TRACES OF VANISHING HÖLDER SPACES

KAUSHIK MOHANTA, CARLOS MUDARRA, AND TUOMAS OIKARI

Abstract. For an arbitrary subset $E \subset \mathbb{R}^n$ and a modulus of continuity ω, we define the subspaces of vanishing jets $\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\Gamma}(E)$ of the jet spaces $\dot{\mathrm{J}}^{m,\omega}(E)$, for the vanishing scales $\Gamma \in \{\mathrm{small}, \mathrm{large}, \mathrm{far}\}$, and up to order $m \in \mathbb{N} \cup \{0\}$. We show that each Γ-vanishing jet of order m is obtained by restricting a *globally* defined function whose mth derivative is in the Γ-vanishing Hölder space. This amounts to proving that the linear Whitney extension operator preserves separately each of the vanishing scales from E to the whole ambient space $\mathbb{R}^n$.

Further results will soon appear in a second version of this manuscript.

1. Introduction and main results

1.1. Introduction. The vanishing Hölder spaces $\dot{\mathrm{V}}\mathrm{C}^{\omega}_{\Gamma}(X, Y)$, for scales $\Gamma \in \{\mathrm{small}, \mathrm{large}, \mathrm{far}\}$, for Banach spaces X, Y, and for moduli of continuity ω (that cover the setting considered in this article), were recognized by the two last named authors [14] as the correct concept that provides a complete description of those Hölder functions $\dot{C}^{0,\omega}(X, Y)$ approximable by Lipschitz or smooth or boundedly supported test functions. In this article we continue to study the scales $\dot{\mathrm{V}}\mathrm{C}^{\omega}_{\Gamma}(\mathbb{R}^n, Y)$, with $X = \mathbb{R}^n$ and Y an arbitrary normed space.

We are interested in understanding when a given vanishing scale restricted to a proper subset $E \subset \mathbb{R}^n$, admits a bounded linear extension operator $L : \dot{\mathrm{V}}\mathrm{C}^{\omega}_{\Gamma}(E, Y) \to \dot{\mathrm{V}}\mathrm{C}^{\omega}_{\Gamma}(\mathbb{R}^n, Y)$ to the whole ambient space. Our answer is that for a completely arbitrary subset $E \subset \mathbb{R}^n$, a single linear bounded extension operator exists and works simultaneously for all the scales $\Gamma \in \{\mathrm{small}, \mathrm{large}, \mathrm{far}\}$. We use the classical Whitney extension operator.

Without much added complexity in the proofs, we do not only consider the possibility of extending the function itself, but also its *putative* derivatives. In particular, we provide a full description of exactly when a jet $\mathcal{A} \in \dot{\mathrm{J}}^{m,\omega}(E, Y)$ is obtained by restricting a function $F \in \dot{C}^{m,\omega}_{\Gamma}(\mathbb{R}^n, Y)$. This amounts to showing that the Whitney extension operator maps the vanishing jet spaces $\dot{\mathrm{J}}^{m,\omega}_{\Gamma}(E, Y)$ to $\dot{C}^{m,\omega}_{\Gamma}(\mathbb{R}^n, Y)$, i.e. preserves separately each of the vanishing scales; see Section 1.2 for the precise definitions and statements of these results.

One perspective to our results is as follows. The scale of Hölder spaces $\dot{C}^{0,\omega}$ can be seen to measure the smoothness of a function relative to any fixed modulus $\omega(t) > 0$, and when $\omega(t) = 0$ the space BMO of bounded mean oscillations is often substituted as the zeroth order smoothness measure. As opposed to real-valued Hölder functions $\dot{C}^{0,\omega}(E)$, with $E \subset \mathbb{R}^n$ arbitrary, being extendable to the whole ambient space $\dot{C}^{0,\omega}(\mathbb{R}^n)$, for example by the infimal convolution formulae

(K.M.) University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyväskylä, Finland

(C.M.) Department of Mathematical Sciences, Norwegian University of Science and Technology, 7941 Trondheim, Norway.

(T.O.) Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C Facultat de Ciències, 08193 Bellaterra (Barcelona), Catalonia

E-mail addresses: kaushik.k.mohanta@jyu.fi, carlos.mudarra@ntnu.no, tuomas.oikari@gmail.com.

Date: January 26, 2024.

2020 *Mathematics Subject Classification.* 26B35, 42B35, 46E35, 26B05, 46T20.

Key words and phrases. Extension operator, trace space, Hölder space, vanishing Hölder space.

$x \mapsto \inf_{y\in E}\{f(y) + M\omega(|x-y|)\}$, this is far from being true for functions in BMO. Indeed, given a connected open set (a domain) $\Omega \subset \mathbb{R}^n$, a classical theorem of Jones [12] tells us that a bounded linear extension operator $L : \mathrm{BMO}(\Omega) \to \mathrm{BMO}(\mathbb{R}^n)$ exists if and only if Ω is uniform. With BMO, geometry of the domain appears naturally. A recent result of Dalia and Gafni [4, Theorem 3.] (see also [5]) extends Jone's result by providing a single bounded extension operator on $\mathrm{BMO}(\Omega)$ and also on several of its subspaces determined by approximability by nice functions, or by having appropriate vanishing mean oscillations. Our results are in the same spirit as Butaev's and Gafni's, but opposed to Jones'; we consider vanishing subspaces, but get rid of all geometry by reintroducing an order of smoothness $\omega(t) > 0$.

Secondly, it follows from our results that a single bounded extension operator exists for the intersection $\dot{\mathrm{VC}}^{\omega}$ of the three vanishing scales. In particular, for the Hölder moduli $\alpha(t) := t^\alpha$, for $\alpha \in (0,1)$, [14, Theorem 1.13.] states that $\dot{\mathrm{VC}}^{\alpha}(\mathbb{R}^n) = \mathrm{VMO}^\alpha(\mathbb{R}^n)$, where the latter fractional vanishing mean oscillation space was originally introduced by Guo et al. [11, Theorem 1.7.] as the space that characterizes fractional compactness of commutators of singular integrals. (We remind that $\mathrm{VMO}^\alpha = \mathrm{CMO}^\alpha$ for Guo et al.) Hence the study of vanishing Hölder scales is also motivated by the needs of singular integrals. The classes $\dot{\mathrm{VC}}^{0,\alpha}_{\mathrm{small}}$ are also called the *little Hölder spaces.* These are fundamental in the study of Lipschitz algebras in metric spaces; we refer to the monographs by Weaver [17, Chapters 4 and 8], and for an exposition of these little Hölder spaces in the setting of Ahlfors regular sets we refer to D. Mitrea, I. Mitrea, and M. Mitrea [13, Chapter 3].

For the extension of $\dot{C}^{0,\omega}$ mappings between subsets of Hilbert spaces, preserving the $\dot{C}^{0,\omega}$-seminorms, see Grünbaum and Zarantonello [10], and the monograph [2] by Benyamini and Lindenstrauss. Concerning classical (and fundamental) results on $C^{m,\omega}$ extension of jets in $\mathbb{R}^n$, we refer to the work of Whitney [18] and Glaeser [9]. For Whitney-type extensions of jets generated by Sobolev functions in $\mathbb{R}^n$, see, for instance, the recent paper of Shvartsman [15]. For infinite dimensional results on $C^{1,\omega}$ extension of jets, see the recent paper [1] by Azagra and the second named author of the present paper. For extension results for functions (instead of jets) of order C^m or $C^{m,\omega}$, or for Sobolev functions, see for instance the papers by Brudnyi and Shvartsman [3], by Fefferman [6, 7], by Fefferman, Israel and Luli [8].

Acknowledgements. K.M. was supported by the Academy of Finland grant no. 21000054221. C.M. was supported by grant no. 334466 of the Research Council of Norway, "Fourier Methods and Multiplicative Analysis". T.O. was supported by the Finnish Academy of Science and Letters, and by the MICINN (Spain) grant no. PID2020-114167GB-I00.

1.2. **Basic definitions and main results.**

Definition 1.1. Let $\omega : (0,\infty) \to (0,\infty)$ be a modulus of continuity, $E \subset \mathbb{R}^n$ an arbitrary set, $m \in \mathbb{N} \cup \{0\}$, and V a normed space. By an m-jet on E (to V) we simply understand a family of $m+1$ functions $\{A_k : E \to \mathcal{L}^k(\mathbb{R}^n, V)\}_{k=0}^m$. For us $\mathcal{L}^k(\mathbb{R}^n, V)$ denotes the vector space of all symmetric k-linear mappings from $\mathbb{R}^n$ to V, and the norm we are using on $\mathcal{L}^k(\mathbb{R}^n, V)$ is the one given by

$$\|T\| := \sup\{\|T(u_1, \ldots, u_k)\|_V \ : \ u_1, \ldots, u_k \in \mathbb{S}^{n-1}\}.$$

Then we define *the trace jet space of* $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ *to* E, denoted by $\dot{\mathrm{J}}^{m,\omega}(E, V)$, as the vector space of all m-jets $\{A_k\}_{k=0}^m$ on E so that

$$\begin{aligned}&\|\{A_k\}_{k=0}^m\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)} \\ &:= \sup\left\{ \frac{\|A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^j\|}{\omega(|x-y|)|x-y|^{m-k}} \ : \ x, y \in E,\ k = 0, \ldots, m \right\} < \infty.\end{aligned} \tag{1}$$

It is clear that $\|\cdot\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)}$ defines a seminorm on $\dot{\mathrm{J}}^{m,\omega}(E,V)$, and it can be made an actual norm if we fix a point $x_0 \in E$ and define

$$|||\{A_k\}_{k=0}^m|||_{\mathrm{J}^{m,\omega}(E,V)} := \|\{A_k\}_{k=0}^m\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)} + \max_{k=0,\ldots,m} \|A_k(x_0)\|.$$

Moreover, if V is a Banach space, this norm is complete, and $\big(\mathrm{J}^{m,\omega}(E,V), |||\cdot|||_{\mathrm{J}^{m,\omega}(E,V)}\big)$ becomes a Banach space.

