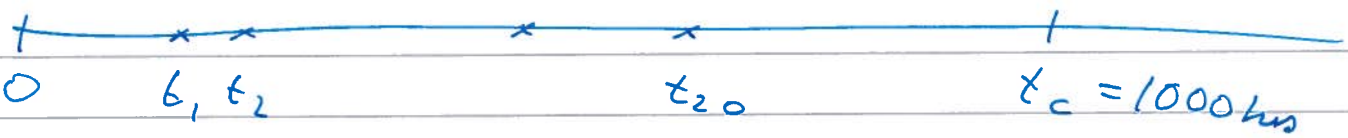
# 2012

## Problem 1

$n = 100$




$r = 20$ has failed

$$F(t;\theta) = 1 - e^{-\frac{t}{\theta}}$$

a) Likelihood: $\displaystyle\prod_{i:\text{failed}} \frac{1}{\theta} e^{-\frac{t_i}{\theta}} \cdot \prod_{i:\text{censored}} e^{-\frac{t_c}{\theta}}$

$$= \frac{1}{\theta^r} e^{-\frac{\sum_{i=1}^{20} t_i}{\theta}} \cdot e^{-\frac{(n-r)t_c}{\theta}}$$

$$= \frac{e^{-\frac{\sum_{i=1}^{20} t_i + (n-r)t_c}{\theta}}}{\theta^r}$$

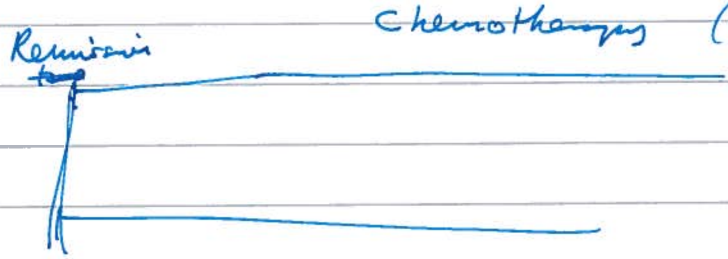
~~So~~ b) log-likelihood $= 0$

$$l(\theta) = -\frac{\sum t_i + (n-r)t_c}{\theta} - r \ln \theta$$

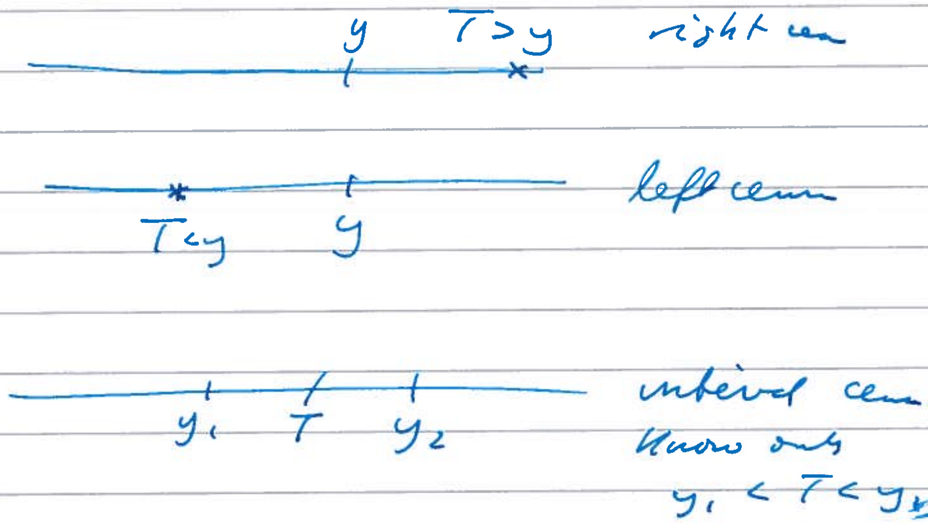
2. $\ell'(\theta) = \boxed{\frac{\sum t_i + (n-r)\, t_c}{\theta^2} - \frac{r}{\theta} = 0}$

$$\Rightarrow \hat{\theta} = \frac{\sum t_i + (n-r)\, t_c}{r} \left( = \frac{\text{tot. time}}{\#\text{failed}} \right)$$

Problem 2:




a) Right cens
Left cens
Interval cens




Causes? • End of trial, event has not occurred
• ~~Moving~~ Dropping out (moving, e.g.)

