SOLUTION TMA4275 V2015

PROBLEM 1

a) $z(t) = \lim_{h \to 0^+} \dfrac{P(t < T \le t+h \mid T > t)}{h}$

$$Z(t) = \int_0^t z(u)\,du$$

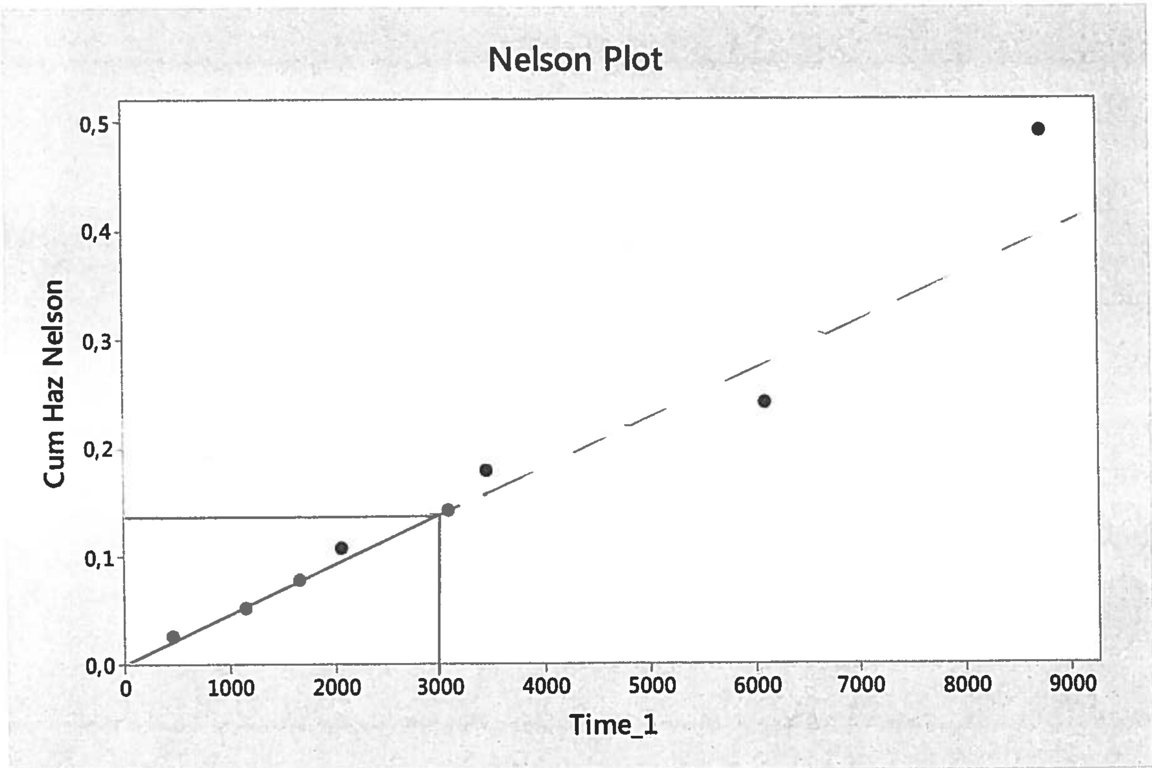
The derivative of $Z(t)$ is $z(t)$, so

$z(t) \nearrow \iff Z(t)$ convex

$z(t) \searrow \iff Z(t)$ concave

$z(t) \cup \iff Z(t)$ S-shaped (bathtub $z(t)$)

#at risk: n_i	C3 Time_1	C4 Haz Nelson	C5 Cum Haz Nelson	C6 Survival Nelson	C7 Survival KM	C8
40	450	0.025000 $1/40$	0.025000	0,975310	0,975000	
39	1150	0.025641 $1/39$	0.050641	0,950620	0,950000	
37	1660	0.027027 $1/37$	0.077668	0,925272	0,924324	
34	2070	0.029412 $1/34$	0,107080	0,898454	0,897138	
29	3100	0,034483 $1/29$	0,141563	0,868001	0,866203	
27	3450	0,037037 $1/27$	0,178600	0,836441	0,834121	
16	6100	0,062500 $1/16$	0,241100	0,785763	0,781988	
14	8750	0,250000 $1/14$	0,491100	0,611953	0,586491	



Plot is fairly linear, but might be an increasing hazard for large values (or a slightly decreasing one if we disregard the highest point 22)

First 3000: $Z(t)$ increases by approx 0.14 from $t=0$ to $t=3000$.