Remark 1.2. In the case $m=0$, the trace space $\dot{\mathrm{J}}^{0,\omega}(E,V)$ coincides with the homogeneous Hölder space $\dot{C}^{0,\omega}(E,V)$, consisting of those $f : E \to V$ such that

$$\|f\|_{\dot{C}^{0,\omega}(E,V)} = \sup_{\substack{x\neq y\\ x,y\in E}} \frac{\|f(x)-f(y)\|}{\omega(|x-y|)} < \infty.$$

Let us now look at functions that are defined everywhere in $\mathbb{R}^n$.

Definition 1.3. Let $\omega : (0,\infty) \to (0,\infty)$ be a modulus of continuity, $m \in \mathbb{N}\cup\{0\}$, and V a normed space. Then $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ consists of those $F : \mathbb{R}^n \to V$ of class $C^m(\mathbb{R}^n, V)$ so that

$$\|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} = \sup_{\substack{x\neq y\\ x,y\in \mathbb{R}^n}} \frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} < \infty.$$

Here $D^mF : \mathbb{R}^n \to \mathcal{L}^m(\mathbb{R}^n, V)$ denotes the mth (total) derivative of F. Under basic assumptions on the modulus ω, a version of Whitney's extension theorem says that every m-jet $\{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V)$ is *the restriction* of some $F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V)$ to E, in the sense that

$$D^kF(y) = A_k(y), \quad y \in E,\ k=0,\ldots,m.$$

We recall the construction of the Whitney extension operator and give appropriate references in Section 3.

In this article, we are insterested in subclasses of the space $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ given by the following vanishing conditions.

Definition 1.4. Let $\omega : (0,\infty) \to (0,\infty)$ be a modulus of continuity, $m \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$, and V a normed space. We define the vanishing scales

$$\dot{\mathrm{VC}}^{m,\omega}_{\mathrm{small}}(\mathbb{R}^n, V) := \Big\{F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V) \ :\ \lim_{\delta\to 0} \sup_{\substack{x\neq y\in E\\ |x-y|\leq\delta}} \frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} = 0\Big\},$$

$$\dot{\mathrm{VC}}^{m,\omega}_{\mathrm{large}}(\mathbb{R}^n, V) := \Big\{F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V) \ :\ \lim_{\delta\to \infty} \sup_{\substack{x\neq y\in E\\ |x-y|\geq\delta}} \frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-z|)} = 0\Big\},$$

$$\dot{\mathrm{VC}}^{m,\omega}_{\mathrm{far}}(\mathbb{R}^n, V) := \Big\{F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V) \ :\ \lim_{\delta\to \infty} \sup_{\min(|x|,|y|)>\delta} \frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} = 0\Big\}.$$

Finally we take the intersection of all the scales,

$$\dot{\mathrm{VC}}^{m,\omega}(\mathbb{R}^n, V) := \dot{\mathrm{VC}}^{m,\omega}_{\mathrm{small}}(\mathbb{R}^n, V) \cap \dot{\mathrm{VC}}^{m,\omega}_{\mathrm{far}}(\mathbb{R}^n, V) \cap \dot{\mathrm{VC}}^{m,\omega}_{\mathrm{large}}(\mathbb{R}^n, V).$$

Problem 1.5. *Given an m-jet $\{A_k : E \to \mathcal{L}^k(\mathbb{R}^n, V)\}_{k=0}^m$ on E, what necessary and sufficient conditions guarantee the existence of $F \in \dot{\mathrm{VC}}^{m,\omega}_{\Gamma}(\mathbb{R}^n, V)$ so that D^kF restricted to E agrees with A_k, for each $k=0,\ldots,m$?*

The answer to this problem is in the following definition. For an m-jet $\{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V)$, let us denote

$$R\left(\{A_k\}_{k=0}^m, x, y\right) := \max_{0\le k\le m} \frac{\|A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^j\|}{|x-y|^{m-k}}, \quad x, y \in E. \tag{2}$$

We show that the solution equals to assuming that the jet $\{A_k\}_{k=0}^m \in \dot{\mathrm{V}}\mathrm{J}^{m,\omega}(E,V)$ vanishes in an appropriate manner, as described in the next definition.

Definition 1.6. Let $\omega : (0,\infty) \to (0,\infty)$ be a modulus of continuity, $E \subset \mathbb{R}^n$ an arbitrary set, $m \in \mathbb{N} \cup \{0\}$, and V a normed space. We define the vanishing subspaces of jets

$$\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\text{small}}(E,V) := \Big\{ \{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V) \,:\, \lim_{\delta\to 0} \sup_{\substack{x\neq y\in E\\ |x-y|\le\delta}} \frac{R\left(\{A_k\}_{k=0}^m, x, y\right)}{\omega(|x-y|)} = 0 \Big\},$$

$$\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\text{large}}(E,V) := \Big\{ \{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V) \,:\, \lim_{\delta\to \infty} \sup_{\substack{x\neq y\in E\\ |x-y|\ge\delta}} \frac{R\left(\{A_k\}_{k=0}^m, x, y\right)}{\omega(|x-y|)} = 0 \Big\},$$

$$\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\text{far}}(E,V) := \Big\{ \{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V) \,:\, \lim_{\delta\to \infty} \sup_{\min(|x|,|y|)\ge\delta} \frac{R\left(\{A_k\}_{k=0}^m, x, y\right)}{\omega(|x-y|)} = 0 \Big\}.$$

And we define the intersection of the three,

$$\dot{\mathrm{V}}\mathrm{J}^{m,\omega}(E,V) := \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\text{small}}(E,V) \cap \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\text{large}}(E,V) \cap \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\text{far}}(E,V).$$

Our assumptions on the moduli are as follows. We always assume that ω is non-decreasing and satisfies

$$\lim_{t\to 0} \omega(t) = 0, \quad \text{and} \quad \frac{s}{\omega(s)} \le C_\omega \frac{t}{\omega(t)}, \quad \text{whenever} \quad 0 < s \le t < \infty, \tag{3}$$

where $C_\omega > 0$ is a fixed constant. We will also eventually assume the conditions

$$\lim_{t\to 0} \frac{t}{\omega(t)} = 0, \tag{4}$$

$$\lim_{t\to \infty} \omega(t) = \infty, \tag{5}$$

to deal with the small, and large and far scales, respectively. The following is our main result.

Theorem 1.7. *Let ω be a modulus of continuity satisfying* (3)*, $E \subset \mathbb{R}^n$ an arbitrary set, $m \in \mathbb{N} \cup \{0\}$, and V a Banach space. For an m-jet $\{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V)$, the following hold.*

(i) Provided that (4) *is satisfied, the jet $\{A_k\}_{k=0}^m$ admits an extension $F \in \dot{\mathrm{V}}\mathrm{C}^{m,\omega}_{\text{small}}(\mathbb{R}^n,V)$ if and only if $\{A_k\}_{k=0}^m \in \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\text{small}}(E,V)$.*

(ii) Provided that (5) *is satisfied, and $\Gamma \in \{\text{large}, \text{far}\}$, the jet $\{A_k\}_{k=0}^m$ admits an extension $F \in \dot{\mathrm{V}}\mathrm{C}^{m,\omega}_{\Gamma}(E,V)$ if and only if $\{A_k\}_{k=0}^m \in \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\Gamma}(\mathbb{R}^n,V)$.*

Consequently:

(iii) Assuming both (4) *and* (5)*, the jet $\{A_k\}_{k=0}^m$ admits an extension $F \in \dot{\mathrm{V}}\mathrm{C}^{m,\omega}(\mathbb{R}^n,V)$ if and only if $\{A_k\}_{k=0}^m \in \dot{\mathrm{V}}\mathrm{C}^{m,\omega}(E,V)$.*

Moreover, when E is closed, the result holds when V is merely a normed space, and these extensions can be defined via the linear Whitney extension operator.

In the particular case $m = 0$, we deduce the following.

Corollary 1.8. *Let ω be a modulus of continuity satisfying* (3)*, $E \subset \mathbb{R}^n$ be an arbitrary set and V a Banach space.*

(i) Provided that (4) *is satisfied, then* $f \in \dot{C}^{0,\omega}(E,V)$ *admits an extension* $F \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(\mathbb{R}^n,V)$ *if and only if* $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(E,V)$.

(ii) Provided that (5) *is satisfied, and* $\Gamma \in \{\text{large}, \text{far}\}$, *then* $f \in \dot{C}^{0,\omega}(E,V)$ *admits and extension* $F \in \dot{\mathrm{VC}}^{0,\omega}_{\Gamma}(\mathbb{R}^n,V)$ *if and only if* $f \in \dot{\mathrm{VC}}^{0,\omega}_{\Gamma}(E,V)$.

Consequently:

(iii) Consequently, assuming both (4) *and* (5), *then* $f \in \dot{C}^{0,\omega}(E,V)$ *admits an extension* $F \in \dot{\mathrm{VC}}^{0,\omega}(\mathbb{R}^n,V)$ *if and only if* $f \in \dot{\mathrm{VC}}^{0,\omega}(E,V)$.

