APPROXIMATION IN HÖLDER SPACES

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Abstract. For a modulus of continuity ω and normed spaces X, Y we introduce and study the subspaces $\dot{\mathrm{VC}}^{0,\omega}_{\Gamma}(X,Y)$ of vanishing scales $\Gamma \in \{\text{small}, \text{large}, \text{far}\}$ of the homogeneous Hölder space $\dot{C}^{0,\omega}(X,Y)$. For many couples X and Y, we characterize the subspaces that admit approximations by smooth, Lipschitz and boundedly supported functions in terms of these three vanishing scales. In the particular case $X = \mathbb{R}^n$, we identify $\dot{\mathrm{VC}}^{0,\omega}_{\Gamma}(\mathbb{R}^n, Y) = \mathrm{VMO}^{\omega}_{\Gamma}(\mathbb{R}^n, Y)$ with the corresponding vanishing mean oscillation spaces, thus transferring these approximations also to these spaces.

Key words: smooth approximation, vanishing mean oscillation, Hölder spaces, Banach spaces

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1. Introduction and main results

We consider normed spaces $X = (X, \|\cdot\|_X)$, $Y = (Y, \|\cdot\|_Y)$ and moduli of continuity $\omega : (0,\infty) \to (0,\infty)$ and study the Hölder spaces $\dot{C}^{0,\omega}(X,Y)$ that consist of those functions f for which

$$\|f\|_{\dot{C}^{0,\omega}(X,Y)} = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\|f(x)-f(y)\|_Y}{\omega(\|x-y\|_X)} < \infty.$$

These spaces are subspaces of uniformly continuous functions, in particular. We establish various full characterizations of approximability in the class $\dot{C}^{0,\omega}(X,Y)$ by Lipschitz and/or smooth and/or boundedly supported test functions. This is done in terms of the three vanishing scales that we define below, but before that let us make a few observations. It is easy to see that, if Y a Banach space, then $\dot{C}^{0,\omega}(X,Y)$ also becomes a Banach space when equipped with the norm

$$|||f|||_{C^{0,\omega}(X,Y)} = \|f\|_{\dot{C}^{0,\omega}(X,Y)} + \|f(0)\|_Y. \tag{1}$$

By dropping the term $\|f(0)\|_Y$, the seminorm $\|\cdot\|_{\dot{C}^{0,\omega}(X,Y)}$ becomes a norm when the functions $f \in \dot{C}^{0,\omega}(X,Y)$ are defined modulo constant equivalence classes. Moreover, in the particular case $\omega(t) = t^\alpha$, $\alpha \in (0,1)$, we recover the Hölder spaces $\dot{C}^{0,\alpha}(X,Y)$. Now, let us notate

$$\mathrm{osc}^{\omega}_{\delta}(f) = \sup_{\substack{x\neq y\in X\\ \|x-y\|_X=\delta}} \frac{\|f(x)-f(y)\|_Y}{\omega(\|x-y\|_X)}, \qquad \mathrm{osc}^{\omega}_{(x,y)}(f) = \frac{\|f(x)-f(y)\|_Y}{\omega(\|x-y\|_X)},$$

and then we define the three vanishing scales

$$\dot{\mathrm{V}}^{0,\omega}_{\text{small}}(X,Y) = \Big\{ f : X \to Y \,:\, \lim_{\delta\to 0} \mathrm{osc}^{\omega}_{\delta}(f) = 0 \Big\},$$

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$$\dot{\mathrm{V}}^{0,\omega}_{\mathrm{large}}(X,Y) = \Big\{ f : X \to Y \,:\, \lim_{\delta\to\infty} \operatorname{osc}^{\omega}_{\delta}(f) = 0 \Big\},$$

$$\dot{\mathrm{V}}^{0,\omega}_{\mathrm{far}}(X,Y) = \Big\{ f : X \to Y \,:\, \lim_{\delta\to\infty} \sup_{\min(\|x\|,\|y\|)>\delta} \operatorname{osc}^{\omega}_{(x,y)}(f) = 0 \Big\}.$$

Then, we define the vanishing subspaces of $\dot{C}^{0,\omega}(X,Y)$.

Definition 1.1. For each scale $\Gamma \in \{\mathrm{small}, \mathrm{large}, \mathrm{far}\}$, let

$$\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\Gamma}(X,Y) = \dot{\mathrm{V}}^{0,\omega}_{\Gamma}(X,Y) \cap \dot{C}^{0,\omega}(X,Y),$$

and

$$\dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X,Y) = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y) \cap \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{far}}(X,Y) \cap \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{large}}(X,Y).$$

Whenever no confusion is possible, we drop the source and target spaces in the norms and simplify notation with $\|x\|_X = \|x\|$ and $\|f(x)\|_Y = |f(x)|$; we also write $\dot{C}^{0,\omega}(X,Y) = \dot{C}^{0,\omega}(X) = \dot{C}^{0,\omega}$.

Remark 1.2. A few remarks concerning the above definitions.

- For $\Gamma \in \{\mathrm{small}, \mathrm{large}, \mathrm{far}\}$, the set $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\Gamma}$ is closed with respect to the $\dot{C}^{0,\omega}$-seminorm. In particular, $(\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\Gamma}, \|\cdot\|_{\dot{C}^{0,\omega}})$ are Banach spaces of modulo constant equivalence classes, provided the target space Y is a Banach space.
- By Lemma 2.1 at the beginning of Section 2 it follows that $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{far}} \subset \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{large}}$. The converse inclusion is not true, as shown by elementary examples. It follows a posteriori that the natural scale $\Gamma = \mathrm{large}$ becomes artificial for some of our results, see e.g. Theorem 1.4 below. Moreover, in general $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\Gamma} \neq \dot{\mathrm{V}}^{0,\omega}_{\Gamma}$ and $\dot{\mathrm{V}}^{0,\omega}_{\Gamma} \neq \dot{\mathrm{V}}^{0,\omega}_{\Gamma'}$, for $\Gamma \neq \Gamma'$. Interestingly then, when we take the intersection of all the scales, we have $\dot{\mathrm{V}}\mathrm{C}^{0,\omega} = \dot{\mathrm{V}}^{0,\omega}_{\mathrm{small}} \cap \dot{\mathrm{V}}^{0,\omega}_{\mathrm{far}} \cap \dot{\mathrm{V}}^{0,\omega}_{\mathrm{large}}$, and for the last inclusion " $\supset$ ", we need the large scale on the right-hand side. These facts are not crucial to our results and we leave their verification to the interested reader.

Our modulus satisfies the following properties

$$\lim_{t\to 0} \omega(t) = 0, \qquad \lim_{t\to\infty} \omega(t) = \infty, \tag{2}$$

$$\lim_{t\to 0} \frac{t}{\omega(t)} = 0, \tag{3}$$

and there exists a doubling constant $C_{\mathrm{db}} > 1$ such that

$$\omega(2t) \le C_{\mathrm{db}}\omega(t), \qquad t \in (0,\infty). \tag{4}$$

We mention explicitly whenever any of the modulus' properties (2), (3), (4) are used. Moreover, we always assume the modulus ω to be non-decreasing.

For the Hölder moduli $\alpha(t) := t^{\alpha}$, the classes $\dot{\mathrm{V}}\mathrm{C}^{0,\alpha}_{\mathrm{small}}$ are sometimes called the *little Hölder spaces*. These are fundamental in the study of Lipschitz algebras in metric spaces, and we refer to the monograph by Weaver [23, Chapters 4 and 8] for background. See also the recent monograph by D. Mitrea, I. Mitrea, and M. Mitrea [18, Chapter 3] for a detailed exposition of the Hölder spaces in the setting of Ahlfors regular sets, their connection with functions of vanishing mean oscillation, and for an introduction to the *little Hölder spaces*.

1.1. **Main results.** During the last half a century, several fundamental results have been established in the theory of uniform approximation of continuous functions by smooth functions. The optimal order of smoothness arrangeable for these approximations is often determined by the regularity of a canonical or an equivalent renorming of the spaces X and Y. For background references on smoothness and renorming on Banach spaces, and the related approximation results, we refer to the monographs [3] by Benyamini and Lindenstrauss; [6] by Deville, Godefroy, and Zizler; and [10] by Hájek and Johanis. Closely related to our resuls are the articles by Azagra, Fry, and Keener [1], Cepedello-Boiso [4], Fry [7], Hájek and Johanis [9, 10], Lasry and Lions [16], Moulis [19] and Pisier [21]. We next clarify that uniform convergence and $\dot{C}^{0,\omega}$ convergence do not follow from each other.

Remark 1.3. Endow $C^{0,\omega}(X,Y)$ with the norm $|||\cdot|||_{C^{0,\omega}}$ defined in (1). The next claims can be verified either directly from the definition or by elementary one-dimensional examples.

- The convergence with respect to the norm $|||\cdot|||_{C^{0,\omega}}$ implies uniform convergence on bounded subsets of X. But, in general, this convergence is not globally uniform on X, even when the sequence of functions involved is uniformly bounded on X.
- The converse to claim (a) is not true: uniform convergence does not imply convergence with respect to the norm $|||\cdot|||_{C^{0,\omega}}$, even if one only looks at bounded subsets of X for the norm $|||\cdot|||_{C^{0,\omega}}$, and even when the sequence of functions is uniformly bounded on X.

Our main result is the recognition of the scales $\{\text{small}, \text{large}, \text{far}\}$ as precisely the concept that allows a complete description of the closure of smooth, Lipschitz and/or boundedly supported functions with respect to the $\dot{C}^{0,\omega}$ seminorm. In order for the reader to familiarize with the conclusions of our theorems, we begin by stating a result (without smoothness) on approximations in $\dot{\mathrm{VC}}^{0,\omega}(X,Y)$ by $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)$ functions with bounded support.

Theorem 1.4. *Let X, Y be normed spaces, the modulus ω satisfy (2), (3) and (4). Then, there holds that*

$$\dot{\mathrm{VC}}^{0,\omega}(X,Y) = \overline{\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y) \cap C_{\mathrm{bs}}(X,Y)}^{\dot{C}^{0,\omega}(X,Y)}.$$

Here and below, for any class of functions $\mathcal{C}(X,Y) \subset \dot{C}^{0,\omega}(X,Y)$, we denote by $\overline{\mathcal{C}(X,Y)}^{\dot{C}^{0,\omega}(X,Y)}$ the closure of $\mathcal{C}(X,Y)$ under the $\dot{C}^{0,\omega}(X,Y)$ seminorm. Naturally, here we refer to the family of those $f \in \dot{C}^{0,\omega}(X,Y)$ for which there exists a sequence $\{f_n\}_n \subset \mathcal{C}(X,Y)$ so that $\lim_n \|f_n - f\|_{\dot{C}^{0,\omega}(X,Y)} = 0$. The subscripted set $\mathcal{C}_{\mathrm{bs}}(X,Y)$ consists of those boundedly supported $h : X \to Y$ in the class $\mathcal{C}$. In Theorem 1.4 the class $C_{\mathrm{bs}}(X,Y)$ denotes the space of continuous functions $f : X \to Y$ with bounded support, in particular. Naturally, the functions in $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)$ are continuous, and the intersection with $C_{\mathrm{bs}}(X,Y)$ indicates that the approximations have bounded support.

While Theorem 1.4 does not provide any smoothness in the approximations, it turns out to be a crucial step for the rest of our approximation theorems. For instance, letting $X = \mathbb{R}^n$ and Y be a Banach space, we combine Theorem 1.4 with elementary mollification arguments to obtain $C^\infty_{\mathrm{bs}}(\mathbb{R}^n, Y)$ approximations in Theorem 1.8 below and both theorems will be proved in Section 2.

Furthermore, our results are not confined to functions with a finite-dimensional source X, and letting X be a normed space and $Y = \mathbb{R}$, we prove smooth and Lipschitz approximation for the classes $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,\mathbb{R})$ and $\dot{\mathrm{VC}}^{0,\omega}(X,\mathbb{R})$. (The corresponding approximations for $\mathbb{C}^n$-valued functions, in place of $\mathbb{R}$-valued, follow on each occassion after splitting them into their n-coordinates and then each to their real and imaginary parts.) In particular, with X being a separable normed space, we have the following general result.

Theorem 1.5. *Let $k \in \mathbb{N} \cup \{\infty\}$ and X be a separable normed space admitting a C^k and Lipschitz bump function. Then, for a modulus ω satisfying (2), (3) and (4), the following hold:*

(i) $\overline{C^k(X) \cap \mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$,
(ii) $\overline{C^k_{\mathrm{bs}}(X) \cap \mathrm{Lip}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X)$.

Notice that, together with the C^k order of smoothness, Theorem 1.5 guarantees Lipschitz approximations. Any separable normed space X with an equivalent norm of class C^k satisfies the assumption of Theorem 1.5, see Remark 3.6. In the particular case where X is a separable Hilbert space, one can arrange C^∞ smooth approximations, thus obtaining an infinite dimensional version of Theorem 1.8, see Corollary 3.7. We will restate Theorem 1.5 and prove it in Subsection 3.1.

Another class of spaces X on which we obtain smooth approximations are super-reflexive Banach spaces. A theorem of Pisier [21] says that any super-reflexive space admits a renorming with *modulus of smoothness of power type* $1+\alpha$, for some $\alpha \in (0,1]$. See the property (33) below for the precise formulation. This class of spaces includes the Hilbert (separable or not) spaces, together with many of the classical Banach function spaces, such as the L^p spaces, with $1 < p < \infty$; see Remark 3.11. The following theorem establishes $C^{1,\alpha}$ and Lipschitz approximations for functions of the classes $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$ and $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$.

Theorem 1.6. *Let X be a super-reflexive space, and $\alpha \in (0,1]$ such that X admits a renorming with modulus of smoothness of power type $1+\alpha$. Then, for a modulus ω satisfying* (2)*,* (3) *and* (4)*, the following hold:*

(i) $\overline{C^{1,\alpha}(X) \cap \mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$,
(ii) $\overline{C^{1,\alpha}_{\mathrm{bs}}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X)$.

In Remark 3.11, we will see that in Hilbert spaces and in L^p spaces, with $p \geq 2$, Theorem 1.6 yields $C^{1,1}$ approximations, i.e., C^1 functions with *Lipschitz* first derivatives.

Finally, for mappings $f : X \to Y$, where Y is an arbitrary Banach space, in Subsection 3.3 we obtain C^∞ smooth approximations in the case where X is a space of the form $X = c_0(\mathcal{A})$.

Theorem 1.7. *For an arbitrary set of indices $\mathcal{A}$, let $X = c_0(\mathcal{A})$, and let Y be a Banach space. Then, for any modulus ω satisfying* (2)*,* (3) *and* (4)*, the following hold:*

(i) $\overline{C^\infty(X,Y) \cap \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y)}^{\dot{C}^{0,\omega}(X,Y)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y)$,
(ii) $\overline{C^\infty_{\mathrm{bs}}(X,Y) \cap \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X,Y)}^{\dot{C}^{0,\omega}(X,Y)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X,Y)$.

This theorem will be restated and proved in Subsection 3.3, see Theorem 3.12. There, we will also show that the approximations can be taken to be Lipschitz in the case $Y = \mathbb{R}$.

1.2. **Applications on $X = \mathbb{R}^n$.** We provide a brief discussion of how our results relate to compactness of commutators of singular integral operators (SIOs). Compact operators are in particular bounded, and for many non-degenerate bounded SIOs T and $p, q \in (1,\infty)$ there holds that

$$\|[b,T]\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \eqsim \|b\|_{X^{p,q}(\mathbb{R}^n)}, \quad X^{p,q}(\mathbb{R}^n) = \begin{cases} \dot{C}^{0,\alpha(p,q)}(\mathbb{R}^n), & q > p, \\ \mathrm{BMO}(\mathbb{R}^n) & q = p, \\ \dot{L}^{r(p,q)}(\mathbb{R}^n), & q < p, \end{cases} \tag{5}$$

where $[b,T]f = bTf - T(bf)$ is the commutator and $b : \mathbb{R}^n \to \mathbb{C}$ is called the symbol of the commutator. The appearing exponents are defined through $\alpha(p,q) = n(1/p - 1/q)$ and $1/q = 1/r(p,q) + 1/p$, and $\dot{C}^{0,\alpha(p,q)}(\mathbb{R}^n)$ is the inhomogeneous Hölder space, $\mathrm{BMO}(\mathbb{R}^n)$ stands for the space of bounded mean oscillations and $\dot{L}^{r(p,q)}(\mathbb{R}^n)$ is $L^{r(p,q)}(\mathbb{R}^n)$ modulo additive constants. Emphasizing the recognition of the correct conditions on the symbol, the bound (5) is attributable to Coifman, Rochberg and Weiss [5], when $q = p$; to Janson [14], when $q > p$; and to Hytönen [12], when $q < p$, to where we also refer for a history leading to (5).

