Problem 1.

a) K-M estimator: $\hat{R}(t) = \sum_{t_i \le t} \frac{n_i - d_i}{n_i}$

~~$\hat{R}$~~ t_i	n_i	d_i	$\frac{n_i-d_i}{n_i}$	$\hat{R}(t_i)$
27	10	1	$\frac{9}{10}$	$\frac{9}{10} = 0.9$
134	8	1	$\frac{7}{8}$	$\frac{9}{10}\cdot\frac{7}{8} = 0.7875$
136	7	1	$\frac{6}{7}$	$\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{6}{7} = 0.675$
170	6	1	$\frac{5}{6}$	$\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{6}{7}\cdot\frac{5}{6} = 0.5625$
210	4	1	$\frac{3}{4}$	$\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{6}{7}\cdot\frac{5}{6}\cdot\frac{3}{4} = 0.421875$

$\hat{R}(180) = \hat{R}(170) = 0.5625$

Greenwood's formula:

$$\widehat{\mathrm{Var}\,\hat{R}(t)} = \left(\hat{R}(t)\right)^2 \cdot \sum_{t_i \le t} \frac{d_i}{n_i(n_i - d_i)}$$

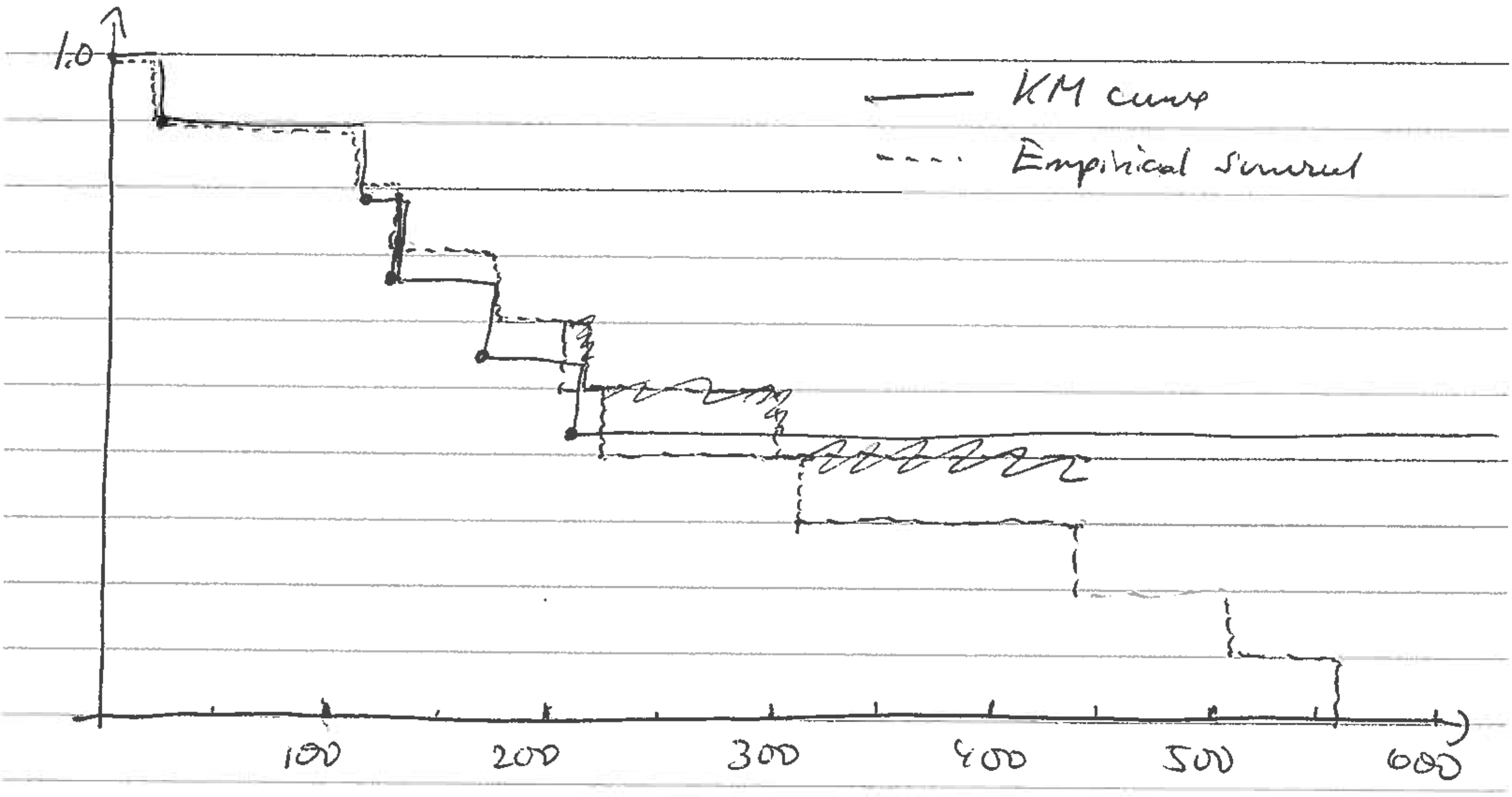
So

$$\widehat{\mathrm{Var}\,\hat{R}(180)} = 0.5625^2 \cdot \left(\frac{1}{10\cdot 9} + \frac{1}{8\cdot 7} + \frac{1}{7\cdot 6} + \frac{1}{6\cdot 5} \cancel{+ \frac{1}{4\cdot 3}} \right)$$

