{0,\omega}(E, V)$, consisting of those $f: E \to V$ such that

$$||f||_{\dot{C}^{0,\omega}(E,V)} = \sup_{\substack{x \neq y \\ x,y \in E}} \frac{||f(x) - f(y)||}{\omega(|x - y|)} < \infty.$$

Let us now look at functions that are defined everywhere in \mathbb{R}^n .

Definition 1.3. Let $\omega : (0, \infty) \to (0, \infty)$ be a modulus of continuity, $m \in \mathbb{N} \cup \{0\}$, and V a normed space. Then $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ consists of those $F : \mathbb{R}^n \to V$ of class $C^m(\mathbb{R}^n, V)$ so that

$$\|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} = \sup_{\substack{x \neq y \\ x,y \in \mathbb{R}^n}} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} < \infty.$$

Here $D^m F : \mathbb{R}^n \to \mathcal{L}^m(\mathbb{R}^n, V)$ denotes the *m*th (total) derivative of *F*. Under basic assumptions on the modulus ω , a version of Whitney's extension theorem says that every *m*-jet $\{A_k\}_{k=0}^m \in \dot{J}^{m,\omega}(E, V)$ is the restriction of some $F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V)$ to *E*, in the sense that

$$D^k F(y) = A_k(y), \quad y \in E, \ k = 0, \dots, m.$$

We recall the construction of the Whitney extension operator and give appropriate references in Section 3.

In this article, we are insterested in subclasses of the space $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ given by the following vanishing conditions.

Definition 1.4. Let $\omega : (0, \infty) \to (0, \infty)$ be a modulus of continuity, $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, and V a normed space. We define the vanishing scales

$$\begin{split} \dot{\mathrm{VC}}_{\mathrm{small}}^{m,\omega}(\mathbb{R}^n, V) &:= \Big\{ F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V) : \lim_{\delta \to 0} \sup_{\substack{x \neq y \in E \\ |x-y| \leq \delta}} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} = 0 \Big\}, \\ \dot{\mathrm{VC}}_{\mathrm{large}}^{m,\omega}(\mathbb{R}^n, V) &:= \Big\{ F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V) : \lim_{\delta \to \infty} \sup_{\substack{x \neq y \in E \\ |x-y| \geq \delta}} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-z|)} = 0 \Big\}, \\ \dot{\mathrm{VC}}_{\mathrm{far}}^{m,\omega}(\mathbb{R}^n, V) &:= \Big\{ F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V) : \lim_{\delta \to \infty} \sup_{\min(|x|, |y|) > \delta} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} = 0 \Big\}. \end{split}$$

Finally we take the intersection of all the scales,

$$\dot{\mathrm{VC}}^{m,\omega}(\mathbb{R}^n,V) := \dot{\mathrm{VC}}^{m,\omega}_{\mathrm{small}}(\mathbb{R}^n,V) \cap \dot{\mathrm{VC}}^{m,\omega}_{\mathrm{far}}(\mathbb{R}^n,V) \cap \dot{\mathrm{VC}}^{m,\omega}_{\mathrm{large}}(\mathbb{R}^n,V).$$

Let us next consider the following questions.

Problem 1.5. Given an m-jet $\{A_k : E \to \mathcal{L}^k(\mathbb{R}^n, V)\}_{k=0}^m$ on E, what necessary and sufficient conditions guarantee the existence of $F \in \mathrm{VC}^{m,\omega}_{\Gamma}(\mathbb{R}^n, V)$ so that $D^k F$ restricted to E agrees with A_k , for each $k = 0, \ldots, m$? Does there exist an extension operator that works simultaneously for all of the three vanishing subspaces? Can such an extension be defined by means of a linear operator?

For an *m*-jet $\{A_k\}_{k=0}^m \in \dot{J}^{m,\omega}(E,V)$, let us denote

$$R\left(\{A_k\}_{k=0}^m, x, y\right) := \max_{0 \le k \le m} \frac{\|A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y) (x-y)^j\|}{|x-y|^{m-k}}, \quad x, y \in E.$$
 (2)

We show that full answers to the questions presented in Problem 1.5 can be obtained by assuming that the jet $\{A_k\}_{k=0}^m \in \dot{\mathrm{VJ}}^{m,\omega}(E,V)$ vanishes in an appropriate manner, as described in the next definition.

Definition 1.6. Let $\omega : (0,\infty) \to (0,\infty)$ be a modulus of continuity, $E \subset \mathbb{R}^n$ an arbitrary set, $m \in \mathbb{N} \cup \{0\}$, and V a normed space. We define the vanishing subspaces of jets

$$\dot{\mathrm{VJ}}_{\mathrm{small}}^{m,\omega}(E,V) := \left\{ \{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V) : \lim_{\delta \to 0} \sup_{\substack{x \neq y \in E \\ |x-y| \leq \delta}} \frac{R\left(\{A_k\}_{k=0}^m, x, y\right)}{\omega(|x-y|)} = 0 \right\},$$
$$\dot{\mathrm{VJ}}_{\mathrm{large}}^{m,\omega}(E,V) := \left\{ \{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V) : \lim_{\delta \to \infty} \sup_{\substack{x \neq y \in E \\ |x-y| \geq \delta}} \frac{R\left(\{A_k\}_{k=0}^m, x, y\right)}{\omega(|x-y|)} = 0 \right\},$$
$$\dot{\mathrm{VJ}}_{\mathrm{far}}^{m,\omega}(E,V) := \left\{ \{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V) : \lim_{\delta \to \infty} \sup_{\min(|x|,|y|) \geq \delta} \frac{R\left(\{A_k\}_{k=0}^m, x, y\right)}{\omega(|x-y|)} = 0 \right\}.$$

And we define the intersection of the three,

$$\dot{\mathrm{VJ}}^{m,\omega}(E,V) := \dot{\mathrm{VJ}}^{m,\omega}_{\mathrm{small}}(E,V) \cap \dot{\mathrm{VJ}}^{m,\omega}_{\mathrm{large}}(E,V) \cap \dot{\mathrm{VJ}}^{m,\omega}_{\mathrm{far}}(E,V).$$

Before stating our main results, let us finally fix the class of moduli that we consider. We always assume that ω is non-decreasing and satisfies

$$\lim_{t \to 0} \omega(t) = 0, \quad \text{and} \quad \frac{s}{\omega(s)} \le C_{\omega} \frac{t}{\omega(t)}, \quad \text{whenever} \quad 0 < s \le t < \infty, \tag{3}$$

where $C_{\omega} > 0$ is a fixed constant. We also assume the conditions

$$\lim_{t \to 0} \frac{t}{\omega(t)} = 0, \tag{4}$$

$$\lim_{t \to \infty} \omega(t) = \infty, \tag{5}$$

to deal with the small, and large and far scales, respectively. Now, the following is our main result. **Theorem 1.7.** Let ω be a modulus of continuity satisfying (3), $E \subset \mathbb{R}^n$ an arbitrary set, $m \in$ $\mathbb{N} \cup \{0\}$, and V a Banach space. For an m-jet $\{A_k\}_{k=0}^m \in \mathbf{j}^{m,\omega}(E,V)$, the following hold.

- Provided that (4) is satisfied, the jet $\{A_k\}_{k=0}^m$ admits an extension $F \in \dot{\mathrm{VC}}_{\mathrm{small}}^{m,\omega}(\mathbb{R}^n, V)$ if
- Provided that (1) is satisfied, and jet [Λ_k]_{k=0} duffied at extension Γ ⊂ VC_{small}(ℝ , V) if and only if {A_k}^m_{k=0} ∈ VJ^{m,ω}_{small}(E, V).
 Provided that (5) is satisfied, and Γ ∈ {large, far}, the jet {A_k}^m_{k=0} admits an extension F ∈ VC_Γ^{m,ω}(E, V) if and only if {A_k}^m_{k=0} ∈ VJ^{m,ω}_Γ(ℝⁿ, V).