Combining Corollary 1.8 with the approximations of globally defined functions [14, Corollary 3.14.] obtained by the last two named authors we deduce the following.

Corollary 1.9. *Let* ω *be a modulus of continuity satisfying* (3), (4) *and* (5). *Let* $E \subset \mathbb{R}^n$ *and* V *a normed space with either* E *closed or* V *Banach.*

- *If* $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(E,V)$ *and* $\varepsilon > 0$, *there exists* $G \in C^\infty(\mathbb{R}^n;V) \cap \dot{C}^{0,\omega}(\mathbb{R}^n,V)$ *such that*
$$\|f - G_{|E}\|_{\dot{C}^{0,\omega}(E,V)} < \varepsilon, \qquad \|G\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,V)} \lesssim \|f\|_{\dot{C}^{0,\omega}(E,V)}.$$
- *If* $f \in \dot{\mathrm{VC}}^{0,\omega}(E,V)$ *and* $\varepsilon > 0$, *there exists* $G \in C^\infty_c(\mathbb{R}^n;V) \cap \dot{C}^{0,\omega}(\mathbb{R}^n,V)$ *such that*
$$\|f - G_{|E}\|_{\dot{C}^{0,\omega}(E,V)} < \varepsilon, \qquad \|G\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,V)} \lesssim \|f\|_{\dot{C}^{0,\omega}(E,V)}.$$

2. Necessity of Theorem 1.7

In this section, we give the proof of the "only if" parts of Theorem 1.7. We begin with the following elementary observation.

Remark 2.1. If $\mathcal{A} \in \dot{\mathrm{J}}^{m,\omega}(E,V)$, then for every $r > 1$, one has
$$\max_{0 \le k \le m} \sup_{z \in E,\, |z| \le r} \|A_k(z)\| \le C(r,m,\omega,\mathcal{A},E) < \infty.$$

Proof. Write, for each $z \in E \cap B(0,r)$ and $z_0 \in E$ be arbitrary,
$$\begin{aligned}\|A_k(z)\| &\le \Big\| A_k(z) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z_0)(z-z_0)^j \Big\| + \Big\| \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z_0)(z-z_0)^j \Big\| \\ &\le \|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)} |z-z_0|^{m-k} \omega(|z-z_0|) + C(\mathcal{A},z_0,m) \max_{0 \le j \le m} |z-z_0|^j \\ &\lesssim (r + |z_0|)^{m-k} \left(\omega(r+|z_0|) + C(\mathcal{A},z_0,m) \right).\end{aligned}$$
□

The following lemma will be used both in this and the next section.

Lemma 2.2. *Let* ω *satisfy* (5), $E \subset \mathbb{R}^n$ *be arbitrary,* $m \in \mathbb{N} \cup \{0\}$, *and* V *a normed space. Then, we have*
$$\dot{\mathrm{VJ}}^{m,\omega}_{\mathrm{far}}(E,V) = \Big\{ \mathcal{A} \in \dot{\mathrm{J}}^{m,\omega}(E,V) \,:\, \lim_{\delta \to \infty} \sup_{\substack{x \ne y \in E \\ \max(|x|,|y|) \ge \delta}} \frac{R(\mathcal{A},x,y)}{\omega(|x-y|)} = 0 \Big\}.$$

Proof. Let $\mathcal{A} = \{A_k\}_{k=0}^m \in \dot{\mathrm{VJ}}^{m,\omega}(E,V)$ and $\varepsilon > 0$. Let $M = M(\varepsilon)$ be such that if $u, v \in E$ and $|u|, |v| > M$, then
$$R(\mathcal{A},u,v) < \varepsilon \omega(|u-v|).$$

Now we consider arbitrary $x, y \in E$, let $K \gg M$ be a large parameter that we will determine later, and let us assume

$$\max(|x|, |y|) \geq K \gg M \geq \min(|x|, |y|). \tag{6}$$

Notice also that the condition $\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\mathrm{far}}(E, V)$ is **not** vaguous if and only if E is unbounded. So for each $M > 0$, there is $z \in E \setminus B(0, M)$ (independent of x and y). Assume further that $K \gg |z|$ from now on. Bearing in mind that x, y satisfy (6), we start by checking the case $|x| \geq K$, and $|y| \leq M$ first. Because of the properties of K, M, x, y, and z, it is clear that

$$|y - z| \leq |x - y| \quad \text{and} \quad C^{-1}|x - y| \leq |x - z| \leq C|x - y|; \tag{7}$$

for some absolute constant C. Now, using that $x, z \in E \setminus B(0, M)$, the relations (7) and Remark 2.1, we have, for every $0 \leq k \leq m$,

$$\begin{aligned}
&\Big\| A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^j \Big\| \\
&\qquad \leq \Big\| A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z)(x-z)^j \Big\| + \Big\| \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(z) \left((y-z)^j - (x-z)^j \right) \Big\| \\
&\qquad \leq \varepsilon |x-z|^{m-k}\, \omega(|x-z|) + C(m) \max_{0 \leq j \leq m-k} \sup_{B(0,|z|)} \|A_{k+j}\| \left(|y-z|^j + |x-z|^j \right), \\
&\qquad \lesssim_{C_\omega, m, E, \|\mathcal{A}\|_{\dot{\mathrm{V}}\mathrm{J}^{m,\omega}(E,V)}} \varepsilon |x-y|^{m-k} \omega(|x-y|) + C(|z|, \mathcal{A}) |x-y|^{m-k}.
\end{aligned} \tag{8}$$

Since K was so that $K \gg |z|$, we assume that K is large to guarantee $C(|z|, \mathcal{A}) \leq \varepsilon \omega(K)$. Thus $C(|z|, \mathcal{A}) \lesssim_{C_\omega} \varepsilon \omega(|x - y|)$, and from (8), we derive the bound

$$\Big\| A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^j \Big\| \lesssim \varepsilon |x-y|^{m-k} \omega(|x-y|), \quad k = 0, \ldots, m. \tag{9}$$

Dividing by the term $|x-y|^{m-k}\omega(|x-y|)$, and taking the maximum among $k = 0, \ldots, m$, we arrive at $R(\mathcal{A}, x, y)/\omega(|x - y|) \lesssim \varepsilon$.

Let us now study the (non-symmetric) case $|y| \geq K$, and $|x| \leq M$. Let us write

$$\begin{aligned}
&\Big\| A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(y)(x-y)^j \Big\| \\
&\leq \Big\| \sum_{j=0}^{m-k} \frac{1}{j!} \Big(A_{k+j}(y) - \sum_{l=0}^{m-k-j} \frac{1}{l!} A_{k+j+l}(x)(y-x)^l \Big) (x-y)^j \Big\| \\
&\qquad + \Big\| A_k(x) - \sum_{j=0}^{m-k} \frac{1}{j!} \sum_{l=0}^{m-k-j} \frac{1}{l!} A_{k+j+l}(x)(y-x)^l (x-y)^j \Big\| \\
&\lesssim_m \sum_{j=0}^{m-k} \Big\| A_{k+j}(y) - \sum_{l=0}^{m-k-j} \frac{1}{l!} A_{k+j+l}(x)(y-x)^l \Big\| |x-y|^j + \max_{1 \leq l \leq m} \sup_{|u| \leq M} \|A_l(u)\| |x-y|^{m-k}.
\end{aligned} \tag{10}$$

For the first term in (10), we use the estimates (9) for each $j = 0, \ldots, m - k$, but swapping the roles of x and y (note that the estimates of (9) were obtained for $|x| \geq K$ and $|y| \leq M$). We thus obtain, for K large enough that

$$\sum_{j=0}^{m-k}\Big\|A_{k+j}(y)-\sum_{l=0}^{m-k-j}\frac{1}{l!}A_{k+j+l}(x)(y-x)^l\Big\||x-y|^j$$
$$\lesssim\sum_{j=0}^{m-k}\varepsilon|x-y|^{m-k-j}\omega(|x-y|)|x-y|^j\lesssim_m\varepsilon|x-y|^{m-k}\omega(|x-y|).$$

As for the second term in (10), recall that $K\gg M$ and so, bearing in mind property (5) and Remark 2.1, we can say that $\max_{1\le l\le m}\sup_{|u|\le M}\|A_l(u)\|\le\varepsilon\omega(K)\lesssim_{C_\omega}\varepsilon\omega(|x-y|)$. Dividing by $|x-y|^{m-k}\omega(|x-y|)$ and combining the above two cases, we conclude that

$$\frac{R(\mathcal{A},x,y)}{\omega(|x-y|)}=\max_{0\le k\le m}\frac{\|A_k(x)-\sum_{j=0}^{m-k}\frac{1}{j!}A_{k+j}(y)(x-y)^j\|}{|x-y|^{m-k}\omega(|x-y|)}\lesssim\varepsilon,$$

as desired. □

So, let us assume that $F\in\dot{\mathrm{V}}\mathrm{C}^{m,\omega}_{\Gamma}(\mathbb{R}^n,V)$ and prove that the restriction of F to E satisfies the properties from Definition 1.6. While the proof in the case $\Gamma=$ small is immediate from Taylor's theorem, for the scales $\Gamma=$ large and $\Gamma=$ far, we need to study a couple of subcases separately.

