NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4275 Lifetime Analysis**

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Examination time (from–to): 09:00-13:00

Permitted examination support material: C: Approved Calculator. One yellow sheet (A4 with stamp) with your own formulae and notes.

Other information:
Tables of the χ^2-distribution with 1 degree of freedom, and of the standard normal distribution are enclosed at the end of the exam.

Language: English

Number of pages: 6

Number pages enclosed: 4

Checked by:

Date Signature

Problem 1 *Discharge from hospital*

At an intensive care unit at a hospital one is interested in whether the presence or no-presence of pneumonia for a patient at admission will have an impact on the length of hospital stay, T, i.e., time from entry to discharge from the hospital unit.

Below are times (in days) to discharge from the hospital for 8 patients *without* pneumonia at admission ($x = 0$), and 7 patients *with* pneumonia at admission ($x = 1$).

Time to discharge (days):

No pneumonia at admission ($x = 0$): 2, 3+, 6, 6, 10, 11, 12+, 23

Pneumonia at admission ($x = 1$): 4+, 9, 12+, 17, 24, 26+, 32

Here + means a right censored observation. (Right censoring occurred in the cases when a patient was still in hospital at the end of the study, or died in hospital.)

It was decided to analyze the data by a Weibull regression model with the single covariate x defined above, representing the status of pneumonia at admission.

The model for discharge time, T, for a patient with pneumonia status x, is then

$$\ln T = \beta_0 + \beta_1 x + (1/\alpha)W, \tag{1}$$

where W is standard Gumbel-distributed.

a) Let U be Weibull-distributed with shape parameter α and scale parameter θ, so that the reliability function of U is

$$R_U(u) = e^{-\left(\frac{u}{\theta}\right)^\alpha} \text{ for } u > 0.$$

Show that the hazard rate function of U can be written

$$z_U(u) = \frac{\alpha u^{\alpha-1}}{\theta^\alpha} \text{ for } u > 0.$$

It can be shown (you are *not* asked to do this) that T defined by (1) is Weibull-distributed with shape parameter α and scale parameter

$$\theta = e^{\beta_0 + \beta_1 x}.$$

Use this to write down an expression for the hazard rate functions of the discharge time T for (i) a patient *without* pneumonia at admission; (ii) a patient *with* pneumonia at admission.

Show that the relative risk (with respect to discharge time) of a patient without pneumonia as compared to a patient with pneumonia is given by $e^{\alpha\beta_1}$.

b) The following is an (edited) MINITAB output for a Weibull regression model using the given data.

```
Regression Table

                        Standard                           95,0% Normal CI
Predictor      Coef        Error        Z       P        Lower      Upper
Intercept   2,53917     0,217702        *       *          *          *
x          0,741551     0,344049        *       *          *          *
Shape       1,91178     0,461130        *       *          *          *

Log-Likelihood = -36,045

Distribution:   Weibull
```

Write down the estimated model for T for a patient with pneumonia status x.

Is there a significant difference between the discharge times for patients without and with pneumonia at admission? Formulate this problem as a testing problem of a parameter in the model, and derive the conclusion when the significance level is set to 5%.

Compute the estimate of the relative risk of a patient without pneumonia as compared to a patient with pneumonia (see subpoint a)). What is the practical interpretation of this number in the current situation?

How would you interpret the estimated value of the shape parameter?

c) Use information from the above MINITAB output to compute a 95% confidence intervals for the parameter β_1.

The data analyst also computed a 95% confidence interval for the parameter α based on the MINITAB output. Do the computation and display the resulting interval.

One reason for the analyst's interest in the latter confidence interval, was to investigate whether an exponential regression model could be sufficient for the data as a simplification of the Weibull model.

Formulate an appropriate null hypothesis and the corresponding alternative hypothesis for this problem. Use the computed confidence interval for α to arrive at a conclusion. What is the significance level of the test that is used?

