Reliability function of T_i in Cox-regression

For T_i: $z_{T_i}(t) = z(t; \underset{\sim}{x}_i) = z_0(t) e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i}$

$$\Rightarrow \quad Z(t; \underset{\sim}{x}_i) \underset{def}{=} \int_0^t z(u; \underset{\sim}{x}_i)\, du$$

$$= \int_0^t z_0(u) e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i} du = Z_0(t) e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i}$$

Thus $$\boxed{R(t; \underset{\sim}{x}_i) = e^{-Z(t; \underset{\sim}{x}_i)} = e^{-Z_0(t) e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i}} = R_0(t)^{e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i}}}$$

Important relation valid for prop. hazards: If $z(t; \underset{\sim}{x}) = z_0(t) g(\underset{\sim}{x})$ then $R(t; \underset{\sim}{x}) = R_0(t)^{g(\underset{\sim}{x})}$

where $R_0(t) = e^{-Z_0(t)}$ is reliab. function of unit with $\underset{\sim}{\beta} = \underset{\sim}{0}$

The Estimated reliab. function of T_i:

Natural estimator $\hat{R}(t; \underset{\sim}{x}_i) = \hat{R}_0(t)^{e^{\hat{\underset{\sim}{\beta}}' \underset{\sim}{x}_i}}$

So we need estimate $\hat{R}_0(t)$.

Or: Since $R_0(t) = e^{-Z_0(t)}$ we may also estimate $Z_0(t)$.

Go through estimator of $Z_0(t)$:

Observe:

$$Z(T_i \,;\, \underset{\sim}{x}_i) = V_i \sim \text{expon}(1)$$

$$\text{so} \quad Z_0(T_i)\, e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i} = V_i \sim \text{expon}(1)$$

$$\text{so} \quad Z_0(T_i) = \frac{V_i}{e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i}}$$

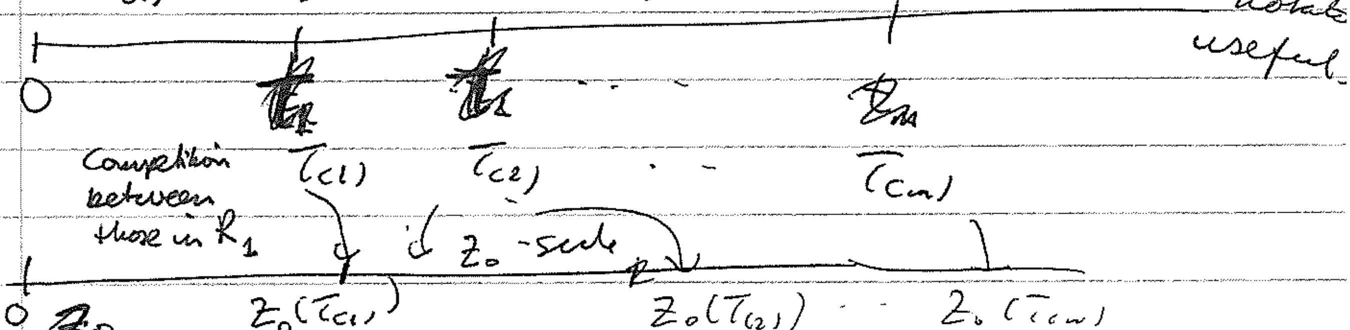
$$\sim \text{expon}\left(e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i} \right)$$

[Recall $T \sim \text{expon}(\lambda)$ where expon(λ): $f(t) = \lambda e^{-\lambda t}$

$\Rightarrow aT \sim \text{expon}\left(\frac{\lambda}{a}\right)$ $\quad E(T) = \frac{1}{\lambda}$

or $\frac{T}{a} \sim \text{expon}(a\lambda)$

Thus - as we did for Nelson plot: Let $T_{(1)} < T_{(2)} < \cdots < T_{(m)}$ be the failure times (new notation useful)



$$Z_0(T_{(1)}) = \min \text{ of } Z_0(T_i) \text{ for all the } i \text{ in } R_1$$

$$\sim \text{expon}\left(\sum_{r \in R_1} e^{\underset{\sim}{\beta}' \underset{\sim}{x}_r} \right)$$

$$\text{So} \quad E[Z_0(T_{(1)})] = \frac{1}{\sum_{r \in R_1} e^{\underset{\sim}{\beta}' \underset{\sim}{x}_r}}$$

Further, $Z_0(\tau_{(2)}) - Z_0(\tau_{(1)})$

= min of ~~the exponential~~ expon$(e^{\beta' x_r})$ for each of $r \in R_2$.