• Provided that both (4), (5) hold, the jet $\{A_k\}_{k=0}^m$ admits an extension $F \in \dot{\mathrm{VC}}^{m,\omega}(\mathbb{R}^n, V)$ if and only if $\{A_k\}_{k=0}^m \in \dot{\mathrm{VC}}^{m,\omega}(E, V)$.

Moreover, when E is closed, the result holds when V is merely a normed space, and these extensions can be defined via the linear Whitney extension operator.

Specifying to the particular case m = 0, we obtain the following.

Corollary 1.8. Let ω be a modulus of continuity satisfying (3), $E \subset \mathbb{R}^n$ be an arbitrary set and V a Banach space.

- Provided that (4) is satisfied, $f \in \dot{C}^{0,\omega}(E,V)$ admits an extension $F \in \dot{VC}^{0,\omega}_{\text{small}}(\mathbb{R}^n,V)$ if
- Provided that (1) is satisfied, f ∈ C ⊂ (E,V) dufines an extension F ∈ VC_{small}(ℝⁿ,V) if and only if f ∈ VC^{0,ω}_{small}(E,V).
 Provided that (5) is satisfied, and Γ ∈ {large, far}, f ∈ C^{0,ω}(E,V) admits and extension F ∈ VC_Γ^{0,ω}(ℝⁿ,V) if and only if f ∈ VC_Γ^{0,ω}(E,V).
 Provided that both (4), (5) hold, f ∈ C^{0,ω}(E,V) admits an extension F ∈ VC^{0,ω}(ℝⁿ,V) if
- and only if $f \in \dot{\mathrm{VC}}^{0,\omega}(E,V)$.

Combining Corollary 1.8 with the approximations of globally defined functions [14, Corollary 3.14., we deduce the following.

Corollary 1.9. Let ω satisfy (3), (4) and (5), let $E \subset \mathbb{R}^n$ be closed and V be a Banach space.

• If $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(E,V)$ and $\varepsilon > 0$, there exists $G \in \mathrm{Lip}(\mathbb{R}^n,V) \cap C^{\infty}(\mathbb{R}^n;V) \cap \dot{C}^{0,\omega}(\mathbb{R}^n,V)$ such that

$$\|f - G_{|E}\|_{\dot{C}^{0,\omega}(E,V)} < \varepsilon, \qquad \|G\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,V)} \lesssim \|f\|_{\dot{C}^{0,\omega}(E,V)}.$$

• If
$$f \in \dot{\mathrm{VC}}^{0,\omega}(E,V)$$
 and $\varepsilon > 0$, there exists $G \in C^{\infty}_{c}(\mathbb{R}^{n};V) \cap \dot{C}^{0,\omega}(\mathbb{R}^{n},V)$ such that

$$\|f - G_{|E}\|_{\dot{C}^{0,\omega}(E,V)} < \varepsilon, \qquad \|G\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,V)} \lesssim \|f\|_{\dot{C}^{0,\omega}(E,V)}.$$

Basic notation. Above C_c^{∞} stands for smooth and compactly supported functions and Lip stands for Lipschitz continuous functions. We denote $A \leq B$, if $A \leq CB$ for some constant C > 0 depending only on parameters like integration exponents or the dimension of \mathbb{R}^n that are not important to track. Furthermore, we set $A \sim B$, if $A \lesssim B$ and $B \lesssim A$. Subscripts or variables on constants and quantifiers, such as, $C_a, C(a)$ and \leq_a , signify their dependence on those subscripts.

2. Necessity of Theorem 1.7

In this section, we give the proof of the "only if" parts of Theorem 1.7. We begin with the following elementary observation.

Remark 2.1. If $\mathcal{A} \in \dot{\mathbf{j}}^{m,\omega}(E,V)$, then for every r > 1, one has $\max_{0 \le k \le m} \sup_{z \in E, |z| \le r} \|A_k(z)\| \le C(r, m, \omega, \mathcal{A}, E) < \infty.$

Proof. Write, for each $z \in E \cap B(0, r)$ and $z_0 \in E$ be arbitrary,

$$\begin{aligned} |A_k(z)| &\leq \left\| A_k(z) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z_0) (z-z_0)^j \right\| + \left\| \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z_0) (z-z_0)^j \right\| \\ &\leq \|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)} |z-z_0|^{m-k} \omega(|z-z_0|) + C(\mathcal{A}, z_0, m) \max_{0 \leq j \leq m} |z-z_0|^j \\ &\lesssim (r+|z_0|)^{m-k} \left(\omega(r+|z_0|) + C(\mathcal{A}, z_0, m) \right). \end{aligned}$$

The following lemma will be used both in this and the next section.

Lemma 2.2. Let ω satisfy (5), $E \subset \mathbb{R}^n$ be arbitrary, $m \in \mathbb{N} \cup \{0\}$, and V a normed space. Then, we have

$$\dot{\mathrm{VJ}}_{\mathrm{far}}^{m,\omega}(E,V) = \left\{ \mathcal{A} \in \dot{\mathrm{J}}^{m,\omega}(E,V) : \lim_{\delta \to \infty} \sup_{\substack{x \neq y \in E \\ \max(|x|,|y|) \ge \delta}} \frac{R\left(\mathcal{A},x,y\right)}{\omega(|x-y|)} = 0 \right\}$$

Proof. Let $\mathcal{A} = \{A_k\}_{k=0}^m \in \dot{\mathrm{VJ}}^{m,\omega}(E,V)$ and $\varepsilon > 0$. Let $M = M(\varepsilon)$ be such that if $u, v \in E$ and |u|, |v| > M, then

$$R\left(\mathcal{A}, u, v\right) < \varepsilon \omega(|u - v|).$$

Now we consider arbitrary $x, y \in E$, let $K \gg M$ be a large parameter that we will determine later, and let us assume

$$\max(|x|, |y|) \ge K \gg M \ge \min(|x|, |y|). \tag{6}$$

Notice also that the condition $\dot{VJ}_{far}^{m,\omega}(E, V)$ is **not** vaguous if and only if E is unbounded. So for each M > 0, there is $z \in E \setminus B(0, M)$ (independent of x and y). Assume further that $K \gg |z|$ from now on. Bearing in mind that x, y satisfy (6), we start by checking the case $|x| \ge K$, and $|y| \le M$ first. Because of the properties of K, M, x, y, and z, it is clear that

$$|y-z| \le |x-y|$$
 and $C^{-1}|x-y| \le |x-z| \le C|x-y|;$ (7)

for some absolute constant C. Now, using that $x, z \in E \setminus B(0, M)$, the relations (7) and Remark 2.1, we have, for every $0 \le k \le m$,

$$\begin{aligned} \left\| A_{k}(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^{j} \right\| \\ &\leq \left\| A_{k}(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z)(x-z)^{j} \right\| + \left\| \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z) \left((y-z)^{j} - (x-z)^{j} \right) \right\| \\ &\leq \varepsilon |x-z|^{m-k} \,\omega(|x-z|) + C(m) \max_{0 \leq j \leq m-k} \sup_{B(0,|z|)} \| A_{k+j} \| \left(|y-z|^{j} + |x-z|^{j} \right), \\ &\lesssim_{C_{\omega},m,E, \|\mathcal{A}\|_{\dot{V}J}^{m,\omega}(E,V)} \varepsilon |x-y|^{m-k} \omega(|x-y|) + C(|z|,\mathcal{A}) |x-y|^{m-k}. \end{aligned}$$
(8)

Since K was so that $K \gg |z|$, we assume that K is large to guarantee $C(|z|, \mathcal{A}) \leq \varepsilon \omega(K)$. Thus $C(|z|, \mathcal{A}) \lesssim_{C_{\omega}} \varepsilon \omega(|x - y|)$, and from (8), we derive the bound

$$\left\|A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y) (x-y)^j\right\| \lesssim \varepsilon |x-y|^{m-k} \omega(|x-y|), \quad k = 0, \dots, m.$$
(9)

Dividing by the term $|x-y|^{m-k}\omega(|x-y|)$, and taking the maximum among $k = 0, \ldots, m$, we arrive at $R(\mathcal{A}, x, y)/\omega(|x-y|) \lesssim \varepsilon$.