If $\alpha(p,q) > 1$, when $q > p$, then $\dot{C}^{0,\alpha(p,q)}(\mathbb{R}^n)$ consists of the constant functions and the commutator is bounded iff it is the zero operator iff the symbol is constant. Therefore, regarding compactness, only the range $\alpha(p,q) \leq 1$ is interesting to study. When $\alpha(p,q) = 1$, Guo et al. [8, Theorem 1.6.] showed that the commutator is compact iff it is the zero operator, again. Now, with $p, q \in (1,\infty)$, and such that if $q > p$, then $\alpha(p,q) < 1$, for a wide class of non-degenerate SIOs there holds that

$$[b,T] \in \mathcal{K}\left(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n)\right) \Longleftrightarrow b \in Y^{p,q}(\mathbb{R}^n) := \overline{C_c^\infty(\mathbb{R}^n)}^{X^{p,q}(\mathbb{R}^n)}. \tag{6}$$

Emphasizing the recognition of the correct conditions on the symbol, (6) is due to Uchiyama [22], when $p = q$; due to Guo, He, Wu and Yang [8], when $q > p$; and due to Hytönen, Li, Tao and Yang [13], when $q < p$. We remark that it was noted recently [20] that in (6) the sufficiencies of $Y^{p,q}(\mathbb{R}^n)$ for commutator compactness in all three cases "$q < p$", "$q = p$" and "$q > p$" all follow easily from each other by classical interpolation of compactness methods. To obtain also the necessity directions on the line (6), it is important to have descriptions of $Y^{p,q}(\mathbb{R}^n)$, in the cases $q \geq p$, in terms of vanishing mean oscillation criteria. These criteria follow immediately from the existence of approximations, but to establish approximations from vanishing mean oscillation criteria takes more work. The existence of nice approximations beginning from vanishing type criteria is exactly the topic of the present article, remarkably. A particular corollary of our Theorem 1.4 is the following theorem.

Theorem 1.8. *Let Y be a Banach space and the modulus ω satisfy* (2), (3) *and* (4)*. Then, there holds that*

$$\dot{\mathrm{V}}\mathrm{C}^{0,\omega}(\mathbb{R}^n, Y) = \overline{C_c^\infty(\mathbb{R}^n, Y)}^{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)}.$$

To connect our pointwise conditions with vanishing mean oscillation type conditions and to establish the two descriptions of these spaces (one through approximations and one through vanishing mean oscillations) we next set some definitions. For simplicity, we let $X = \mathbb{R}^n$ be the Euclidean space, but Y is still allowed to be an arbitrary Banach space.

Definition 1.9. Let ω be a modulus and Y a Banach space. Then, $\mathrm{BMO}^\omega(\mathbb{R}^n, Y)$ consists of those $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, Y)$ locally Bochner integrable functions for which

$$\|f\|_{\mathrm{BMO}^\omega(\mathbb{R}^n)} = \sup_{Q\in\mathcal{Q}} \mathcal{O}^\omega(f;Q) < \infty, \qquad \mathcal{O}^\omega(f;Q) := \frac{1}{\omega(\ell(Q))} \fint_Q |f - \langle f\rangle_Q|_Y.$$

Definition 1.10. Let ω be a modulus and Y a Banach space and define the vanishing scales

$$\mathrm{VMO}^\omega_{\mathrm{small}}(\mathbb{R}^n, Y) := \Big\{ f \in \mathrm{BMO}^\omega(\mathbb{R}^n, Y) \,:\, \lim_{\delta\to 0} \sup_{\substack{Q\in\mathcal{Q},\\ \ell(Q)=\delta}} \mathcal{O}^\omega(f;Q) = 0 \Big\},$$

$$\mathrm{VMO}^\omega_{\mathrm{large}}(\mathbb{R}^n, Y) := \Big\{ f \in \mathrm{BMO}^\omega(\mathbb{R}^n, Y) \,:\, \lim_{\delta\to \infty} \sup_{\substack{Q\in\mathcal{Q},\\ \ell(Q)=\delta}} \mathcal{O}^\omega(f;Q) = 0 \Big\},$$

$$\mathrm{VMO}^\omega_{\mathrm{far}}(\mathbb{R}^n, Y) := \Big\{ f \in \mathrm{BMO}^\omega(\mathbb{R}^n, Y) \,:\, \lim_{\delta\to \infty} \sup_{\substack{Q\in\mathcal{Q},\\ d(Q,0)>\delta}} \mathcal{O}^\omega(f;Q) = 0 \Big\},$$

$$\mathrm{VMO}^\omega(\mathbb{R}^n, Y) := \mathrm{VMO}^\omega_{\mathrm{small}}(\mathbb{R}^n, Y) \cap \mathrm{VMO}^\omega_{\mathrm{far}}(\mathbb{R}^n, Y) \cap \mathrm{VMO}^\omega_{\mathrm{large}}(\mathbb{R}^n, Y).$$

One of the most interesting results in Guo et al. [8, Theorem 1.7.] states that $\mathrm{VMO}^{\alpha}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathrm{BMO}^{\alpha}(\mathbb{R}^n)}$, for $\alpha \in (0,1)$; this is exactly the equivalence between approximations and vanishing mean oscillations mentioned above. (We also note that $\mathrm{CMO}^{\alpha} = \mathrm{VMO}^{\alpha}$ for Guo et al.) Importantly for $\alpha = 0$, the approximation $\mathrm{VMO}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)}$ was already present in the work of Uchiyama [22]. Uchiyama's construction does not easily translate to the case $\alpha \in (0,1)$, and Guo et al. [8] provided a non-trivial new approximation that depends on the existence of a dyadic grid on the underlying space $\mathbb{R}^n$ (as does Uchiyama's original approximation) and a careful construction using this grid. One fact of our result 1.8, and in particular of its proof, is that it provides a conceptually easier approximation using only basic mollification and a (somewhat) basic truncation of the support.

Finally, to make the connection with the mean oscillations of [8, Theorem 1.7] and the pointwise conditions of Theorem 1.8 we assume the following summability condition

$$[\omega]_* := \sup_{s>0} \frac{1}{\omega(s)} \int_0^s \omega(t) \frac{\mathrm{d}t}{t} < \infty. \tag{7}$$

Theorem 1.11. *Let Y be a Banach space and the modulus ω satisfy $\omega(\infty) = \infty$, (4) and (7). Then, there holds that*

$$\frac{\|\cdot\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)}}{\|\cdot\|_{\mathrm{BMO}^{\omega}(\mathbb{R}^n,Y)}} + \frac{\|\cdot\|_{\mathrm{BMO}^{\omega}(\mathbb{R}^n,Y)}}{\|\cdot\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)}} \lesssim_{n,C_{\mathrm{db}}} [\omega]_*. \tag{8}$$

Moreover, for each $\Gamma \in \{\mathrm{small}, \mathrm{far}, \mathrm{large}\}$, there holds that

$$\dot{\mathrm{V}\mathrm{C}}^{0,\omega}_{\Gamma}(\mathbb{R}^n, Y) = \mathrm{VMO}^{\omega}_{\Gamma}(\mathbb{R}^n, Y), \qquad \dot{\mathrm{V}\mathrm{C}}^{0,\omega}(\mathbb{R}^n, Y) = \mathrm{VMO}^{\omega}(\mathbb{R}^n, Y). \tag{9}$$

The proof-idea of Theorem 1.11 of linking a pointwise difference with a telescoping sum of differences of averages traces back to Meyers [17], who proved a version of (8) with the modulus t^α, for $\alpha > 0$. It seems to us that the condition (7) is new, but it is natural summability condition, after all. To our best knowledge, the identification (9) is new. Combining Theorems 1.8 and 1.11, we obtain the following.

Theorem 1.12. *Let Y be a Banach space and the modulus ω satisfy (2), (3), (4) and (7). Then, there holds that*

$$\mathrm{VMO}^{\omega}(\mathbb{R}^n, Y) = \overline{C_c^\infty(\mathbb{R}^n, Y)}^{\mathrm{BMO}^{\omega}(\mathbb{R}^n,Y)}.$$

2. Bounded support approximation

In this section we prove Theorem 1.4 and then show how to easily obtain from this Theorem 1.8. At the end of this section, we also prove Theorem 1.11. Recall that we simplify notation by denoting $\|x\|_X = \|x\|$ and $\|f(x)\|_Y = |f(x)|$. We begin with two simple lemmas.

Lemma 2.1. *Let X, Y be arbitrary normed spaces. Let ω satisfy (2) and (4). Then, we have*

$$\dot{\mathrm{V}\mathrm{C}}^{0,\omega}_{\mathrm{far}}(X,Y) = \Big\{ f \in \dot{C}^{0,\omega}(X,Y) \,:\, \lim_{\delta\to\infty} \sup_{\max(\|x\|,\|y\|)>\delta} \mathrm{osc}^{\omega}_{(x,y)}(f) = 0 \Big\}.$$

Proof. Only the inclusion $\subset$ is not immediate; let $f \in \dot{\mathrm{V}\mathrm{C}}^{0,\omega}_{\mathrm{far}}(X,Y)$ and $\varepsilon > 0$. Let $M = M(\varepsilon)$ be such that if $y, z \in X$ and $\|y\|, \|z\| \geq M$, then $|f(y) - f(z)| \leq \varepsilon\omega(\|y - z\|)$. Now we consider arbitrary $x, y \in X$, and assume $\|y\| > R$, for certain $R \gg M$ to be specified later. If $\|x\| > M$, then we are done by how we fixed M above. So suppose that $x \in B(0, M)$. Take a point $z \in [x, y]$ with

$\|z\| = M$. By $\|x\|, \|z\| \le M$ and $\|y\|, \|z\| \ge M$,

$$\begin{aligned} \frac{|f(x)-f(y)|}{\omega(\|x-y\|)} &\le \frac{|f(x)-f(z)|}{\omega(\|x-y\|)} + \frac{|f(z)-f(y)|}{\omega(\|z-y\|)} \cdot \frac{\omega(\|z-y\|)}{\omega(\|x-y\|)} \\ &\le \frac{\|f\|_{\dot C^{0,\omega}(X)}\omega(\|x-z\|)}{\omega(\|x-y\|)} + \varepsilon \le \frac{\|f\|_{\dot C^{0,\omega}(X)}\omega(2M)}{\omega(\|x-y\|)} + \varepsilon. \end{aligned} \tag{10}$$

To control the above term, note that if $\|y\| > R \gg M$, then we have $\|x-y\| \ge R/2$ because $\|x\| \le M$. Also, by condition $\lim_{t\to\infty}\omega(t) = \infty$, we find $R = R(M,\varepsilon)$ so that $\omega(2M) \le \varepsilon\omega(R)$. Therefore, since $\|y\| \ge R$, using the doubling condition (4) we conclude

$$\mathrm{RHS}\,(10) \le \frac{\|f\|_{\dot C^{0,\omega}(X)}\omega(2M)}{\omega(R/2)} + \varepsilon \le \frac{\|f\|_{\dot C^{0,\omega}(X)}\omega(2M)}{C_{\mathrm{db}}^{-1}\omega(R)} + \varepsilon \le (1 + C_{\mathrm{db}}\|f\|_{\dot C^{0,\omega}(X)})\varepsilon.$$

□

Lemma 2.2. *Let X, Y be arbitrary normed spaces. Let ω be non-decreasing and satisfy (4). Let $\tau : X \to X$ be Lipschitz. Then,*

$$\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y) \circ \tau \subset \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y).$$

Proof. Let $\varepsilon > 0$ and we need to show that if r is taken sufficiently small, then

$$\|f\circ\tau\|_{\dot C^{0,\omega}_r(X,Y)} := \sup_{\substack{x\ne y\in X\\ \|x-y\|<r}} \frac{|(f\circ\tau)(x)-(f\circ\tau)(y)|}{\omega(\|x-y\|)} \le \varepsilon. \tag{11}$$

Let us denote $\varepsilon(r) := \|f\|_{\dot C^{0,\omega}_r}$ so that $\varepsilon(r) \to 0$ as $r \to 0$. Let $\|x-y\| < r$ and by ω being non-decreasing and doubling we bound

$$\begin{aligned} \frac{|(f\circ\tau)(x)-(f\circ\tau)(y)|}{\omega(\|x-y\|)} &= \frac{|f(\tau(x))-f(\tau(y))|}{\omega(\|\tau(x)-\tau(y)\|)} \frac{\omega(\|\tau(x)-\tau(y)\|)}{\omega(\|x-y\|)} \\ &\le \varepsilon(\mathrm{Lip}(\tau)r)\cdot C(C_{\mathrm{db}}, \mathrm{Lip}(\tau)), \end{aligned}$$

and the right-hand side tends to zero as r does. □

Proof of Theorem 1.4. First, note that if $f \in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y)$ and has bounded support, then, by property (3) of ω, f belongs to $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X,Y)$. Since $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X,Y)$ is closed under limits with respect to the seminorm $\dot C^{0,\omega}$, the inclusion $\overline{\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y) \cap C_{\mathrm{bs}}(X,Y)} \subset \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X,Y)$ holds true.

To prove the reverse (much more complicated) inclusion, let $f \in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}$ and $\varepsilon > 0$. By Lemma 2.1 we find two parameters $0 < r \ll R$ such that

$$\sup_{\substack{x\ne y\in X\\ \|x-y\|\le r}} \frac{|f(x)-f(y)|}{\omega(\|x-y\|)} \le \varepsilon, \tag{12}$$

and

$$\sup_{x\in B(0,R)^c}\sup_{y\in X} \frac{|f(x)-f(y)|}{\omega(\|x-y\|)} \le \varepsilon. \tag{13}$$

We consider an auxiliary parameter $M \gg R$ and the following function

$$\tau = \tau_M : X \to B(0,M), \qquad \tau(x) = \begin{cases} x, & \|x\| < M, \\ \left(\frac{2M-\|x\|}{M}\right)^2 x, & M \le \|x\| < 2M, \\ 0, & \|x\| \ge 2M. \end{cases} \tag{14}$$

We next show that if r is small enough and R, M are large enough then

$$f\circ\tau \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}, \qquad \|f - f\circ\tau\|_{\dot{C}^{0,\omega}} \lesssim \varepsilon. \tag{15}$$

Notice that $f\circ\tau - f(0)$ is zero outside $B(0,2M)$, and hence $f\circ\tau$ has bounded support. (Recall that $\dot{C}^{0,\omega}$ is defined modulo constant equivalence classes.) To actually verify the claims of (15) we begin checking some properties of the function τ.

First, we verify

$$\|\tau(x)-\tau(z)\| \le 5\|x-z\| \quad \text{for all} \quad x,z\in X, \tag{16}$$

and in particular by doing so, Lemma 2.2 implies the left claim on the line (15). Indeed, for points $x,z\in B(0,2M)\setminus B(0,M)$, the definition of τ and the triangle inequality give

$$\begin{aligned}
\|\tau(x)-\tau(z)\| &\le \Big| \Big(2-\frac{\|x\|}{M}\Big)^2 - \Big(2-\frac{\|z\|}{M}\Big)^2 \Big| \|z\| + \Big(2-\frac{\|x\|}{M}\Big)^2 \|x-z\| \\
&\le \Big(2-\frac{\|x\|}{M}+2-\frac{\|z\|}{M}\Big)\frac{\|x-z\|}{M}\|z\| + \Big(2-\frac{\|x\|}{M}\Big)^2\|x-z\| \\
&\le \Big(4-\frac{2M}{M}\Big)\frac{\|x-z\|}{M}(2M) + \Big(2-\frac{M}{M}\Big)^2\|x-z\| = 5\|x-z\|,
\end{aligned} \tag{17}$$

where in the second bound we used the basic identity $a^2-b^2=(a-b)(a+b)$. As τ is obviously 1-Lipschitz in the sets $B(0,M)$ and $B(0,2M)^c$, and τ is continuous in X, we deduce (16) for all $x,z\in X$.