$\sim$ expon$\left(\sum_{r \in R_2} e^{\beta' x_r}\right)$

so $E\left[Z_0(\tau_{(2)}) - Z_0(\tau_{(1)})\right]$

$$= \frac{1}{\sum_{r \in R_2} e^{\beta' x_r}}$$

leading to

$$E(Z_0(\tau_{(2)})) = \frac{1}{\sum_{r \in R_1}} + \frac{1}{\sum_{r \in R_2}}$$

etc.

Thus (Breslow - estimator) put

$$\hat{Z}_0(t) = \sum_{t_i \le t} \frac{1}{\sum_{r \in R_i} e^{\hat{\beta}' x_r}}$$

Note - $\hat{\beta}$ here!

[This is ~~without~~ similar to Nelson-Aalen estimator

~~without~~ when no covariates, simp
then $\beta = 0$ and we get

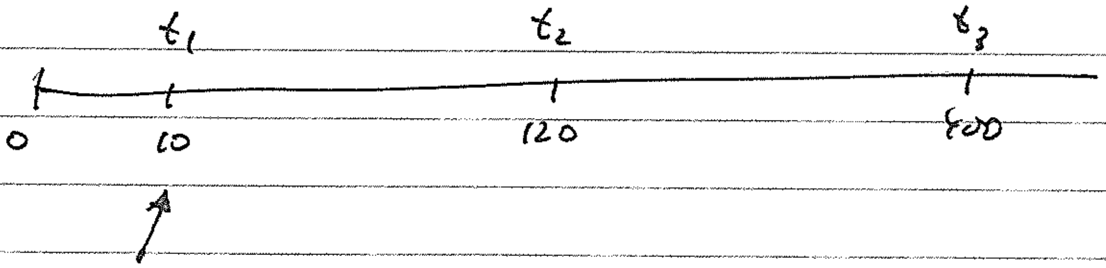
$$\sum_{t_i \le t} \frac{1}{\# R_i}$$

[Also a counterpart of KM-estimator:

$$\hat{R}_0(t) = \prod_{i : t_i \le t} \left(1 - \frac{e^{\hat{\beta}' x_{l_i}}}{\sum_{r \in R_i} e^{\hat{\beta}' x_r}} \right)^{e^{-\hat{\beta}' x_{l_i}}}$$

[For $\hat{\beta} = 0$: $\prod_{i : t_i \le t} \left(1 - \frac{1}{\# R_i} \right)$]

In simple example:



at 10: $$\frac{1}{e^{10\hat{\beta}}+e^{3\hat{\beta}}+e^{5\hat{\beta}}+e^{3\hat{\beta}}+e^{4\hat{\beta}}+e^{\hat{\beta}}} = 4.57\cdot 10^{-4}$$

at 120: $$\frac{1}{e^{3\hat{\beta}}+e^{4\hat{\beta}}+e^{\hat{\beta}}} = \cancel{0.0304}\ 0.0299$$

at 400: $$\frac{1}{e^{4\hat{\beta}}+e^{\hat{\beta}}} = \cancel{0.0730}\ 0.0426$$

$$\Rightarrow \hat{\Lambda}_0(10) = 4.57\cdot 10^{-4}$$

$$\hat{\Lambda}_0(120) = 0.0304$$

$$\hat{\Lambda}_0(400) = 0.0730$$

MODEL CHECKING:

Cox-Snell Residuals (Called "Generalized residual" in A&Ph)

Recall that

$$-\ln R(T_i; \underset{\sim}{x}_i) = Z(T_i; \underset{\sim}{x}_i)$$

$$= Z_0(T_i) \cdot e^{\underset{\sim}{\beta}' \underset{\sim}{x}_i} \sim \text{expon}(1)$$

Define Cox-Snell residuals:

$$\hat{V}_i = \hat{Z}_0(Y_i)\, e^{\hat{\underset{\sim}{\beta}}' \underset{\sim}{x}_i}$$

which is a censored set from expon (1)

[Some add 1 for the censored ones to have "uncensored" set. Because if $V \sim$ expon (1) then $E[V \mid V > y] = y + E(V) = y+1$]

Example: A&P Fig. 3.5.