$$= \cancel{0.1238} \; 0.1651^2$$

So $\widehat{\mathrm{SD}\,\hat{R}(180)} = 0.1651$

$\hat{R}$ $R^*(210)$ ~~$R^*(180) =$~~



b) Order the times: 27, 124, 136, 170, 191, 210, 308, 441, 511, 559

$R^*(180) = \frac{6}{10}$

Then $X \sim \text{bin}(10, R(180))$

So $\text{Var}(R^*(180)) = \frac{R(180)(1-R(180))}{10}$

which is estimated by $\widehat{\text{Var}\, R^*(180)} = \frac{0.6 \cdot 0.4}{10}$

$= 0.1549^2$

~~So SD~~ So $\widehat{SD\, R^*(180)} = 0.1549$

~~The ex~~ It is seen that the estimates are fairly similar, ~~but~~ as is reasonable, the estimated st. dev. of the estimates for $R(180)$ is lower when we have complete data.

c) Know that $E(T) = \int_0^\infty R(t)\,dt$ in general

So ideally one would use the are under the KM-curve from 0 to ∞ to estimate $E(T)$. But this is ∞, so it could be reasonable to integrate only to the maximum observed time, here 410 for the censored data.

$$\underline{\text{Area}} = 27 \cdot 1 + 97 \cdot 0.9 + 12 \cdot 0.7875 + 34 \cdot 0.675$$

$$+ 40 \cdot 0.5625 + 200 \cdot 0.421875$$

$$= \underline{\underline{253.575}} \leftarrow \widehat{E(T)}$$

From complete data: $\widehat{E(T)} = \overline{T} = \underline{\underline{267.7}}$

Problem 2

a) Curves are $\cancel{\hat{W}_j(t) =}$ $\hat{N}_j(t)$ = # failures in $(0,t]$, for $0 < t \le 30$

which are unbiased estimators for

$$W_j(t) = E[N_j(t)] \quad \text{for } j = 1,2.$$

$W_j'(t) = w_j(t)$ so increasing (decreasing) trend is equivalent to convex (concave) plot.
There seems to be a convexity here, indicating increasing trends.

$\hat{W}_j(30)$ is expected # failures in a month for machine j.

$\hat{W}_j(t) \sim \text{Poisson}(W_j(t))$

so $\hat{W}_j(30) \sim \text{Poisson}(W_j(30))$

$\hat{W}_1(30) = 4$, $SD(\hat{W}_1(30)) = \sqrt{4} = 2$ since $\hat{W}_j(t) \sim$ Poisson.

$\hat{W}_2(30) = 7$, $SD(\hat{W}_2(30)) = \sqrt{7} = 2.65$ – · –

b) H_0: $w_j(t)$ is constant vs. H_1: $w_j(t)$ is increasing

Test statistics: $Z_j = 2 \sum_{i=1}^{N_j} \ln \frac{\tau_j}{S_{ij}}$

$$Z_1 = 2\left[\ln\frac{30}{11} + \ln\frac{30}{19} + \ln\frac{30}{21} + \ln\frac{30}{28}\right]$$

$= 3.77$

Under H_0 is $Z_1 \sim \chi^2_{2\cdot 4=8}$ so reject if

$$\underline{\underline{Z_1 \le \chi^2_{0.95,8} = 2.73}}$$

Thus: Do not reject H_0 for Machine 1.

$$Z_2 = 2\left[\ln\frac{30}{8} + \ln\frac{30}{13} + \ln\frac{30}{18} + \ln\frac{30}{21} + \ln\frac{30}{25} + \ln\frac{30}{26} + \ln\frac{30}{29}\right]$$

$$= 6.77$$

Under H_0 is $Z_2 \sim \chi^2_{2\cdot 7=14}$, so reject if

$$Z_2 \le \chi^2_{0.95,14} = 6.57$$

Thus: Do not reject H_0 for Machine 2.

<u>Pooled Mil Hbk test:</u>

H_0: both $w_1(t)$ and $w_2(t)$ are constant, but not necessarily equal

vs

H_1: at least one of them is increasing.