Then we show the following contraction property:

$$\Big\{x\in B(0,M)^c, \quad \tau(x),\tau(z)\in B(0,R)\Big\} \Longrightarrow \|\tau(x)-\tau(z)\| \lesssim R\,\theta(M)\|x-z\|, \tag{18}$$

where $\theta:(0,\infty)\to(0,\infty)$ is a function satisfying $\lim_{M\to\infty}\theta(M)=0$, whose definition will be specified while we next check the truth of the implication[1]. If $\|x-z\|>M/2$, then we have

$$\|\tau(x)-\tau(z)\| \le 2R < 2R\frac{\|x-z\|}{M/2} = 4R\,\theta_1(M)\|x-z\|, \qquad \theta_1(M) := \frac{1}{M}.$$

So next assume that $z\in B(x,M/2)$. Since $R\ll M$ and $\tau(z)\in B(0,R)$, by the definition of τ we must have necessarily $\|z\|>M$. Thus, it remains to show that

$$\Big\{x,z\in B(0,M)^c, \quad \tau(x),\tau(z)\in B(0,R)\Big\} \Longrightarrow \|\tau(x)-\tau(z)\| \lesssim R\,\theta(M)\|x-z\|, \tag{19}$$

which is symmetric with respect to both variables x,z. Notice that if $x,z\in B(0,2M)^c$, then the left-hand side of the claimed estimate is zero. Assume that $x,z\in B(0,2M)\setminus B(0,M)$ and we make the following observation. By $\tau(x)\in B(0,R)$ and $x\in B(0,M)^c$, there holds that

$$\Big\| \Big(\frac{2M-\|x\|}{M}\Big)^2 x\Big\| \le R \Longrightarrow \Big|\frac{2M-\|x\|}{M}\Big| \le \sqrt{\frac{R}{\|x\|}} \le \sqrt{\frac{R}{M}} = \sqrt{R}\,\theta_2(M), \qquad \theta_2(M) := \frac{1}{\sqrt{M}},$$

and the same bound is valid for the variable z. Thus, by (17) we obtain

$$\begin{aligned}
\|\tau(x)-\tau(z)\| &\le \Big(2-\frac{\|x\|}{M}+2-\frac{\|z\|}{M}\Big)\frac{\|x-z\|}{M}\|z\| + \Big(2-\frac{\|x\|}{M}\Big)^2\|x-z\| \\
&\le \Big(4\sqrt{\frac{R}{M}}+\frac{R}{M}\Big)\|x-z\| \le 5R\,\theta_2(M)\|x-z\|.
\end{aligned}$$

[1]We remark that (18) would not be true, if we replaced the power 2 in the definition (14) of τ by the power 1.

As the last case, suppose that $z \in B(0,2M)^c$. This case follows by the continuity of τ and the previous case. Indeed, taking a point $y \in [x,z]$ with $\|y\| = 2M$, by the previous case applied for x and y, and bearing in mind that $\tau(z) = \tau(y) = 0$, we deduce

$$\|\tau(x) - \tau(z)\| = \|\tau(x) - \tau(y)\| \leq 5R\,\theta_2(M)\|x-y\| \leq 5R\,\theta_2(M)\|x-z\|.$$

All in all, we have now shown that (18) is valid with the function $\theta : (0,\infty) \to (0,\infty)$ given by

$$\theta(M) := \frac{1}{\sqrt{M}} \geq \frac{1}{2}(\theta_1(M) + \theta_2(M)). \tag{20}$$

Now, let us denote $g := f \circ \tau \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}$ and let $r' > 0$ be such that

$$\|g\|_{\dot{C}^{0,\omega}_{r'}} := \sup_{\substack{x \neq y \in X \\ \|x-y\| < r'}} \frac{|g(x) - g(y)|}{\omega(\|x-y\|)} \leq \varepsilon. \tag{21}$$

By (12), the fact that $\mathrm{Lip}(\tau) \leq 5$ (see (16)), and the proof of Lemma 2.2, this new $r' > 0$ can be taken to be an absolute multiple of the r in (12). Therefore, we may and do assume that both (12) and (21) hold for the same $r = r(\varepsilon)$, which is of course independent of R and M.

Next we show that

$$\sup_{x \in B(0,M)^c} \sup_{z \in X} \frac{|g(x) - g(z)|}{\omega(\|x-z\|)} \lesssim \varepsilon, \tag{22}$$

provided M is large enough. By (21), it is enough to estimate (22) only at those couples x,z with $\|x-z\| \geq r$. Fix a point $x \in B(0,M)^c$ and we distinguish into cases. If $\tau(x) \in B(0,R)^c$, then by (13) and ω being non-decreasing and doubling we obtain

$$\begin{aligned} |g(x) - g(z)| &= |f(\tau(x)) - f(\tau(z))| \\ &\leq \left(\sup_{u \in B(0,R)^c} \sup_{v \in X} \frac{|f(u) - f(v)|}{\omega(\|u-v\|)} \right) \omega(\|\tau(x) - \tau(z)\|) \lesssim_{C_{\mathrm{db}}} \varepsilon\omega(\|x-z\|). \end{aligned} \tag{23}$$

The case $\tau(z) \in B(0,R)^c$ is symmetrical.

Now consider $\tau(x), \tau(z) \in B(0,R)$ and we split into subcases. But first observe that for those x,z with $\|x-z\| \geq M^{1/4}$, using that $f \in \dot{C}^{0,\omega}$ we can write

$$|g(x) - g(z)| = |f(\tau(x)) - f(\tau(z))| \leq \|f\|_{\dot{C}^{0,\omega}} \omega(|\tau(x) - \tau(z)|) \leq \|f\|_{\dot{C}^{0,\omega}} \omega(2R).$$

Provided that $M = M(\varepsilon, R)$ is large enough, the condition $\omega(\infty) = \infty$ says that the last term can be made smaller than $\varepsilon\omega(M^{1/4}) \leq \varepsilon\omega(\|x-z\|)$, and thus (22) holds for these x and z. Therefore, during the rest of the verification of (22) we can assume that $\|x-z\| \leq M^{1/4}$, whenever needed.

Notice also that in the case $x,z \in X \setminus B(0,2M)$ there is nothing to prove, as then $g(x) = g(z) = 0$.

Thus, we can assume that both $x,z \in B(0,10M)$. In this case we bound

$$\begin{aligned} |g(x) - g(z)| &= |f(\tau(x)) - f(\tau(z))| \\ &\lesssim \|f\|_{\dot{C}^{0,\omega}(B(0,2R))} \omega(\|\tau(x) - \tau(z)\|) \lesssim \omega(\|\tau(x) - \tau(z)\|). \end{aligned} \tag{24}$$

Let $\theta_{x,z}(M)$ be defined by the identity

$$\|\tau(x) - \tau(z)\| =: \theta_{x,z}(M)\|x-z\|, \qquad \theta_{x,z}(M) \lesssim R/\sqrt{M},$$

where the bound follows by (18) and (20). By the condition $\lim_{t\to 0} \omega(t) = 0$, we choose $M = M(\varepsilon, r, R)$ sufficiently large so that $\omega(RM^{-1/4}) \leq \varepsilon\omega(r)$. Since we are assuming $r \leq \|x-z\| \leq M^{1/4}$, we obtain

$$\begin{aligned}\mathrm{RHS}\,(24) = \omega(\theta_{x,z}(M)\|x-z\|) &\lesssim_{C_{\mathrm{db}}} \omega\left(\frac{R}{\sqrt{M}}\|x-z\|\right) \\ &\le \omega\left(\frac{R}{\sqrt{M}}M^{1/4}\right) \le \varepsilon\omega(r) \le \varepsilon\omega(\|x-z\|).\end{aligned} \tag{25}$$

Chaining the estimates (24) and (25), we have shown (22) for x, z as in the present case.

Finally we consider the case $\tau(x), \tau(z) \in B(0,R)$, and $x \in B(0,2M) \setminus B(0,M)$ and $z \in X \setminus B(0,10M)$. We pick an auxiliary point y such that $|y| = 2M$. Using $\tau(z) = \tau(y) = 0$, the bound (25) (valid by $x, y \in B(0,10M)$) and that ω is non-decreasing we obtain

$$|g(x)-g(z)| = |g(x)-g(y)| \le \|f\|_{\dot{C}^{0,\omega}(X)}\omega(\|\tau(x)-\tau(y)\|) \lesssim \varepsilon\omega(\|x-y\|) \le \varepsilon\omega(\|x-z\|),$$

which is a bound of the correct form. The case $\tau(x), \tau(z) \in B(0,R)$ and $z \in B(0,2M)$ and $x \in X \setminus B(0,10M)$ is symmetrical to the last one.

We now fix the parameters $M \gg R \gg r$ so that both (21) and (22) hold. Then, we show the right claim on the line (15). If $x, z \in B(0,M)$ there is nothing to prove, since $\tau(x) = x$ and $\tau(z) = z$. Finally, we check that

$$\begin{aligned}&\sup_{x\in B(0,M)^c}\sup_{z\in\mathbb{R}^n}\frac{|(f-g)(x)-(f-g)(z)|}{\omega(\|x-z\|)} \\ &\quad\le \sup_{x\in B(0,M)^c}\sup_{z\in X}\frac{|f(x)-f(z)|}{\omega(\|x-z\|)} + \sup_{x\in B(0,M)^c}\sup_{z\in X}\frac{|g(x)-g(z)|}{\omega(\|x-z\|)} \lesssim \varepsilon,\end{aligned}$$

and hence have now shown both claims on the line (15). □

We next show how to easily upgrade the above bounded support approximation into a compact and smooth approximation on $X = \mathbb{R}^n$, but Y is allowed to be an arbitrary Banach space.

Proof of Theorem 1.8. Let $f \in \dot{\mathrm{VC}}^{0,\omega}(\mathbb{R}^n, Y)$ and by Theorem 1.4 find $g \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(\mathbb{R}^n, Y)$ with bounded support such that $\|f-g\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)} < \varepsilon$. Let $r > 0$ and $0 \le \eta_r \in C_c^\infty(B(0,r))$ be a standard real-valued smooth bump in $\mathbb{R}^n$ with $\int_{\mathbb{R}^n}\eta_r = 1$, and we define $h_r := g * \eta_r$, where the integral is understood in the Bochner sense. Notice that h_r is compactly supported, as g is, and smooth $h_r \in C^\infty(\mathbb{R}^n, Y)$ as a smooth mollification. Provided that $|u| \le \delta$, there holds uniformly in r that

$$|h_r(x+u) - h_r(x)| = \left|\int_{\mathbb{R}^n}\eta_r(y)(g(x+u+y)+g(x+y))\,\mathrm{d}y\right| \le \|g\|_{\dot{C}^{0,\omega}_\delta}\omega(|u|),$$

and hence we find $\delta > 0$ so that, uniformly in r, we have $\|g-h_r\|_{\dot{C}^{0,\omega}_\delta} \le \|g\|_{\dot{C}^{0,\omega}_\delta} + \|h_r\|_{\dot{C}^{0,\omega}_\delta} \le \varepsilon$. To deal with the scales $\ge \delta$ we argue as follows. Observe that $g \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}$ is uniformly continuous, by $\lim_{t\to 0}\omega(t) = 0$, and thus the approximation $h_r := g * \eta_r$ converges uniformly to g, as $r \to 0$. We let $r = r(\delta)$ be so small that $\|g-h_r\|_\infty \le \frac{1}{2}\varepsilon\omega(\delta)$. Then, for $|x-y| > \delta$, there holds that

$$\|(g-h_r)(x)-(g-h_r)(y)\| \le 2\|g-h_r\|_\infty \le \varepsilon\omega(\delta) \le \varepsilon\omega(|x-y|),$$

by ω being non-decreasing. This concludes the proof of $\dot{\mathrm{VC}}^{0,\omega}(\mathbb{R}^n, Y) \subset \overline{C_c^\infty(\mathbb{R}^n, Y)}^{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)}$.

For the reverse inclusion " $\supset$ ", simply notice that by the Mean Value Inequality, $C_c^\infty(\mathbb{R}^n, Y) \subset \mathrm{Lip}_{\mathrm{bs}}(\mathbb{R}^n, Y)$. We remind that this inequality holds for smooth Y-valued functions, as a consequence of the Hahn-Banach theorem on Y. Also, by (3) there holds that $\mathrm{Lip}_{\mathrm{bs}}(\mathbb{R}^n, Y) \subset \dot{\mathrm{VC}}^{0,\omega}(\mathbb{R}^n, Y)$. As $\dot{\mathrm{VC}}^{0,\omega}(\mathbb{R}^n, Y)$ is closed with respect to $\dot{C}^{0,\omega}$ limits, we are done. □

Proving approximations in a more general normed space X and $Y = \mathbb{R}$ (or $Y = \mathbb{C}$) is a much more delicate task and this is the content of the next Section 3. We finish this section by establishing the connection between pointwise and mean oscillation type conditions.

Proof of Theorem 1.11. We first prove the norms equivalent. So fix an arbitrary cube $Q \in \mathcal{Q}$ and estimate

$$\fint_Q |f(x) - \langle f\rangle_Q|_Y \,\mathrm{d}x \leq \fint_Q \fint_Q |f(x) - f(y)|_Y \,\mathrm{d}x\,\mathrm{d}y \leq \|f\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)} \fint_Q \fint_Q \omega(|x-y|)\,\mathrm{d}x\,\mathrm{d}y$$

and we continue the bound with

$$\fint_Q \fint_Q \omega(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \lesssim_{n,C_{\mathrm{db}}} \omega(\ell(Q)), \tag{26}$$

thus we obtain $\|f\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)} \gtrsim_{n,C_{\mathrm{db}}} \|f\|_{\mathrm{BMO}^\omega(\mathbb{R}^n,Y)}$.

For the other direction, let $x, z \in \mathbb{R}^n$ be arbitrary and let $Q_0 \subset \mathbb{R}^n$ be a cube containing both x and y such that $\ell(Q_0) = |x-y|$. Consider the dyadic descendants of Q_0 (achieved by iteratively halving the sides) and for every $k \in \mathbb{N} \cup \{0\}$ let $Q_k(x)$ be the descendant of Q_0 of sidelength $2^{-k}\ell(Q_0)$ that contains the point x. Similarly, we define $Q_k(y)$ for each k. By the continuity of $f : \mathbb{R}^n \to Y$, we write

$$f(x) - f(y) = \Big(\sum_{k=0}^{\infty} \langle f\rangle_{Q_{k+1}(x)} - \langle f\rangle_{Q_k(x)}\Big) - \Big(\sum_{k=0}^{\infty} \langle f\rangle_{Q_{k+1}(y)} - \langle f\rangle_{Q_k(y)}\Big). \tag{27}$$

Both sums are estimated identically, so consider the first:

$$\Big|\sum_{k=0}^{\infty} \langle f\rangle_{Q_{k+1}(x)} - \langle f\rangle_{Q_k(x)}\Big|_Y \lesssim_n \sum_{k=0}^{\infty} \fint_{Q_k(x)} |f - \langle f\rangle_{Q_k(x)}|_Y \leq \|f\|_{\mathrm{BMO}^\omega(\mathbb{R}^n,Y)} \sum_{k=0}^{\infty} \omega(\ell(Q_k(x)))$$

and we continue the bound, using that ω is non-decreasing in the first bound, with

$$\sum_{k=0}^{\infty} \omega(\ell(Q_k(x))) = \sum_{k=0}^{\infty} \ell(Q_k(x)) \frac{\omega(\ell(Q_k(x)))}{\ell(Q_k(x))} \leq 2\int_0^{\ell(Q_0)} \frac{\omega(t)}{t}\,\mathrm{d}t \lesssim [\omega]_* \omega(\ell(Q_0)),$$

and, since $\omega(\ell(Q_0)) = \omega(|x-y|)$, we have shown that $\|f\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)} \lesssim_n [\omega]_* \|f\|_{\mathrm{BMO}^\omega(\mathbb{R}^n,Y)}$.