Let us now study the (non-symmetric) case $|y| \ge K$, and $|x| \le M$. Let us write

$$\begin{aligned} \left\| A_{k}(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^{j} \right\| \\ &\leq \left\| \sum_{j=0}^{m-k} \frac{1}{j!} \left(A_{k+j}(y) - \sum_{l=0}^{m-k-j} \frac{1}{l!} A_{k+j+l}(x)(y-x)^{l} \right) (x-y)^{j} \right\| \\ &+ \left\| A_{k}(x) - \sum_{j=0}^{m-k} \frac{1}{j!} \sum_{l=0}^{m-k-j} \frac{1}{l!} A_{k+j+l}(x)(y-x)^{l} (x-y)^{j} \right\| \\ &\lesssim_{m} \sum_{j=0}^{m-k} \left\| A_{k+j}(y) - \sum_{l=0}^{m-k-j} \frac{1}{l!} A_{k+j+l}(x)(y-x)^{l} \right\| \|x-y\|^{j} + \max_{1 \le l \le m} \sup_{\|u\| \le M} \|A_{l}(u)\| \|x-y\|^{m-k}. \end{aligned}$$
(10)

For the first term in (10), we use the estimates (9) for each j = 0, ..., m - k, but swapping the roles of x and y (note that the estimates of (9) were obtained for $|x| \ge K$ and $|y| \le M$). We thus obtain, for K large enough that

$$\sum_{j=0}^{m-k} \left\| A_{k+j}(y) - \sum_{l=0}^{m-k-j} \frac{1}{l!} A_{k+j+l}(x)(y-x)^l \right\| |x-y|^j \\ \lesssim \sum_{j=0}^{m-k} \varepsilon |x-y|^{m-k-j} \omega(|x-y|) |x-y|^j \lesssim_m \varepsilon |x-y|^{m-k} \omega(|x-y|).$$

As for the second term in (10), recall that $K \gg M$ and so, bearing in mind property (5) and Remark 2.1, we can say that $\max_{1 \le l \le m} \sup_{|u| \le M} ||A_l(u)|| \le \varepsilon \omega(K) \lesssim_{C_\omega} \varepsilon \omega(|x-y|)$. Dividing by $|x-y|^{m-k} \omega(|x-y|)$ and combining the above two cases, we conclude that

$$\frac{R(\mathcal{A}, x, y)}{\omega(|x-y|)} = \max_{0 \le k \le m} \frac{\|A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^j\|}{|x-y|^{m-k} \omega(|x-y|)} \lesssim \varepsilon,$$

as desired.

So, let us assume that $F \in \dot{\mathrm{VC}}_{\Gamma}^{m,\omega}(\mathbb{R}^n, V)$ and prove that the restriction of F to E satisfies the properties from Definition 1.6. While the proof in the case $\Gamma =$ small is immediate from Taylor's theorem, for the scales $\Gamma =$ large and $\Gamma =$ far, we need to study a couple of subcases separately.

The case $\Gamma = \text{small.}$ Because $F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V)$ we use Taylor's theorem to write, for each couple $x, y \in \mathbb{R}^n$ of distinct points, and each $k = 0, \ldots, m$:

$$\frac{\|D^k F(x) - \sum_{j=0}^{m-k} \frac{1}{j!} D^{k+j} F(y)(x-y)^j\|}{\omega(|x-y|)|x-y|^{m-k}} \le \frac{1}{(m-k)!} \sup_{z \in [x,y]} \frac{\|D^m F(z) - D^m F(y)\|}{\omega(|x-y|)}.$$
 (11)

One can continue the estimate (11) by using that $D^m F \in \dot{C}^{0,\omega}(\mathbb{R}^n, V)$, thus deducing that the resctriction of $\{F, DF, \ldots, D^m F\}$ to E defines an m-jet in $\dot{J}^{m,\omega}(E, V)$. To show that those restrictions actually belong to $\dot{VJ}^{m,\omega}_{small}(E, V)$ is also very easy: because $F \in \dot{VC}^{m,\omega}_{small}(\mathbb{R}^n, V)$, given $\varepsilon > 0$ there exists $\delta > 0$ so that

$$||D^m F(u) - D^m F(v)|| \le \varepsilon \omega (|u - v|), \quad |u - v| \le \delta.$$

Thus, assuming that $|x - y| \leq \delta$, and as obviously then $|z - y| \leq \delta$ for each $z \in [x, y]$, the right hand side of (11) is bounded from above by $C(m)\varepsilon$.

The case $\Gamma = \text{large.}$ Here we assume (5) for ω and that $F \in \dot{\mathrm{VC}}_{\mathrm{large}}^{m,\omega}(\mathbb{R}^n, V)$, and estimate (11) in the following manner. For any $\varepsilon > 0$, let R > 0 so that $|u - v| \ge R$ implies $||D^m F(u) - D^m F(v)|| \le \varepsilon \omega(|u - v|)$. Let us assume that $|x - y| \ge M$, where $M \gg R$ and its value will be specified in a moment. For those $z \in [x, y]$ so that $|z - y| \ge R$, it is enough to write

$$\frac{\|D^m F(z) - D^m F(y)\|}{\omega(|x - y|)} \le \varepsilon \frac{\omega(|z - y|)}{\omega(|x - y|)} \le \varepsilon.$$

For those $z \in [x, y]$ with |z - y| < R, we see that

$$\frac{|D^m F(z) - D^m F(y)||}{\omega(|x-y|)} \le ||F||_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} \frac{\omega(|z-y|)}{\omega(|x-y|)} \le ||F||_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} \frac{\omega(R)}{\omega(|x-y|)}.$$

Due to the assumption (5), if $M \gg R$ is large enough, then $|x - y| \ge M$ implies that the last term can be made smaller than ε . This shows that the jet given by the restriction of $\{F, DF, \ldots, D^mF\}$ to E belongs to $\dot{VJ}_{large}^{m,\omega}(E, V)$.

The case $\Gamma = \text{far.}$ Again we assume (5) for ω and that $F \in \dot{\mathrm{VC}}_{\mathrm{far}}^{m,\omega}(\mathbb{R}^n, V)$. Applying Lemma 2.2 for $D^m F \in \dot{\mathrm{VC}}_{\mathrm{far}}^{0,\omega}(\mathbb{R}^n, \mathcal{L}^m(\mathbb{R}^n, V))$, it follows that for every $\varepsilon > 0$ there exists R > 0 so that if $u, v \in \mathbb{R}^n$ with $|u| \geq R$, then

$$\frac{\|D^m F(u) - D^m F(v)\|}{\omega(|u - v|)} \le \varepsilon.$$
(12)

Let $x, y \in \mathbb{R}^n$ are such that $|x|, |y| \ge M$, for certain $M \gg R$ that we will determine later. Let $z \in [x, y]$ be as on the right-hand side of (11). In the case where $\max(|y|, |z|) \ge R$, then

$$|D^m F(z) - D^m F(y)|| \le \varepsilon \omega(|y - z|) \le \varepsilon \omega(|x - y|),$$

by the bound (12). And when |y|, |z| < R, we can estimate as in the case $\Gamma =$ large:

$$\frac{\|D^m F(z) - D^m F(y)\|}{\omega(|x-y|)} \le \|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} \frac{\omega(|z-y|)}{\omega(|x-y|)} \le \|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} \frac{\omega(2R)}{\omega(|x-y|)}.$$

Since $|x - y| \ge M - R$, using condition (5), we can choose $M \gg R$ large enough so that the last term is smaller than ε . Thus, we have the desired estimate for all possible $z \in [x, y]$, and so (11) gives that the restriction of $\{F, DF, \ldots, D^mF\}$ to E belongs to $VJ_{far}^{m,\omega}(E, V)$.