$Z_{pooled} = Z_1 + Z_2 = \cancel{3.77} + \cancel{6}.77 = 10.54$

Under H_0 is $Z_{pooled} \sim \chi^2_{8+14=22}$, so reject H_0 if

$$Z_{pooled} \leq \chi^2_{0.95, 22} = \cancel{12}.34$$

10.98 for 0.975

Thus we reject H_0, and conclude that at least one of the $w_j(t)$ is increasing.

~~b)~~

Problem 3

a) $N_j \sim \text{Poisson}(W_j(\cancel{30}))$; $j = 1, 2$.

Here $W_1(t) = \lambda t^\beta \equiv E(N_1)$

$W_2(t) = \delta \lambda t^\beta \equiv E(N_2)$

See that $\delta = \dfrac{E(N_2)}{E(N_1)}$. ~~so δ is the~~

b) General:

$$L = \left\{ \prod_{i=1}^{N(\tau)} w(S_i) \right\} e^{-W(\cancel{\beta}\,\tau)}$$

so log-likelihood is

$$\ell = \left\{ \sum_{i=1}^{N(\tau)} \ln w(S_i) \right\} - W(\tau)$$

$$\ell(\lambda,\beta,\delta) = \sum_{i=1}^{N_1} \left(\ln\lambda + \ln\beta + (\beta-1)\ln S_{i1} \right) - \lambda\cdot 30^\beta$$

$$+ \sum_{i=1}^{N_2} \left(\ln\delta + \ln\lambda + \ln\beta + (\beta-1)\ln S_{i2} \right) - \delta\cdot\lambda\cdot 30^\beta$$

$$= (N_1+N_2)\ln\lambda + (N_1+N_2)\ln\beta$$

$$+ N_2\ln\delta + (\beta-1)\,U - \lambda(1+\delta)\cdot 30^\beta$$

where $U = \sum_{i=1}^{N_1} \ln S_{i1} + \sum_{i=1}^{N_2} \ln S_{i2}$

c) $$\hat{\delta} = \frac{N_2}{N_1} = \frac{7}{4} = \underline{1.75}$$

$$\hat{\beta} = \frac{11}{11\cdot\ln 30 - 32.1426} = \underline{2.0871}$$

$$\hat{\lambda} = \frac{11}{(1+1.75)\cdot 30^{2.0871}} = \underline{0.003305}$$

The parameters are all positive, so we use the standard interval for positive parameters given as (general)

$$\hat{\theta}\, e^{\pm 1.96 \frac{\widehat{SD(\theta)}}{\hat{\theta}}}$$

So for λ: $0.003305 \cdot e^{\pm 1.96 \frac{\sqrt{5.278 \cdot 10^{-5}}}{0.003305}}$

$(4.449 \cdot 10^{-5},\ 0.245)$

For β: $2.0871 \cdot e^{\pm 1.96 \cdot}$

$(1.156,\ 3.769)$

Only this needs to be done.

For δ: $(0.5123,\ 5.978)$

Conf int. can be used to test $H_0: \beta = 1$ vs $H_1: \beta \neq 1$ with sign level 5%: Reject H_0 if $1 \notin$ conf int. This is the case, so we reject!

$H_0: \beta = 1$ corresponds to $w_1(t) = \lambda$, $w_2(t) = \delta\lambda$ which is exactly that both $w_1(t)$ and $w_2(t)$ are constant, but possibly different

d) Test statistic $W = 2\,[\,\ell(\hat\lambda, \hat\beta, \hat\delta) - \ell(\hat\lambda(1), 1, \hat\delta(1))\,]$

is approx χ^2_1 under H_0, where $\hat\lambda(1), \hat\delta(1)$ are the maximum likelihood estimates of λ and δ when $\beta = 1$.

We need to find $\hat\lambda(1), \hat\delta(1)$:

When $\beta = 1$ is the log-likelihood

$$\ell(\lambda, 1, \delta) = (N_1 + N_2)\ln\lambda + N_2 \ln\delta - \lambda(1+\delta)\cdot 30$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{N_1 + N_2}{\lambda} - (1+\delta)\cdot 30$$

$$\frac{\partial \ell}{\partial \delta} = \frac{N_2}{\delta} - \lambda \cdot 30$$

Solve likelihood equations $\frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell}{\partial \beta} = 0$
gives

$$\hat\lambda(1) = \frac{N_1}{30} = \frac{4}{30} = 0.1333$$

$$\hat\delta(1) = \frac{N_2}{N_1} = \frac{7}{4} = 1.75$$

and $\ell(\hat\lambda(1), 1, \hat\delta(1)) = 11 \cdot \ln\left(\frac{4}{30}\right) + 7 \ln\left(\frac{7}{4}\right) - \frac{4}{30}\left(1 + \frac{7}{4}\right) 30$

$$= -29.24662$$

So $W = 2\,[-26.88 - (-29.25)] = 4.73$

Reject at 5% if $W \geq 3.84$, so we reject!