b) ~~Maintenance~~

$$\hat{R}_{KM}(t) = \prod_{t_i \le t} \frac{n_i - d_i}{n_i}$$

~~Maintenance gm~~

| $R_M(t)$ | $n_i$ | $d_i$ | $\frac{n_i - d_i}{n_i}$ | $P_m$ | Prod. ~~Total~~ |
|---|---|---|---|---|---|
| 9 | ~~9~~ | 1 | $\frac{9-1}{9} = \frac{8}{9} = 0.89$ | | 0.889 |
| 13 | 8 | 1 | $\frac{8-1}{8} = \frac{7}{8}$ | $\frac{8}{9} \cdot \frac{7}{8} = \frac{7}{9}$ | 0.778 |
| 23 | 6 | 1 | $\frac{6-1}{6} = \frac{5}{6}$ | Multiply | 0.648 |
| 34 | 4 | 1 | $\frac{4-1}{4} = \frac{3}{4}$ | | 0.486 |
| 55 | 2 | 1 | $\frac{2-1}{2} = \frac{1}{2}$ | | 0.243 |

For the other:

## Distribution Analysis: Y by t

Variable: Y
t = 1

| Censoring Information | Count |
|---|---|
| Uncensored value | 5 |
| Right censored value | 4 |

Censoring value: d = 0

Nonparametric Estimates

Characteristics of Variable

| | Standard | 95,0% Normal CI | |
|---|---|---|---|
| Mean(MTTF) | Error | Lower | Upper |
| 63,4352 | 26,5864 | 11,3268 | 115,544 |

Median = 34
IQR = 32  Q1 = 23  Q3 = 55

Kaplan-Meier Estimates

| | Number | Number | Survival | Standard | 95,0% Normal CI | |
|---|---|---|---|---|---|---|
| Time | at Risk | Failed | Probability | Error | Lower | Upper |
| 9 | 9 | 1 | 0,888889 | 0,104757 | 0,683570 | 1,00000 |
| 13 | 8 | 1 | 0,777778 | 0,138580 | 0,506166 | 1,00000 |
| 23 | 6 | 1 | 0,648148 | 0,165347 | 0,324074 | 0,97222 |
| 34 | 4 | 1 | 0,486111 | 0,187271 | 0,119066 | 0,85316 |
| 55 | 2 | 1 | 0,243056 | 0,195718 | 0,000000 | 0,62666 |

## Distribution Analysis: Y by t

Variable: Y
t = 2

| Censoring Information | Count |
|---|---|
| Uncensored value | 7 |
| Right censored value | 1 |

Censoring value: d = 0

Nonparametric Estimates

Characteristics of Variable

| | Standard | 95,0% Normal CI | |
|---|---|---|---|
| Mean(MTTF) | Error | Lower | Upper |
| 24,0313 | 5,74427 | 12,7727 | 35,2898 |

Median = 20
IQR = 30  Q1 = 13  Q3 = 43

Kaplan-Meier Estimates

/.

| Time | Number at Risk | Number Failed | Survival Probability | Standard Error | 95,0% Normal CI Lower | Upper |
|---|---|---|---|---|---|---|
| 5 | 8 | 1 | 0,87500 | 0,116927 | 0,645828 | 1,00000 |
| 13 | 7 | 2 | 0,62500 | 0,171163 | 0,289526 | 0,96047 |
| 20 | 4 | 1 | 0,46875 | 0,186521 | 0,103176 | 0,83432 |
| 21 | 3 | 1 | 0,31250 | 0,178152 | 0,000000 | 0,66167 |
| 43 | 2 | 1 | 0,15625 | 0,141921 | 0,000000 | 0,43441 |
| 45 | 1 | 1 | 0,00000 | 0,000000 | 0,000000 | 0,00000 |