3. Sufficiency of Theorem 1.7

This section is devoted to the "if parts" of Theorem 1.7. We extend a jet from $\dot{J}^{m,\omega}(E,V)$ to a $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ function while simultaneously preserving a given vanishing scale. As before, our modulus $\omega : (0, \infty) \to (0, \infty)$ is non-decreasing, with $\lim_{t\to 0} \omega(t) = 0$, and such that

$$\frac{s}{\omega(s)} \le C_{\omega} \frac{t}{\omega(t)}, \quad \text{whenever} \quad 0 < s \le t < \infty, \tag{13}$$

for some $C_{\omega} > 0$; or equivalently that

$$\omega(\lambda t) \le C_{\omega} \lambda \omega(t), \quad \text{for all} \quad \lambda \ge 1, \, t > 0.$$
(14)

The first reduction in the extension Problem 1.5 is to notice that the unique continuous extension $\bar{\mathcal{A}}$ of $\mathcal{A} \in \dot{\mathrm{VJ}}_{\Gamma}^{m,\omega}(E,V)$ to the closure \bar{E} of E satisfies

$$\|\bar{\mathcal{A}}\|_{\mathbf{j}^{m,\omega}(\overline{E},V)} = \|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)}, \quad \text{and} \quad \bar{\mathcal{A}} \in \dot{\mathrm{VJ}}_{\Gamma}^{m,\omega}(\bar{E},V),$$
(15)

for each scale $\Gamma \in \{\text{small}, \text{far}, \text{large}\}.$

Indeed, this is possible since we are assuming that V is a Banach space, and thus the functions $A_k \in (A_1, \ldots, A_m) \in \mathcal{A}$ of a jet map Cauchy sequences of E to Cauchy sequences of $\mathcal{L}^k(\mathbb{R}^n, V)$.

Since V is Banach, so is $\mathcal{L}^m(\mathbb{R}^n, V)$. Thus if $\{x_j\}$ is Cauchy in E with a limit $\bar{x} \in \bar{E}$, then $\{A_m(x_j)\}$ is Cauchy in $\mathcal{L}^m(\mathbb{R}^n, V)$, by the estimate

$$\|A_m(x_j) - A_m(x_i)\| \lesssim \|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)} \omega(|x_j - x_i|) \to 0, \quad \min(i,j) \to \infty.$$

Thus the limit $\bar{A}_m(\bar{x}) := A_m(\bar{x}) \in \mathcal{L}^m(\mathbb{R}^n, V)$ is uniquely determined and in this way we extend A_m to \bar{E} . Now the extension of the case k = m - 1, i.e. A_{m-1} as \bar{A}_{m-1} , follows by recursing from the case k = m above and the definition of (1). Thus for the extension problem, we can assume that $E \subset \mathbb{R}^n$ is closed. In fact, only here we need the completeness of V. Thus, if the given set $E \subset \mathbb{R}^n$ is closed, Theorem 1.7 holds for V a normed space.

We next describe the linear extension operator defined by the classical Whitney decomposition and the associated partition of unity. We remark that even if we restrict ourselves to the particular case m = 0 and $V = \mathbb{R}$, any simple variation of the infimal convolution formula, as mentioned in the introduction, does not seem to provide an extension operator that maps $\dot{\mathrm{VC}}_{\Gamma}^{0,\omega}(E,\mathbb{R}) \rightarrow$ $\dot{\mathrm{VC}}_{\Gamma}^{0,\omega}(\mathbb{R}^n,\mathbb{R})$, for any of the scales $\Gamma \in \{\text{small}, \text{far}, \text{large}\}$. Moreover, it is an obvious benefit that the extension operator is linear, and of course any operator defined through an infimal convolution formula is automatically non-linear.

Whitney partition of unity. We begin by recalling the main properties of the Whitney decomposition of an open set into cubes. For a closed set E, the Whitney cubes associated with the open set $\mathbb{R}^n \setminus E$ is a collection \mathcal{Q} of dyadic cubes with the following properties:

- There holds that $\bigcup_{Q \in \mathcal{Q}} Q = \mathbb{R}^n \setminus E$.
- For every $Q \in \mathcal{Q}$, there holds that $d(Q, E) \leq \operatorname{diam}(Q) \leq 4d(Q, E)$.
- If $Q, Q' \in \mathcal{Q}$ are two distinct cubes, then $int(Q) \cap int(Q') = \emptyset$.

To construct a C^{∞} partition of unity associated with these cubes, for each $Q \in Q$ denote $Q^* := 9/8Q$ and let $p_Q \in E$ be a point (not necessarily unique) for which $d(E,Q) = d(p_Q,Q)$. A C^{∞} -Whitney partition of unity is, in particular, a collection of functions $\{\varphi_Q : Q \in Q\}$ such that each φ_Q is supported on Q^* . Many relevant properties hold for these families of cubes and functions, and we refer to [16, Chapter VI] for a detailed exposition of this topic.

Proposition 3.1. The Whitney partition of unity satisfies the following properties.

(i) There holds that

$$\bigcup_{Q \in \mathcal{Q}} Q = \bigcup_{Q \in \mathcal{Q}} Q^* = \mathbb{R}^n \setminus E$$

(ii) There exists a dimensional constant N(n) > 0 so that

$$\sum_{Q \in \mathcal{Q}} 1_{Q^*} \le N(n);$$

i.e. every $x \in \mathbb{R}^n \setminus E$ is contained in a neighbourhood that intersects at most N(n) cubes of the family $\{Q^* : Q \in Q\}$.

(iii) There exists an absolute constant C > 0 so that for all cubes $Q \in \mathcal{Q}$ there holds that

 $|p_Q - y| \le C|x - y|$, whenever $x \in Q^*, y \in E$.

(iv) For each $Q \in \mathcal{Q}$, there holds that

$$0 \le \varphi_Q \le 1_{Q^*}, \qquad \varphi_Q \in C^\infty(\mathbb{R}^n).$$

(v) Partition of unity: for all $x \in \mathbb{R}^n \setminus E$, there holds that

$$\sum_{Q \in \mathcal{Q}} \varphi_Q(x) = \sum_{Q : Q^* \ni x} \varphi_Q(x) = 1, \qquad \sum_{Q \in \mathcal{Q}} D^k \varphi_Q(x) = 0, \quad k \in \mathbb{N}.$$

Notice that the latter follows from the first and (ii).

(vi) For each $k \in \mathbb{N}$, there exists a constant C(n,k) so that for all $Q \in \mathcal{Q}$ and $z \in Q^*$, there holds that

$$||D^k \varphi_Q(z)|| \le C(n,k) d(z,E)^{-k}.$$

Now, given a jet $\mathcal{A} = \{A_k\}_{k=0}^m \in \dot{\mathbf{J}}^{m,\omega}(E,V)$ and a point $y \in E$, we define the polynomial

$$P_y^{\mathcal{A}} : \mathbb{R}^n \to V, \qquad P_y^{\mathcal{A}}(x) = \sum_{k=0}^m \frac{1}{k!} A_k(y) (x-y)^k, \quad x \in \mathbb{R}^n.$$

And with these, the Whitney extension operator

$$\mathcal{W}(\mathcal{A})(x) = \begin{cases} A_0(x), & x \in E, \\ \sum_{Q \in \mathcal{Q}} \varphi_Q(x) P_{p_Q}^{\mathcal{A}}(x), & x \in \mathbb{R}^n \setminus E. \end{cases}$$
(16)

Using the properties (ii) and (iv), it is easy to see that $\mathcal{W}f \in C^{\infty}(\mathbb{R}^n \setminus E)$, and for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$, the α -partial derivative $D^{\alpha}(\mathcal{W}f) : \mathbb{R}^n \setminus E \to V$ is given by

$$D^{\alpha}(\mathcal{W}(\mathcal{A}))(x) = \sum_{Q \in \mathcal{Q}} \sum_{\beta \le \alpha} {\alpha \choose \beta} D^{\beta} \varphi_Q(x) D^{\alpha - \beta} P^{\mathcal{A}}_{p_Q}(x), \quad x \in \mathbb{R}^n \setminus E.$$
(17)

Naturally, for any function $G: \mathbb{R}^n \to V$, the β -partial derivative $D^{\beta}G(x)$ at $x \in \mathbb{R}^n$ is defined as

$$D^{\beta}G(x) = \frac{\partial^{|\beta|}G}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(x) = D^{|\beta|}G(x)(e_1, \stackrel{\beta_1}{\ldots}, e_1, \dots, e_n, \stackrel{\beta_n}{\ldots}, e_n),$$

where e_i is the *i*th unit vector.