The case $\Gamma=$ small. Because $F\in\dot{C}^{m,\omega}(\mathbb{R}^n,V)$ we use Taylor's theorem to write, for each couple $x,y\in\mathbb{R}^n$ of distinct points, and each $k=0,\dots,m$:

$$\frac{\|D^kF(x)-\sum_{j=0}^{m-k}\frac{1}{j!}D^{k+j}F(y)(x-y)^j\|}{\omega(|x-y|)|x-y|^{m-k}}\le\frac{1}{(m-k)!}\sup_{z\in[x,y]}\frac{\|D^mF(z)-D^mF(y)\|}{\omega(|x-y|)}.\tag{11}$$

One can continue the estimate (11) by using that $D^mF\in\dot{C}^{0,\omega}(\mathbb{R}^n,V)$, thus deducing that the resctriction of $\{F,DF,\dots,D^mF\}$ to E defines an m-jet in $\dot{\mathrm{J}}^{m,\omega}(E,V)$. To show that those restrictions actually belong to $\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\mathrm{small}}(E,V)$ is also very easy: because $F\in\dot{\mathrm{V}}\mathrm{C}^{m,\omega}_{\mathrm{small}}(\mathbb{R}^n,V)$, given $\varepsilon>0$ there exists $\delta>0$ so that

$$\|D^mF(u)-D^mF(v)\|\le\varepsilon\omega(|u-v|),\quad |u-v|\le\delta.$$

Thus, assuming that $|x-y|\le\delta$, and as obviously then $|z-y|\le\delta$ for each $z\in[x,y]$, the right hand side of (11) is bounded from above by $C(m)\varepsilon$.

The case $\Gamma=$ large. Here we assume (5) for ω and that $F\in\dot{\mathrm{V}}\mathrm{C}^{m,\omega}_{\mathrm{large}}(\mathbb{R}^n,V)$, and estimate (11) in the following manner. For any $\varepsilon>0$, let $R>0$ so that $|u-v|\ge R$ implies $\|D^mF(u)-D^mF(v)\|\le\varepsilon\omega(|u-v|)$. Let us assume that $|x-y|\ge M$, where $M\gg R$ and its value will be specified in a moment. For those $z\in[x,y]$ so that $|z-y|\ge R$, it is enough to write

$$\frac{\|D^mF(z)-D^mF(y)\|}{\omega(|x-y|)}\le\varepsilon\frac{\omega(|z-y|)}{\omega(|x-y|)}\le\varepsilon.$$

For those $z\in[x,y]$ with $|z-y|<R$, we see that

$$\frac{\|D^mF(z)-D^mF(y)\|}{\omega(|x-y|)}\le\|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)}\frac{\omega(|z-y|)}{\omega(|x-y|)}\le\|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)}\frac{\omega(R)}{\omega(|x-y|)}.$$

Due to the assumption (5), if $M\gg R$ is large enough, then $|x-y|\ge M$ implies that the last term can be made smaller than ε. This shows that the jet given by the restriction of $\{F,DF,\dots,D^mF\}$ to E belongs to $\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\mathrm{large}}(E,V)$.

The case $\Gamma = $ far. Again we assume (5) for ω and that $F \in \dot{\mathrm{V}}\mathrm{C}^{m,\omega}_{\mathrm{far}}(\mathbb{R}^n, V)$. Applying Lemma 2.2 for $D^m F \in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{far}}(\mathbb{R}^n, \mathcal{L}^m(\mathbb{R}^n, V))$, it follows that for every $\varepsilon > 0$ there exists $R > 0$ so that if $u, v \in \mathbb{R}^n$ with $|u| \geq R$, then

$$\frac{\|D^m F(u) - D^m F(v)\|}{\omega(|u - v|)} \leq \varepsilon. \tag{12}$$

Let $x, y \in \mathbb{R}^n$ are such that $|x|, |y| \geq M$, for certain $M \gg R$ that we will determine later. Let $z \in [x, y]$ be as on the right-hand side of (11). In the case where $\max(|y|, |z|) \geq R$, then

$$\|D^m F(z) - D^m F(y)\| \leq \varepsilon\omega(|y - z|) \leq \varepsilon\omega(|x - y|),$$

by the bound (12). And when $|y|, |z| < R$, we can estimate as in the case $\Gamma = $ large:

$$\frac{\|D^m F(z) - D^m F(y)\|}{\omega(|x - y|)} \leq \|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n, V)} \frac{\omega(|z - y|)}{\omega(|x - y|)} \leq \|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n, V)} \frac{\omega(2R)}{\omega(|x - y|)}.$$

Since $|x - y| \geq M - R$, using condition (5), we can choose $M \gg R$ large enough so that the last term is smaller than ε. Thus, we have the desired estimate for all possible $z \in [x, y]$, and so (11) gives that the restriction of $\{F, DF, \ldots, D^m F\}$ to E belongs to $\dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\mathrm{far}}(E, V)$.

3. Sufficiency of Theorem 1.7

This section is devoted to the "if parts" of Theorem 1.7. We extend a jet from $\dot{\mathrm{J}}^{m,\omega}(E, V)$ to a $\dot{C}^{m,\omega}(\mathbb{R}^n, V)$ function while simultaneously preserving a given vanishing scale. As before, our modulus $\omega : (0, \infty) \to (0, \infty)$ is non-decreasing, with $\lim_{t \to 0} \omega(t) = 0$, and such that

$$\frac{s}{\omega(s)} \leq C_\omega \frac{t}{\omega(t)}, \quad \text{whenever} \quad 0 < s \leq t < \infty, \tag{13}$$

for some $C_\omega > 0$; or equivalently that

$$\omega(\lambda t) \leq C_\omega \lambda \omega(t), \quad \text{for all} \quad \lambda \geq 1,\, t > 0. \tag{14}$$

The first reduction in the extension Problem 1.5 is to notice that the unique continuous extension $\bar{\mathcal{A}}$ of $\mathcal{A} \in \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\Gamma}(E, V)$ to the closure $\bar{E}$ of E satisfies

$$\|\bar{\mathcal{A}}\|_{\dot{\mathrm{J}}^{m,\omega}(\bar{E}, V)} = \|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E, V)}, \quad \text{and} \quad \bar{\mathcal{A}} \in \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\Gamma}(\bar{E}, V), \tag{15}$$

for each scale $\Gamma \in \{\mathrm{small}, \mathrm{far}, \mathrm{large}\}$.

Indeed, this is possible since we are assuming that V is a Banach space, and thus the functions $A_k \in (A_1, \ldots, A_m) \in \mathcal{A}$ of a jet map Cauchy sequences of E to Cauchy sequences of $\mathcal{L}^k(\mathbb{R}^n, V)$.

Since V is Banach, so is $\mathcal{L}^m(\mathbb{R}^n, V)$. Thus if $\{x_j\}$ is Cauchy in E with a limit $\bar{x} \in \bar{E}$, then $\{A_m(x_j)\}$ is Cauchy in $\mathcal{L}^m(\mathbb{R}^n, V)$, by the estimate

$$\|A_m(x_j) - A_m(x_i)\| \lesssim \|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E, V)} \omega(|x_j - x_i|) \to 0, \quad \min(i, j) \to \infty.$$

Thus the limit $\bar{A}_m(\bar{x}) := A_m(\bar{x}) \in \mathcal{L}^m(\mathbb{R}^n, V)$ is uniquely determined and in this way we extend A_m to $\bar{E}$. Now the extension of the case $k = m - 1$, i.e. A_{m-1} as $\bar{A}_{m-1}$, follows by recursing from the case $k = m$ above and the definition of (1).

Thus for the extension problem, we can assume that $E \subset \mathbb{R}^n$ is closed. In fact, only here we need the completeness of V. Thus, if the given set $E \subset \mathbb{R}^n$ is closed, Theorem 1.7 holds for V a normed space.

Whitney partition of unity. We begin by recalling the main properties of the Whitney decomposition of an open set into cubes. For a closed set E, the Whitney cubes associated with the open set $\mathbb{R}^n \setminus E$ is a collection $\mathcal{Q}$ of dyadic cubes with the following properties:

- There holds that $\bigcup_{Q\in\mathcal{Q}} Q = \mathbb{R}^n \setminus E$.
- For every $Q \in \mathcal{Q}$, there holds that $d(Q,E) \le \operatorname{diam}(Q) \le 4d(Q,E)$.
- If $Q, Q' \in \mathcal{Q}$ are two distinct cubes, then $\operatorname{int}(Q) \cap \operatorname{int}(Q') = \varnothing$.

To construct a C^∞ partition of unity associated with these cubes, for each $Q \in \mathcal{Q}$ denote $Q^* := 9/8Q$ and let $p_Q \in E$ be a point (not necessarily unique) for which $d(E,Q) = d(p_Q,Q)$. A C^∞-Whitney partition of unity is, in particular, a collection of functions $\{\varphi_Q : Q \in \mathcal{Q}\}$ such that each φ_Q is supported on Q^*. Many relevant properties hold for these families of cubes and functions, and we refer to [16, Chapter VI] for a detailed exposition of this topic.

Proposition 3.1. *The Whitney partition of unity satisfies the following properties.*

(i) *There holds that*
$$\bigcup_{Q\in\mathcal{Q}} Q = \bigcup_{Q\in\mathcal{Q}} Q^* = \mathbb{R}^n \setminus E.$$
(ii) *There exists a dimensional constant $N(n) > 0$ so that*
$$\sum_{Q\in\mathcal{Q}} 1_{Q^*} \le N(n);$$
i.e. every $x \in \mathbb{R}^n \setminus E$ is contained in a neighbourhood that intersects at most $N(n)$ cubes of the family $\{Q^* : Q \in \mathcal{Q}\}$.

