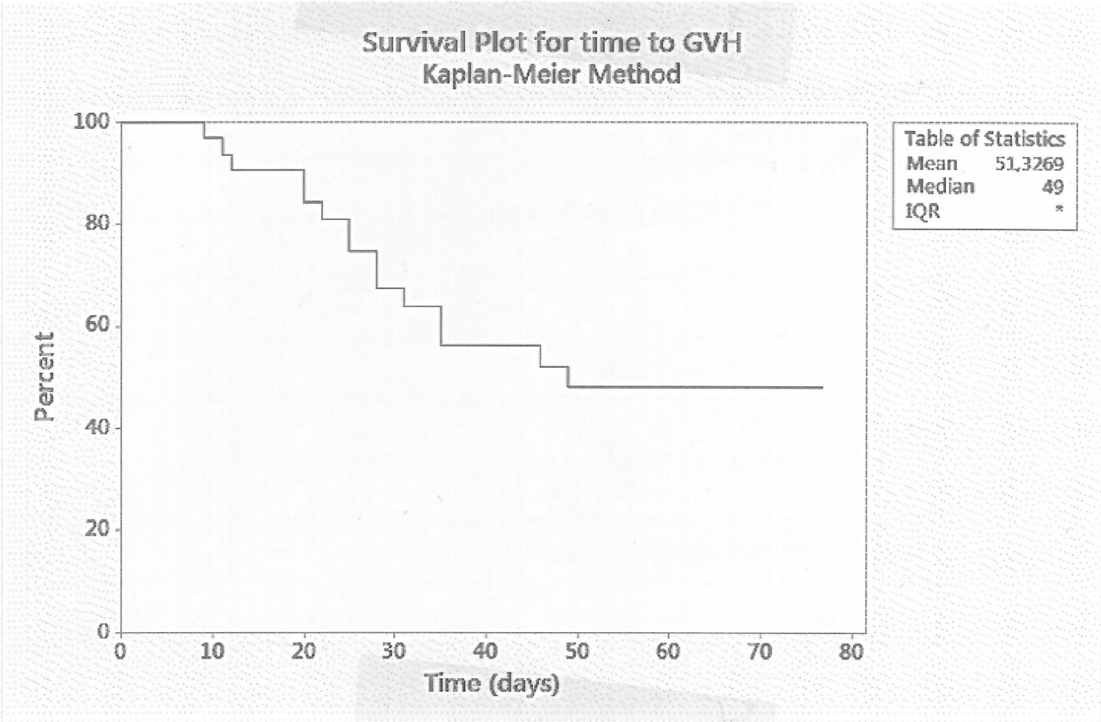
SOLUTION

Problem 1

(a)



$\hat{R}_{KM}(21)$ from MINITAB-output:

$$= \hat{R}_{KM}(20) = \underline{0.843750}$$

Calculation:

$$\hat{R}_{KM}(21) = \prod_{T_{(i)} \le 21} \frac{n_i - d_i}{n_i}$$

$$= \frac{32-1}{32} \cdot \frac{31-1}{31} \cdot \frac{30-1}{30} \cdot \frac{29-2}{29}$$

$$= 0.84375 \text{ (as from MINITAB!)}$$

$$\widehat{\mathrm{Var}\, \hat{R}(t)} = \left(\hat{R}(t)\right)^2 \cdot \sum \left(\frac{1}{32 \cdot 31} + \frac{1}{31 \cdot 30} + \frac{1}{30 \cdot 29} + \frac{2}{29 \cdot 27} \right)$$

$$= 0.064^2, \text{ so } \underline{SE = 0.064}$$

MINITAB uses standard CI:

$$0.84375 \pm 1.96 \cdot 0.064$$

$(0.7179,\ 0.9696)$.

Standard C.I. for positive parameters:

$$0.84375 \cdot e^{\pm 1.96 \cdot \frac{0.064}{0.84375}}$$

$(0.7272,\ 0.9790)$

b) Lower quartile:

We must find the failure time for which the estimate for the first time is below 0.75, i.e. $t_{0.25} = \underline{\underline{25}}$.

Median:

First failure time with estimate ≤ 0.50, i.e. $t_{0.5} = \text{median} = \underline{\underline{49}}$

Upper quartile:

Not estimable, since estimates are never ≤ 0.25.

Mean time: Is the area under the KM-curve until the largest censored value: Here 77.

Motivation: Theoretically is $E(T) = \int_0^\infty R(t)\,dt$.

c) $\hat{R}_{KM}(t) = \prod_{t_{(i)} \le t} \frac{n_i - d_i}{n_i}$

If last time is a failure time then necessarily $n_i = d_i$, so the last factor becomes 0.

If last time is a censoring time, then the number at risk at the last failure time must be ~~as~~ larger than the number failing, and hence all factors of $R_{KM}(t)$ are positive and hence positive for all t.

In the present situation, $\hat{R}(t) = 0.482364$ for all $t \ge 49$.