A classical result is that for a $V = \mathbb{R}$ valued *m*-jet $\mathcal{A} \in \dot{J}^{m,\omega}(E, V)$, there holds that

$$\mathcal{W}(\mathcal{A}) \in \dot{C}^{m,\omega}(\mathbb{R}^n, V), \qquad \|\mathcal{W}(\mathcal{A})\|_{\dot{C}^{m,\omega}(\mathbb{R}^n, V)} \le \kappa(n, m, C_{\omega})\|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E, V)}, \tag{18}$$

for a constant $\kappa(n, m, C_{\omega})$ depending only on n, m, and the constant C_{ω} from (13). A proof can be found in [16, Chapter VI, p.175]. However, a small modification of the arguments therein shows that (18) is satisfied for all $\mathcal{A} \in \dot{J}^{m,\omega}(E, V)$, with an arbitrary normed space V.

Nevertheless, it seems very far from elementary to determine the action of the Whitney extension operator (16) over the three vanishing scales. For each scale, we study pairs (x, y) and their relative position with respect to E and |x - y|. Although this separation into cases will be the same for each scale, we split the proof of Theorem 1.7 into three parts, with each of the scales presenting its own particular difficulties.

During the proof we will fix an *m*-jet $\mathcal{A} \in \dot{\mathrm{VJ}}_{\Gamma}^{m,\omega}(E,V)$, whose norm $\|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)} < \infty$ is considered to be an *absolute* constant. Moreover, we indicate the dependence on the dimension or on the order of smoothness *m* in some estimates by including subscripts, e.g. $\leq_n, \leq_m, \leq_{n,m}$. The same applies for the constant C_{ω} associated with the modulus ω from (13). Thus, when using the notation $A \leq B$, the constant *C* involved in the estimate $A \leq C \cdot B$ is allowed to depend on *n*, *m*, $\|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)}$ and C_{ω} .

To simplify notation, we denote by $F := \mathcal{W}(\mathcal{A})$ the Whitney Extension (16) and also $P_y = P_y^{\mathcal{A}}$, for each $y \in E$.

We will use several times the estimates of the following Lemma 3.2, which are implicitly proved in [16].

Lemma 3.2. Let $y \in E$. Let $x \in \mathbb{R}^n \setminus E$ and $\xi = \xi(x) \in E$ be a point that minimizes $d(x, E) = |x - \xi|$. Then,

(a)
$$||D^m F(x) - A_m(y)|| \lesssim_{m,n} \sum_{Q^* \ni x} (R(\mathcal{A}, y, p_Q) + R(\mathcal{A}, \xi, p_Q)),$$

(b) $||D^{m+1}F(x)|| \lesssim_{m,n} \sum_{Q^* \ni x} R(\mathcal{A}, \xi, p_Q)|x - \xi|^{-1}.$

Proof.

(a) Using multi-index notation, the polynomials $P_y, y \in E$, can be expressed as

$$P_{y}(x) := \sum_{k=0}^{m} \frac{1}{k!} A_{k}(y) (x-y)^{k} = \sum_{|\beta| \le m} \frac{1}{\beta!} A_{\beta}(y) (x-y)^{\beta},$$
(19)

where the A_{β} s are defined as follows. For any multi-index $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$ of order $|\beta| := \sum_i \beta_j \leq m$, and $y \in E$, we set

$$A_{\beta}(y) := A_{|\beta|}(y)(e_1, \stackrel{\beta_1}{\dots}, e_1, \dots, e_n, \stackrel{\beta_n}{\dots}, e_n);$$

$$(20)$$

where e_i is the *i*th unit vector. Also, for any such β and $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, we denote

$$z^{\beta} = z_1^{\beta_1} \cdots z_n^{\beta_n}$$

The second identity in (19) follows from the symmetry and the k-linearity of the A_k s' and a basic combinatorial argument. Thus, for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$ of order $|\alpha| = m$, one has $D^{\alpha}P_y(x) = A_{\alpha}(y)$, for every $x \in \mathbb{R}^n$ and $y \in E$. Now, to show the point (a) it is enough to show that for each multi-index α of order $|\alpha| = m$, there holds that

$$\|D^{\alpha}F(x) - A_{\alpha}(y)\| \lesssim_{m,n} \sum_{Q^* \ni x} \Big(R(\mathcal{A}, y, p_Q) + R(\mathcal{A}, \xi, p_Q) \Big).$$
⁽²¹⁾

Then, using the multilinearity of $D^m F(x) - A_m(y)$, the bound (a) follows.

Also, by property (v), $\sum_{Q} D^{\gamma} \varphi_{Q}(x) = 0$ for any multi-index γ with $|\gamma| \ge 1$. These observations and formula (17) permit to write, for $x \in \mathbb{R}^{n} \setminus E$, $y \in E$, and for a multi-index α of order $|\alpha| = m$:

$$D^{\alpha}F(x) - A_{\alpha}(y) = \sum_{Q} D^{\alpha}(\varphi_{Q} \cdot P_{p_{Q}})(x) - A_{\alpha}(y) = \sum_{Q} \varphi_{Q}(x) \left(A_{\alpha}(p_{Q}) - A_{\alpha}(y)\right) + \sum_{Q} \sum_{\beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\varphi_{Q}(x) \left(D^{\beta}P_{p_{Q}}(x) - D^{\beta}P_{\xi}(x)\right).$$

The first term is estimated using the definition of $R(\mathcal{A}, \cdot, \cdot)$ (2):

$$\left\|\sum_{Q}\varphi_{Q}(x)\left(A_{\alpha}(p_{Q})-A_{\alpha}(y)\right)\right\| \leq \sum_{Q^{*}\ni x}\left\|A_{\alpha}(p_{Q})-A_{\alpha}(y)\right\| \leq \sum_{Q^{*}\ni x}R(\mathcal{A},y,p_{Q}).$$

For the second term, we take into account that the polynomials $P_{p_Q} - P_{\xi}$ have degree up to m, and so, for $|\beta| \leq m$,

$$D^{\beta}P_{pQ}(x) - D^{\beta}P_{\xi}(x) = \sum_{|\gamma| \le m - |\beta|} \frac{1}{\gamma!} \left[D^{\beta+\gamma}(P_{pQ} - P_{\xi})(\xi) \right] (x - \xi)^{\gamma}$$
$$= \sum_{|\gamma| \le m - |\beta|} \frac{1}{\gamma!} \left[D^{\beta+\gamma}P_{pQ}(\xi) - A_{\beta+\gamma}(\xi) \right] (x - \xi)^{\gamma}$$
$$= \sum_{|\gamma| \le m - |\beta|} \frac{1}{\gamma!} \left[\sum_{|\delta| \le m - |\beta+\gamma|} \frac{1}{\delta!} A_{\delta+\beta+\gamma}(pQ)(\xi - pQ)^{\delta} - A_{\beta+\gamma}(\xi) \right] (x - \xi)^{\gamma}.$$

Now from the formula (2), it follows that

$$\|D^{\beta}P_{p_Q}(x) - D^{\beta}P_{\xi}(x)\| \le \sum_{j=0}^{m-|\beta|} R(\mathcal{A},\xi,p_Q)|\xi - p_Q|^{m-|\beta|-j}|x-\xi|^j.$$
(22)

