Statistical modeling and analysis of repairable systems: The trend-renewal process

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Repairable systems and recurrent events
Basic models for repairable systems: nonhomogeneous Poisson process (NHPP) and renewal process (RP)
The trend-renewal process (TRP)
The heterogeneous TRP: HTRP and its submodels
Statistical inference and model-checking in HTRP models
  Parametric
  Nonparametric

Concluding remarks

This is joint work with Georg Elvebakk and Knut Heggland
Ascher and Feingold (1984):

“A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, other than replacement of the entire system”.

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The trend-renewal process
Definition of repairable system

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The trend-renewal process
Ascher and Feingold presented the following example of a “happy” and “sad” system:
Ascher and Feingold’s mission in 1984

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Their claim:

*Reliability engineers do not recognize the difference between these cases since they always treat times between failures as i.i.d. and fit probability models like Weibull.*
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Reliability engineers do not recognize the difference between these cases since they always treat times between failures as i.i.d. and fit probability models like Weibull.

Use nonstationary stochastic point process models to analyze repairable systems data!
Today: Recurrent events extensively studied

- Observe events occurring in time

\[ 0 \quad S_1 \quad S_2 \quad \ldots \quad S_N \quad \tau \]
Today: Recurrent events extensively studied

Observe events occurring in time

Applications: engineering and reliability studies, public health, clinical trials, politics, finance, insurance, sociology, etc.
Today: Recurrent events extensively studied

- Observe events occurring in time
- Applications: engineering and reliability studies, public health, clinical trials, politics, finance, insurance, sociology, etc.

Reliability applications:
- breakdown or failure of a mechanical or electronic system
- discovery of a bug in an operating system software
- the occurrence of a crack in concrete structures
- the breakdown of a fiber in fibrous composites
- Warranty claims of manufactured products
Important aspects for modelling and analysis

- Trend in times between events?
- Renewals at events?
- “Randomness” of events?
- Dependence on covariates?
- Unobserved heterogeneity (“frailty”, “random effects”) among individual processes?
- Dependence between event process and the censoring at $\tau$?
Typical data format

\[ 0 \quad S_{11} \quad S_{21} \quad \cdots \quad S_{N_{1}1} \quad \tau_{1} \]

\[ \vdots \]

\[ 0 \quad S_{1j} \quad S_{2j} \quad \cdots \quad S_{N_{j}j} \quad \tau_{j} \]

\[ \vdots \]

\[ 0 \quad S_{1m} \quad S_{2m} \quad \cdots \quad S_{N_{m}m} \quad \tau_{m} \]
Proschan (1963): The classical “aircondition data”

Times of failures of aircondition system in a fleet of Boeing 720 airplanes

Times of valve-seat replacements in a fleet of 41 diesel engines

Event Plot for Valve Seat Replacements

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Failure times for tractor engines

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The trend-renewal process
Basic models for repairable systems

$0 \quad S_1 \quad S_2 \quad \cdots \quad S_N \quad \tau$

- RP($F$): Renewal process with interarrival distribution $F$.

**Defining property:**
- Times between events are i.i.d. with distribution $F$
Basic models for repairable systems

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- A *censored* observation may occur at the end of the observation time
Basic models for repairable systems

- **RP\( (F) \):** Renewal process with interarrival distribution \( F \).

**Defining property:**
- Times between events are i.i.d. with distribution \( F \)
- A *censored* observation may occur at the end of the observation time
- The data can be analysed as an ordinary set of i.i.d. random variables, except for the censored observation which needs to be handled by methods from *survival analysis*
Basic models for repairable systems

- NHPP(\(\lambda(\cdot)\)): Nonhomogeneous Poisson process with intensity \(\lambda(t)\).