(iii) *There exists an absolute constant $C > 0$ so that for all cubes $Q \in \mathcal{Q}$ there holds that*
$$|p_Q - y| \le C|x-y|, \quad \text{whenever} \quad x \in Q^*,\, y \in E.$$
(iv) *For each $Q \in \mathcal{Q}$, there holds that*
$$0 \le \varphi_Q \le 1_{Q^*}, \qquad \varphi_Q \in C^\infty(\mathbb{R}^n).$$
(v) *Partition of unity: for all $x \in \mathbb{R}^n \setminus E$, there holds that*
$$\sum_{Q\in\mathcal{Q}} \varphi_Q(x) = \sum_{Q\,:\,Q^*\ni x} \varphi_Q(x) = 1, \qquad \sum_{Q\in\mathcal{Q}} D^k\varphi_Q(x) = 0, \quad k \in \mathbb{N}.$$
Notice that the latter follows from the first and (ii).

(vi) *For each $k \in \mathbb{N}$, there exists a constant $C(n,k)$ so that for all $Q \in \mathcal{Q}$ and $z \in Q^*$, there holds that*
$$\|D^k\varphi_Q(z)\| \le C(n,k)d(z,E)^{-k}.$$

Now, given a jet $\mathcal{A} = \{A_k\}_{k=0}^m \in \dot{\mathrm{J}}^{m,\omega}(E,V)$ and a point $y \in E$, we define the polynomial
$$P_y^{\mathcal{A}} : \mathbb{R}^n \to V, \qquad P_y^{\mathcal{A}}(x) = \sum_{k=0}^m \frac{1}{k!} A_k(y)(x-y)^k, \quad x \in \mathbb{R}^n.$$
And with these, the Whitney extension operator
$$\mathcal{W}(\mathcal{A})(x) = \begin{cases} A_0(x), & x \in E, \\ \sum_{Q\in\mathcal{Q}} \varphi_Q(x) P_{p_Q}^{\mathcal{A}}(x), & x \in \mathbb{R}^n \setminus E. \end{cases} \tag{16}$$

Using the properties (ii) and (iv), it is easy to see that $\mathcal{W}f \in C^\infty(\mathbb{R}^n \setminus E)$, and for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$, the α-partial derivative $D^\alpha(\mathcal{W}f) : \mathbb{R}^n \setminus E \to V$ is given by

$$D^\alpha(\mathcal{W}(\mathcal{A}))(x) = \sum_{Q\in\mathcal{Q}} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \varphi_Q(x) D^{\alpha-\beta} P^{\mathcal{A}}_{p_Q}(x), \quad x \in \mathbb{R}^n \setminus E. \tag{17}$$

Naturally, for any function $G : \mathbb{R}^n \to V$, the β-partial derivative $D^\beta G(x)$ at $x \in \mathbb{R}^n$ is defined as

$$D^\beta G(x) = \frac{\partial^{|\beta|} G}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(x) = D^{|\beta|} G(x)(e_1, \overset{\beta_1}{\ldots}, e_1, \ldots, e_n, \overset{\beta_n}{\ldots}, e_n),$$

where e_i is the ith unit vector.

A classical result is that for a $V = \mathbb{R}$ valued m-jet $\mathcal{A} \in \dot{\mathrm{J}}^{m,\omega}(E, V)$, there holds that

$$\mathcal{W}(\mathcal{A}) \in \dot{C}^{m,\omega}(\mathbb{R}^n, V), \qquad \|\mathcal{W}(\mathcal{A})\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} \leq \kappa(n, m, C_\omega) \|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)}, \tag{18}$$

for a constant $\kappa(n, m, C_\omega)$ depending only on n, m, and the constant C_ω from (13). A proof can be found in [16, Chapter VI, p.175]. However, a small modification of the arguments therein shows that (18) is satisfied for all $\mathcal{A} \in \dot{\mathrm{J}}^{m,\omega}(E, V)$, with an arbitrary normed space V.

Nevertheless, it seems very far from elementary to determine the action of the Whitney extension operator (16) over the three vanishing scales. For each scale, we study pairs (x, y) and their relative position with respect to E and $|x - y|$. Although this separation into cases will be the same for each scale, we split the proof of Theorem 1.7 into three parts, with each of the scales presenting its own particular difficulties.

During the proof we will fix an m-jet $\mathcal{A} \in \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_\Gamma(E, V)$, whose norm $\|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)} < \infty$ is considered to be an *absolute* constant. Moreover, we indicate the dependence on the dimension or on the order of smoothness m in some estimates by including subscripts, e.g. $\lesssim_n$, $\lesssim_m$, $\lesssim_{n,m}$. The same applies for the constant C_ω associated with the modulus ω from (13). Thus, when using the notation $A \lesssim B$, the constant C involved in the estimate $A \leq C \cdot B$ is allowed to depend on n, m, $\|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)}$ and C_ω.

To simplify notation, we denote by $F := \mathcal{W}(\mathcal{A})$ the Whitney Extension (16) and also $P_y = P^{\mathcal{A}}_y$, for each $y \in E$.

We will use several times the estimates of the following Lemma 3.2, which are implicitly proved in [16].

Lemma 3.2. *Let $y \in E$. Let $x \in \mathbb{R}^n \setminus E$ and $\xi = \xi(x) \in E$ be a point that minimizes $d(x, E) = |x - \xi|$. Then,*

(a) $\|D^m F(x) - A_m(y)\| \lesssim_{m,n} \sum_{Q^*\ni x} \Big(R(\mathcal{A}, y, p_Q) + R(\mathcal{A}, \xi, p_Q) \Big)$,

(b) $\|D^{m+1} F(x)\| \lesssim_{m,n} \sum_{Q^*\ni x} R(\mathcal{A}, \xi, p_Q) |x - \xi|^{-1}$.

Proof.

(a) Using multi-index notation, the polynomials P_y, $y \in E$, can be expressed as

$$P_y(x) := \sum_{k=0}^{m} \frac{1}{k!} A_k(y)(x - y)^k = \sum_{|\beta|\leq m} \frac{1}{\beta!} A_\beta(y)(x - y)^\beta, \tag{19}$$

where the A_βs are defined as follows. For any multi-index $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$ of order $|\beta| := \sum_j \beta_j \leq m$, and $y \in E$, we set

$$A_\beta(y) := A_{|\beta|}(y)(e_1, \overset{\beta_1}{\ldots}, e_1, \ldots, e_n, \overset{\beta_n}{\ldots}, e_n); \tag{20}$$

where e_i is the ith unit vector. Also, for any such β and $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, we denote

$$z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}.$$

The second identity in (19) follows from the symmetry and the k-linearity of the A_ks' and a basic combinatorial argument. Thus, for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$ of order $|\alpha| = m$, one has $D^\alpha P_y(x) = A_\alpha(y)$, for every $x \in \mathbb{R}^n$ and $y \in E$. Now, to show the point (a) it is enough to show that for each multi-index α of order $|\alpha| = m$, there holds that

$$\|D^\alpha F(x) - A_\alpha(y)\| \lesssim_{m,n} \sum_{Q^* \ni x} \Big(R(\mathcal{A}, y, p_Q) + R(\mathcal{A}, \xi, p_Q) \Big). \tag{21}$$

Then, using the multilinearity of $D^m F(x) - A_m(y)$, the bound (a) follows.

Also, by property (v), $\sum_Q D^\gamma \varphi_Q(x) = 0$ for any multi-index γ with $|\gamma| \geq 1$. These observations and formula (17) permit to write, for $x \in \mathbb{R}^n \setminus E$, $y \in E$, and for a multi-index α of order $|\alpha| = m$:

$$D^\alpha F(x) - A_\alpha(y) = \sum_Q D^\alpha(\varphi_Q \cdot P_{p_Q})(x) - A_\alpha(y) = \sum_Q \varphi_Q(x) \left(A_\alpha(p_Q) - A_\alpha(y) \right) +$$
$$+ \sum_Q \sum_{\beta \leq \alpha,\, \beta \neq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} \varphi_Q(x) \left(D^\beta P_{p_Q}(x) - D^\beta P_\xi(x) \right).$$

The first term is estimated using the definition of $R(\mathcal{A}, \cdot, \cdot)$ (2):

$$\Big\| \sum_Q \varphi_Q(x) \left(A_\alpha(p_Q) - A_\alpha(y) \right) \Big\| \leq \sum_{Q^* \ni x} \|A_\alpha(p_Q) - A_\alpha(y)\| \leq \sum_{Q^* \ni x} R(\mathcal{A}, y, p_Q).$$

For the second term, we take into account that the polynomials $P_{p_Q} - P_\xi$ have degree up to m, and so, for $|\beta| \leq m$,

$$\begin{aligned} D^\beta P_{p_Q}(x) - D^\beta P_\xi(x) &= \sum_{|\gamma| \leq m - |\beta|} \frac{1}{\gamma!} \Big[D^{\beta+\gamma}(P_{p_Q} - P_\xi)(\xi) \Big] (x - \xi)^\gamma \\ &= \sum_{|\gamma| \leq m - |\beta|} \frac{1}{\gamma!} \Big[D^{\beta+\gamma} P_{p_Q}(\xi) - A_{\beta+\gamma}(\xi) \Big] (x - \xi)^\gamma \\ &= \sum_{|\gamma| \leq m - |\beta|} \frac{1}{\gamma!} \left[\sum_{|\delta| \leq m - |\beta+\gamma|} \frac{1}{\delta!} A_{\delta+\beta+\gamma}(p_Q)(\xi - p_Q)^\delta - A_{\beta+\gamma}(\xi) \right] (x - \xi)^\gamma. \end{aligned}$$