This estimate and property (vi), and also $|\alpha| = m$, lead us to

$$\left\|\sum_{Q}\sum_{\beta\leq\alpha,\beta\neq\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\varphi_{Q}(x) \left(D^{\beta}P_{p_{Q}}(x) - D^{\beta}P_{\xi}(x)\right)\right\|$$

$$\leq C(n,m)\sum_{Q^{*}\ni x}\sum_{\beta\leq\alpha,\beta\neq\alpha} |x-\xi|^{-(m-|\beta|)} \|D^{\beta}P_{p_{Q}}(x) - D^{\beta}P_{\xi}(x)\|$$

$$\leq C(n,m)\sum_{Q^{*}\ni x} R(\mathcal{A},\xi,p_{Q})\sum_{k=0}^{m-1}\sum_{j=0}^{m-k} \left(\frac{|\xi-p_{Q}|}{|x-\xi|}\right)^{m-k-j} \leq C(n,m)\sum_{Q^{*}\ni x} R(\mathcal{A},\xi,p_{Q}), \quad (23)$$

where in the last bound we used that $|\xi - p_Q| \leq C|x - \xi|$, for an absolute constant C, by (iii). (b) The proof is very similar to that of (a). Since the polynomials have degree up to m, if α is a multi-index with order $|\alpha| = m + 1$, then

$$D^{\alpha}F(x) = \sum_{Q} \sum_{\beta \le \alpha, |\beta| \le m} {\alpha \choose \beta} D^{\alpha-\beta}\varphi_{Q}(x) D^{\beta}P_{p_{Q}}(x)$$
$$= \sum_{Q} \sum_{\beta \le \alpha, |\beta| \le m} {\alpha \choose \beta} D^{\alpha-\beta}\varphi_{Q}(x) \left(D^{\beta}P_{p_{Q}}(x) - D^{\beta}P_{\xi}(x) \right).$$

By the estimate (22), the arguments that led us to (23) and to the conclusion of (a), we derive

$$\|D^{\alpha}F(x)\| \leq C(n,m) \sum_{Q^* \ni x} \sum_{k=0}^m \sum_{|\beta|=k} |x-\xi|^{-(m+1-k)} \|D^{\beta}P_{pQ}(x) - D^{\beta}P_{\xi}(x)\|$$

$$\leq C(n,m) \sum_{Q^* \ni x} R(\mathcal{A},\xi,p_Q) |x-\xi|^{-1}.$$

Then, using the multilinearity of $D^{m+1}F(x)$ the bound (b) follows.

Proof of Theorem 1.7 for $\Gamma = \text{small.}$ Given $\mathcal{A} \in \dot{\mathrm{VJ}}_{\mathrm{small}}^{m,\omega}(E,V)$, we show that $F \in \dot{\mathrm{VC}}_{\mathrm{small}}^{m,\omega}(\mathbb{R}^n,V)$, i.e. that

$$\lim_{|x-y|\to 0} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} = 0,$$

uniformly on $x, y \in \mathbb{R}^n$. Denote

$$S(t) := \sup\left\{\frac{R(\mathcal{A}, u, v)}{\omega(|u - v|)} : u, v \in E, \ 0 < |u - v| \le t\right\},\tag{24}$$

understanding that S(t) = 0 if there is no such couple $u, v \in E$ with $0 < |u - v| \le t$. We have that $S(t) \to 0$ as $t \to 0$ and $S(t) \le ||\mathcal{A}||_{j^{m,\omega}(E,V)}$. We distinguish three possible situations for any couple of distinct points $x, y \in \mathbb{R}^n$, where at least one of them is outside E.

Case 1. Assume $x \in \mathbb{R}^n \setminus E$ and $y \in E$.

We will use property (iii), which tells us that if $x \in Q^*$, then $|p_Q - y| \leq C|x - y|$, for some absolute constant C > 0. Let $\xi = \xi(x) \in E$ minimize $d(x, E) = |x - \xi|$. Together with Lemma 3.2,

and the definition of S, we find that

$$\begin{split} \|D^m F(x) - D^m F(y)\| &= \|D^m F(x) - A_m(y)\| \\ &\lesssim \sum_{Q^* \ni x} \left(R(\mathcal{A}, y, p_Q) + R(\mathcal{A}, \xi, p_Q) \right) \\ &\leq \sum_{Q^* \ni x} \left(S(|y - p_Q|) \omega(|y - p_Q|) + S(|\xi - p_Q|) \omega(|\xi - p_Q|) \right) \\ &\lesssim_n S(C|x - y|) \omega(|x - y|), \end{split}$$

where we used property (14) of ω in the last inequality. Therefore

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \mathcal{S}(C|x-y|), \tag{25}$$

and the right-hand side tends to zero as $|x - y| \rightarrow 0$.

Case 2. Assume that $x, y \in \mathbb{R}^n \setminus E$ and $d([x, y], E) \leq |x - y|$. Pick $z \in [x, y]$ and $p \in E$ minimizing d([x, y], E) = |z - p|. Then,

$$|x - p| \le |x - z| + |z - p| \le |x - y| + |x - y| = 2|x - y|$$

and the same holds for |y - p|. Then $|x - p|, |y - p| \le 2|x - y|$ and applying the estimate (25) from Case 1 (for $x \in \mathbb{R}^n \setminus E$ and $p \in E$, and for $y \in \mathbb{R}^n \setminus E$ and $p \in E$) we obtain,

$$||D^m F(x) - D^m F(y)|| \le ||D^m F(x) - D^m F(p)|| + ||D^m F(p) - D^m F(y)|| \le S(2C|x - y|)\omega(2|x - y|),$$

which is a bound of the correct form.

Case 3. Assume $x, y \in \mathbb{R}^n \setminus E$, and $d([x, y], E) \geq |x - y|$. Here we use the assumption $\lim_{t\to 0} t/\omega(t) = 0$.

Let us begin with a computation for arbitrary $z \in \mathbb{R}^n \setminus E$. Let $\xi = \xi(z) \in E$ minimize $d(z, E) = |z - \xi|$. Then, by the formula (16) and the properties (iii), (v) and (vi) we have

$$\|D^{m+1}F(z)\| \lesssim \sum_{Q^* \ni z} \frac{R(\mathcal{A}, \xi, p_Q)}{|z - \xi|} \le \sum_{Q^* \ni z} \frac{\omega(|p_Q - \xi|) \,\mathrm{S}(|p_Q - \xi|)}{d(z, E)} \lesssim_n \frac{\omega(d(z, E)) \,\mathrm{S}(C \, d(z, E))}{d(z, E)}.$$

In the last inequality, together with (iii), we used the fact that every point of $\mathbb{R}^n \setminus E$ is contained in at most N(n) cubes Q^* as well as property (14) of ω . As $D^m F$ is differentiable on $\mathbb{R}^n \setminus E$ and the segment [x, y] lies outside E, by the mean value inequality (for normed-valued mappings) and the above estimate we obtain

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \le \frac{|x-y|}{\omega(|x-y|)} \sup_{z \in [x,y]} \|D^{m+1} F(z)\| \lesssim_n \sup_{z \in [x,y]} U_z(x,y);$$
(26)

where

$$U_z(x,y) := \frac{|x-y|}{\omega(|x-y|)} \frac{\omega(d(z,E))}{d(z,E)} \mathcal{S}(C \ d(z,E)).$$

Now we fix $z \in [x, y]$ and we consider the behaviour of d(z, E) as |x - y| tends to zero. By $\lim_{t\to 0} S(t) = 0$, we have the following: for any $\varepsilon > 0$, let $\delta > 0$ be so that $S(t) \le \varepsilon$, whenever $t \le \delta$. First, if $C d(z, E) \le \delta$, then by $|x - y| \le d(z, E)$ and property (13) for ω , we obtain

$$U_z(x,y) \lesssim S(C d(z,E)) \le \varepsilon.$$

On the other hand, if $C d(z, E) \ge \delta$, we use the property (13) of ω and that $S \le ||\mathcal{A}||_{\mathbf{j}^{m,\omega}(E,V)}$ to bound

$$U_z(x,y) \lesssim_{\delta} \frac{|x-y|}{\omega(|x-y|)} \cdot 1 \cdot \|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)} \lesssim \frac{|x-y|}{\omega(|x-y|)}$$

now the right-hand side tends to zero as |x - y| tends to 0, by the condition (4). We conclude from the above cases and (26) that $||D^m F(x) - D^m F(y)|| / \omega(|x - y|) \to 0$ as $|x - y| \to 0$, uniformly. \Box

Proof of Theorem 1.7 for $\Gamma = \text{large. Let } \mathcal{A} \in \dot{\text{VJ}}_{\text{large}}^{m,\omega}(E,V)$ and we show that $F \in \dot{\text{VC}}_{\text{large}}^{m,\omega}(\mathbb{R}^n,V)$. Denote

$$\mathcal{L}(t) := \sup\left\{\frac{R(\mathcal{A}, u, v)}{\omega(|u - v|)} : u, v \in E, |u - v| \ge t\right\},\$$

so that $L(t) \leq \|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)}$, L is non-increasing, and $L(t) \to 0$ as $t \to \infty$. We understand S(t) = 0 when there are no $u, v \in E$ with $|u - v| \geq t$.