It is clear that the above argument gives $\dot{\mathrm{VC}}^{0,\omega}_\Gamma(\mathbb{R}^n,Y) = \mathrm{VMO}^\omega_\Gamma(\mathbb{R}^n,Y)$ when $\Gamma = \text{small}$, as well as the inclusion $\dot{\mathrm{VC}}^{0,\omega}_\Gamma(\mathbb{R}^n,Y) \subset \mathrm{VMO}^\omega_\Gamma(\mathbb{R}^n,Y)$ for $\Gamma = \text{far}$. The reverse inclusion " $\supset$ " for $\Gamma = \text{far}$ can be seen as follows. Let $R > 0$ be such that if $d(Q,0) > R$, then $\mathcal{O}^\omega(f,Q) \leq \varepsilon$. Let $|x|, |y| > R$ both be far from the origin. Using the structure of the Euclidean space, clearly there exists a dimensional constant $N(n)$ and points and cubes $\{z_k, Q_k\}_{k=1}^N$ such that $z_0 = x$, $z_N = y$ and $z_k, z_{k+1} \in Q_k$ and $d(Q_k, 0) \geq R$ and $\ell(Q_k) = |z_k - z_{k+1}|$. Thus, by the chain of inequalities that led us to the right estimate for (27), we obtain

$$|f(x) - f(y)|_Y \leq \sum_{k=1}^{N} |f(z_k) - f(z_{k+1})|_Y \lesssim [\omega]_* \varepsilon \sum_{k=1}^{N} \omega(\ell(Q_k)) \lesssim \varepsilon\omega(|x-y|),$$

which concludes the proof of the inclusion $\dot{\mathrm{VC}}^{0,\omega}_\Gamma(\mathbb{R}^n,Y) \supset \mathrm{VMO}^\omega_\Gamma(\mathbb{R}^n,Y)$.

For $\Gamma = \text{large}$, a further argument is needed for both inclusions. We begin by showing that $\dot{\mathrm{VC}}^{0,\omega}_{\text{large}}(\mathbb{R}^n,Y) \supset \mathrm{VMO}^\omega_{\text{large}}(\mathbb{R}^n,Y)$. Let $f \in \mathrm{VMO}^\omega_{\text{large}}(\mathbb{R}^n,Y)$, and $\varepsilon > 0$, and let $K \in \mathbb{N}$ be such that if $\ell(Q) \geq 2^K$, then $\mathcal{O}^\omega(f;Q) \leq \varepsilon$. Let $N \gg K$ be a large integer to be specified later, and suppose $x, y \in \mathbb{R}^n$ are so that $|x-y| \geq 2^N$. Let $M \geq N$ be so that $2^M \leq |x-y| \leq 2^{M+1}$, and let

Q_0 be a cube containing both x, y and with $\ell(Q_0) = |x-y|$. Then, again considering the left sum in the expansion (27), we bound

$$
\begin{aligned}
&\Big|\sum_{k=0}^{\infty} \langle f\rangle_{Q_{k+1}(x)} - \langle f\rangle_{Q_k(x)}\Big|_Y \leq \Big(\sum_{k=0}^{M-K} + \sum_{k=M-K+1}^{\infty}\Big) |\langle f\rangle_{Q_{k+1}(x)} - \langle f\rangle_{Q_k(x)}|_Y \\
&\lesssim_n \varepsilon \sum_{k=0}^{M-K} \omega(\ell(Q_k(x))) + \sup_{\ell(Q)\leq 2^K} \mathcal{O}^\omega(f;Q) \sum_{k=M-K+1}^{\infty} \omega(\ell(Q_k(x))) \\
&\lesssim_{C_{\mathrm{db}}} \varepsilon \int_{2^K}^{2^{M+1}} \frac{\omega(t)}{t}\,\mathrm{d}t + \sup_{\ell(Q)\leq 2^K} \mathcal{O}^\omega(f;Q) \int_0^{2^K} \frac{\omega(t)}{t}\,\mathrm{d}t \\
&\lesssim_{C_{\mathrm{db}}} \Big[\varepsilon\Big(\frac{1}{\omega(2^{M+1})}\int_{2^K}^{2^{M+1}} \frac{\omega(t)}{t}\,\mathrm{d}t\Big) \\
&\qquad\qquad + \frac{\omega(2^K)}{\omega(2^{M+1})} \sup_{\ell(Q)\leq 2^K} \mathcal{O}^\omega(f;Q)\Big(\frac{1}{\omega(2^K)}\int_0^{2^K} \frac{\omega(t)}{t}\,\mathrm{d}t\Big)\Big]\omega(\ell(Q_0)) \\
&\lesssim \Big[\varepsilon[\omega]_* + \frac{\omega(2^K)}{\omega(2^{N+1})}\|f\|_{\mathrm{BMO}^\omega(\mathbb{R}^n,Y)}[\omega]_*\Big]\omega(|x-y|).
\end{aligned}
$$

By the condition $\omega(\infty)=\infty$, we choose N large enough and bound the bracketed term from above by $\leq 2\varepsilon[\omega]_*$. Repeating the same proof with y in place of x, we control the right sum on the line (27) and thus obtain $|f(x)-f(y)| \leq 4\varepsilon[\omega]_*\omega(|x-y|)$, that is, $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{large}}(\mathbb{R}^n, Y)$.

Then, we show that $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{large}}(\mathbb{R}^n, Y) \subset \mathrm{VMO}^\omega_{\mathrm{large}}(\mathbb{R}^n, Y)$. Let $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{large}}(\mathbb{R}^n, Y)$, $\varepsilon > 0$, and $R > 0$ be such that $\mathrm{osc}^\omega_\delta(f) \leq \varepsilon$, if $\delta \geq R$. Let $M \gg R$ be a large constant, which we will specify later, and $Q \in \mathcal{Q}$ a cube such that $\ell(Q) \geq M$. For every $y \in Q$, denote by $Q_R(y)$ the cube centered at y and of $\mathrm{diam}(Q_R(y)) = R$. Then, we have

$$
\begin{aligned}
&\fint_Q \int_Q |f(x)-f(y)|_Y\,\mathrm{d}x\,\mathrm{d}y = \fint_Q \Big(\int_{Q\setminus Q_R(y)} + \int_{Q\cap Q_R(y)}\Big)|f(x)-f(y)|_Y\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \varepsilon \fint_Q \int_{Q\setminus Q_R(y)} \omega(|x-y|)\,\mathrm{d}x\,\mathrm{d}y + \sup_{\delta\leq R}\mathrm{osc}^\omega_\delta(f)\fint_Q\int_{Q\cap Q_R(y)} \omega(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \Big[\varepsilon \fint_Q\fint_Q \omega(|x-y|)\,\mathrm{d}x\,\mathrm{d}y + \sup_{\delta\leq R}\mathrm{osc}^\omega_\delta(f)\fint_Q \frac{1}{|Q|}\int_{Q_R(y)} \omega(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\Big]|Q|.
\end{aligned}
$$

By repeating the bound (26), the left term in the square bracket is bounded from above by $\lesssim_{n,C_{\mathrm{db}}} \varepsilon\omega(\ell(Q))$, which is a bound of the correct form. For the right term we bound

$$
\lesssim_n \|f\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)}\left(\frac{R}{\ell(Q)}\right)^n \omega(R) \leq \|f\|_{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)}\omega(R) \leq \varepsilon\omega(\ell(Q));
$$

as $\omega(\infty)=\infty$, and M is chosen sufficiently large. □

3. Approximation in infinite dimensional spaces

In this section we study smooth approximations in Banach spaces with respect to the $\dot{C}^{0,\omega}$ seminorm. In Subsections 3.1 and 3.2 we establish results for real-valued functions $f : X \to \mathbb{R}$, where the approximating functions are not only C^k or $C^{1,\alpha}$ smooth but also Lipschitz and with bounded support. In the last Subsection 3.3 we obtain C^∞ smooth approximations of Banach-valued mappings $f : c_0(\mathcal{A}) \to Y$, that are not necessarily Lipschitz, but have bounded support and belong to $\dot{\mathrm{VC}}^{0,\omega}(X,Y)$. As a corollary, we deduce C^∞ and $\dot{\mathrm{VC}}^{0,\omega}$ and approximations with bounded support for all $\dot{\mathrm{VC}}^{0,\omega}$ functions defined on $\mathbb{R}^n$ and with values in any Banach space.

As we mentioned in the introduction, for general background on smooth analysis on Banach spaces, including smooth renomings and approximations, we refer the reader to the monographs [3] by Benyamini and Lindenstrauss; [6] by Deville, Godefroy, and Zizler; and [10] by Hájek and Johanis.

Assumptions on the modulus. Throughout this whole Section 3 we assume that the modulus satisfies all the assumptions of Theorem 1.4, i.e. ω is non-decreasing and satisfies (2), (3) and (4).

Notation and basic definitions. By $\mathrm{Lip}(X)$ we denote the class of real-valued Lipschitz functions (not necessarily bounded) on X. We say that $g : X \to \mathbb{R}$ is L-Lipschitz provided that $|g(x) - g(y)| \leq L\|x - y\|$ for every $x, y \in X$, and we denote the minimal Lipschitz constant of g by $\mathrm{Lip}(g)$.

We will denote by X^* the (continuous) dual of X, and the dual norm in X^* by $\|\cdot\|_*$, that is,

$$\|\Lambda\|_* := \sup\{|\Lambda(v)| \,:\, \|v\| \leq 1\},$$

for every $\Lambda \in X^*$.

When speaking of differentiability of functions or mappings $f : X \to Y$, we always mean differentiability in the Fréchet sense. We say, of course, that f is of class $C^k(X,Y)$, when f has Fréchet derivatives up to order k, and those derivatives are continuous on X.

Also, we say that a norm $\|\cdot\|$ on X is *of class* C^k if $\|\cdot\| \in C^k(X \setminus \{0\})$, for $k \in \mathbb{N} \cup \{\infty\}$.

For any class of functions $\mathcal{C}(X,Y)$, the subscripted set $\mathcal{C}_{\mathrm{bs}}(X,Y)$ consists of functions $h : X \to Y$ of class $\mathcal{C}$ that have bounded support, meaning that there exists $R > 0$ so that $h(x) = 0$ for every $x \in X \setminus B(0,R)$.

Finally, a *bump* function $h : X \to Y$ is a non-zero function with bounded support.

3.1. **Lipschitz and smooth approximations.** In a real normed space X, we consider real-valued functions "f" of the class $\dot{\mathrm{VC}}^{0,\omega}(X,\mathbb{R})$, which we abbreviate by $\dot{\mathrm{VC}}^{0,\omega}(X)$. By Theorem 1.4 these functions "f" can be approximated by functions, say "g", of class $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$ with bounded support, in the $\dot{C}^{0,\omega}(X)$ seminorm. Our goal is to further approximate these functions "g" by Lipschitz functions (whose regularity will vary depending on the smoothness properties of X) with bounded support. In approximating "g" a crucial step is to reduce the problem to the problem of approximating Lipschitz functions by smooth Lipschitz functions with good control on the Lipschitz constants. The main results of this subsection are Theorem 3.3, valid for arbitrary normed spaces, and Theorem 3.5, valid for separable normed spaces X with a fixed degree of smoothness. The proofs of these two Theorems 3.3 and 3.5 employ several technical lemma, which will be reused in the later Subsections 3.2 and 3.3.

We begin by proving these technical lemmas and for this purpose, it will be useful to consider the *minimal modulus of continuity* of a function $f : X \to \mathbb{R}$, i.e.,

$$\omega_f(t) := \sup\{|f(x) - f(y)| \,:\, \|x - y\| \leq t\}, \quad t > 0.$$

Note that $\omega_f : [0,\infty] \to [0,\infty]$ is non-decreasing, and if f is uniformly continuous, then $\lim_{t\to 0} \omega_f(t) = 0$. Moreover, if $f \in \dot{C}^{0,\omega}(X)$, then $\omega_f \leq \|f\|_{\dot{C}^{0,\omega}(X)}\omega$, and especially $\omega_f(t)$ is finite at every t, but, in general, ω and ω_f are incomparable. Furthermore, since X is a normed space, it is easy to see that ω_f is sub-additive, i.e. $\omega_f(s+t) \leq \omega_f(t) + \omega_f(s)$ for all $t, s > 0$. This implies

$$\omega_f(t) \leq 2t\,\omega_f(1), \quad \text{for all} \quad t \geq 1. \tag{28}$$

Lemma 3.1. *Let X be a normed space and $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$. Then, there exists a sequence $(f_n)_n \subset \mathrm{Lip}(X)$ of functions converging to f in the $\dot{C}^{0,\omega}(X)$-seminorm. Moreover, if f has bounded support, the sequence can be taken $(f_n)_n \subset \mathrm{Lip}_{\mathrm{bs}}(X)$ to have bounded supports.*

Proof. As $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$, by the comments preceding the present lemma, the minimal modulus of continuity ω_f is sub-additive and satisfies $\omega_f(t) \to 0$, as $t \to 0$; then, it is a known result that the sequence

$$f_n(x) = \inf_{y \in X}\{f(y) + n\|x-y\|\}, \quad x \in X,$$

converges uniformly to f in X and each f_n is n-Lipschitz; for a proof see e.g. [11, p. 408].

Now, let us see that $\|f - f_n\|_{\dot{C}^{0,\omega}(X)} \to 0$, as $n \to \infty$. Indeed, given $\varepsilon > 0$ there exists $\delta > 0$ so that $\|x-z\| \le \delta$ implies $|f(x) - f(z)| \le \varepsilon\omega(\|x-z\|)$. Let us take $N \in \mathbb{N}$ large enough so that $\sup_X |f - f_n| \le \varepsilon\omega(\delta)$ for each $n \ge N$. Using that ω is non-decreasing, this gives the estimate

$$|(f-f_n)(x) - (f-f_n)(z)| \le 2\|f-f_n\|_\infty \le \varepsilon\omega(\delta) \le \varepsilon\omega(\|x-z\|),$$
$$\text{whenever} \quad \|x-z\| \ge \delta,\ n \ge N.$$

Now, for fixed $x, z \in X$ such that $\|x-z\| \le \delta$, we proceed as follows. Given $\eta > 0$, let $y = y(x,n,f,\eta)$ be so that

$$f_n(x) \ge f(y) + n\|x-y\| - \eta.$$

This choice of y and the definition of f_n yield

$$\begin{aligned}(f-f_n)(x) - (f-f_n)(z) &\le f(x) - f(y) - n\|x-y\| + \eta - f(z) + f_n(z)\\ &\le f(x) - f(y) - n\|x-y\| + \eta - f(z) + f(y+z-x) + n\|z-(y+z-x)\|\\ &\le f(x) - f(z) - (f(y) - f(y+z-x)) + \eta \le \varepsilon\omega(\|x-z\|) + \varepsilon\omega(\|x-z\|) + \eta.\end{aligned}$$

Letting $\eta \to 0$, we get

$$(f-f_n)(x) - (f-f_n)(z) \le 2\varepsilon\omega(\|x-z\|).$$

The same argument swapping x and z,gives $|(f-f_n)(x) - (f-f_n)(z)| \le 2\varepsilon\omega(\|x-z\|)$, for every $n \in \mathbb{N}$. We conclude that $\|f-f_n\|_{\dot{C}^{0,\omega}(X)} \le 2\varepsilon$, for $n \ge N$.

For the assertion concerning the boundedness of the supports, we may assume that $f(y) = 0$, whenever $\|y\| \ge R$, for some $R > 0$. We will localize the infimum defining f_n. For each $y \in X$, it follows from the definition of f_n, and the fact that $f \in \dot{C}^{0,\omega_f}(X)$, that

$$\begin{aligned} f(y) + n\|x-y\| &\ge f(x) - \|f\|_{\dot{C}^{0,\omega_f}(X)}\omega_f(\|x-y\|) + n\|x-y\|\\ &\ge f_n(x) - \|f\|_{\dot{C}^{0,\omega_f}(X)}\omega_f(\|x-y\|) + n\|x-y\|. \end{aligned} \tag{29}$$

Using property (28) of the minimal modulus of continuity ω_f, it follows that, for n large enough (but depending only on $\|f\|_{\dot{C}^{0,\omega_f}(X)}$), RHS(29) $> f_n(x)$, provided that $\|x-y\| > 1$. These observations show that the infimum defining $f_n(x)$ is restricted to $B(x,1)$:

$$f_n(x) = \inf_{y \in B(x,1)}\{f(y) + n\|x-y\|\}, \quad x \in X.$$

Now, if $\|x\| \ge R+1$, then $\|y\| \ge R$, and so $f(y) = 0$, whenever $y \in B(x,1)$. Thus, by the previous formula, we deduce $f_n(x) = \inf_{y \in B(x,1)}\{n\|x-y\|\} = 0$, which shows that f_n has bounded support, for n large enough.