A colleague of the analyst suggested to run another MINITAB analysis, where an *exponential* regression model is fitted to the data. The output of this run is displayed below. Explain how one can use information from this output together with information from the earlier output to construct an alternative test for the model choice problem. Use significance level 5% in the testing. Compare the conclusion to the one already obtained.

```
Regression Table

                          Standard                        95,0% Normal CI
Predictor       Coef         Error     Z       P        Lower     Upper
Intercept    2,49870      0,408248  6,12   0,000      1,69855   3,29885
x           0,935287      0,645497  1,45   0,147    -0,329864   2,20044
Shape              1

Log-Likelihood = -38,728

Distribution:   Exponential
```

d) One finally did a logrank test to compare in a *nonparametric* manner the two groups of patients with respect to the distribution of discharge time.

Write down the relevant null hypothesis and the alternative hypothesis for such a test, and explain how the test statistic is computed, without necessarily doing all the intermediate computations.

Compute the final test statistic by using that the computed expected number of discharges under the null hypothesis are, respectively, 3.20 and 6.80 for the patients without and with pneumonia at admission.

What is the conclusion of the test when the significance level is 5%? How does this conclusion fit with the conclusion of the corresponding problem in subpoint b)? Give a comment.

Problem 2 *Nonparametric estimation with left-censored data*

15 units of a certain mechanical equipment were put on a laboratory test. Failures were automatically recorded, but due to an error in the monitoring device, the exact failure times of the units failing during the first 2 hours were not recorded. 8 units failed during these 2 hours, while the exactly recorded failure times (hours) of the remaining 7 units were:

$$2.1,\ 2.5,\ 3.9,\ 5.6,\ 16.2,\ 22.5,\ 28.8.$$

a) What does it mean that a lifetime T is, respectively, *right*-censored, *left*-censored or *interval* censored? Give a simple example to illustrate each of these three types of censoring.

Explain why the data from the laboratory test described above can be considered as a left-censored set of data.

b) Let T be the time to failure of a unit of the described mechanical equipment. Consider the transformation

$$V = 1/T.$$

Verify that this transformation, when applied to the above given data, changes the data set from a left-censored set to a *right*-censored set, with 7 exactly observed "lifetimes"

$$\begin{aligned} 1/28.8 &= \mathbf{0.035},\ 1/22.5 = \mathbf{0.044},\ 1/16.2 = \mathbf{0.062},\ 1/5.6 = \mathbf{0.179}, \\ 1/3.9 &= \mathbf{0.256},\ 1/2.5 = \mathbf{0.400},\ 1/2.1 = \mathbf{0.476} \end{aligned}$$

and 8 right-censored observations $1/2.0 = \mathbf{0.500}$.

Compute the Kaplan-Meier estimate $\hat{R}_V(v)$ for $R_V(v) = P(V > v)$ by using the transformed data (bold numbers).

c) Verify the connection

$$R_T(t) = 1 - R_V(1/t) \text{ for all } t > 0,$$

where $R_T(t) = P(T > t)$ and it is assumed that T has a continuous distribution.

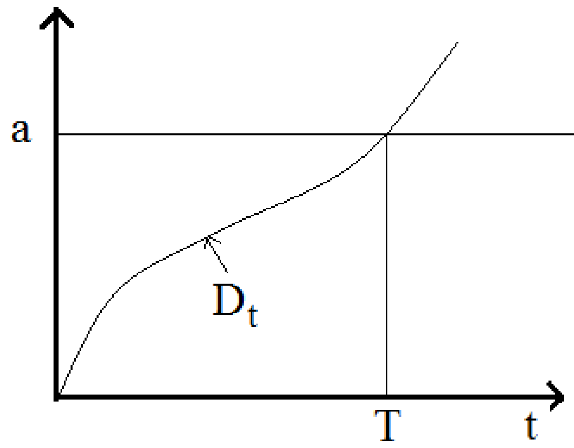
Use this to compute an estimate $\hat{R}_T(t)$ for $R_T(t)$ and draw the corresponding plot.

How would you estimate MTTF for the equipment with lifetime T? (You need not do the complete calculation).

Problem 3 *Modeling of deterioration and failure*

Consider again the mechanical equipment of Problem 2. In the present problem we first establish a parametric model for the equipment's lifetime under the laboratory test, and then we estimate the parameter of this model based on the data given in Problem 2.