Now from the formula (2), it follows that

$$\|D^\beta P_{p_Q}(x) - D^\beta P_\xi(x)\| \leq \sum_{j=0}^{m-|\beta|} R(\mathcal{A}, \xi, p_Q) |\xi - p_Q|^{m-|\beta|-j} |x - \xi|^j. \tag{22}$$

This estimate and property (vi), and also $|\alpha| = m$, lead us to

$$\begin{aligned} &\Big\| \sum_Q \sum_{\beta \leq \alpha,\, \beta \neq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} \varphi_Q(x) \left(D^\beta P_{p_Q}(x) - D^\beta P_\xi(x) \right) \Big\| \\ &\leq C(n,m) \sum_{Q^* \ni x} \sum_{\beta \leq \alpha,\, \beta \neq \alpha} |x - \xi|^{-(m-|\beta|)} \|D^\beta P_{p_Q}(x) - D^\beta P_\xi(x)\| \\ &\leq C(n,m) \sum_{Q^* \ni x} R(\mathcal{A}, \xi, p_Q) \sum_{k=0}^{m-1} \sum_{j=0}^{m-k} \left(\frac{|\xi - p_Q|}{|x - \xi|} \right)^{m-k-j} \leq C(n,m) \sum_{Q^* \ni x} R(\mathcal{A}, \xi, p_Q), \end{aligned} \tag{23}$$

where in the last bound we used that $|\xi - p_Q| \leq C|x - \xi|$, for an absolute constant C, by (iii).

(b) The proof is very similar to that of (a). Since the polynomials have degree up to m, if α is a multi-index with order $|\alpha| = m+1$, then

$$\begin{aligned} D^\alpha F(x) &= \sum_Q \sum_{\beta\le\alpha, |\beta|\le m} \binom{\alpha}{\beta} D^{\alpha-\beta}\varphi_Q(x) D^\beta P_{p_Q}(x) \\ &= \sum_Q \sum_{\beta\le\alpha, |\beta|\le m} \binom{\alpha}{\beta} D^{\alpha-\beta}\varphi_Q(x) \left(D^\beta P_{p_Q}(x) - D^\beta P_\xi(x)\right). \end{aligned}$$

By the estimate (22), the arguments that led us to (23) and to the conclusion of (a), we derive

$$\begin{aligned} \|D^\alpha F(x)\| &\le C(n,m) \sum_{Q^*\ni x} \sum_{k=0}^m \sum_{|\beta|=k} |x-\xi|^{-(m+1-k)} \|D^\beta P_{p_Q}(x) - D^\beta P_\xi(x)\| \\ &\le C(n,m) \sum_{Q^*\ni x} R(\mathcal{A}, \xi, p_Q)|x-\xi|^{-1}. \end{aligned}$$

Then, using the multilinearity of $D^{m+1}F(x)$ the bound (b) follows. □

Proof of Theorem 1.7 for $\Gamma = \text{small}$. Given $\mathcal{A} \in \dot{\mathrm{VJ}}^{m,\omega}_{\text{small}}(E,V)$, we show that $F \in \dot{\mathrm{VC}}^{m,\omega}_{\text{small}}(\mathbb{R}^n, V)$, i.e. that

$$\lim_{|x-y|\to 0} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} = 0,$$

uniformly on $x, y \in \mathbb{R}^n$. Denote

$$\mathrm{S}(t) := \sup\left\{\frac{R(\mathcal{A}, u, v)}{\omega(|u-v|)} \,:\, u, v \in E,\ 0 < |u-v| \le t\right\}, \tag{24}$$

understanding that $S(t) = 0$ if there is no such couple $u, v \in E$ with $0 < |u-v| \le t$. We have that $\mathrm{S}(t) \to 0$ as $t \to 0$ and $\mathrm{S}(t) \le \|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)}$. We distinguish three possible situations for any couple of distinct points $x, y \in \mathbb{R}^n$, where at least one of them is outside E.

Case 1. Assume $x \in \mathbb{R}^n \setminus E$ and $y \in E$.

We will use property (iii), which tells us that if $x \in Q^*$, then $|p_Q - y| \le C|x-y|$, for some absolute constant $C > 0$. Let $\xi = \xi(x) \in E$ minimize $d(x, E) = |x - \xi|$. Together with Lemma 3.2, and the definition of S, we find that

$$\begin{aligned} \|D^m F(x) - D^m F(y)\| &= \|D^m F(x) - A_m(y)\| \\ &\lesssim \sum_{Q^*\ni x} \Big(R(\mathcal{A}, y, p_Q) + R(\mathcal{A}, \xi, p_Q)\Big) \\ &\le \sum_{Q^*\ni x} (\mathrm{S}(|y-p_Q|)\omega(|y-p_Q|) + \mathrm{S}(|\xi-p_Q|)\omega(|\xi-p_Q|)) \\ &\lesssim_n \mathrm{S}(C|x-y|)\omega(|x-y|), \end{aligned}$$

where we used property (14) of ω in the last inequality. Therefore

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \mathrm{S}(C|x-y|), \tag{25}$$

and the right-hand side tends to zero as $|x-y| \to 0$.

Case 2. Assume that $x, y \in \mathbb{R}^n \setminus E$ and $d([x,y], E) \le |x-y|$. Pick $z \in [x,y]$ and $p \in E$ minimizing $d([x,y], E) = |z-p|$. Then,

$$|x-p| \le |x-z| + |z-p| \le |x-y| + |x-y| = 2|x-y|$$

and the same holds for $|y-p|$. Then $|x-p|, |y-p| \le 2|x-y|$ and applying the estimate (25) from Case 1 (for $x \in \mathbb{R}^n \setminus E$ and $p \in E$, and for $y \in \mathbb{R}^n \setminus E$ and $p \in E$) we obtain,

$$\begin{aligned}\|D^m F(x) - D^m F(y)\| &\le \|D^m F(x) - D^m F(p)\| + \|D^m F(p) - D^m F(y)\| \\ &\lesssim \mathrm{S}(2C|x-y|)\omega(2|x-y|),\end{aligned}$$

which is a bound of the correct form.

Case 3. Assume $x, y \in \mathbb{R}^n \setminus E$, and $d([x,y], E) \ge |x-y|$. Here we use the assumption $\lim_{t\to 0} t/\omega(t) = 0$.

Let us begin with a computation for arbitrary $z \in \mathbb{R}^n \setminus E$. Let $\xi = \xi(z) \in E$ minimize $d(z,E) = |z-\xi|$. Then, by the formula (16) and the properties (iii), (v) and (vi) we have

$$\|D^{m+1}F(z)\| \lesssim \sum_{Q^* \ni z} \frac{R(\mathcal{A}, \xi, p_Q)}{|z-\xi|} \le \sum_{Q^* \ni z} \frac{\omega(|p_Q - \xi|)\,\mathrm{S}(|p_Q-\xi|)}{d(z,E)} \lesssim_n \frac{\omega(d(z,E))\,\mathrm{S}(C\,d(z,E))}{d(z,E)}.$$

In the last inequality, together with (iii), we used the fact that every point of $\mathbb{R}^n \setminus E$ is contained in at most $N(n)$ cubes Q^* as well as property (14) of ω. As $D^m F$ is differentiable on $\mathbb{R}^n \setminus E$ and the segment $[x,y]$ lies outside E, by the mean value inequality (for normed-valued mappings) and the above estimate we obtain

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \le \frac{|x-y|}{\omega(|x-y|)} \sup_{z\in[x,y]} \|D^{m+1}F(z)\| \lesssim_n \sup_{z\in[x,y]} U_z(x,y); \tag{26}$$

where

$$U_z(x,y) := \frac{|x-y|}{\omega(|x-y|)} \frac{\omega(d(z,E))}{d(z,E)} \mathrm{S}(C\,d(z,E)).$$

Now we fix $z \in [x,y]$ and we consider the behaviour of $d(z,E)$ as $|x-y|$ tends to zero. By $\lim_{t\to 0} S(t) = 0$, we have the following: for any $\varepsilon > 0$, let $\delta > 0$ be so that $\mathrm{S}(t) \le \varepsilon$, whenever $t \le \delta$. First, if $C\,d(z,E) \le \delta$, then by $|x-y| \le d(z,E)$ and property (13) for ω, we obtain

$$U_z(x,y) \lesssim \mathrm{S}(C\,d(z,E)) \le \varepsilon.$$

On the other hand, if $C\,d(z,E) \ge \delta$, we use the property (13) of ω and that $\mathrm{S} \le \|\mathcal{A}\|_{\mathrm{J}^{m,\omega}(E,V)}$ to bound

$$U_z(x,y) \lesssim_\delta \frac{|x-y|}{\omega(|x-y|)} \cdot 1 \cdot \|\mathcal{A}\|_{\mathrm{J}^{m,\omega}(E,V)} \lesssim \frac{|x-y|}{\omega(|x-y|)};$$

now the right-hand side tends to zero as $|x-y|$ tends to 0, by the condition (4). We conclude from the above cases and (26) that $\|D^m F(x) - D^m F(y)\|/\omega(|x-y|) \to 0$ as $|x-y| \to 0$, uniformly. □

Proof of Theorem 1.7 for $\Gamma =$ large. Let $\mathcal{A} \in \dot{\mathrm{V}}\mathrm{J}^{m,\omega}_{\mathrm{large}}(E,V)$ and we show that $F \in \dot{\mathrm{V}}\mathrm{C}^{m,\omega}_{\mathrm{large}}(\mathbb{R}^n, V)$. Denote

$$\mathrm{L}(t) := \sup\left\{ \frac{R(\mathcal{A},u,v)}{\omega(|u-v|)} \,:\, u,v \in E,\ |u-v| \ge t \right\},$$

so that $L(t) \le \|\mathcal{A}\|_{\mathrm{J}^{m,\omega}(E,V)}$, L is non-increasing, and $L(t) \to 0$ as $t \to \infty$. We understand $S(t) = 0$ when there are no $u, v \in E$ with $|u-v| \ge t$.