**Defining property:**

1. \(Pr(\text{one event in } (t, t + \Delta t]) = \lambda(t)\Delta t + o(\Delta t)\)
2. \(Pr(\text{more than one event in } (t, t + \Delta t]) = o(\Delta t)\)
3. Number of events in disjoint time intervals are stochastically independent
Basic models for repairable systems

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**Equivalently:**

1. Number of events in $(s, t]$ is Poisson-distributed with expectation $\int_s^t \lambda(u)du$
2. Number of events in disjoint time intervals are stochastically independent
The homogeneous Poisson process (HPP) and the NHPP

The HPP($\rho$) is defined for a constant $\rho > 0$ by

1. $\Pr(\text{one event in } (t, t + \Delta t]) = \rho \Delta t + o(\Delta t)$
2. $\Pr(\text{more than one event in } (t, t + \Delta t]) = o(\Delta t)$
3. Number of events in disjoint time intervals are stochastically independent

Then for $\Lambda(t) = \int_0^t \lambda(u)du$ we have the following connection:
Let $T$ be the time to failure of an item

The *hazard rate* of $T$ is defined by

$$z(t) = \lim_{\Delta t \downarrow 0} \frac{Pr(t < T \leq t + \Delta t \mid T > t)}{\Delta t}$$

so that

$$Pr(t < T \leq t + \Delta t \mid T > t) \approx z(t) \Delta t$$
Hazard rate of a time to failure

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$$z(t) = \lim_{\Delta t \downarrow 0} \frac{Pr(t < T \leq t + \Delta t|T > t)}{\Delta t}$$

so that

$$Pr(t < T \leq t + \Delta t|T > t) \approx z(t)\Delta t$$

Furthermore, the density of $T$ is given by

$$f(t) = z(t)e^{Z(t)}$$

where $Z(t) = \int_0^t z(u)du$. 
Point process modelling of recurrent event processes

\[ \phi(t | \mathcal{F}_{t-}) \]

- \( \mathcal{F}_{t-} \) = history of events until time \( t \).
- Conditional intensity at \( t \) given history until time \( t \),

\[
\phi(t | \mathcal{F}_{t-}) = \lim_{\Delta t \downarrow 0} \frac{Pr(\text{failure in } [t, t + \Delta t] | \mathcal{F}_{t-})}{\Delta t}
\]

- ... so that

\[
Pr(\text{failure in } [t, t + \Delta t] | \mathcal{F}_{t-}) \approx \phi(t | \mathcal{F}_{t-}) \Delta t
\]
Special cases: The basic models

- NHPP($\lambda(\cdot)$):

  \[ \phi(t|\mathcal{F}_{t-}) = \lambda(t) \]

  so conditional intensity is independent of history.

  *Interpreted as “minimal repair” at failures*
Special cases: The basic models

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*Interpreted as “minimal repair” at failures*

- RP($F$) (where $F$ has hazard rate $z(\cdot)$):

$$\phi(t|\mathcal{F}_{t-}) = z(t - S_{N(t-)})$$

so conditional intensity depends (only) on time since last event, $t - S_{N(t-)}$.

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- Between minimal and perfect repair? *Imperfect repair* models.
Trend Renewal Process – TRP
(BL, Elvebakk and Heggland 2003)

- Trend function: $\lambda(t)$ (cumulative $\Lambda(t) = \int_0^t \lambda(u)du$)
- Renewal distribution: $F$ with expected value 1 (for uniqueness)

SPECIAL CASES:
- NHPP: $F$ is standard exponential distribution
- RP: $\lambda(t)$ is constant in $t$
Conditional intensity of TRP($F, \lambda(\cdot)$):

$$\phi(t|\mathcal{F}_{t-}) = z(\Lambda(t) - \Lambda(S_{N(t-)}))\lambda(t)$$

where $z(\cdot)$ is hazard rate of $F$
Conditional intensity of TRP($F, \lambda(\cdot)$):

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*Recall special cases:*
  - NHPP: $z(\cdot) \equiv 1$, implies $\phi(t|\mathcal{F}_{t-}) = \lambda(t)$
Point process formulation of TRP:

Conditional intensity of TRP\((F, \lambda(\cdot))\):

\[
\phi(t|\mathcal{F}_{t^-}) = z(\Lambda(t) - \Lambda(S_{N(t^-)}))\lambda(t)
\]

where \(z(\cdot)\) is hazard rate of \(F\)

Recall special cases:

- NHPP: \(z(\cdot) \equiv 1\), implies \(\phi(t|\mathcal{F}_{t^-}) = \lambda(t)\)
- RP: \(\lambda(\cdot) \equiv 1\), implies \(\phi(t|\mathcal{F}_{t^-}) = z(t - S_{N(t^-)})\)
Comparison of NHPP, TRP and RP: Conditional intensities

Example: Failures observed at times 1.0 and 2.25. Conditional intensities for NHPP (solid); TRP (dashed); RP (dotted)
Heterogeneity ("frailty") of systems

Systems of the same kind may exhibit different failure intensities because of external unobserved sources:

- Differing environmental conditions
- Differing maintenance philosophies
- Differing quality of operators
- Differing quality of equipment
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**Example with NHPP:**

- Failure intensity of \( j \)th system: \( a_j \lambda(t) \)
  where \( a_1, \ldots, a_n \) are i.i.d. *unobservable* random variables from some positive distribution with expected value 1.
Including unobserved heterogeneity in the TRP – the heterogeneous TRP (HTRP)

Definition of HTRP\((F, \lambda(\cdot), H)\)

- \(m\) systems are observed
- \(j\)th system observed in \([0, \tau_j]\), with \(N_j\) observed failures

\[
0 \quad S_{1j} \quad S_{2j} \quad \cdots \quad S_{N_{jj}} \quad \tau_j
\]

- Conditional on \(a_j\) is process \(j\) a TRP\((F, \lambda_j(\cdot))\) where

\[
\lambda_j(t) = a_j \lambda(t)
\]

- The \(a_j\) are i.i.d. (unobserved) random variables with d.f. \(H\), expected value 1.
The seven submodels of $\text{HTRP}(F, \lambda(\cdot), H)$

<table>
<thead>
<tr>
<th>Submodel</th>
<th>$HTRP$-formulation</th>
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<tbody>
<tr>
<td>$\text{HPP}(\nu)$</td>
<td>$\text{HTRP}(\text{exp}, \nu, 1)$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$\text{HNHPP}(\lambda(\cdot), H)$</td>
<td>$\text{HTRP}(\text{exp}, \lambda(\cdot), H)$</td>
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</table>
HTRP and its submodels – the model cube

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The trend-renewal process
Example model: HTRP($F, \lambda(\cdot), H$) and submodels

- $F$ (renewal distribution):
  - Weibull distribution (expected value 1, shape parameter $s$)
  - Submodel: Standard exponential distribution

- $\lambda(\cdot)$ (trend function): Power law,
  - $\lambda(t) = cbt^{b-1}$
  - Submodel: $\lambda(t) = c$

- $H$ (heterogeneity distribution):
  - Gamma distribution (expected value 1)
  - Submodel: $H$ is degenerate at 1
Likelihood function for TRP\((F, \lambda(\cdot))\) observed on \([0, \tau]\), with events at \(S_1, S_2, \ldots, S_{N(\tau)}\)

The likelihood function for a counting process is generally given as

\[
L = \left\{ \prod_{i=1}^{N(\tau)} \gamma(S_i) \right\} \exp\left\{ - \int_0^\tau \gamma(u)du \right\}.
\]  

(1)

where \(\gamma\) is the conditional intensity. Hence we get for the TRP:

\[
L = \left\{ \prod_{i=1}^{N(\tau)} \left[ z[\Lambda(S_i) - \Lambda(S_{i-1})] \lambda(S_i) \right] \right\} \times \exp\left\{ - \int_0^\tau z[\Lambda(u) - \Lambda(S_{N(u)})] \lambda(u)du \right\}
\]

where the last line can be computed as

\[
\times \exp\left\{ - \sum_{i=1}^{N(\tau)+1} Z[\Lambda(S_i) - \Lambda(S_{i-1})] \right\}
\]

with \(S_{N(\tau)+1} \equiv \tau\) and \(Z(t) = \int_0^t z(u)\)
By definition, $\Lambda(S_i) - \Lambda(S_{i-1})$ are i.i.d. $\sim z(t)e^{-Z(t)}$. Hence

$$ L = \left\{ \prod_{i=1}^{N(\tau)} z[\Lambda(S_i) - \Lambda(S_{i-1})] \lambda(S_i) \right\} $$

$$ \times \exp \left\{ - \sum_{i=1}^{N(\tau)} Z[\Lambda(S_i) - \Lambda(S_{i-1})] - Z[\Lambda(\tau) - \Lambda(S_{N(\tau-)})] \right\} $$

where $\Lambda(\tau) - \Lambda(S_{N(\tau-)})$ is a censored observation.