□

In the following lemma we show that a uniform approximation by Lipschitz functions with controlled Lipschitz constants yields a $\dot{C}^{0,\omega}(X)$ approximation.

Lemma 3.2. *Let $f : X \to \mathbb{R}$ and $(f_n)_n \subset \mathrm{Lip}(X)$ be a sequence such that $\limsup_n \mathrm{Lip}(f_n) < \infty$ and f_n converges to f uniformly on X. Then,* $\limsup_n \|f_n - f\|_{\dot{C}^{0,\omega}(X)} = 0$.

Proof. By passing to a subsequence, we may assume that $L := \sup_n \mathrm{Lip}(f_n) < \infty$. Then, f is Lipschitz on X, with $\mathrm{Lip}(f) \leq L$. Given $\varepsilon > 0$, by $\lim_{t\to 0} t/\omega(t) = 0$, we find $\delta > 0$ so that

$$\frac{t}{\omega(t)} \leq \frac{\varepsilon}{1+L}, \quad \text{whenever} \quad t \leq \delta.$$

Let $N \in \mathbb{N}$ be such that $\sup_X |f_n - f| \leq \varepsilon\omega(\delta)$ for all $n \geq N$. Because f and each f_n is L-Lipschitz, for any two distinct points $x, z \in X$ with $\|x - z\| \leq \delta$, we have

$$\max\left\{\frac{|f(x) - f(z)|}{\omega(\|x-z\|)}, \frac{|f_n(x) - f_n(z)|}{\omega(\|x-z\|)}\right\} \leq L\frac{\|x-z\|}{\omega(\|x-z\|)} \leq \varepsilon;$$

then, triangle inequality gives

$$\frac{|(f-f_n)(x) - (f-f_n)(z)|}{\omega(\|x-z\|)} \leq \frac{|f(x) - f(z)|}{\omega(\|x-z\|)} + \frac{|f_n(x) - f_n(z)|}{\omega(\|x-z\|)} \leq 2\varepsilon.$$

And for scales $\|x - z\| \geq \delta$, observe that for $n \geq N$, using that ω is non-decreasing, we have

$$|(f-f_n)(x) - (f-f_n)(z)| \leq 2\sup_X |f - f_n| \leq 2\varepsilon\omega(\delta) \leq 2\varepsilon\omega(\|x-z\|).$$

□

We are now ready to prove our first main Theorem 3.3 on Lipschitz approximations. It does not involve smoothness, but holds in any normed space and will be a key ingredient in the proofs of Theorems 3.10 and 3.12 of later sections.

Theorem 3.3 (Lipschitz approximation of $\dot{\mathrm{VC}}^{0,\omega}$)**.** *Let X be a normed space. Then, the following hold:*

(i) $\overline{\mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$,

(ii) $\overline{\mathrm{Lip}_{\mathrm{bs}}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{VC}}^{0,\omega}(X)$.

Proof.

(i) The inclusion " $\subset$ " follows immediately from Lemma 3.1. For the converse inclusion " $\supset$ ", we observe that condition (3) gives $\mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X) \subset \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$, and then recall that $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$ is closed.

(ii) By Theorem 1.4 we know that if $f \in \dot{\mathrm{VC}}^{0,\omega}(X)$, and $\varepsilon > 0$, we can find $g \in \dot{\mathrm{VC}}^{0,\omega}(X)$ with bounded support and $\|f - g\|_{\dot{C}^{0,\omega}(X)} \leq \varepsilon$. Then, Lemma 3.1 provides us with $h \in \mathrm{Lip}_{\mathrm{bs}}(X)$ so that $\|h - g\|_{\dot{C}^{0,\omega}(X)} \leq \varepsilon$, and therefore $\|f - h\|_{\dot{C}^{0,\omega}(X)} \leq 2\varepsilon$; this shows the inclusion " $\supset$ ".

For the converse implication " $\subset$ ", it is enough to observe that $\mathrm{Lip}_{\mathrm{bs}}(X) \subset \dot{\mathrm{VC}}^{0,\omega}(X)$ and that $\dot{\mathrm{VC}}^{0,\omega}(X)$ is a closed subspace of $\dot{C}^{0,\omega}(X)$. □

Performing smooth approximations in the appropriate Banach spaces will require more work. We first show that if a normed space X has a C^k and Lipschitz bump function, from now on a $C^k \cap \mathrm{Lip}$ bump, then a $C^k \cap \mathrm{Lip}$ approximation with uniformly bounded Lipschitz constants, of a Lipschitz function g with bounded support, can be upgraded to a $C^k \cap \mathrm{Lip}$ approximation with uniformly bounded Lipschitz constants *and bounded supports.*

Lemma 3.4. *Let $k \in \mathbb{N} \cup \{\infty\}$ be fixed and let X be a normed space that admits a $C^k \cap \mathrm{Lip}$ bump. Let $g \in \mathrm{Lip}(X)$ and $(g_n)_n \subset C^k(X) \cap \mathrm{Lip}(X)$ be a sequence of functions converging uniformly to g on X, and with $\limsup_n \mathrm{Lip}(g_n) < \infty$.*

Then, if g has bounded support, there exists a sequence $(h_n)_n \subset C^k(X) \cap \mathrm{Lip}(X)$ with bounded supports that converge to g uniformly (on X) and moreover $\limsup_n \mathrm{Lip}(h_n) < \infty$.

Proof. Let g be as in the assumption, and let $R > 0$ be so that $g = 0$ outside the ball $B(0, R)$. We first construct a suitable bump that decays from one to zero over the annular region $B(0, \lambda R) \setminus B(0, 2R)$, for a certain $\lambda > 2$ to be fixed later. Since X admits a $C^k \cap \mathrm{Lip}$ bump, by [6, Proposition II, 5.1] there exist a constant $0 < a < 1$ and a function $\psi : X \to [0, \infty)$ so that $\psi \in \mathrm{Lip}(X) \cap C^k(X \setminus \{0\})$ and

$$a\|x\| \leq \psi(x) \leq \|x\|, \quad x \in X. \tag{30}$$

The statement and proof of [6, Proposition II, 5.1] are written for $k = 1$, but they clearly hold true for $k \in \mathbb{N} \cup \{\infty\}$ as well. Now, pick a function $\theta : \mathbb{R} \to [0, 1]$ so that $\theta \in C^\infty(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, and $\theta(t) = 1$, whenever $t \leq 2R$, and $\theta(t) = 0$, whenever $t \geq 4R$. Then, define a bump function by

$$\varphi : X \to \mathbb{R}, \qquad \varphi(x) = \theta(\psi(x)), \quad x \in X. \tag{31}$$

By the properties of θ and ψ, it is immediate that $\varphi \in C^k(X) \cap \mathrm{Lip}(X)$, and that φ takes values in $[0, 1]$. Also, if $\|x\| \leq 2R$, one has $\psi(x) \leq 2R$ by (30), and thus $\varphi(x) = \theta(\psi(x)) = 1$. Similary, we deduce that $\varphi(x) = \theta(\psi(x)) = 0$ for those $x \in X$ such that $\|x\| \geq 4R/a =: \lambda$.

Now, by assumption we can find a sequence $(g_n)_n \subset C^k(X) \cap \mathrm{Lip}(X)$ converging uniformly to g on X, and, after passing to a subsequence, with the property $\sup_n \mathrm{Lip}(g_n) < \infty$. Define $h_n := \varphi g_n$ for each $n \in \mathbb{N}$, where φ is as in (31). Since φ vanishes outside the ball $B(0, 4R/a)$, the function g_n has bounded support. Also,

$$|h_n(x) - g(x)| = |\varphi(x) g_n(x) - g(x)| \leq |\varphi(x) - 1||g_n(x)| + |g_n(x) - g(x)|. \tag{32}$$

The second term in the sum converges to 0 uniformly on x. For the first term, note that if $\|x\| \leq 2R$, then $\varphi(x) = 1$, and so $|\varphi(x) - 1||g_n(x)| = 0$. When $\|x\| \geq 2R$, we have $\lim_n |g_n(x)| = |g(x)| = 0$, uniformly on those x, by the bounded support of g. Since also φ takes values in $[0, 1]$, we conclude that the first term tends uniformly to 0 as $n \to \infty$. Now we have shown that

$$\lim_{n \to \infty} \sup_{x \in X} |h_n(x) - g(x)| \to 0.$$

Concerning the regularity of h_n, note that obviously $h_n \in C^k(X)$ because both φ and g_n are of class C^k. To verify that $\limsup_n \mathrm{Lip}(h_n) < \infty$, we estimate the derivative of h_n by

$$\|Dh_n(x)\|_* \leq \varphi(x)\|Dg_n(x)\|_* + |g_n(x)|\|D\varphi(x)\|_* \leq \mathrm{Lip}(g_n) + \mathrm{Lip}(\varphi)(\sup_X |g_n - g| + \sup_X |g|).$$

Then $\mathrm{Lip}(g_n) \leq \sup_m \mathrm{Lip}(g_m)$, $\sup_X |g_n - g| \to 0$, and $\sup_X |g| < \infty$ because g is Lipschitz with bounded support. This shows $\limsup_n \mathrm{Lip}(h_n) < \infty$. ☐

Now, we combine Lemmas 3.1, 3.2, Theorem 3.3, and Lemma 3.4, with an approximation result for separable spaces that admit a $C^k \cap \mathrm{Lip}$ bump, and obtain our most general theorem concerning approximation of $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}$ functions by $C^k \cap \mathrm{Lip}$ functions in separable spaces.

Theorem 3.5 (Smooth approximation in separable spaces)**.** *Let $k \in \mathbb{N} \cup \{\infty\}$ and let X be a* ***separable*** *normed space admitting a $C^k \cap \mathrm{Lip}$ bump. Then, the following hold:*

(i) $\overline{C^k(X) \cap \mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$,

(ii) $\overline{C^k_{\mathrm{bs}}(X) \cap \mathrm{Lip}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X)$.

Proof.

(i) Let $f \in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$ and $\varepsilon > 0$. By Lemma 3.1 we find an approximation $g \in \mathrm{Lip}(X)$ so that $\|f - g\|_{\dot{C}^{0,\omega}(X)} \leq \varepsilon$. Now, since $g : X \to \mathbb{R}$ is a Lipschitz function, and X has a $C^k \cap \mathrm{Lip}$ bump, as a consequence of a result of Hájek and Johanis [10, Corollary 15] (here we use that X is separable), we find a sequence $(g_n)_n$ of $C^k \cap \mathrm{Lip}$ functions converging uniformly to g such that $\mathrm{Lip}(g_n) \leq \lambda \mathrm{Lip}(g)$, for all n, and for some absolute constant $\lambda > 0$ that may depend on the space X. Since obviously the function g and the sequence $(g_n)_n$ satisfy all the assumptions of Lemma 3.2, and hence $\lim_n \|g_n - g\|_{\dot{C}^{0,\omega}(X)} = 0$, completing the proof of the inclusion " $\supset$ ". For the reverse inclusion recall that the assumption $t/\omega(t) \to 0$, as $t \to 0$, guarantees that $\mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X) \subset \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$; and $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X)$ is a closed subspace of $\dot{C}^{0,\omega}(X)$.

(ii) Assume that $f \in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X)$ and let $\varepsilon > 0$. According to Theorem 1.4 we find $g \in \mathrm{Lip}(X)$ with bounded support so that $\|f - g\|_{\dot{C}^{0,\omega}(X)} \leq \varepsilon$. Now, let $(g_n)_n$ be the sequence we used in part (i) (but associated with this new g), and then Lemma 3.4 guarantees a $C^k \cap \mathrm{Lip}$ approximation with bounded support $(h_n)_n$ so that $\lim_n \sup_X |h_n - g| = 0$ and $\limsup_n \mathrm{Lip}(h_n) < \infty$. Now, we apply Lemma 3.2 to deduce that $\lim_n \|h_n - g\|_{\dot{C}^{0,\omega}(X)} = 0$ as well. Thus we pick some $h \in (h_n)_n \subset C^k(X) \cap \mathrm{Lip}_{\mathrm{bs}}(X)$ such that $\|g - h\|_{\dot{C}^{0,\omega}(X)} \leq \varepsilon$, which in turn implies

$$\|f - h\|_{\dot{C}^{0,\omega}(X)} \leq \|f - g\|_{\dot{C}^{0,\omega}(X)} + \|g - h\|_{\dot{C}^{0,\omega}(X)} \leq 2\varepsilon.$$

Thus we have shown the inclusion " $\supset$ ". For the converse inclusion " $\subset$ ", it is enough to observe again (as we did at the end of the proof of Theorem 3.3) that $\mathrm{Lip}_{\mathrm{bs}}(X) \subset \dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X)$ and that $\dot{\mathrm{V}}\mathrm{C}^{0,\omega}(X)$ is a closed subspace of $\dot{C}^{0,\omega}(X)$.

□

Remark 3.6. Clarifications and remarks concerning Theorem 3.5 and its proof.

(1) If X is a separable Banach space with an equivalent norm of class C^k, then Theorem 3.5 applies for X and k. Indeed, denote such an equivalent norm by $\psi : X \to \mathbb{R}$; and we can assume that ψ satisfies the inequalities on the line (30). As $\psi \in C^k(X \setminus \{0\})$, and since ψ is subadditive in X, it is clear that ψ is 1-Lipschitz (with respect to the original norm $\|\cdot\|$) on X. If we now pick $\theta : \mathbb{R} \to [0,1]$ of class $C^\infty \cap \mathrm{Lip}$, with $\theta = 1$ on $(-\infty, 1/2]$ and $\theta = 0$ on $[1, +\infty)$, it is easy to see that the composition $\theta \circ \psi$ defines a $C^k \cap \mathrm{Lip}$ bump function on X. Thus, we are in assumptions of Theorem 3.5 for X and k.

(2) Let us now recall the smoothness of the canonical, or equivalent, norms of some classical separable spaces. The canonical norms of the spaces $X = \ell_p$ or L^p, for $1 < p < \infty$, have the following order of smoothness $k = k(p)$:

$$k(p) = \begin{cases} \infty & \text{if } p \text{ is even,} \\ p - 1, & \text{if } p \text{ is odd,} \\ \lfloor p \rfloor, & \text{if } p \text{ is not an integer.} \end{cases}$$

Also, for a compact metric space K which is *scattered*, let $C(K)$ be the Banach space consisting of real-valued continuous functions on K, equipped with the supremum norm. Then $C(K)$ admits an equivalent C^∞-norm. Recall that a set S is scattered if every nonempty subset of S contains a (relatively) isolated point. We refer the reader to [6, Theorems V.1.1 and V.1.8] for detailed proofs and statements of these theorems. Then, according to point (1) above, Theorem 3.5 applies for $X = L^p$, ℓ_p with k_p as above, and for $X = C(K)$ with K countable and $k = \infty$.

(3) For $X = c_0$, see Section 3.3 below, Theorem 3.5 also holds for $k = \infty$, but for this particular X, we will obtain much more in Theorem 3.12 below.

(4) In the proof of Theorem 3.5 we employed [10, Corollary 15] for our real-valued functions, but this result holds for target spaces $Y = B_0(V)$ (real-valued functions f in a topological space V with $f(v) \to 0$ as $v \to v_0$, for some fixed $v_0 \in V$); or $Y = C_u(P)$ (uniformly continuous bounded functions on a metric space P). Naturally, the particular case $Y = \mathbb{R}$ is covered by these spaces, e.g., when $Y = C_u(P)$ for $P = \{0\} \subset \mathbb{R}$.