Case 1. Let $x \in \mathbb{R}^n \setminus E$ and $y \in E$.

For each cube $Q \in \mathcal{Q}$ so that $x \in Q^*$, we have $|p_Q - y| \le C|x-y|$ for an absolute constant C; see (iii). Let $\xi = \xi(x) \in E$ minimize $d(x,E) = |x-\xi|$. In this case, although we are letting $|x-y| \to \infty$, we do not know whether $|p_Q - y|$ or $|p_Q - \xi|$ are large or not. Therefore, we need to study the behaviour of $|p_Q - z|$, for $z \in \{y, \xi\}$ as $|x-y| \to \infty$. If $|p_Q - z| \le M$ (for a constant M that we will specify in a moment), then we have the bound

$$\frac{R(\mathcal{A}, z, p_Q)}{\omega(|x-y|)} \le \|\mathcal{A}\|_{\dot{\mathrm{J}}^{m,\omega}(E,V)} \frac{\omega(|z-p_Q|)}{\omega(|x-y|)} \lesssim \frac{\omega(M)}{\omega(|x-y|)}. \tag{27}$$

On the other hand, if $|p_Q - z| \ge M$, then $\mathrm{L}(|p_Q - z|) \le \mathrm{L}(M)$ and so

$$\frac{R(\mathcal{A}, z, p_Q)}{\omega(|x-y|)} \le \mathrm{L}(|p_Q - z|) \frac{\omega(|p_Q - z|)}{\omega(|x-y|)} \le \mathrm{L}(M) \frac{\omega(C|x-y|)}{\omega(|x-y|)} \lesssim \mathrm{L}(M). \tag{28}$$

We have used that if $z \in \{y, \xi\}$, then $|p_Q - z| \le C|x-z| \le C|x-y|$, by virtue of (iii). Now, given $\varepsilon > 0$, we choose $M > 0$ large enough so that $\mathrm{L}(M) \le \varepsilon$. Using $\omega(\infty) = \infty$, we find $K > M$ such that $\omega(|x-y|) \ge \varepsilon^{-1}\omega(M)$, provided that $|x-y| \ge K$. Thus the bounds (27) and (28) tell us that regardless of the size of $|p_Q - z|$, we have

$$\frac{R(\mathcal{A}, z, p_Q)}{\omega(|x-y|)} \lesssim \varepsilon, \quad z \in \{y, \xi\}. \tag{29}$$

From the bound (29) and Lemma 3.2, we obtain

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \sum_{Q^* \ni x} \left(\frac{R(\mathcal{A}, y, p_Q)}{\omega(|x-y|)} + \frac{R(\mathcal{A}, \xi, p_Q)}{\omega(|x-y|)} \right) \lesssim_n \varepsilon,$$

whenever $|x-y| \ge K$.

Case 2. Assume $x, y \in \mathbb{R}^n \setminus E$, and $d([x,y], E) \le |x-y|$. Let $p = p(x,y) \in E$ minimize $d([x,y], E)$. Then, again, $|x-p|, |y-p| \le 2|x-y|$. Similarly, we saw already in Case 1 the following: given $\varepsilon > 0$, there exists $M > 0$ (independent of $x \in \mathbb{R}^n \setminus E$ and $p \in E$) such that: if $|x-p| \ge M$, then

$$\|D^m F(x) - D^m F(p)\| \le \varepsilon\omega(|x-p|) \le \varepsilon\omega(2|x-y|) \lesssim \varepsilon\omega(|x-y|). \tag{30}$$

Now, consider $|x-p| \le M$ and let $K > 0$ be so large that $\varepsilon\omega(K) > \omega(M)$. Now if also $|x-y| \ge K$, then we have

$$\|D^m F(x) - D^m F(p)\| \le \|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} \omega(|x-p|) \lesssim \omega(M) \le \varepsilon\omega(K) \le \varepsilon\omega(|x-y|). \tag{31}$$

We used above the fact that the Whitney extension is a bounded operator. Combining (30) and (31), we obtain

$$\sup_{|x-y|>K} \frac{\|D^m F(x) - D^m F(p)\|}{\omega(|x-y|)} \lesssim \varepsilon. \tag{32}$$

And the same bound holds with y in place of x; thus by triangle inequality,

$$\begin{aligned} &\sup_{|x-y|>K} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x-y|)} \\ &\le \sup_{|x-y|>K} \frac{\|D^m F(x) - D^m F(p)\|}{\omega(|x-y|)} + \sup_{|x-y|>K} \frac{\|D^m F(p) - D^m F(y)\|}{\omega(|x-y|)} \lesssim \varepsilon. \end{aligned}$$

Case 3. Assume $x, y \in \mathbb{R}^n \setminus E$ and $d([x,y], E) > |x-y|$.

For any $\varepsilon > 0$, let $M > 0$ be so that $L(M) \le \varepsilon$ and also choose $K \gg M$ so that $\varepsilon\omega(K) \ge \omega(M)$. For any two points $u, v \in E$, we have either $R(\mathcal{A}, u, v) \le \varepsilon\omega(|u-v|)$ (when $|u-v| \ge M$) or $R(\mathcal{A}, u, v) \le \|\mathcal{A}\|_{\dot{\mathrm{VJ}}^{m,\omega}(E,V)} \omega(M)$ (when $|u-v| \le M$). In other words,

$$R(\mathcal{A}, u, v) \lesssim \max\{\varepsilon\,\omega(|u-v|), \omega(M)\}, \qquad \text{for all} \quad u, v \in E. \tag{33}$$

Let $z \in [x,y]$ and let $\xi = \xi(z) \in E$ minimize $d(z,E) = |z-\xi|$. Employing first Lemma 3.2, then (33), and finally property (ii), we derive

$$\|D^{m+1}F(z)\| \lesssim_n \sum_{Q^*\ni z} \frac{R(\mathcal{A},\xi,p_Q)}{d(z,E)} \lesssim_n \max_{Q\in\mathcal{Q}\,:\,Q^*\ni z} \frac{R(\mathcal{A},\xi,p_Q)}{d(z,E)} \tag{34}$$

$$\le \max_{Q\in\mathcal{Q}\,:\,Q^*\ni z} \frac{\max\{\varepsilon\,\omega(|p_Q-\xi|),\omega(M)\}}{d(z,E)} \lesssim_n \frac{\max\{\varepsilon\,\omega(d(z,E)),\omega(M)\}}{d(z,E)}. \tag{35}$$

We used the fact that when $z \in Q^*$, then $|p_Q - \xi| \le C|z-\xi|$. Then (35) and the mean value inequality (for normed-valued mappings) give

$$\begin{aligned}\frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} &\le \frac{|x-y|}{\omega(|x-y|)} \sup_{z\in[x,y]} \|D^{m+1}F(z)\| \\ &\lesssim_n \sup_{z\in[x,y]} \frac{|x-y|}{\omega(|x-y|)} \frac{\max\{\varepsilon\,\omega(d(z,E)),\omega(M)\}}{d(z,E)}.\end{aligned} \tag{36}$$

By property (13) of ω and $d(z,E) \ge |x-y|$, for all $z \in [x,y]$, we continue bounding,

$$\text{RHS}\,(36) \lesssim \sup_{z\in[x,y]} \max\left\{\varepsilon, \frac{|x-y|}{\omega(|x-y|)}\frac{\omega(M)}{d(z,E)}\right\} \le \max\left\{\varepsilon, \frac{\omega(M)}{\omega(|x-y|)}\right\}. \tag{37}$$

Recalling that $\varepsilon\omega(K) \ge \omega(M)$ we conclude

$$\sup_{|x-y|>K} \frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} \lesssim \sup_{|x-y|>K} \max\left\{\varepsilon, \frac{\omega(M)}{\omega(|x-y|)}\right\} \le \varepsilon.$$

□

Proof of Theorem 1.7 when $\Gamma = \text{far}$. Here we assume that $\omega(\infty) = \infty$. We denote

$$\mathrm{D}(t) := \sup\left\{\frac{R(\mathcal{A},u,v)}{\omega(|u-v|)} \,:\, u,v\in E,\ \max(|u|,|v|) \ge t\right\},$$

so that $\mathrm{D}(t) \le \|\mathcal{A}\|_{\mathrm{J}^{m,\omega}(E,V)}$ and D is non-increasing, and moreover, thanks to Lemma 2.2, that $\mathrm{D}(t) \to 0$ as $t \to \infty$. We show that

$$\lim_{K\to\infty} \sup_{\substack{x,y\in\mathbb{R}^n \\ \min(|x|,|y|)\ge K}} \frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} = 0.$$