- But this is exactly the same expression as we got on the previous slide.
- For HTRP($F, \lambda(\cdot), H$) we need to replace $\lambda(\cdot)$ by $a\lambda(\cdot)$ and integrate out the frailty $a$ w.r.t. $H$
F is Weibull-distribution with expected value 1 and shape parameter s

\[ \lambda(t) = cbt^{b-1} \] is a power function of t

H is gamma-distribution with expected value 1 and variance v.

l is maximum value of the log likelihood.
Proschan (1963):

“... it seems safe to accept the exponential distribution as describing the failure interval, although to each plane may correspond a different failure rate”
Figure 5. The Log-Likelihood Cube of the HTRP\((F_g, \lambda_p(\cdot), H_g)\) Model for the Tractor Engine Data.
Basic property: $T > 0$, hazard rate $z(t)$

$$\Rightarrow \int_0^T z(y)dy = Z(T)$$

is unit exponentially distributed.

**HTRP($F, \lambda(\cdot), H$):**

Thus $R_{ij} = Z(\hat{a}_j \hat{\Lambda}(S_{ij}) - \hat{a}_j \hat{\Lambda}(S_{i-1,j})) \sim$ unit exponential

so we define “Cox-Snell” RESIDUALS by

$$\hat{R}_{ij} = \hat{Z}(\hat{a}_j \hat{\Lambda}(S_{ij}) - \hat{a}_j \hat{\Lambda}(S_{i-1,j}))$$
Residual processes

\[ \hat{R}_{1j} \quad \hat{R}_{2j} \quad \hat{R}_{3j} \]

for \( j = 1, 2, \ldots, m \)

- Expected to behave like independent HPP(1)
- Check against
  - distribution
  - trend
  - serial dependence
Figure 6. TTT Plots for Distribution of Residuals for Models NHPP($\lambda(x)$) (a), RP($F_g(x)$) (b), and TRP($F_g, \lambda(x)$) (c), for Tractor Engine Data.
Figure 7. TTT Plots for Time Trend in Residual Processes for Models NHPP(λₚ(·)) (a), RP(Fₔ₁, v) (b), and TRP(Fₔ₁, λₚ(·)) (c), for Tractor Engine Data.
Non-parametric estimation of the intensity $\lambda(t)$ in NHPPs is well developed (e.g. Gamiz et al., Ch. 3, 2011)

likewise nonparametric estimation of the inter-event distribution $F$ in RPs (e.g. Pena, Strawderman and Hollander, 2001)
Non-parametric estimation of the intensity $\lambda(t)$ in NHPPs is well developed (e.g. Gamiz et al., Ch. 3, 2011)

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For TRP($F, \lambda(\cdot)$):

- May estimate $\lambda(t)$ or $F$ or both nonparametrically
- If $\lambda(t)$ is of most interest, one may estimate it nonparametrically while letting $F$ be parametric.

(Remember: NHPP corresponds to $F$ being standard exponential. May therefore use $F$ as Weibull or gamma distributed to generalize the NHPP).
Heggland and BL (2007) derived nonparametric maximum likelihood estimator for $\lambda(t)$ under the assumptions:

- $\lambda(t)$ is monotone
- $F$ is Weibull with shape parameter $s$

**Idea:** Extend the approach by Bartożyński et al. (1981) for NHPPs.

Estimation uses theory of *isotonic regression* and the *minimum lower sets algorithm*; alternatively maximizing the likelihood with respect to the shape parameter $s$ in $z(\cdot)$ and the $\lambda(\cdot)$
Example: Analysis of USS Halfbeak data

- **Left plot:** Failure number vs time for 71 unscheduled maintenance actions for diesel engine, USS Halfbeak.