Case 1. Let $x \in \mathbb{R}^n \setminus E$ and $y \in E$.

For each cube $Q \in \mathcal{Q}$ so that $x \in Q^*$, we have $|p_Q - y| \leq C|x - y|$ for an absolute constant C; see (iii). Let $\xi = \xi(x) \in E$ minimize $d(x, E) = |x - \xi|$. In this case, although we are letting $|x - y| \to \infty$, we do not know whether $|p_Q - y|$ or $|p_Q - \xi|$ are large or not. Therefore, we need to study the behaviour of $|p_Q - z|$, for $z \in \{y, \xi\}$ as $|x - y| \to \infty$. If $|p_Q - z| \leq M$ (for a constant M that we will specify in a moment), then we have the bound

$$\frac{R(\mathcal{A}, z, p_Q)}{\omega(|x-y|)} \le \|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)} \frac{\omega(|z-p_Q|)}{\omega(|x-y|)} \lesssim \frac{\omega(M)}{\omega(|x-y|)}.$$
(27)

On the other hand, if $|p_Q - z| \ge M$, then $L(|p_Q - z|) \le L(M)$ and so

$$\frac{R(\mathcal{A}, z, p_Q)}{\omega(|x-y|)} \le \mathcal{L}(|p_Q - z|) \frac{\omega(|p_Q - z|)}{\omega(|x-y|)} \le \mathcal{L}(M) \frac{\omega(C|x-y|)}{\omega(|x-y|)} \lesssim \mathcal{L}(M).$$
(28)

We have used that if $z \in \{y,\xi\}$, then $|p_Q - z| \leq C|x - z| \leq C|x - y|$, by virtue of (iii). Now, given $\varepsilon > 0$, we choose M > 0 large enough so that $L(M) \leq \varepsilon$. Using $\omega(\infty) = \infty$, we find K > M such that $\omega(|x - y|) \geq \varepsilon^{-1}\omega(M)$, provided that $|x - y| \geq K$. Thus the bounds (27) and (28) tell us that regardless of the size of $|p_Q - z|$, we have

$$\frac{R(\mathcal{A}, z, p_Q)}{\omega(|x - y|)} \lesssim \varepsilon, \quad z \in \{y, \xi\}.$$
(29)

From the bound (29) and Lemma 3.2, we obtain

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \sum_{Q^* \ni x} \left(\frac{R(\mathcal{A}, y, p_Q)}{\omega(|x-y|)} + \frac{R(\mathcal{A}, \xi, p_Q)}{\omega(|x-y|)} \right) \lesssim_n \varepsilon,$$

whenever $|x - y| \ge K$.

Case 2. Assume $x, y \in \mathbb{R}^n \setminus E$, and $d([x, y], E) \leq |x - y|$. Let $p = p(x, y) \in E$ minimize d([x, y], E). Then, again, $|x - p|, |y - p| \leq 2|x - y|$. Similarly, we saw already in Case 1 the following: given $\varepsilon > 0$, there exists M > 0 (independent of $x \in \mathbb{R}^n \setminus E$ and $p \in E$) such that: if $|x - p| \geq M$, then

$$\|D^m F(x) - D^m F(p)\| \le \varepsilon \omega(|x-p|) \le \varepsilon \omega(2|x-y|) \le \varepsilon \omega(|x-y|).$$
(30)

Now, consider $|x - p| \le M$ and let K > 0 be so large that $\varepsilon \omega(K) > \omega(M)$. Now if also $|x - y| \ge K$, then we have

$$\|D^m F(x) - D^m F(p)\| \le \|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n, V)}\omega(|x-p|) \lesssim \omega(M) \le \varepsilon \omega(K) \le \varepsilon \omega(|x-y|).$$
(31)

We used above the fact that the Whitney extension is a bounded operator. Combining (30) and (31), we obtain

$$\sup_{|x-y|>K} \frac{\|D^m F(x) - D^m F(p)\|}{\omega(|x-y|)} \lesssim \varepsilon.$$
(32)

And the same bound holds with y in place of x; thus by triangle inequality,

$$\sup_{\substack{|x-y|>K}} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \le \sup_{\substack{|x-y|>K}} \frac{\|D^m F(x) - D^m F(p)\|}{\omega(|x-y|)} + \sup_{\substack{|x-y|>K}} \frac{\|D^m F(p) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \varepsilon$$

Case 3. Assume $x, y \in \mathbb{R}^n \setminus E$ and d([x, y], E) > |x - y|.

For any $\varepsilon > 0$, let M > 0 be so that $L(M) \leq \varepsilon$ and also choose $K \gg M$ so that $\varepsilon \omega(K) \geq \omega(M)$. For any two points $u, v \in E$, we have either $R(\mathcal{A}, u, v) \leq \varepsilon \omega(|u - v|)$ (when $|u - v| \geq M$) or $R(\mathcal{A}, u, v) \leq ||\mathcal{A}||_{\mathrm{VJ}^{m,\omega}(E,V)}\omega(M)$ (when $|u - v| \leq M$). In other words,

$$R(\mathcal{A}, u, v) \lesssim \max\{\varepsilon \,\omega(|u - v|), \omega(M)\}, \quad \text{for all} \quad u, v \in E.$$
(33)

Let $z \in [x, y]$ and let $\xi = \xi(z) \in E$ minimize $d(z, E) = |z - \xi|$. Employing first Lemma 3.2, then (33), and finally property (ii), we derive

$$\|D^{m+1}F(z)\| \lesssim_n \sum_{Q^* \ni z} \frac{R(\mathcal{A}, \xi, p_Q)}{d(z, E)} \lesssim_n \max_{Q \in \mathcal{Q} : Q^* \ni z} \frac{R(\mathcal{A}, \xi, p_Q)}{d(z, E)}$$
(34)

$$\leq \max_{Q \in \mathcal{Q}: Q^* \ni z} \frac{\max\{\varepsilon \,\omega(|p_Q - \xi|), \omega(M)\}}{d(z, E)} \lesssim_n \frac{\max\{\varepsilon \,\omega(d(z, E)), \omega(M)\}}{d(z, E)}.$$
 (35)

We used the fact that when $z \in Q^*$, then $|p_Q - \xi| \leq C|z - \xi|$. Then (35) and the mean value inequality (for normed-valued mappings) give

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \le \frac{|x-y|}{\omega(|x-y|)} \sup_{z \in [x,y]} \|D^{m+1} F(z)\|$$

$$\lesssim_n \sup_{z \in [x,y]} \frac{|x-y|}{\omega(|x-y|)} \frac{\max\{\varepsilon \,\omega(d(z,E)), \omega(M)\}}{d(z,E)}.$$
(36)

By property (13) of ω and $d(z, E) \ge |x - y|$, for all $z \in [x, y]$, we continue bounding,

$$\operatorname{RHS}(36) \lesssim \sup_{z \in [x,y]} \max\left\{\varepsilon, \frac{|x-y|}{\omega(|x-y|)} \frac{\omega(M)}{d(z,E)}\right\} \le \max\left\{\varepsilon, \frac{\omega(M)}{\omega(|x-y|)}\right\}.$$
(37)

Recalling that $\varepsilon \omega(K) \ge \omega(M)$ we conclude

$$\sup_{|x-y|>K} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \sup_{|x-y|>K} \max\left\{\varepsilon, \frac{\omega(M)}{\omega(|x-y|)}\right\} \le \varepsilon.$$