Now, as a corollary of Theorem 3.5, we provide $C^\infty \cap \mathrm{Lip}$ approximation of $\dot{\mathrm{V}\mathrm{C}}^{0,\omega}(X)$ functions for separable Hilbert spaces X. In particular, notice that this extends Theorem 1.8 to infinite dimensions.

Corollary 3.7. *Let X be a separable **Hilbert** space. Then, the following hold:*

(i) $\overline{C^\infty(X) \cap \mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}\mathrm{C}}^{0,\omega}_{\mathrm{small}}(X)$,

(ii) $\overline{C^\infty_{\mathrm{bs}}(X) \cap \mathrm{Lip}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{V}\mathrm{C}}^{0,\omega}(X)$.

Proof. Both statements (i) and (ii) follow immediately from Theorem 3.5, since a Hilbert space always admits a $C^\infty \cap \mathrm{Lip}$ bump. For the sake of completeness, we exhibit an elementary construction of such a bump.

The function $\psi : X \to \mathbb{R}$ given by $x \mapsto \psi(x) = \|x\|^2$ is of class $C^\infty(X)$ with the Fréchet derivative $D\psi(x) : X \to \mathbb{R}$, at every $x \in X$, given by $D\psi(x)(v) = \langle 2x, v\rangle$ for all $v \in X$. Take a function $\theta : \mathbb{R} \to [0,1]$ of class $C^\infty(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ with $\theta(t) = 1$ for all $t \leq 1/2$ and $\theta(t) = 0$ for all $t \geq 1$. Then, the map $\varphi = \theta \circ \psi : X \to [0,1]$ is of class $C^\infty(X)$ with $\varphi(x) = 1$ for all $\|x\| \leq 1/2$ and $\varphi(x) = 0$ for all $\|x\| \geq 1$. Also, the Fréchet derivative of φ satisfies

$$\|D\varphi(x)\|_* = \|\theta'(\psi(x))\mathcal{X}_{B(0,1)}(x) 2x\|_* \leq 2\,\mathrm{Lip}(\theta),$$

showing that φ is Lipschitz on X. Therefore, φ is a $C^\infty \cap \mathrm{Lip}$ bump on X. $\square$

3.2. **Approximation in super-reflexive spaces.** Let us now approximate real-valued functions of the classes $\dot{\mathrm{V}\mathrm{C}}^{0,\omega}_{\mathrm{small}}(X)$ and $\dot{\mathrm{V}\mathrm{C}}^{0,\omega}(X)$ over a super-reflexive Banach space X. We remind that X is super-reflexive if every Banach space Y that is *finitely representable* into X is reflexive. According to Pisier's renorming theorem [21], a Banach space X is super-reflexive if and only if X admits a renorming (which we keep denoting by $\|\cdot\|$), an exponent $\alpha \in (0,1]$, and a constant $C > 0$ for which

$$\|x+h\|^{1+\alpha} + \|x-h\|^{1+\alpha} - 2\|x\|^{1+\alpha} \leq C\|h\|^{1+\alpha}, \quad \text{for all} \quad x, h \in X. \tag{33}$$

See also [3, pp. 412–413] for a proof of this equivalence. This property is often rephrased by saying that X admits a renorming with *modulus of smoothness of power type* $1+\alpha$. Since the function $x \mapsto \psi(x) = \|x\|^{1+\alpha}$ is convex and continuous on X, then (33) implies that ψ is of class $C^{1,\alpha}(X)$; and with $\|D\psi\|_{\dot{C}^{0,\alpha}(X,X^*)} \lesssim_{\alpha,C} 1$; see [6, Lemma V.3.5].

We remind that a function $f : X \to \mathbb{R}$ belongs to the class $C^{1,\alpha}(X)$ if f is Fréchet differentiable at every point $x \in X$ and the Fréchet derivative $Df : X \to X^*$ is α-Hölder continuous on X, namely, that

$$\|Df\|_{\dot{C}^{0,\alpha}(X,X^*)} := \sup\left\{ \frac{\|Df(x) - Df(y)\|_*}{\|x-y\|^\alpha} \ : \ x, y \in X, \, x \neq y \right\} < \infty.$$

In Hilbert spaces the Lasry–Lions regularization theorem (see [16] or [10, p. 408]) provides uniform approximation of Lipschitz functions by $C^{1,1} \cap \mathrm{Lip}$ functions. Cepedello-Boiso [4, Theorem 1] used an ingenious variant of the Lasry-Lions technique to obtain uniform approximation of Lipschitz functions in super-reflexive spaces by C^1 functions whose derivatives are α-Hölder on bounded subsets, but not globally on X. These approximations are, of course, Lipschitz on bounded sets, but not globally Lipschitz. However, for our purposes, we need to approximate Lipschitz functions uniformly by *globally* $C^{1,\alpha} \cap \mathrm{Lip}$ functions, and we need a good control on the Lipschitz constants of the approximations. In Theorem 3.8(2) below, we have obtained uniform approximations that are *globally* $C^{1,\alpha}$ and *globally* Lipschitz on X, preserving the Lipschitz constants up to a multiplicative factor depending only on α and X. In the proof, we use a recent result on $C^{1,\omega}$ extensions for jets [2] to construct $C^{1,\alpha}$ *bump functions* with all the properties stated in part (1) of Theorem 3.8. This is combined with a method of Hájek and Johanis [10, Proposition 1] to *glue up* a suitable sequence of these bumps. Both parts (1) and (2) in Theorem 3.8 are interesting to the theory of smooth approximation, and also will be essential for the approximation of $\dot{\mathrm{VC}}^{0,\omega}$ functions.

Let us mention that, after making the first version of this article public, it has come to our attention that part (2) of Theorem 3.8 was very recently discovered by Johanis [15] too, by means of a different proof. A benefit of Johanis' proof is that it provides the sharp constant $\kappa = 1$. Since we only need an absolute control on κ for our purposes, and since we consider part (1) of independent interest as well, we have chosen to include our original proof of Theorem 3.8(2).

Theorem 3.8. *Let X be a super-reflexive space, let $\alpha \in (0,1]$ and $C > 0$ be so that a renorming of $\|\cdot\|$ of X satisfies* (33). *Then, there exists a constant $\kappa \geq 1$ depending only on α and C for which the following hold.*

(1) For every set $S \subset X$ there exists $h_S : X \to [0,1]$ of class $C^{1,\alpha}(X) \cap \mathrm{Lip}(X)$ so that:
$h_S(x) = 0$, for every $x \in S$;
$h_S(x) = 1$, whenever $d(x,S) \geq 1$;
$Dh_S(x) = 0$, for every $x \in S \cup \{y \in X \,:\, d(y,S) \geq 1\}$; and

$$\mathrm{Lip}(h_S) + \|Dh_S\|_{\dot{C}^{0,\alpha}(X,X^*)} \leq \kappa.$$

(2) Given an L-Lipschitz function $g : X \to \mathbb{R}$ and $\varepsilon > 0$, there exists $h \in C^{1,\alpha}(X)$ so that h is κL-Lipschitz, and $\sup_X |g - h| \leq \varepsilon$.

Proof of Theorem 3.8.
(1) In the cases $S = \varnothing$ or $\{d(\cdot,S) \geq 1\} = \varnothing$, we simply take $h_S \equiv 1$ in the first case, and $h_S \equiv 0$ in the latter. Assume from now on that both sets are nonempty. On the set

$$E := S \cup \{x \in X \,:\, d(x,S) \geq 1\},$$

we define a 1-jet $(f,G) : E \to \mathbb{R} \times X^*$ by setting

$$f(x) = 0, \quad \text{if} \quad x \in S; \qquad f(x) = 1, \quad \text{if} \quad d(x,S) \geq 1; \qquad G(x) = 0, \quad \text{for all} \quad x \in E.$$

By separating into cases, it is immediate to verify the existence of $M = M(\alpha) > 0$ for which

$$f(y) + G(y)(x-y) - f(z) - G(z)(x-z) \leq \frac{M}{1+\alpha}\left(\|x-y\|^{1+\alpha} + \|x-z\|^{1+\alpha}\right),$$

for all $y, z \in E$, $x \in X$; indeed, $G(y) = G(z) = 0$ is the zero functional, and for $f(y) - f(z)$, by symmetry, it is enough to verify the case $y \in S$ and $d(z,S) \geq 1$ and $x \in X$ arbitrary, but this case is clear by inspection, since

$$f(y) - f(z) = 1 \leq \|y-z\| \leq \|y-z\|^{1+\alpha} \leq \frac{M}{1+\alpha}\Big(\|x-y\|^{1+\alpha} + \|x-z\|^{1+\alpha}\Big).$$

Thus, applying [2, Theorem 4.1], there exists a constant $\kappa_0 = \kappa_0(\alpha, C)$, and a function $F : X \to \mathbb{R}$ of class $C^{1,\alpha}$, bounded and Lipschitz, with

$$\|F\|_\infty + \mathrm{Lip}(F) + \|DF\|_{\dot{C}^{0,\alpha}(X,X^*)} \leq \kappa_0 M =: \kappa_1, \tag{34}$$

and so that $F = f$ and $DF = G = 0$ on E. Although [2, Theorem 4.1] is formulated for Hilbert spaces, it generalizes immediately to super-reflexive Banach spaces satisfying property (33), as pointed out in [2, Remark 4.7]; see also [2, Theorem 1.9]. Let us examine the extension formula F, since we need some of its properties. At the same time, for the reader's convenience, we sketch some of the key steps in the proof of [2, Theorem 4.1]. With M as above,

$$M^* := \max\Big\{3\big(\|f\|_\infty + \|G\|_\infty\big),\, \frac{M}{1+\alpha}\Big\},$$

and the functions $m, g : X \to \mathbb{R}$ by

$$m(x) := \max\Big\{ -2\Big(\|f\|_\infty + \|G\|_\infty\Big), \sup_{z\in E}\Big\{f(z) + G(z)(x-z) - M^*\|x-z\|^{1+\alpha}\Big\}\Big\},$$
$$g(x) := \min\Big\{2\Big(\|f\|_\infty + \|G\|_\infty\Big), \inf_{y\in E}\Big\{f(y) + G(y)(x-y) + M^*\|x-y\|^{1+\alpha}\Big\}\Big\}.$$

For a suitable number A (depending only on α and C), F is defined to be the $AM^*t^{1+\alpha}$-*strongly paraconvex envelope of* g, that is,

$$F(x) := \sup\{h(x) \,:\, h \text{ is } AM^*t^{1+\alpha}\text{-strongly paraconvex, and } h \le g \text{ on } X\}, \quad x \in X.$$

As defined in [2], a function $h : \psi \to \mathbb{R}$ is $Lt^{1+\alpha}$-strongly paraconvex, for $L > 0$, if

$$\psi(\lambda u + (1-\lambda)v) - \lambda\psi(u) - (1-\lambda)\psi(v) \le \lambda(1-\lambda)L\|u-v\|^{1+\alpha}, \quad u, v \in X,\ \lambda \in [0,1].$$

By the comment just after formula (33), the function $u \mapsto \psi(u) := -\|u\|^{1+\alpha}$ is $C^{1,\alpha}$, and their Fréchet derivative $D\psi$ satisfies $\|D\psi\|_{\dot{C}^{0,\alpha}(X,X^*)} \le B(\alpha, C)$. Thus, using the Fundamental Theorem of Calculus and some elementary computations, ψ is $At^{1+\alpha}$-strongly paraconvex for some $A = A(\alpha, C)$; see the argument near the end of [2, the proof of Lemma 3.6]. It turns out that then both m and $(-g)$ are AM^*t^α-strongly paraconvex. Also, $m \le g$ on X, and these properties permit to prove the regularity $F \in C^{1,\alpha}(X)$, along with (34) and that $(F, DF) = (f, G)$ on E. See [2, Theorems 1.9 and 4.1] for further explanations and details of these concepts and their proofs. Now, by the definition of F, the pointwise relation

$$m(x) \le F(x) \le g(x), \quad x \in X,$$

holds true. Since $G \equiv 0$ and $\|f\|_\infty = 1$, the definition of m and g and this estimate imply that $-2 \le F \le 2$ on X. It only remains to slightly modify F so that the final function h_S takes values in $[0,1]$.

To do so, we pick a bump $\theta : \mathbb{R} \to [0,1]$ of class $C^\infty(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ with $\theta(t) = 0$ for $t \le 0$ and $t \ge 2$, and $\theta(1) = 1$. Define $h_S := \theta \circ F : X \to [0,1]$. It is immediate that $h_S(x) = 0$ for every $x \in S$, $h_S(x) = 1$ if $d(x, S) \ge 1$, and that $Dh_S = 0$ on E. Since F is κ_1-Lipschitz, h_S is $\mathrm{Lip}(\theta)\kappa_1$-Lipschitz. Finally, using the facts that $\mathrm{Lip}(F) + \|DF\|_{\dot{C}^{0,\alpha}(X,X^*)} \le \kappa_1$ and that $\mathrm{Lip}(\theta) + \|\theta'\|_{\dot{C}^{0,\alpha}(\mathbb{R})} \le c(\alpha)$ for some $c(\alpha) > 0$, it is an easy exercise to verify that

$$\|Dh_S\|_{\dot{C}^{0,\alpha}(X,X^*)} \le \|\theta'\|_{\dot{C}^{0,\alpha}(\mathbb{R})}\,\mathrm{Lip}(F)^\alpha + \mathrm{Lip}(\theta)\|DF\|_{\dot{C}^{0,\alpha}(X,X^*)} \le \kappa(\alpha, \kappa_1).$$

Therefore, we can find $\kappa = \kappa(\alpha, C)$ for which h_S satisfies all the properties stated in (1).

(2) Let $g : X \to \mathbb{R}$ be L-Lipschitz, and $\varepsilon > 0$. We will now apply part (1) for a suitable sequence of sets to obtain the desired approximation. In order to do so, we use the same construction as in [10, Proposition 1], replacing the C^1-*separating functions* from there with those we just obtained in part (1).

Define $\tilde{g}(x) = \varepsilon^{-1}g(\varepsilon x/L)$, for all $x \in X$, and note that $\tilde{g}$ is 1-Lipschitz. Now, we define the set $S_n = \{x \in X \,:\, \tilde{g}(x) \ge n\}$, for each integer $n \in \mathbb{Z}$. Note that $S_{n+1} \subset S_n$, and also, because $\tilde{g}$ is 1-Lipschitz, one has that $d(X \setminus S_n, S_{n+1}) \ge 1$ for every $n \in \mathbb{Z}$. Now, for each subset S_n, let $h_{S_n} : X \to [0,1]$ be the function from (1). Define now $h_n = 1 - h_{S_{n+1}}$ for every n, and also

$$h(x) = \sum_{n=0}^{\infty} h_n(x) + \sum_{n=-\infty}^{-1}(h_n(x) - 1), \quad x \in X.$$

Each h_n is C^1 and κ-Lipschitz, $h_n = 1$ on S_{n+1}, and $h_n(x) = 0$ if $d(x, S_{n+1}) \ge 1$. As proven in [10, Proposition 1], $h : X \to \mathbb{R}$ is a well-defined κ-Lipschitz and C^1 function, with $\sup_X |h - \tilde{g}| \le 1$.

Moreover, the sums defining h are locally finite, meaning that for every $x \in X$, there exists $N_x \in \mathbb{N}$, and a ball $B(x, r_x)$ so that

$$h(z) = \sum_{n=0}^{\max\{|m|,N_x\}} h_n(z) + \sum_{n=-\max\{|m|,N_x\}}^{-1} (h_n(z) - 1), \quad z \in B(x, r_x),\, m \in \mathbb{Z}. \tag{35}$$

The relation (35) will allow us to differentiate the sums defining h term by term locally around every $x \in X$.