So - Cox-Snell residuals in simple example:

Y		δ
5	0	0
10	0.9593	1
40	0.0045	0
80	0.0209	0
120	0.3017	1
400	1.5567	1
600	0.1569	0

$\hat{\Lambda}_0(t_1) \cdot e^{\hat{\beta} \cdot 10} = 4.5666 \cdot 10^{-4} \cdot e^{7.65} = 0.9593$

$\hat{\Lambda}_0(t_1) \cdot e^{\hat{\beta} \cdot 3} = 4.5666 \cdot 10^{-4} \cdot e^{3 \cdot 0.765}$

$\hat{\Lambda}_0(t_2) \cdot e^{3 \cdot \hat{\beta}} = 0.0304 \cdot e^{3 \cdot 0.765} =$

$\hat{\Lambda}_0(t_2) \cdot e^{4 \hat{\beta}}$

$e^{\hat{\beta}} \dots$

~~Alternativ med å legge til 1:~~

Testing av prop. hazards-egenskapen i Cox-modellen

~~dvs~~ $\lambda(t|\underset{\sim}{x}) = \cancel{\lambda_0(t)\, g(\underset{\sim}{x})}$

$= \cancel{\lambda_0(t)\, e^{\underset{\sim}{\beta}'\underset{\sim}{x}}}$

$$= \lambda_0(t)\, e^{\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k}$$

Dette sier at effekten av en økning på en enhet i ~~residual~~ kovariat x_1 gir en faktor e^{β_1} på hazardrate for alle t

Kunne tenkes at β_1 var en funksjon av t, slik at f.eks. $\beta_1(t) = 0$ for store t, dvs kov x_1 har ingen betydning for store t.

Schoenfeld-residualet (3.5.2 s.77 i A&P) defineres slik. Anta for enkelthets skyld bare en kovariat x.

$$x_{l_i} - \underbrace{\frac{\sum_{r \in R(t_i)} x_r\, e^{\hat{\beta}^* x_r}}{\sum_{r \in R(t_i)} e^{\hat{\beta}^* x_r}}}$$

Kovariat verdi for den enhet som feiler ved t_i

Forventet verdi på kovariat for den enhet som feiler ved t_i

<u>Nemlig:</u>

t_i

~~De~~

$R(t_i)$

enhet nr. k har en sanns. for å feile som er

$$\frac{e^{\beta x_k}}{\sum_{j \in R(t_i)} e^{\beta x_j}}$$

Forventet kovariatverdi er da

$$\sum_{k \in R(t_i)} x_k \cdot P(\text{enhet nr. } k \text{ feiler})$$

= det som står!

Se Fig. 3.2

Ideen er at disse residualene skal ha "tyngdepunkt i 0"

SIMPLE EXAMPLE:

$t_1 = 10$: 2 feile $x_{\cancel{m}} = 10$. Mean values

$$\frac{10 \cdot e^{10\hat{\beta}} + 3 \cdot e^{3\hat{\beta}} + \cdots + 1 \cdot e^{\hat{\beta}}}{e^{10\hat{\beta}} + e^{3\hat{\beta}} + \cdots + e^{\hat{\beta}}}$$

$$= \underline{9.7646}$$

$\hat{\beta} = 0.765$

des Scheenhed: $10 - 9.7646 = \underline{0.2354}$

$t_2 = 120$: $\underline{x_{\cancel{m}} = 3}$

"Ex" $= 3.5099$ Sch: $3 - 3.5099 = -0.5099$

$t_3 = 400$: $x = 4$.

"Ex" $= 3.7254$ Sch: $4 - 3.7254 = \underline{\underline{0.2746}}$