It is rather arbitrary to stop the integration under the curve at 77. Since there seems to be a long, heavy tail, the expected value might be much bigger

d) This is by the construction of the KM estimator (see slides) where we start at time 0 and look for survivors as time goes.

Let $\hat{q} = 0.482364$ (the limit as $t \to \infty$ of the KM estimator. In (1), $q = \lim_{t \to \infty} P(X > t)$.

We have $P(X > x_p) = 1-p$ for all $0<p<1$

But by (1),

$$P(X > x_p) = q + (1-q)\, R_0(x_p)$$

so

$$q + (1-q)\, R_0(x_p) = 1-p$$

$$(*) \quad R_0(x_p) = \frac{1-p-q}{1-q} = 1 - \frac{p}{1-q}$$

But then $x_p = t_{\frac{p}{1-q}}$

We must have $p < 1-q$ since $R_0(x_p) > 0$.
(see (*))

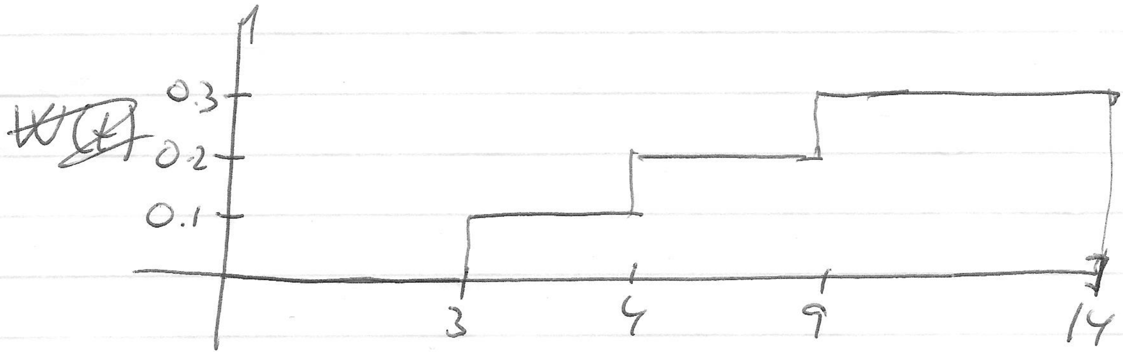
<u>Problem 2</u>. (next page)

Problem 2

$W(t) = E\,N(t)$

a) $W(t)$ is estimated by NA-estimator.
For 10 processes we project to a single axis

~~A~~ $\hat{W}(t)$ jumps by $1/10$ at each event:

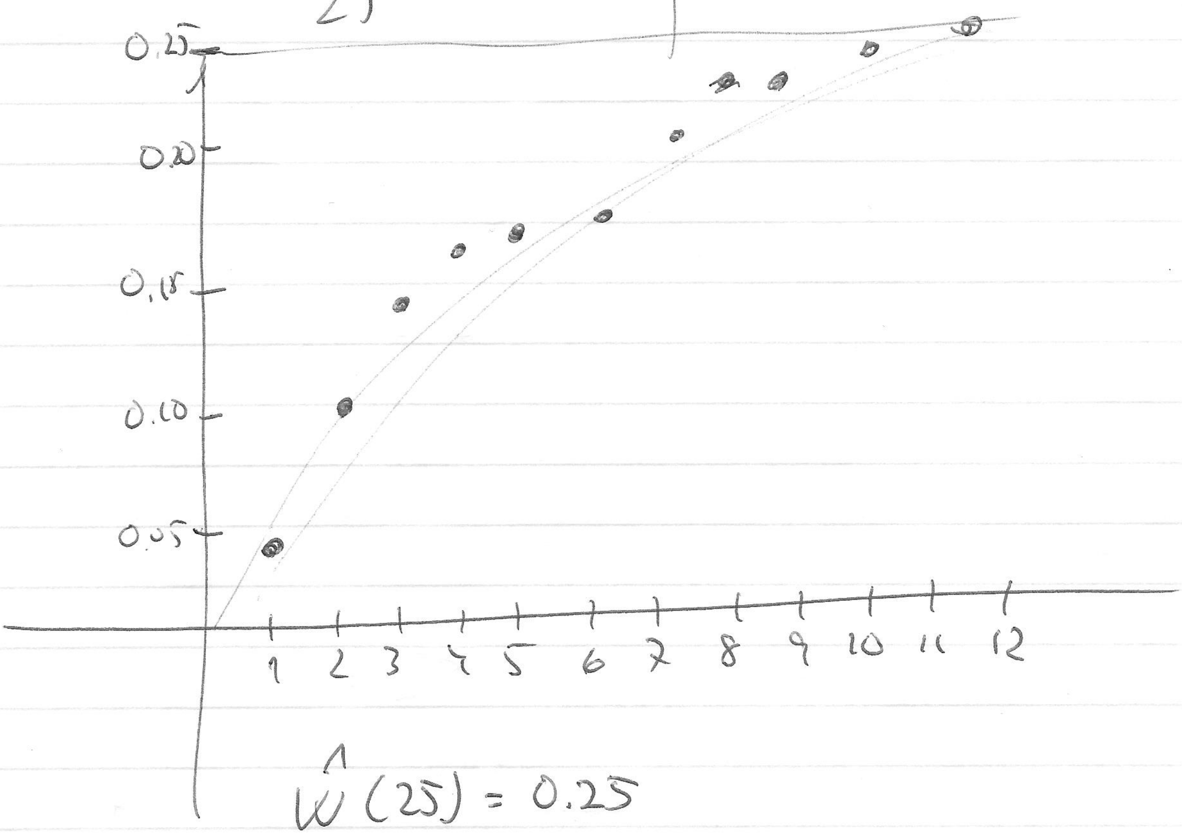


$\hat{W}(14) = 0.3$

b) ~~W~~ There are 100 units sold.
Project all failures down to a single axis. Then

$$\hat{W}(t) = \sum_{\substack{\text{all event}\\ \text{times} \le t}} \frac{1}{100} = \frac{1}{100}\ \#\text{event times} \le t.$$

Time t	# event times at t	Cum. no	$\hat{W}(t)$	[illegible]
1	4	4	0.04	4
2	6	10	0.10	12
3	4	14	0.14	12
4	2	16	0.16	8
5	1	17	0.17	5
6	1	18	0.18	6
7	3	21	0.21	21
8	0			[illegible]
9	2	23	0.23	18
10	1	24	0.24	10
11	0			
12	1	25	0.25	12
	25			108



$\underline{\hat{W}(25) = 0.25}$

Concave shape $\Leftrightarrow$ decreasing trend (ROCOF) $w(t)$.

The general expression for the pooled Laplace statistic is simpler when all the τ_j are equal:

$$W_{pooled} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{n_j} S_{ij} - \frac{n\tau}{2}}{\tau\sqrt{\frac{n}{12}}}$$

where $n = \sum_{j=1}^{m} n_j$

Here: $W_{pooled} = \frac{108 - \frac{25 \cdot 7}{}}{14\sqrt{\frac{25}{12}}}$

$= \underline{-3.315}$

H_0: all ~~the~~ units have HPP ~~the~~ process for complaints.

Natural H_1: decreasing trend

P-value $= P(Z \leq -3.315)$ where $Z \sim N(0,1)$

This is 0.000.

[Must have normal table].

Conclusion: There is a significant decreasing trend in complaints.
for any reasonable significance value.

Problem 3

a) $R(t) = \dfrac{1}{1+\left(\frac{t}{\theta}\right)^{\alpha}}$; $t>0, \alpha>0, \theta>0.$

$$f(t) = -R'(t) = \frac{\frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}}{\left[1+\left(\frac{t}{\theta}\right)^{\alpha}\right]^2}$$

$$z(t) = \frac{f(t)}{R(t)} = \frac{\frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}}{1+\left(\frac{t}{\theta}\right)^{\alpha}} = \frac{\alpha t^{\alpha-1}}{\theta^{\alpha}+t^{\alpha}}$$

QED.

b) $(R(t))^{-1} = 1+\left(\frac{t}{\theta}\right)^{\alpha}$

$$\left(\frac{t}{\theta}\right)^{\alpha} = (R(t))^{-1} - 1$$

$$\alpha \ln t - \alpha \ln \theta = \ln\left((R(t))^{-1} - 1\right)$$

Thus the points

$$\left(\ln t, \ln\left((R(t))^{-1} - 1\right)\right)$$

are all on the straight line "$y = \alpha x - \alpha \ln \theta$"

Thus plot, for the failure times $T_{(i)}$,

$$\left(\ln T_{(i)}, \ln\left((\hat{R}(T_{(i)}))^{-1} - 1\right)\right) \qquad (**)$$

where $\hat{R}$ is the KM-estimator (or possibly a modification of it).

Note also: $\underline{\underline{R(t)^{-1} - 1}} = \frac{1}{R(t)} - 1$

$$= \underline{\underline{\frac{1 - R(t)}{R(t)}}}$$

If a line is fitted by some method to the points (**), then α can be estimated by the <u>slope</u> and θ can be estimated by putting

$-\alpha \ln \theta$ = intercept of line.

c) $z'(t) = \dfrac{\alpha(\alpha-1)t^{\alpha-2}(\theta^{\alpha}+t^{\alpha}) - \alpha t^{\alpha-1} \cdot \alpha t^{\alpha-1}}{(\theta^{\alpha}+t^{\alpha})^2}$

The sign of $z'(t)$ is determined by the numerator, which after cancelling the term $\alpha^2 t^{2\alpha-2}$ can be written

$$t^{\alpha-2}(\alpha\theta^{\alpha} - \theta^{\alpha} - t^{\alpha})$$

which is > 0 if and only if

$$t^{\alpha} < (\alpha - 1)\theta^{\alpha}$$

If $\alpha \le 1$, then this is trivially not satisfied

If $\alpha > 1$, then this happens if and only if $t < \theta(\alpha-1)^{1/\alpha}$ (which is hence the max.)