So we estimate a constant hazard to

$$\frac{0.14}{3000} \approx 4.7 \cdot 10^{-5}$$

$$MTTF \approx \frac{1}{4.7 \cdot 10^{-5}} \approx 21429$$

b) $$\hat{Z}(1800) = \sum_{T_{(i)} \le 1800} \frac{d_i}{n_i} = \frac{1}{40} + \frac{1}{39} + \frac{1}{37} = 0.077668$$

$$\widehat{SE\,\hat{Z}(1800)} = \sqrt{\frac{1}{40^2} + \frac{1}{39^2} + \frac{1}{37^2}} = 0.04487$$

95% standard CI:

$$0.077668 \pm 1.96 \cdot 0.04487$$

$$(0, 0.1656)$$

We know that $R(t) = e^{-Z(t)}$

Thus we can estimate $\hat{R}(1800) = e^{-\hat{Z}(1800)}$

$$= e^{-0.077668} = 0.925272$$

(see output on p. -1-)

A 95% conf int is obtained by computing

$e^{-[CI \text{ for } Z(1800)]}$, i.e. $[e^{-0.1656}, e^{0}] = [0.8473, 1]$

c) $T(t)$ er den totale tid under risiko opp til tid t for alle de 40 viftene.

$t = 450$: 40 fans survived until 450 hours
total time = $40 \cdot 450 = 18000$

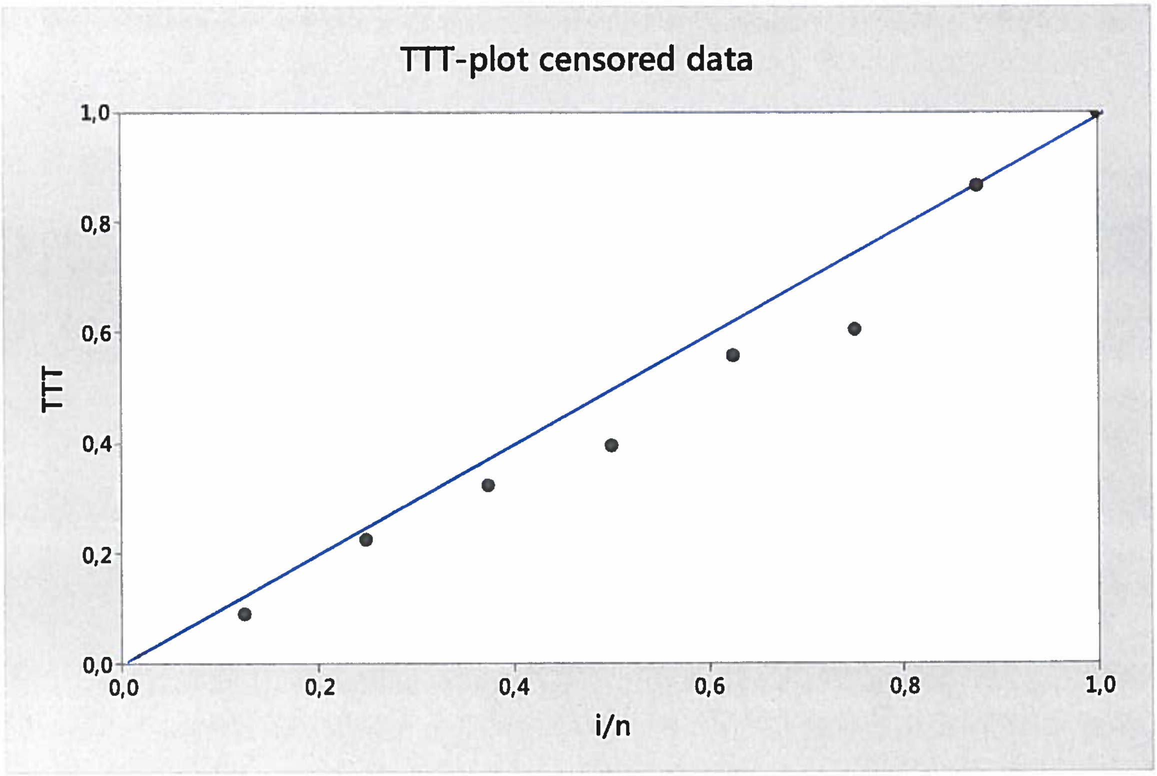
$t = 1150$: 39 fans survived an additional
$1150 - 450 = 700$ hours;
totally $39 \cdot 700 = 27300$
~~Thus T(t) =~~
Thus $T(1150) = 18000 + 27300 = 45300$.