Assume that the deterioration (wear) of the equipment at time $t \geq 0$ can be measured by D_t, where $D_0 = 0$. As time goes on, D_t develops randomly (stochastic) in a strictly increasing way until it reaches a fixed critical level $a > 0$, at which time the equipment fails. Let T be the time of failure (see figure below).



a) Use the figure to explain that for any given $t \geq 0$,

$$P(T > t) = P(D_t < a). \tag{2}$$

Then assume that the D_t-curve is linear in t, given as

$$D_t = Bt \tag{3}$$

for all $t \geq 0$, where the slope B is random and exponentially distributed with hazard rate $\theta > 0$, i.e. $P(B \leq b) = 1 - e^{-\theta b}$ for $b > 0$.

Show that the cumulative distribution function of T under this assumption is

$$F(t) = P(T \leq t) = e^{-\frac{a\theta}{t}} \text{ for } t > 0.$$

In the following we assume that (3) holds, and that the critical level a is set to 1 (this can always be achieved by scaling the θ accordingly).

Then the distribution of the time to failure, T, belongs to the parametric one-parameter family with cumulative distribution function

$$F(t;\theta) = P(T \leq t) = e^{-\frac{\theta}{t}} \text{ for } t > 0, \tag{4}$$

where $\theta > 0$ is the parameter. This distribution is called the *inverse exponential distribution.*

b) Let $t_p(\theta)$ be the $100p$-percentile of the distribution (4), for $0 < p < 1$, so that $F(t_p(\theta);\theta) = p$.

Find an expression for $t_p(\theta)$ for $0 < p < 1$.

Show in particular that the median of T is $\theta/\ln 2$.

Also, find an expression for the interquartile range of T.

c) Assume now that the failure data of the equipment given in the beginning of Problem 2 follow the parametric model (4).

Write down the likelihood function of the data and find the maximum likelihood estimate of θ.

Estimate the median lifetime of the equipment.

Table of cumulative probabilities of the χ^2-distribution with 1 degree of freedom.

x	$P(X \leq x)$	x	$P(X \leq x)$
0.01	0.079656	2.7	0.899652
0.02	0.112463	2.8	0.905736
0.03	0.137510	2.9	0.911420
0.04	0.158519	3.0	0.916735
0.05	0.176937	3.1	0.921708
0.06	0.193504	3.2	0.926362
0.07	0.208663	3.3	0.930720
0.08	0.222703	3.4	0.934804
0.09	0.235823	3.5	0.938631
0.10	0.248170	3.6	0.942220
0.20	0.345279	3.7	0.945588
0.30	0.416118	3.8	0.948747
0.40	0.472911	3.9	0.951714
0.50	0.520500	4.0	0.954500
0.60	0.561422	4.1	0.957117
0.70	0.597216	4.2	0.959576
0.80	0.628907	4.3	0.961888
0.90	0.657218	4.4	0.964061
1.00	0.682689	4.5	0.966105
1.10	0.705734	4.6	0.968028
1.20	0.726678	4.7	0.969837
1.30	0.745787	4.8	0.971540
1.40	0.763276	4.9	0.973143
1.50	0.779329	5.0	0.974653
1.60	0.794097	5.1	0.976074
1.70	0.807712	5.2	0.977413
1.80	0.820288	5.3	0.978675
1.90	0.831922	5.4	0.979863
2.00	0.842701	5.5	0.980984
2.10	0.852701	5.6	0.982040
2.20	0.861989	5.7	0.983035
2.30	0.870626	5.8	0.983974
2.40	0.878665	5.9	0.984859
2.50	0.886154	6.0	0.985694
2.60	0.893136	6.1	0.986482

Standard normalfordeling

$$\Phi(z) = P(Z \leq z)$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
-.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard normalfordeling

$$\Phi(z) = P(Z \leq z)$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999

Kritiske verdier i standard normalfordelingen

$$P(Z > z_\alpha) = \alpha$$

α	z_α
.2	0.842
.15	1.036
.1	1.282
.075	1.440
.05	1.645
.04	1.751
.03	1.881
.025	1.960
.02	2.054
.01	2.326
.005	2.576
.001	3.090
.0005	3.291
.0001	3.719
.00005	3.891
.00001	4.265
.000005	4.417
.000001	4.753