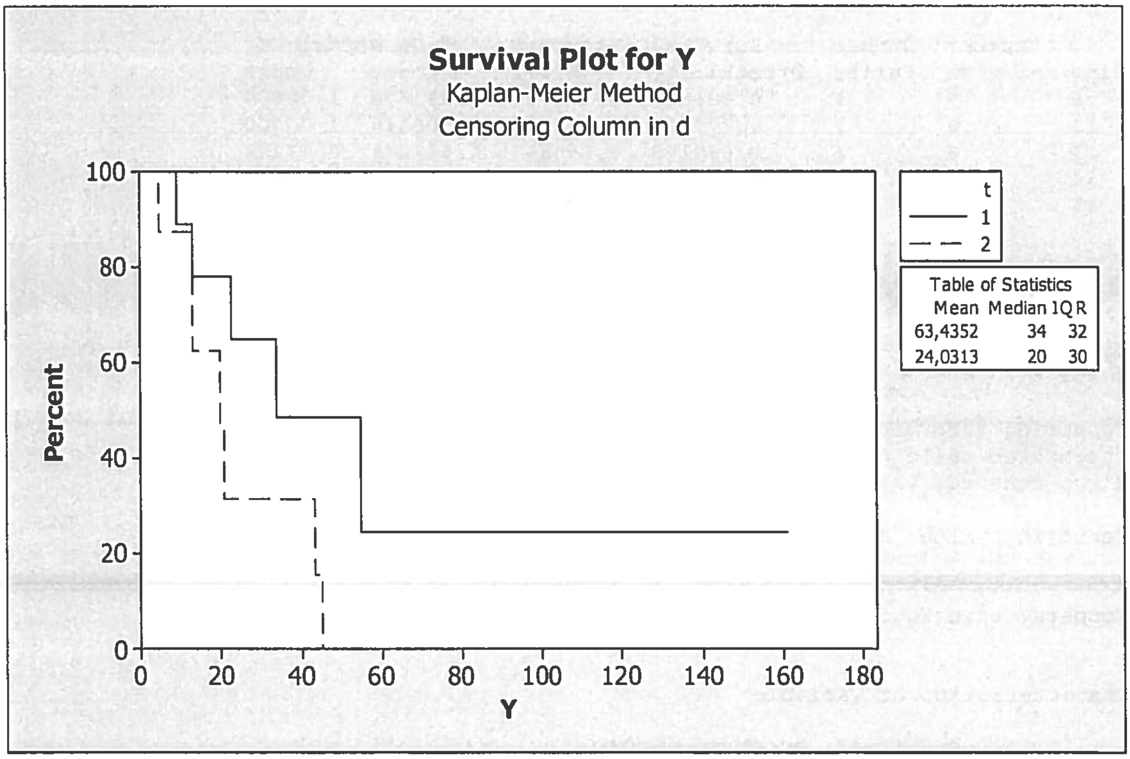
## Distribution Analysis: Y by t

Comparison of Survival Curves

Test Statistics

| Method | Chi-Square | DF | P-Value |
|---|---|---|---|
| Log-Rank | 3,05050 | 1 | 0,081 |
| Wilcoxon | 1,78571 | 1 | 0,181 |

## Nonparametric Survival Plot for Y



Logrank test:

$H_0: R_M(t) = R_C(t)$ for all $t$

vs. $H_1:$ $\neq$ (at least for some $t$).

Must compute

Slides 2:

| Time | #Risk$_M$ | #Risk$_C$ | #Risk | Fail$_M$ | Fail$_C$ | Fail | EM | EC |
|---|---|---|---|---|---|---|---|---|
| ~~3~~ | | | | | | | | |
| 5 | 9 | 8 | 17 | 0 | 1 | 1 | $\frac{1}{17}\cdot 9$ | $\frac{1}{17}\cdot 8$ |
| 9 | 9 | 7 | 16 | 1 | 0 | 1 | $\frac{1}{16}\cdot 9$ | $\frac{1}{16}\cdot 7$ |
| 13 | 8 | 7 | 15 | 1 | 2 | 3 | $\frac{3}{15}\cdot 8$ | $\frac{3}{15}\cdot 7$ |
| ⋮ | | | | | | | | |
| | | | | | | | 7.73 | 4.27 |

So: ~~#~~ Log rank statistic is

$$\frac{(5-7.73)^2}{7.73} + \frac{(7-4.27)^2}{4.27} = 2.71$$

Compare to $\chi^2_1$:

$P(\chi^2_1 > 2.71) = 0.10$

so do not reject

# TMA4275 Lifetime analysis
# Spring 2009
*
# The logrank test for comparison of survival functions

Bo Lindqvist

The logrank test is a test for equality of survival functions of two or more groups of units.