$t = 1660$: 38 fans survived an additional
$(1600 - 1150) = \cancel{4}50$ hours
total $38 \cdot \cancel{4}50 = \cancel{22300}$ $\cancel{20000}$ 17100
& 37 survived additional 60:
$37 \cdot 60 = 2220$

So
$T(1660) = 45300 + \cancel{20000}\,17100 + 2220$
$= \underline{\underline{64620}}$

Plot total time	i/n	TTT
18000	0,125	0,08961
45300	0,250	0,22553
64620	0,375	0,32172
79120	0,500	0,39391
111910	0,625	0,55715
121460	0,750	0,60470
174010	0,875	0,86632
200860	1,000	1,00000

this column is first column/200860



Is not easy to see how this ~~d~~ indicates a significant deviance from a straight line. But is slightly convex, so could be a decreasing trend (?)

BP-test: From table:

$$W = 0.08961 + \cdots + 0.86632 = 3.05895$$

H_0: T is exponentially distr vs H_1: not so.

Here $n=8$

$$Z = \frac{W - \frac{n-1}{2}}{\sqrt{\frac{n-1}{12}}} = \frac{3.05895 - 3.5}{\sqrt{\frac{7}{12}}} = -0.58$$

With level 5% we reject when $|Z| \geq 1.96$, so we do not reject. [A negative value of Z indicates an alternative in direction of a decreasing ~~...~~ z(t).]

d) $T \sim \text{expon}(\lambda)$

~~Known from~~

~~Known that~~

$$L(\lambda) = \prod_{i:\delta_i=1} \lambda e^{-\lambda y_i} \cdot \prod_{i:\delta_i=0} e^{-\lambda y_i}$$

$$= \left(\prod_{i=1}^{n} e^{-\lambda y_i}\right) \cdot \lambda^{\sum \delta_i}$$

~~where $r = \sum_{i=1}^{n} \delta_i = \#$ failures~~

$$= e^{-\lambda s} \cdot \lambda^{r}$$

~~where~~

where $r = \sum \delta_i = \#$ fail.
$s = \sum y_i =$ total time on test.

So

$$\ell(\lambda) = -\lambda s + r \ln \lambda$$

$$\ell'(\lambda) = -s + \frac{r}{\lambda} = 0$$

$$\lambda = \frac{r}{s}$$

So: $\hat{\lambda} = \frac{r}{s} = \frac{8}{201510} = \underline{\underline{3.97 \cdot 10^{-5}}}$

(a little lower than the one in a), but of the same order of magnitude.

Now $z(t) = \lambda$, so $Z(t) = \lambda t$

Hence $Z(1800) = 1800\lambda$

so $\hat{Z}(1800) = 1800 \cdot 3.97 \cdot 10^{-5} \approx \underline{0.0715}$

To find a CI, we need a CI for λ:

R

Now $l''(\lambda) = -\frac{r}{\lambda^2}$

so $\widehat{\operatorname{Var} \hat{\lambda}} = \frac{\hat{\lambda}^2}{r}$

and $\widehat{SE \hat{\lambda}} = \frac{\hat{\lambda}}{\sqrt{r}} = \frac{3.97 \cdot 10^{-5}}{\sqrt{8}}$

Thus the standard CI for pos. parameter for λ is

$$\hat{\lambda} e^{\pm 1.96 \cdot \frac{SE(\hat{\lambda})}{\hat{\lambda}}}$$

For λ:

$$3.97 \cdot 10^{-5} \cdot e^{\pm 1.96 \cdot \frac{1}{\sqrt{8}}}$$

For $Z(1800)$:

$$1800 \cdot 3.97 \cdot 10^{-5} \cdot e^{\pm 1.96 \cdot \frac{1}{\sqrt{8}}}$$

$$0.0715 \cdot e^{\pm 1.96 \cdot \frac{1}{\sqrt{8}}}$$

$$\underline{(0.0357,\ 0.1429)}$$

which is slightly shorter than the one obtained earlier in b)

[The standard interval is

$$1800\left(\hat{\lambda} \pm 1.96 \cdot SE_{\hat{\lambda}}\right)$$

$$1800\left(3.97 \cdot 10^{-5} \pm 1.96 \cdot \frac{3.97 \cdot 10^{-5}}{\sqrt{8}}\right)$$

$$0.0715 \pm 0.0715 \cdot \frac{1.96}{\sqrt{8}}$$

$$\underline{(0.022,\ 0.1210)}$$

SOLUTION PROBLEM 2

~~1~~a) Minimal repair: Restored to condition immediately before failure.

Perfect repair: Restored to as good as new condition.