Let us now prove that $h \in C^{1,\alpha}(X)$ with $\|Dh\|_{\dot{C}^{0,\alpha}(X,X^*)} \leq 2\kappa$. Recalling that by definition $Dh_{S_{n+1}}(x) = 0$ whenever $x \in S_{n+1} \cup \{x \in X \,:\, d(x, S_{n+1}) \geq 1\}$, it is immediate from the definition $h_n := 1 - h_{S_{n+1}}$ that

$$Dh_n = 0 \quad \text{on} \quad S_{n+1} \cup \{x \in X \,:\, d(x, S_{n+1}) \geq 1\}, \quad n \in \mathbb{Z}.$$

Note that this implies $Dh_n = 0$ on $X \setminus S_n$, as $d(X \setminus S_n, S_{n+1}) \geq 1$. These observations together with the continuity of Dh_n give

$$Dh_n = 0 \quad \text{on} \quad S_{n+1} \cup \overline{X \setminus S_n}, \quad n \in \mathbb{Z}. \tag{36}$$

Also, we claim that

$$Dh(x) = Dh_m(x), \quad \text{whenever} \quad x \in S_m \setminus S_{m+1}, \quad m \in \mathbb{Z}. \tag{37}$$

Indeed, let $x \in S_m \setminus S_{m+1}$. For those $n \geq m+1$, we have $x \notin S_n$ and thus $Dh_n(x) = 0$, e.g., by virtue of (36). And for those $n \leq m-1$, we have $x \in S_{n+1}$ and so $Dh_n(x) = 0$. In other words, $Dh_n(x) = 0$ for $n \neq m$. It then follows by (35) and evaluating at $x \in S_m \setminus S_{m+1}$ that

$$Dh(x) = \sum_{n=0}^{\max\{|m|,N_x\}} Dh_n(x) + \sum_{n=-\max\{|m|,N_x\}}^{-1} Dh_n(x) = Dh_m(x).$$

This proves the claim (37).

Now given $x, y \in X$, let $m \in \mathbb{Z}$ and $l \in \mathbb{N} \cup \{0\}$ be so that $x \in S_m \setminus S_{m+1}$ and $y \in S_{m+l} \setminus S_{m+l+1}$, and we show the α-Hölder estimate for $\|Dh(x) - Dh(y)\|_*$.

In the case $l = 0$, we have $x, y \in S_m \setminus S_{m+1}$, and so it suffices to apply (37) and the fact that $\|Dh_m\|_{\dot{C}^{0,\alpha}(X,X^*)} \leq \kappa$.

Assume now $l \geq 1$. By the connectedness of $[x, y]$ and $x \notin S_{m+l}$ and $y \in S_{m+l}$, the segment $[x, y]$ must intersect the boundary of S_{m+l}. Let $z \in [x, y] \cap S_{m+l} \cap \overline{X \setminus S_{m+l}}$. In particular, $z \in \overline{X \setminus S_{m+l}}$, and by (36), this implies $Dh_{m+l}(z) = 0$. Also, observe that $y \in S_{m+l} \subset S_{m+1}$, and so $Dh_m(y) = 0$, again by (36). Using first (37), then $Dh_m(y) = Dh_{m+l}(z) = 0$, and finally that $\|Dh_n\|_{\dot{C}^{0,\alpha}(X,X^*)} \leq \kappa$ for all $n \in \mathbb{Z}$, we conclude that

$$\begin{aligned} \|Dh(x) - Dh(y)\|_* &= \|Dh_m(x) - Dh_{m+l}(y)\|_* \\ &\leq \|Dh_m(x) - Dh_m(y)\|_* + \|Dh_{m+l}(z) - Dh_{m+l}(y)\|_* \\ &\leq \kappa \left(\|x - y\|^\alpha + \|z - y\|^\alpha\right) \leq 2\kappa \|x - y\|^\alpha. \end{aligned}$$

By symmetry, we can swap the roles of x and y, and thus we have shown that $h \in C^{1,\alpha}(X)$.

Finally, rescaling the function h by $\tilde{h}(x) = \varepsilon h(xL/\varepsilon)$ and noting that $g(x) = \varepsilon \tilde{g}(xL/\varepsilon)$, we see that $\tilde{h} \in C^{1,\alpha}(X)$ with $\sup_X |\tilde{h} - g| \leq \varepsilon$ and $\mathrm{Lip}(\tilde{h}) \leq L \,\mathrm{Lip}(h) \leq \kappa L$. ☐

We will use the following elementary fact.

Remark 3.9. If U, V are normed spaces, and $\psi : U \to V$ is Lipschitz and bounded, then ψ is α-Hölder for every $\alpha \in (0, 1]$.

A consequence of Theorem 3.8, in combination with our results from previous sections, is the following $C^{1,\alpha}\cap \mathrm{Lip}$ approximation of $\dot{\mathrm{VC}}^{0,\omega}$ functions in super-reflexive spaces. Note that, unlike in Theorem 3.5, X may be nonseparable.

Theorem 3.10 (Approximation in super-reflexive spaces)**.** *Let X be a super-reflexive space, and let $\alpha\in(0,1]$ and $C>0$ be so that a renorming of $\|\cdot\|$ of X satisfies* (33)*. Then the following hold:*

(i) $\overline{C^{1,\alpha}(X)\cap \mathrm{Lip}(X)\cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$,

(ii) $\overline{C^{1,\alpha}_{\mathrm{bs}}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{VC}}^{0,\omega}(X)$.

Notice that the modulus ω is independent of α. Also, the approximations in part (ii) are Lipschitz. Indeed, as the derivative is α-Hölder and boundedly supported, it is in particular bounded and this implies that the function is Lipschitz.

Proof.

(i) Let $f\in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$ and $\varepsilon>0$. Applying Lemma 3.1, we can find a function $g\in \mathrm{Lip}(X)$ so that $\|f-g\|_{\dot{C}^{0,\omega}(X)}\leq\varepsilon$. Now, by Theorem 3.8(2), there is a sequence (g_n) of $\mathrm{Lip}(X)\cap C^{1,\alpha}(X)$ functions with $\sup_n \mathrm{Lip}(g_n)\leq \kappa\,\mathrm{Lip}(g)$, and $\lim_n \sup_X |g-g_n|=0$. Therefore Lemma 3.2 yields that $\lim_n \|g_n-g\|_{\dot{C}^{0,\omega}(X)}=0$, and taking some g_n from the sequence with $\|g_n-g\|_{\dot{C}^{0,\omega}(X)}\leq\varepsilon$, we conclude the inclusion " $\supset$ " of (i). The reverse inclusion follows by noting that $\mathrm{Lip}(X)\cap\dot{C}^{0,\omega}(X)\subset \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$, by the condition (3), and that $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$ is a closed subspace of $\dot{C}^{0,\omega}(X)$.

(ii) Let $f\in\dot{\mathrm{VC}}^{0,\omega}(X)$ and $\varepsilon>0$. By Theorem 3.3 we find $g\in\mathrm{Lip}(X)$ with bounded support so that $\|f-g\|_{\dot{C}^{0,\omega}(X)}\leq\varepsilon$. With $R>0$ so that $g(x)=0$ for $\|x\|\geq R$, we use Theorem 3.8(1) to obtain a $\mathrm{Lip}(X)\cap C^{1,\alpha}(X)$ function $\varphi: X\to[0,1]$ with $\varphi=1$ on $B(0,2R)$ and $\varphi=0$ on $X\setminus B(0,2R+1)$. If $(g_n)_n$ is the approximating function for g from part (i), we define $h_n=\varphi g_n$. Obviously each $h_n\in C^1(X)$ and also has bounded support, and moreover the properties $\|h_n-g\|_\infty\to 0$ and $\limsup_n \mathrm{Lip}(h_n)<\infty$ hold and are checked exactly as in the proof of Lemma 3.4, see the line (32). So, it only remains to verify that h_n has α-Hölder derivative, at least, for n large enough. Write,

$$Dh_n(x)=g_n(x)D\varphi(x)+\varphi(x)Dg_n(x)$$

and let us prove that both terms define an α-Hölder function.

For the first function $x\mapsto g_n(x)D\varphi(x)$, let us observe that, for n large enough, one has $\sup_X|g_n|\leq 1+\sup_X|g|$, as g_n converges uniformly on X to g. Hence, by Remark 3.9, g_n is α-Hölder continuous on X. Then, we write

$$\begin{aligned}\|g_n(x)D\varphi(x)&-g_n(z)D\varphi(z)\|_*\\ &\leq |g_n(x)|\|D\varphi(x)-D\varphi(z)\|_*+|g_n(x)-g_n(z)|\|D\varphi(z)\|_*\\ &\leq \sup_X|g_n|\|D\varphi\|_{\dot{C}^{0,\alpha}(X,X^*)}\|x-z\|^\alpha+\|g_n\|_{\dot{C}^{0,\alpha}(X,\mathbb{R})}\|x-z\|^\alpha\sup_X\|D\varphi\|_*.\end{aligned}$$

All the factors that multiply the term $\|x-z\|^\alpha$ are finite by the properties of φ and g_n.

As concerns the second function $x\mapsto\varphi(x)Dg_n(x)$, its α-Hölder continuity is verified in a very similar way, this time using Remark 3.9 for φ and that Dg_n is bounded (as g_n is Lipschitz) and α-Hölder continuous.

We conclude that $(h_n)_n\to g$ uniformly, with $(h_n)_n\in C^{1,\alpha}_{\mathrm{bs}}(X)$ and $\limsup_n\mathrm{Lip}(h_n)<\infty$. By Lemma 3.2, we have $\|h_n-g\|_{\dot{C}^{0,\omega}(X)}\to 0$ as well. Therefore, we can find $h\in C^{1,\alpha}_{\mathrm{bs}}(X)$ with $\|h-g\|_{\dot{C}^{0,\omega}(X)}\leq\varepsilon$, and thus $\|h-f\|_{\dot{C}^{0,\omega}(X)}\leq 2\varepsilon$.

We have shown the inclusion $\dot{\mathrm{VC}}^{0,\omega}(X)\subset\overline{C^{1,\alpha}_{\mathrm{bs}}(X)}^{\dot{C}^{0,\omega}(X)}$. For the reverse inclusion, observe that if $f\in C^{1,\alpha}_{\mathrm{bs}}(X)$, then $Df: X\to X^*$ is α-Hölder and with bounded support, and so Df is bounded in X. Because Lipschitz functions with bounded support are contained in $\dot{\mathrm{VC}}^{0,\omega}(X)$, which is closed under limits with respect to the $\dot{C}^{0,\omega}(X)$ seminorm, the reverse inclusion holds. □

Remark 3.11. We exhibit some examples of classical spaces that are covered by Theorem 3.10.

(1) If $X = L^p$, for $1 < p < \infty$, then (33) holds for $\alpha_p = p - 1$ when $p \leq 2$ and for $\alpha_p = 1$ when $p \geq 2$; for a proof see [6, Corollary V.1.2]. By Theorem 3.10, this gives rise to approximations of $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$ and $\dot{\mathrm{VC}}^{0,\omega}(X)$ by Lipschitz functions of class $C^{1,\alpha_p}(X)$.

(2) If X is a Hilbert space (separable or not), then (33) holds as an identity for $\alpha = 1$ and $C = 2$ (the parallelogram law), and so Theorem 3.10 gives $C^{1,1}$ approximations:

(i) $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X) = \overline{C^{1,1}(X) \cap \mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)}$,

(ii) $\dot{\mathrm{VC}}^{0,\omega}(X) = \overline{C^{1,1}_{\mathrm{bs}}(X)}^{\dot{C}^{0,\omega}(X)}$.

3.3. **Approximation of Banach-valued mappings from c_0.** For an arbitrary set of indices $\mathcal{A}$, the space $c_0(\mathcal{A})$ consists of those elements $x = (x_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ such that for every $\varepsilon > 0$, there exists a finite subset $S = S_x \subset \mathcal{A}$ so that $|x_\alpha| \leq \varepsilon$, whenever $\alpha \in \mathcal{A} \setminus S$. We equip $c_0(\mathcal{A})$ with the norm $\|(x_\alpha)_{\alpha\in\mathcal{A}}\|_\infty := \sup_{\alpha\in\mathcal{A}} |x_\alpha|$ and this results in a Banach space.

Given $x = (x_\alpha)_{\alpha\in\mathcal{A}}$ and $\alpha \in \mathcal{A}$, we denote the projection $(x)_\beta = ((x_\alpha)_{\alpha\in\mathcal{A}})_\beta := x_\beta \in \mathbb{R}$; and for every finite subset $S \subset \mathcal{A}$, we denote

$$P_S : c_0(\mathcal{A}) \to c_{00}(\mathcal{A}), \qquad P_S(x) = \sum_{\alpha\in S} x_\alpha e_\alpha,$$

where e_α is the element of $c_0(\mathcal{A})$ that satisfies $(e_\alpha)_\beta = \delta_{\alpha,\beta}$, for every $\beta \in \mathcal{A}$.

Theorem 3.12 (Approximation in $c_0(\mathcal{A})$)**.** *For an arbitrary set of indices $\mathcal{A}$, let $X = c_0(\mathcal{A})$, and let Y be a Banach space. Then, the following hold:*

(i) $\overline{C^\infty(X,Y) \cap \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)}^{\dot{C}^{0,\omega}(X,Y)} = \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)$,

(ii) $\overline{C^\infty_{\mathrm{bs}}(X,Y) \cap \dot{\mathrm{VC}}^{0,\omega}(X,Y)}^{\dot{C}^{0,\omega}(X,Y)} = \dot{\mathrm{VC}}^{0,\omega}(X,Y)$.

Moreover, in the particular case $Y = \mathbb{R}$, the approximation can be taken to be Lipschitz:

(i) $\overline{C^\infty(X) \cap \mathrm{Lip}(X) \cap \dot{C}^{0,\omega}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$,

(ii) $\overline{C^\infty_{\mathrm{bs}}(X) \cap \mathrm{Lip}(X)}^{\dot{C}^{0,\omega}(X)} = \dot{\mathrm{VC}}^{0,\omega}(X)$.

One step in our proof of Theorem 3.12 will rely on the construction by Hájek and Johanis [9, Theorem 1] (stated for Lipschitz functions), to obtain the C^∞ approximation on $c_0(\mathcal{A})$. However, the verification $\dot{C}^{0,\omega}$ convergence after our $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}$ condition will require more work.

Following [9] we first construct an approximation that locally depends only on a finite number of coordinates.

Lemma 3.13. *Let $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)$ and $\varepsilon > 0$. Then, there exists $g \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)$ and $r > 0$ so that $\|g - f\|_{\dot{C}^{0,\omega}(X,Y)} \leq \varepsilon$, and for every $x \in X$, there exists a finite subset $S = S_x$ of $\mathcal{A}$ for which $g(z) = g(P_S(z))$, whenever $z \in B(x,r)$.*

Moreover, if f has bounded support (resp. Lipschitz), then the above g has bounded supported (resp. Lipschitz) too.

Proof. Let $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)$ and $\varepsilon > 0$, let $\delta > 0$ be so that

$$\sup_{\|u-v\|\leq\delta} \frac{\|f(u) - f(v)\|_Y}{\omega(\|u - v\|)} \leq \frac{\varepsilon}{2}. \tag{38}$$

By $\omega(0)=0$, we find $r>0$ so that $2\omega(r)\|f\|_{\dot C^{0,\omega}(X,Y)} \le \varepsilon\omega(\delta)$. Let us define

$$\varphi:\mathbb{R}\to\mathbb{R}, \qquad \varphi(t)=\begin{cases} t+r & \text{if} \quad t\le -r,\\ 0 & \text{if} \quad -r\le t\le r,\\ t-r & \text{if} \quad t\ge r,\end{cases}$$

and

$$\phi: X\to X, \qquad \phi(x)=\phi((x_\alpha)_{\alpha\in\mathcal{A}})=(\varphi(x_\alpha))_{\alpha\in\mathcal{A}}.$$

As clearly φ is 1-Lipschitz, it is immediate that ϕ is 1-Lipschitz. Also, it is immediate that $\|f-f\circ\phi\|_Y\le\omega(r)\|f\|_{\dot C^{0,\omega}(X,Y)}$. Defining $g:=f\circ\phi$, we have $g\in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y)$, by virtue of Lemma 2.2. If $x,z\in X$ are such that $\|x-z\|\le\delta$, then $\|\phi(x)-\phi(z)\|\le\delta$ and so

$$\|g(x)-g(z)\|_Y=\|f(\phi(x))-f(\phi(z))\|_Y\le\frac{\varepsilon}{2}\omega(\|\phi(x)-\phi(z)\|)\le\frac{\varepsilon}{2}\omega(\|x-z\|).$$

This estimate, in combination with (38), yields,

$$\sup_{\|x-z\|\le\delta}\frac{\|(f-g)(x)-(f-g)(z)\|_Y}{\omega(\|x-z\|)}\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

On the other hand, the choice of r permits to estimate by

$$\sup_{\|x-z\|\ge\delta}\frac{\|(f-g)(x)-(f-g)(z)\|_Y}{\omega(\|x-z\|)}\le\frac{2\|f-g\|_\infty}{\omega(\delta)}\le\frac{2\omega(r)\|f\|_{\dot C^{0,\omega}(X,Y)}}{\omega(\delta)}\le\varepsilon.$$

We have shown that $\|f-g\|_{\dot C^{0,\omega}(X,Y)}\le\varepsilon$.