Consider here the case of two groups. Suppose that we have censored data of the usual form $(Y_{1i}, \delta_{1i})$ for group 1 and similarly $(Y_{2i}, \delta_{2i})$ for group 2. Let $R_1(t)$ and $R_2(t)$ be the true underlying survival functions for the two groups. The logrank test is a test for the null hypothesis

$$H_0 : \; R_1(t) = R_2(t) \text{ for all } t$$

versus the alternative hypothesis

$$H_1 : \; R_1(t) \neq R_2(t) \text{ for at least one } t$$

Let $T_{(j)}$ for $j = 1, \ldots, k$ be the distinct times of observed failures when considering the two groups together. For each time $T_{(j)}$ let $N_{1j}$ and $N_{2j}$ be the number of subjects "at risk" (have not yet had a failure or been censored) immediately before $T_{(j)}$ in the two groups. Let $N_j = N_{1j} + N_{2j}$ be the total number at risk at time $T_{(j)}$.

Now let $O_{1j}$ and $O_{2j}$ be the observed number of failures in the two groups at time $T_{(j)}$, and define $O_j = O_{1j} + O_{2j}$.

Under the null hypothesis the probability of failure at $T_{(j)}$ would be the same for the two groups, reasonably estimated by $O_j/N_j$. Hence since the number at risk for the two groups are respectively $N_{1j}$ and $N_{2j}$, the expected number of failures in the two groups at $T_{(j)}$ would be respectively

$$E_{1j} = \frac{O_j}{N_j} N_{1j}$$

and

$$E_{2j} = \frac{O_j}{N_j} N_{2j}$$

Let now $O_1 = \sum_{j=1}^{k} O_{1j}$ and $O_2 = \sum_{j=1}^{k} O_{2j}$ be the total observed number of failures for each of the two groups, and let $E_1 = \sum_{j=1}^{k} E_{1j}$ and $E_2 = \sum_{j=1}^{k} E_{2j}$ be the corresponding total expected number of failures.

The logrank statistic compares expected and observed values in the two groups, and is given by

$$\frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2}$$

which is approximately chi-square distributed with 1 degree of freedom if the null hypothesis holds and the number of observations is large enough. We therefore reject at significance level $\alpha$ if this is larger than the upper $\alpha$ quantile of this distribution (e.g. equal to 3.84 if $\alpha = 0.05$).

Note that the logrank statistic for the case of two groups is often given in a slightly different way in the literature. However, the above statistic has the advantage of being easily generalized to the case of comparing more than two survival functions (degrees of freedom is then always the number of groups minus 1).

The logrank test is based on the same assumptions as the Kaplan-Meier estimator.

An example is given in the slides (page 68 in 2009).

d) Cox-model

$x = 1$ for Marix
$x = 0$ for Control.

$$z(t; x) = z_0(t) \cdot e^{\beta x}$$

~~The~~ R was used to estimate

$\hat{\beta} = -1.06$, Testing $H_0: \beta = 0$ vs. $\beta \neq 0$ with p-value 0.096 & likelihood ratio test 2.92.