NHPP models minimal repair:
after repair, the behavior is independent of the behavior before failure.

RP models perfect repair
Times between failures are i.i.d., so starts after failure in perfect condition.

b) It is known that the number of failures in an interval is Poisson-distributed with exp. value equal to the integral of $w(t)$ over the interval. Also, what happens in disjoint intervals are independent.
Thus the ~~ea~~ NHPP-assumptions implies that the D_i are ① independent ② Poisson $(\underbrace{W(h_i;\theta) - W(h_{i-1};\theta)}_{W(h_{i-1},h_i;\theta)})$

Thus

$$L(\theta) = \prod_{i=1}^{r} \left\{ \frac{W(h_{i-1},h_i;\theta)^{d_i}}{d_i!} e^{-W(h_{i-1},h_i;\theta)} \right\}$$

Now (1) follows since

$$\prod_{i=1}^{r} e^{-W(h_{i-1},h_i;\theta)} = e^{-W(\tau;\theta)}$$

c) Now $h_i = i$; $i = 0, 1, \ldots, r.$

$$L(\lambda,\beta) = \prod_{i=1}^{10} \frac{(\lambda i^\beta - \lambda(i-1)^\beta)^{d_i}}{d_i!} \cdot e^{-\lambda 10^\beta}$$

$$~~\text{So}~~ \quad ~~\ell(\lambda,\beta)~~ = \left(\prod_{i=1}^{10} \frac{\lambda^{d_i}(i^\beta-(i-1)^\beta)^{d_i}}{d_i!}\right) \cdot e^{-\lambda 10^\beta}$$

so

$$\ell(\lambda,\beta) = \sum_{i=1}^{10}\left[d_i \ln\lambda + d_i(i^\beta-(i-1)^\beta) - \ln(d_i!)\right]$$

$$-\lambda 10^\beta$$

$$= (\ln\lambda)\sum_{i=1}^{10} d_i + \sum_{i=1}^{10} d_i(i^\beta-(i-1)^\beta) - \sum_{i=1}^{10}\ln(d_i!)$$

$$-\lambda \cdot 10^\beta$$

β known:

$$\boxed{\frac{\partial \ell}{\partial \lambda} = \frac{1}{\lambda}\sum d_i - 10^\beta = 0}$$

$$\boxed{\lambda = \frac{\sum_{i=1}^{10} d_i}{10^\beta}}$$

$$\hat\lambda(\beta) = \frac{\sum d_i}{10^\beta} \quad \text{so} \quad \hat\lambda(1.2) = \frac{35}{10^{\beta}} = 1.75$$

Prob of failure in ~~first week~~ ith week:

Exp. value:
~~Paramete~~: $\lambda i^{\beta} - \lambda (i-1)^{\beta}$

In the first week: $e^{-\hat{\lambda}}$
$= e^{-1.75} = \underline{0.174}$

In the 10th week:
$e^{-\hat{\lambda}(10^{1.3} - 9^{1.3})} = \underline{\underline{0.011}}$

$=$

d) $\tilde{\ell}(\beta) = \ell(\hat{\lambda}(\beta), \beta)$

$= \left(\ln(\Sigma d_i) - \beta \ln 10\right) \Sigma d_i$

$+ \Sigma d_i \ln(i^{\beta} - (i-1)^{\beta})$

$\underline{- \Sigma \ln(d_i!) - \Sigma d_i}$

1. Read off $\hat{\beta} \approx 1.49$

2. Read off CI: $[1.04, 2.04]$

Computation; 1.92 - interval:

Max value -18.60

$$\begin{array}{r} -18.60 \\ -\ 1.92 \\ \hline -20.52 \end{array}$$

So: Solve $\tilde{\ell}(\beta) = -20.52$

to get $\hat{\beta}_L = 1.04$

$\hat{\beta}_U = 2.04$

3. Test $H_0: \beta = 1$ vs. $H_1: \beta \neq 1$.

Log likelihood test:

$$W = 2[\tilde{\ell}(\hat{\beta}) - \tilde{\ell}(1)]$$

$$= 2[-18.60 - (\underbrace{-21.00}_{\text{read off}})]$$

$$= 2 \cdot 2.4 = \underline{4.8}$$

Under H_0 is $W \approx \chi^2_1$, so 5% test rejects when $W \geq 3.84$. So reject!

P-value: $P(W \geq 4.8) = 1 - P(W < 4.8)$

$= 1 - 0.9715 = \underline{\underline{0.0285}}$

$$\hat{\lambda} = \hat{\lambda}(1.49) = \frac{35}{10^{1.49}} = \underline{1.13}$$