- **Right plot:** *parametric* (based on a power law function for $\lambda(t)$) and *nonparametric* estimates of $\Lambda(t)$ based on increasing trend function $\lambda(t)$.

Estimates of shape parameter $s$ in renewal distribution (Weibull):
- Nonparametric model: $\hat{s} = 0.937$
- Parametric model (power law): $\hat{s} = 0.762$ (*unreasonable!*)

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The trend-renewal process
Consider estimating $\lambda(t)$ (not necessarily monotone) by kernel estimation, while $F$ is parametric (e.g. Weibull).

- It is suggested to use a weighted kernel estimator

$$
\lambda(t; a) = \frac{1}{h} \sum_{i=1}^{N(\tau)} w \left( \frac{t - S_i}{h} \right) a_i
$$

with positive weights $a = (a_i)$ to be estimated from the data together with the parameters of the renewal distribution $F$

- Constant weights $a_i \equiv 1$ are used for NHPPs in Gamiz et al. (2011), as in ordinary density estimation

- Jones and Henderson (2005, 2009) consider density estimation using weighted kernels and maximizing the likelihood of the data

- $h$ has to be determined externally
Recall that the likelihood for data from a single system is

\[ L = \left\{ \prod_{i=1}^{N(\tau)} z[\Lambda(S_i) - \Lambda(S_{i-1})] \lambda(S_i) \right\} \]

\[ \times \exp\left\{ - \sum_{i=1}^{N(\tau)+1} Z[\Lambda(S_i) - \Lambda(S_{i-1})] \right\} \]

Now let \( z(t) \equiv z(t; \theta) \) be a parametric function, e.g. Weibull with expected value 1, while

\[ \lambda(t) \equiv \lambda(t; a) = \frac{1}{h} \sum_{i=1}^{N(\tau)} w \left( \frac{t - S_i}{h} \right) a_i \]

and maximize w.r.t. \( \theta \) and \( a = (a_i) \) (for given \( h \)).
Weighted kernel estimation for USS Halfbeak data

Estimates of $\lambda(\cdot)$ for $h = 2$ (solid), $h = 5$ (dotted), $h = 10$ (dot-dash).

Estimates of shape parameter $s$ of renewal distribution:

<table>
<thead>
<tr>
<th></th>
<th>$h = 2$</th>
<th>$h = 5$</th>
<th>$h = 10$</th>
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<tbody>
<tr>
<td>$s$</td>
<td>0.957</td>
<td>0.915</td>
<td>0.870</td>
</tr>
</tbody>
</table>
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Concluding remarks

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  - Several other classes are studied in the literature, often under the name of “imperfect repair models”
  - TRP is one such model, interpreted as the “least common multiple” of NHPP and RP
Concluding remarks

- Ascher and Feingold advocated use of nonstationary stochastic point processes
  - A & F studied mainly NHPP and RP
  - Several other classes are studied in the literature, often under the name of “imperfect repair models”
  - TRP is one such model, interpreted as the “least common multiple” of NHPP and RP
- Three “dimensions” of recurrent events properties are illustrated in model cube for TRP:
  - Randomness or renewal behavior
  - No trend or trend
  - No heterogeneity or heterogeneity
Parametric estimation:
- Maximum likelihood
- Likelihood ratio tests
- Residual plots

Nonparametric estimation:
- Estimate trend function nonparametrically
- Use Weibull or gamma distribution for renewal distribution
Concluding remarks (cont.)

- Parametric estimation:
  - Maximum likelihood
  - Likelihood ratio tests
  - Residual plots

- Nonparametric estimation:
  - Estimate trend function nonparametrically
  - Use Weibull or gamma distribution for renewal distribution

- Future work: More detailed studies of convergence and asymptotic properties of the nonparametric methods. Two types of asymptotics:
  - Few processes, number of events in each tend to infinity
  - Few events in each process, number of processes tends to infinity.