Proof of Theorem 1.7 when $\Gamma = \text{far.}$ Here we assume that $\omega(\infty) = \infty$. We denote

$$D(t) := \sup\left\{\frac{R(\mathcal{A}, u, v)}{\omega(|u - v|)} : u, v \in E, \max(|u|, |v|) \ge t\right\},\$$

so that $D(t) \leq \|\mathcal{A}\|_{j^{m,\omega}(E,V)}$ and D is non-increasing, and moreover, thanks to Lemma 2.2, that $D(t) \to 0$ as $t \to \infty$. We show that

$$\lim_{K \to \infty} \sup_{\substack{x,y \in \mathbb{R}^n \\ \min(|x|,|y|) \ge K}} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} = 0$$

Case 1. Let $x \in \mathbb{R}^n \setminus E$, $y \in E$. Given $\varepsilon > 0$, let M > 0 be such that $D(M) \leq \varepsilon$. We consider $|x| \geq K \gg M$ for a large constant K to be soon determined. Let $\xi = \xi(x) \in E$ minimize

 $d(x, E) = |x - \xi|$. Applying Lemma 3.2 and bearing in mind the property (ii), we find a cube $Q_x \in \mathcal{Q}$ with $x \in (Q_x)^*$, and $z \in \{\xi, y\}$ so that

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \leq \frac{\sum_{Q^* \ni x} R(\mathcal{A}, y, p_Q) + R(\mathcal{A}, \xi, p_Q)}{\omega(|x-y|)} \lesssim_n \frac{R(\mathcal{A}, z, p_{Q_x})}{\omega(|x-y|)}$$
$$\leq \min\{ \mathbf{D}(|p_{Q_x}|), \mathbf{D}(|z|), \|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)} \} \frac{\omega(|p_{Q_x} - z|)}{\omega(|x-y|)}.$$
(38)

The last inequality is a consequence of the definition of D(t). Now, if either $|p_{Q_x}| \geq M$ or $|z| \geq M$, the minimum in the term (38) is smaller than ε due to the choice of M. And because $|p_{Q_x}-z| \leq C|x-y|$ for an absolute constant C > 0 (see (iii)); we conclude that (38) is bounded by an absolute multiple of ε in this particular case. And if $|p_{Q_x}| \leq M$ and $|z| \leq M$, then $|p_{Q_x}-z| \leq 2M$, and so $\omega(|p_{Q_x}-z|) \lesssim \omega(M)$ and we bound

RHS (38)
$$\lesssim \frac{\omega(M)}{\omega(||x| - M|)} \lesssim \frac{\omega(M)}{\omega(K - M)}.$$

By $\omega(\infty) = \infty$ the right-hand side is smaller than ε provided that $K \gg M$ is taken sufficiently large.

Case 2. Assume $x, y \in \mathbb{R}^n \setminus E$ and $d([x, y], E) \leq |x - y|$.

Let $\varepsilon > 0$ and let R > 0 be as in Case 1. Suppose that $|x|, |y| \ge K$. Let $p \in E$ be such that d([x, y], E) = d([x, y], p), and then $|x - p|, |y - p| \le 2|x - y|$, as we have already seen several times. By Case 1 applied to the pairs of points $x \in \mathbb{R}^n \setminus E$, $p \in E$ and $y \in \mathbb{R}^n \setminus E$, $p \in E$, we have

$$\|D^m F(x) - D^m F(p)\| \lesssim \varepsilon \omega(|x-p|), \qquad \|D^m F(y) - D^m F(p)\| \lesssim \varepsilon \omega(|y-p|)$$

Thus the claim follows by triangle inequality.

Case 3. Assume $x, y \in \mathbb{R}^n \setminus E$ and $d([x, y], E) \ge |x - y|$.

Let $\varepsilon > 0$, and $K \gg M$ be the parameters we used in Case 1. Note that we can enlarge K if necessary so as to satisfy

$$K \ge 2M$$
 and $\omega(M) \le \varepsilon \omega(K)$. (39)

Note that condition (5) guarantees the existence of such K. Let us also assume that $|x| \ge K$. Following the lines (34) and (36), we find $z \in [x, y]$, a point $\xi = \xi(z) \in E$ minimizing $d(z, E) = |z - \xi|$, and a cube $Q_z \in \mathcal{Q}$ with $z \in Q^*$ so that

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim_n \frac{|x-y|}{\omega(|x-y|)} \frac{R(\mathcal{A},\xi,p_{Q_z})}{d(z,E)}.$$
(40)

First of all, observe that when $|\xi| \ge M$ or $|p_{Q_z}| \ge M$, then $R(\mathcal{A}, \xi, p_{Q_z}) \le \varepsilon \omega(|\xi - p_{Q_z}|)$, with $|\xi - p_{Q_z}| \le C|\xi - z| = Cd(z, E)$. Thus, in this particular subcase, using (13) we can estimate

$$\operatorname{RHS}(40) \lesssim \frac{|x-y|}{\omega(|x-y|)} \frac{\omega(d(z,E))}{d(z,E)} \varepsilon \lesssim_{C_{\omega}} \varepsilon.$$
(41)

Therefore, we will assume from now on that $|\xi|, |p_{Q_z}| \leq M$. Now, we look at the last numerator in (40); observe that, by the definition of $R(\mathcal{A}, \xi, p_{Q_z})$, and the fact that F extends the jet \mathcal{A} from E to \mathbb{R}^n , we can find some $k \in \{0, \ldots, m\}$ and some $\tilde{\xi} \in [\xi, p_{Q_z}]$ so that

$$R(\mathcal{A},\xi,p_{Q_{z}}) \leq \frac{\|A_{k}(\xi) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(p_{Q_{z}})(\xi - p_{Q_{z}})\|}{|\xi - p_{Q_{z}}|^{m-k}} = \frac{\|D^{k}F(\xi) - \sum_{j=0}^{m-k} \frac{1}{j!} D^{k+j}F(p_{Q_{z}})(\xi - p_{Q_{z}})\|}{|\xi - p_{Q_{z}}|^{m-k}} \lesssim \|D^{m}F(\widetilde{\xi}) - D^{m}F(p_{Q_{z}})\|.$$
(42)

Now, since $F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V)$ with $||F||_{\dot{C}^{m,\omega}(\mathbb{R}^n, V)} \lesssim_{n,m,C_{\omega}} ||\mathcal{A}||_{\mathbf{j}^{m,\omega}(E,V)}$, and $|\tilde{\xi} - p_{Q_z}| \le |\xi - p_{Q_z}| \le 2M$, we can combine the estimates (40) and (42) to derive

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim_{n,m,C_{\omega}} \frac{|x-y|}{\omega(|x-y|)} \frac{\|D^m F(\widetilde{\xi}) - D^m F(p_{Q_z})\|}{d(z,E)} \\ \lesssim_{\|\mathcal{A}\|_{\mathbf{j}^{m,\omega}(E,V)}} \frac{|x-y|}{\omega(|x-y|)} \frac{\omega(|\widetilde{\xi} - p_{Q_z}|)}{d(z,E)} \lesssim_{C_{\omega}} \frac{|x-y|}{\omega(|x-y|)} \frac{\omega(M)}{d(z,E)}.$$
(43)

To complete the proof, observe that (39) gives

 $d(z, E) = |z - \xi| \ge |z| - |\xi| \ge K - M \ge K/2,$

and $\omega(M) \leq \varepsilon \omega(K) \lesssim_{C_{\omega}} \varepsilon \omega(d(z, E))$. By plugging this estimate into RHS(43) we obtain $\frac{\|D^m F(x) - D^m F(y)\|}{\|||_{c \leq |w|}} \leq \frac{|x - y|}{|w|} \frac{\omega(d(z, E))}{|w|} \leq \frac{|w|}{|w|}$

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \frac{|x-y|}{\omega(|x-y|)} \frac{\omega(d(z,E))}{d(z,E)} \varepsilon \lesssim_{C_\omega} \varepsilon_{z_\omega}$$

the last bound being a consequence of (13) and $|x - y| \le d(z, E)$.

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