MINITAB; Weibull regression

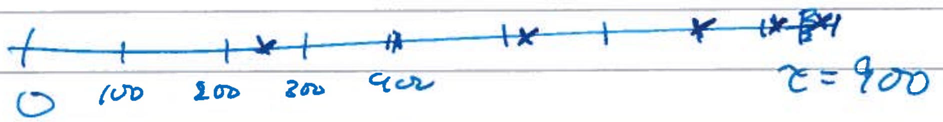
$\hat{\alpha} = 1.26691$

$\hat{\beta} = 1.06502$

But $\hat{\beta} = -\alpha\hat{\beta} = -1.26691 \cdot 1.06502$

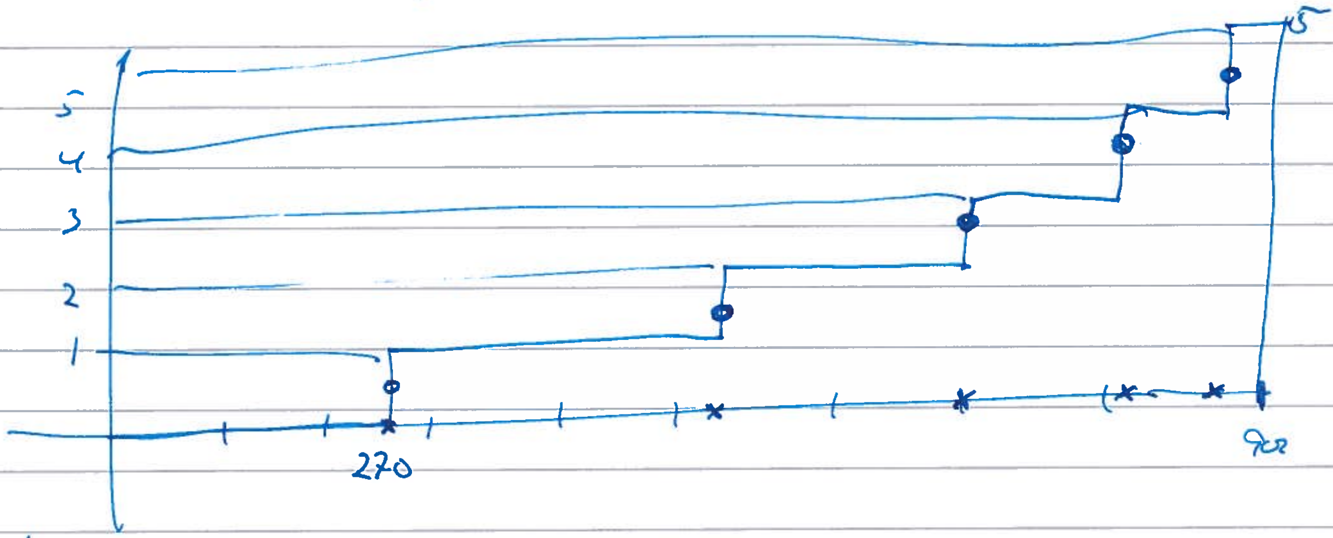
$= \underline{\underline{-1.34}}$

## Problem 4.




a)

$\hat{W}(t) = N(t)$




Laplace test:

$$W = \frac{\sum \left(S_i - \frac{\tau}{2}\right)}{\tau\sqrt{\frac{n}{12}}} = \frac{(270-450)+(520-450)+(700-450)+(810-450)+(860-450)}{900\sqrt{\frac{5}{12}}}$$

$$= \frac{-180+70+250+360+410}{900\sqrt{\frac{5}{12}}} = 1.57$$

Not reject

b) $w(t) = e^{\alpha + \beta t}$

Log-linear model

$$W(t) = \int_0^t e^{\alpha + \beta u}\, du = e^{\alpha} \int_0^t e^{\beta u}\, du$$

$$= e^{\alpha} \cdot \Big|_0^t \frac{1}{\beta} e^{\beta u} = \underline{\underline{\frac{e^{\alpha}}{\beta}\left(e^{\beta t} - 1\right)}}$$

If $\beta = 0$: HPP with rate $e^{\alpha}$

$$\underline{\underline{W(t) = e^{\alpha} t}}$$

c) General

$$\ell = \sum_{i=1}^{N} \ln w(s_i) - W(\tau)$$

Here:

$$\ell(\alpha, \beta) = \sum_{i=1}^{N} (\alpha + \beta s_i) - \frac{e^{\alpha}}{\beta}\left(e^{\beta \tau} - 1\right)$$

$$= N\alpha + \beta \underbrace{\sum s_i}_{S} - \frac{e^{\alpha}}{\beta}\left(e^{\beta \tau} - 1\right)$$

$$\frac{\partial \ell}{\partial \alpha} = N - \frac{e^{\alpha}}{\beta}\left(e^{\beta\tau}-1\right) \qquad (1)$$

$$\frac{\partial \ell}{\partial \beta} = S - e^{\alpha}\cdot \frac{\tau e^{\beta\tau}\beta - \left(e^{\beta\tau}-1\right)}{\beta^{2}} \qquad (2)$$

$$\cancel{= S - \frac{e^{\alpha}}{\beta^2}\Big[}$$

(1) = 0 gives

$$\beta N = e^{\alpha}\left(e^{\beta\tau}-1\right)$$

$$e^{\alpha} = \frac{\beta N}{e^{\beta\tau}-1} \qquad \text{so ok}$$

$$\hat{\alpha} = \ln\left[\frac{5\cdot 0.003}{e^{0.003\cdot 900}-1}\right]$$

Put this into (ii) to solve.
Or put $\alpha = \ln(\ )$ into log-likelihood to get the profile likelihood.

$$= -6.83$$

$$\hat{W}(t;\hat{\alpha},\hat{\beta}) = \frac{e^{-6.83}}{0.003}\left(e^{0.003\,t}-1\right)$$

| $t$ | ~~270~~ |
|---|---|
| 270 | 0.450 |
| 520 | 1.31 |
| 700 | 2.58 |
| 810 | 3.72 |
| 860 | 4.35 |

900 : 5.0