Now, if $x\in X=c_0(\mathcal{A})$, then (by definition) there exists a finite subset $S=S_x$ such that if $\alpha\in\mathcal{A}\setminus S$, then $|x_\alpha|\le r/2$. Then, for $z\in B(x,r/2)$ and for each $\alpha\in\mathcal{A}\setminus S$ it follows that $|z_\alpha|\le r$, implying that $\varphi(z_\alpha)=0$. It follows for $z\in B(x,r/2)$ that $\phi(z)=\phi(P_S(z))$ and hence that $g(z)=g(P_S(z))$.

For the second part, suppose there exists $R>0$ so that $f(x)=0$ for all $\|x\|\ge R$. If $\|z\|\ge R+r$, then $|z_{\alpha_0}|\ge R+r$ for some $\alpha_0\in\mathcal{A}$. Consequently

$$\|\phi(z)\|=\sup_{\alpha\in\mathcal{A}}|\varphi(z_\alpha)|\ge|\varphi(z_{\alpha_0})|\ge|z_{\alpha_0}|-r\ge R,$$

and thus $g(z)=f(\phi(z))=0$. Finally, if f is Lipschitz, then clearly $g=f\circ\phi$ is Lipschitz.

□

Proof of Theorem 3.12. In the proof of part (i), by Lemma 3.13, given a fixed $r>0$, it is enough to approximate functions $g\in\dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X,Y)$ with the property that for every $x\in X$, there exists finite $S_x\subset\mathcal{A}$ such that $g(z)=g(P_{S_x}(z))$, for all $z\in B(x,2r)$.

Given ε, let $\delta>0$ be so that

$$\|g(u)-g(v)\|_Y\le\varepsilon\omega(\|u-v\|), \quad \text{whenever} \quad \|u-v\|\le\delta. \tag{39}$$

Moreover, for $0<\eta<\min(r,\varepsilon)$ small enough (more precisely, so that $\omega(\eta)\le\varepsilon\omega(\delta)/(1+\|g\|_{\dot C^{0,\omega}})$), let $\theta:\mathbb{R}\to\mathbb{R}$ be an even C^∞ smooth non-negative function such that $\int_{\mathbb{R}}\theta=1$ and $\theta=0$ on $\mathbb{R}\setminus[-\eta,\eta]$. For every finite set $F\subset\mathcal{A}$, and $x\in X$, we define the Bochner integral

$$h_F(x):=\int_{[-\eta,\eta]^{|F|}} g\Big(x-\sum_{\alpha\in F}t_\alpha e_\alpha\Big)\prod_{\alpha\in F}\theta(t_\alpha)\,\mathrm{d}\lambda_{|F|}(t), \tag{40}$$

where $\lambda_{|F|}$ is the Lebesgue measure on $\mathbb{R}^{|F|}$.

Because $g(z) = g(P_{S_x}(z))$ for all $z \in B(x, 2r)$, the fact that $\eta < r$ implies that $h_{S_x} = h_{S_x} \circ P_{S_x}$ on the ball $B(x, r)$. Therefore

$$h_{S_x}(z) = \int_{[-\eta,\eta]^{|S_x|}} g\Big(P_{S_x}(z) - \sum_{\alpha\in S_x} t_\alpha e_\alpha\Big) \prod_{\alpha\in S_x} \theta(t_\alpha)\,\mathrm{d}\lambda_{|S_x|}(t), \quad z \in B(x, r).$$

Since $g \in \dot{C}^{0,\omega}(X, Y)$ and P_{S_x} is 1-Lipschitz, the function $z \mapsto (g \circ P_{S_x})(z)$ belongs to $\dot{C}^{0,\omega}(X, Y)$, implying that $z \mapsto (g \circ P_{S_x})(z)$ is uniformly continuous on X. Therefore, by the previous formula, $B(x, r) \ni z \mapsto h_{S_x}(z)$ is a finite dimensional smooth mollification (in the Bochner sense) with a uniformly continuous mapping, which shows that $h_{S_x} \in C^\infty(B(x, r), Y)$. Moreover, using Fubini's theorem, one can easily check that if $F \subset \mathcal{A}$ is finite and $F \supset S_x$, then also $h_F = h_{S_x}$ on $B(x, r)$. This enables us to define the desired mapping $h : X \to Y$ in the following manner. Given $x \in X$, let $S_x \subset \mathcal{A}$ be the subset described above, and define $h(x) := h_{S_x}(x)$. Moreover, ordering the collection of finite subsets $\mathcal{F}(\mathcal{A})$ of $\mathcal{A}$ by inclusion, the pointwise limit of the net $\{h_S\}_{S\in\mathcal{F}(\mathcal{A})}$ is exactly h.

Now, for every fixed ball $B(x, r)$ and $z \in B(x, r)$, let $F = S_x \cup S_z$. By the definition of h and the above mentioned properties, we derive

$$h(z) = h_{S_z}(z) = h_F(z) = h_{S_x}(z).$$

Since $z \mapsto h_{S_x}(z)$ is of class $C^\infty(B(x, r), Y)$, the above yields $h \in C^\infty(X, Y)$. All these properties were stated and proved in [9, Lemma 6].

We next show that $h \in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X, Y)$. Given any two points $x, z \in X$, let $S_x, S_z \subset \mathcal{A}$ be the finite subsets associated with x and z, and set $S = S_x \cup S_z$. Since $h(x) = h_S(x)$ and $h(z) = h_S(z)$ we have

$$\begin{aligned} \|h(x) - h(z)\|_Y &= \Big\| \int_{[-\eta,\eta]^{|S|}} \Big(g\big(x - \sum_{\alpha\in S} t_\alpha e_\alpha\big) - g\big(z - \sum_{\alpha\in S} t_\alpha e_\alpha\big)\Big) \prod_{\alpha\in S} \theta(t_\alpha)\,\mathrm{d}\lambda_{|S|}(t) \Big\|_Y \\ &\le \int_{[-\eta,\eta]^{|S|}} \Big\| g\big(x - \sum_{\alpha\in S} t_\alpha e_\alpha\big) - g\big(z - \sum_{\alpha\in S} t_\alpha e_\alpha\big)\Big\|_Y \prod_{\alpha\in S} \theta(t_\alpha)\,\mathrm{d}\lambda_{|S|}(t) \\ &\le \sup_{\substack{u,v\in X,\\ \|u-v\|=\|x-z\|}} \|g(u) - g(v)\|_Y, \end{aligned} \tag{41}$$

bearing in mind that $\int_{[-\eta,\eta]^{|S|}} \prod_{\alpha\in S} \theta(t_\alpha)\,\mathrm{d}\lambda_{|S|}(t) = 1$. It follows immediately from the bound (41) that $\|h\|_{\dot{C}^{0,\omega}(X,Y)} \le \|g\|_{\dot{C}^{0,\omega}(X,Y)}$ and $h \in \dot{\mathrm{V}}\mathrm{C}^{0,\omega}_{\mathrm{small}}(X, Y)$. Now if $\|x - z\| \le \delta$, then together (41) and (39) give

$$\|(g - h)(x) - (g - h)(z)\|_Y \le \|g(x) - g(z)\|_Y + \|h(x) - h(z)\|_Y \le 2\varepsilon\omega(\|x - z\|).$$

Hence to verify $\|g - h\|_{\dot{C}^{0,\omega}(X,Y)} \le 2\varepsilon$, it remains to check the case $\|x - z\| \ge \delta$. To do so, let us see that $\sup_X \|g - h\| \le \varepsilon\omega(\delta)$. Indeed, if $x \in X$ and $S = S_x$, then again $h(x) = h_{S_x}(x)$, and we obtain

$$\begin{aligned} \|h(x) - g(x)\|_Y &\le \int_{[-\eta,\eta]^{|S|}} \Big\| g\Big(x - \sum_{\alpha\in S} t_\alpha e_\alpha\Big) - g(x)\Big\|_Y \prod_{\alpha\in S} \theta(t_\alpha)\,\mathrm{d}\lambda_{|S|}(t) \\ &\le \|g\|_{\dot{C}^{0,\omega}(X,Y)} \int_{[-\eta,\eta]^{|S|}} \omega\Big(\Big\| \sum_{\alpha\in S} t_\alpha e_\alpha \Big\|\Big) \prod_{\alpha\in S} \theta(t_\alpha)\,\mathrm{d}\lambda_{|S|}(t) \\ &\le \|g\|_{\dot{C}^{0,\omega}(X,Y)}\omega(\eta) \le \varepsilon\omega(\delta), \end{aligned}$$

where we chose η small enough in the beginning of the proof for the last bound to hold. Now, if $x, z \in X$ are such that $\|x - z\| \ge \delta$, then

$$\|(g - h)(x) - (g - h)(z)\| \le 2 \sup_X \|g - h\| \le 2\varepsilon\omega(\delta) \le 2\varepsilon\omega(\|x - z\|).$$

We conclude that $\|g-h\|_{\dot{C}^{0,\omega}(X,Y)} \le 2\varepsilon$ and complete the proof of Theorem 3.12(i).

For part (ii), given $f \in \dot{\mathrm{VC}}^{0,\omega}(X,Y)$, by Theorem 1.4, we approximate f, in the $\dot{C}^{0,\omega}(X,Y)$-seminorm by a $g \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X,Y)$ with bounded support. By Lemma 3.13, we can assume that for every $x \in X$ there exists a finite $S = S_x$ such that $g = g \circ P_S$ on $B(x,r)$, and for a uniform fixed $r > 0$. Let us next check that the approximation h from part (i) has bounded support. Let $R > 0$ be so that g is zero on $X \setminus B(0,R)$. Assume that $\|x\| \ge R+\eta$. There exists a finite set $S = S_x \subset \mathcal{A}$ so that $h(x) = h_S(x)$. Now, for $t = (t_\alpha)_{\alpha\in S} \in [-\eta,\eta]^{|S|}$, there holds that

$$\Big\|x - \sum_{\alpha\in S} t_\alpha e_\alpha\Big\| \ge \|x\| - \Big\|\sum_{\alpha\in S} t_\alpha e_\alpha\Big\| \ge \|x\| - \eta \ge R$$

and thus

$$h(x) = h_S(x) = \int_{[-\eta,\eta]^{|S|}} g\Big(x - \sum_{\alpha\in S} t_\alpha e_\alpha\Big) \prod_{\alpha\in S} \theta(t_\alpha)\,\mathrm{d}\lambda_{|S|}(t) = 0.$$

The proof of Theorem 3.12(ii) is now complete.

Finally, in the particular case $Y = \mathbb{R}$, let us establish the desired Lipschitz approximations. Let $f \in \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$ be the function to be approximated. By Theorem 3.3 we can assume that f is Lipschitz, and by Lemma 3.13, there is a Lipschitz approximation g of f, with g locally depending only on a finite number of coordinates around each ball of fixed radius $r > 0$. Repeating the construction of $h \in C^\infty(X,)$, we immediately see from the estimate (41), that h is Lipschitz, and in fact $\mathrm{Lip}(h) \le \mathrm{Lip}(g)$. Because any Lipschitz mapping in $\dot{C}^{0,\omega}(X)$ belongs to $\dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(X)$, by the condition (3), the identity in (i) now follows at once.

As for approximation of an $f \in \dot{\mathrm{VC}}^{0,\omega}(X)$, we apply Theorem 3.3 to reduce matters to $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$. Then the function g of Lemma 3.13 can be taken Lipschitz and with bounded support, and so can the function h we constructed in (i) and (ii) of the present theorem. Thus $h \in C^\infty_{\mathrm{bs}}(X) \cap \mathrm{Lip}(X)$ approximates f in the $\dot{C}^{0,\omega}(X)$ seminorm. Because any Lipschitz mapping with bounded support belongs to $\dot{\mathrm{VC}}^{0,\omega}(X)$, the identity in (ii) now follows at once.

□

For the particular case of $\mathcal{A} = \{1,\dots,n\}$, $c_0(\mathcal{A})$ becomes $\mathbb{R}^n$ with the supremum norm, which is of course equivalent with the usual Euclidean setting. Theorem 3.12 then has the following consequence.

Corollary 3.14. *Let Y be an arbitrary Banach space. Then:*

(i) $\overline{C^\infty(\mathbb{R}^n,Y) \cap \dot{C}^{0,\omega}(\mathbb{R}^n,Y)}^{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)} = \dot{\mathrm{VC}}^{0,\omega}_{\mathrm{small}}(\mathbb{R}^n,Y)$,

(ii) $\overline{C^\infty_c(\mathbb{R}^n,Y)}^{\dot{C}^{0,\omega}(\mathbb{R}^n,Y)} = \dot{\mathrm{VC}}^{0,\omega}(\mathbb{R}^n,Y)$.

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References

[1] D. Azagra, R. Fry, and L. Keener, *Real analytic approximation of Lipschitz functions on Hilbert space and other Banach spaces*, J. Funct. Anal. **262** (2012), no. 1, 124–166.

[2] D. Azagra and C. Mudarra, *$C^{1,\omega}$ extension formulas for 1-jets on Hilbert space*, Adv. Math. **389** (2021), Paper No. 107928, 44 pp.

[3] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[4] M. Cepedello, *On regularization in superreflexive Banach spaces by infimal convolution formulas*, Studia Math. **129** (1998), no. 3, 265–284.

[5] R. R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635.

[6] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993.

[7] R. Fry, *Approximation by Lipschitz, C^p smooth functions on weakly compactly generated Banach spaces*, J. Funct. Anal. **252** (2007), no. 1, 34–41.

[8] W. Guo, J. He, H. Wu, and D. Yang, *Boundedness and compactness of commutators associated with Lipshitz functions*, Anal. Appl. **20** (2021), no. 1, 1–37.

[9] P. Hájek and M. Johanis, *Uniformly Gâteaux smooth approximations on* $c_0(\Gamma)$, J. Math. Anal. Appl. **350** (2009), no. 2, 623–629.

[10] ———, *Smooth approximations*, J. Funct. Anal. **259** (2010), no. 3, 561–582.

[11] ———, *Smooth analysis in Banach spaces*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 19, De Gruyter, Berlin, 2014.

[12] T. Hytönen, *The L^p-to-L^q boundedness of commutators with applications to the Jacobian operator*, J. Math. Pures Appl. **156** (2021), 351–391.

[13] T. Hytönen, K. Li, J. Tao, and Y. Dachun, *The L^p-to-L^q compactness of commutators with $p > q$*, Studia Math. **271** (2023), no. 1, 85–105.

[14] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16** (1978), no. 2, 263–270.

[15] M. Johanis, *A note on $C^{1,\alpha}$-smooth approximation of Lipschitz functions*, Proc. Amer. Math. Soc. in press (2024).

[16] J.-M. Lasry and P.-L. Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math. **55** (1986), no. 3, 257–266.

[17] N. G. Meyers, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc. **15** (1964), no. 5, 717–721.

[18] D. Mitrea, I. Mitrea, and M. Mitrea, *Geometric harmonic analysis II. Function spaces measuring size and smoothness on rough sets*, Developments in Mathematics, vol. 73, Springer, Cham, 2022.

[19] N. Moulis, *Approximation de fonctions différentiables sur certains espaces de Banach*, Ann. Inst. Fourier (Grenoble) **21** (1971), no. 4, 293–345.

[20] T. Oikari, *Note on compactness of commutators* (2024), available at `2402.04993`.

[21] G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **20** (1975), no. 3-4, 236–350.

[22] A. Uchiyama, *On the compactness of operators of Hankel type*, Tohoku Math. J. **30** (1978), no. 1, 163 –171.

[23] N. Weaver, *Lipschitz algebras*, Second Edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.