Case 1. Let $x \in \mathbb{R}^n \setminus E$, $y \in E$. Given $\varepsilon > 0$, let $M > 0$ be such that $\mathrm{D}(M) \le \varepsilon$. We consider $|x| \ge K \gg M$ for a large constant K to be soon determined. Let $\xi = \xi(x) \in E$ minimize $d(x,E) = |x-\xi|$. Applying Lemma 3.2 and bearing in mind the property (ii), we find a cube $Q_x \in \mathcal{Q}$ with $x \in (Q_x)^*$, and $z \in \{\xi, y\}$ so that

$$\begin{aligned}\frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} &\le \frac{\sum_{Q^*\ni x} R(\mathcal{A},y,p_Q) + R(\mathcal{A},\xi,p_Q)}{\omega(|x-y|)} \lesssim_n \frac{R(\mathcal{A},z,p_{Q_x})}{\omega(|x-y|)} \\ &\le \min\{\mathrm{D}(|p_{Q_x}|),\mathrm{D}(|z|),\|\mathcal{A}\|_{\mathrm{J}^{m,\omega}(E,V)}\}\frac{\omega(|p_{Q_x}-z|)}{\omega(|x-y|)}.\end{aligned} \tag{38}$$

The last inequality is a consequence of the definition of $\mathrm{D}(t)$. Now, if either $|p_{Q_x}| \ge M$ or $|z| \ge M$, the minimum in the term (38) is smaller than ε due to the choice of M. And because $|p_{Q_x} - z| \le C|x-y|$ for an absolute constant $C > 0$ (see (iii)); we conclude that (38) is bounded by an absolute multiple of ε in this particular case. And if $|p_{Q_x}| \le M$ and $|z| \le M$, then $|p_{Q_x} - z| \le 2M$, and so $\omega(|p_{Q_x} - z|) \lesssim \omega(M)$ and we bound

$$\text{RHS}\,(38) \lesssim \frac{\omega(M)}{\omega(||x| - M|)} \lesssim \frac{\omega(M)}{\omega\,(K - M)}.$$

By $\omega(\infty) = \infty$ the right-hand side is smaller than ε provided that $K \gg M$ is taken sufficiently large.

Case 2. Assume $x, y \in \mathbb{R}^n \setminus E$ and $d([x, y], E) \le |x - y|$.

Let $\varepsilon > 0$ and let $R > 0$ be as in Case 1. Suppose that $|x|, |y| \ge K$. Let $p \in E$ be such that $d([x, y], E) = d([x, y], p)$, and then $|x - p|, |y - p| \le 2|x - y|$, as we have already seen several times. By Case 1 applied to the pairs of points $x \in \mathbb{R}^n \setminus E$, $p \in E$ and $y \in \mathbb{R}^n \setminus E$, $p \in E$, we have

$$\|D^m F(x) - D^m F(p)\| \lesssim \varepsilon\omega(|x - p|), \qquad \|D^m F(y) - D^m F(p)\| \lesssim \varepsilon\omega(|y - p|).$$

Thus the claim follows by triangle inequality.

Case 3. Assume $x, y \in \mathbb{R}^n \setminus E$ and $d([x, y], E) \ge |x - y|$.

Let $\varepsilon > 0$, and $K \gg M$ be the parameters we used in Case 1. Note that we can enlarge K if necessary so as to satisfy

$$K \ge 2M \quad \text{and} \quad \omega(M) \le \varepsilon\omega(K). \tag{39}$$

Note that condition (5) guarantees the existence of such K. Let us also assume that $|x| \ge K$. Following the lines (34) and (36), we find $z \in [x, y]$, a point $\xi = \xi(z) \in E$ minimizing $d(z, E) = |z - \xi|$, and a cube $Q_z \in \mathcal{Q}$ with $z \in Q^*$ so that

$$\frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x - y|)} \lesssim_n \frac{|x - y|}{\omega(|x - y|)} \frac{R(\mathcal{A}, \xi, p_{Q_z})}{d(z, E)}. \tag{40}$$

First of all, observe that when $|\xi| \ge M$ or $|p_{Q_z}| \ge M$, then $R(\mathcal{A}, \xi, p_{Q_z}) \le \varepsilon\omega(|\xi - p_{Q_z}|)$, with $|\xi - p_{Q_z}| \le C|\xi - z| = Cd(z, E)$. Thus, in this particular subcase, using (13) we can estimate

$$\text{RHS}\,(40) \lesssim \frac{|x - y|}{\omega(|x - y|)} \frac{\omega(d(z, E))}{d(z, E)} \varepsilon \lesssim_{C_\omega} \varepsilon. \tag{41}$$

Therefore, we will assume from now on that $|\xi|, |p_{Q_z}| \le M$. Now, we look at the last numerator in (40); observe that, by the definition of $R(\mathcal{A}, \xi, p_{Q_z})$, and the fact that F extends the jet $\mathcal{A}$ from E to $\mathbb{R}^n$, we can find some $k \in \{0, \ldots, m\}$ and some $\widetilde{\xi} \in [\xi, p_{Q_z}]$ so that

$$\begin{aligned} R(\mathcal{A}, \xi, p_{Q_z}) &\le \frac{\|A_k(\xi) - \sum_{j=0}^{m-k} \frac{1}{j!} A_{k+j}(p_{Q_z})(\xi - p_{Q_z})\|}{|\xi - p_{Q_z}|^{m-k}} \\ &= \frac{\|D^k F(\xi) - \sum_{j=0}^{m-k} \frac{1}{j!} D^{k+j} F(p_{Q_z})(\xi - p_{Q_z})\|}{|\xi - p_{Q_z}|^{m-k}} \lesssim \|D^m F(\widetilde{\xi}) - D^m F(p_{Q_z})\|. \end{aligned} \tag{42}$$

Now, since $F \in \dot{C}^{m,\omega}(\mathbb{R}^n, V)$ with $\|F\|_{\dot{C}^{m,\omega}(\mathbb{R}^n,V)} \lesssim_{n,m,C_\omega} \|\mathcal{A}\|_{\mathrm{J}^{m,\omega}(E,V)}$, and $|\widetilde{\xi} - p_{Q_z}| \le |\xi - p_{Q_z}| \le 2M$, we can combine the estimates (40) and (42) to derive

$$\begin{aligned} \frac{\|D^m F(x) - D^m F(y)\|}{\omega(|x - y|)} &\lesssim_{n,m,C_\omega} \frac{|x - y|}{\omega(|x - y|)} \frac{\|D^m F(\widetilde{\xi}) - D^m F(p_{Q_z})\|}{d(z, E)} \\ &\lesssim_{\|\mathcal{A}\|_{\mathrm{J}^{m,\omega}(E,V)}} \frac{|x - y|}{\omega(|x - y|)} \frac{\omega(|\widetilde{\xi} - p_{Q_z}|)}{d(z, E)} \lesssim_{C_\omega} \frac{|x - y|}{\omega(|x - y|)} \frac{\omega(M)}{d(z, E)}. \end{aligned} \tag{43}$$

To complete the proof, observe that (39) gives

$$d(z, E) = |z - \xi| \ge |z| - |\xi| \ge K - M \ge K/2,$$

and $\omega(M) \le \varepsilon\omega(K) \lesssim_{C_\omega} \varepsilon\omega(d(z,E))$. By plugging this estimate into RHS(43) we obtain

$$\frac{\|D^mF(x)-D^mF(y)\|}{\omega(|x-y|)} \lesssim \frac{|x-y|}{\omega(|x-y|)}\frac{\omega(d(z,E))}{d(z,E)}\varepsilon \lesssim_{C_\omega} \varepsilon;$$

the last ibound being a consequence of (13) and $|x-y| \le d(z,E)$.

□

References

[1] D. Azagra and C. Mudarra, $C^{1,\omega}$ *extension formulas for* 1*-jets on Hilbert space*, Adv. Math. **389** (2021), Paper No. 107928, 44 pp.

[2] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[3] Y. Brudnyi and P. Shvartsman, *Whitney's extension problem for multivariate* $C^{1,\omega}$*-functions*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2487–2512.

[4] A. Butaev and G. Dafni, *Approximation and extension of functions of vanishing mean oscillation*, J. Geom. Anal. **31** (2021), no. 7, 6892–6921.

[5] ———, *Locally uniform domains and extension of bmo functions*, Ann. Fenn. Math. **48** (2023), no. 2, 567–594.

[6] C. Fefferman, *A sharp form of Whitney's extension theorem*, Ann. of Math. (2) **161** (2005), no. 1, 509–577.

[7] ———, *Whitney's extension problem for* C^m, Ann. of Math. (2) **164** (2006), no. 1, 313–359.

[8] C. Fefferman, A. Israel, and G.K. Luli, *Sobolev extension by linear operators*, J. Amer. Math. Soc. **27** (2014), no. 1, 69–145.

[9] G. Glaeser, *Étude de quelques algèbres tayloriennes*, J. Analyse Math. **6** (1958).

[10] F. Grünbaum and E.H. Zarantonello, *On the extension of uniformly continuous mappings*, Michigan Math. J. **15** (1968), 65–74.

[11] W. Guo, J. He, H. Wu, and D. Yang, *Boundedness and compactness of commutators associated with Lipshitz functions*, Anal. Appl. **20** (2021), no. 1, 1–37.

[12] P. W. Jones, *Extension theorems for BMO*, Indiana University Mathematics Journal **29** (1980), no. 1, 41–66.

[13] D. Mitrea, I. Mitrea, and M. Mitrea, *Geometric harmonic analysis II. Function spaces measuring size and smoothness on rough sets*, Developments in Mathematics, vol. 73, Springer, Cham, 2022.

[14] C. Mudarra and T. Oikari, *Approximation in Hölder spaces* (2024), available at `https://arxiv.org/abs/2401.01695`.

[15] P. Shvartsman, *Whitney-type extension theorems for jets generated by Sobolev functions*, Adv. Math. **313** (2017), 379–469.

[16] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[17] N. Weaver, *Lipschitz algebras*, Second Edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

[18] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89.