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Contact during exam:
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EXAM IN LIFETIME ANALYSIS (TMA4275)

Tuesday 1 June 2010
Time: 09:00 – 13:00 Grading: 22 Juni 2010

Permitted Aids:
Statistiske tabeller og formler (Tapir forlag)
Approved Calculator (HP30S)
K. Rotman: Matematisk formelsamling
One yellow paper (A4 with stamp) with your own formula and notes

Problem 1

The ordered lifetimes of $n = 10$ identical items are as follows:

i	T_i	δ_i
1	2.8	1
2	3.2	1
3	3.4	0
4	3.7	0
5	4.0	0
6	4.1	1
7	4.6	0
8	4.6	0
9	4.8	1
10	5.0	0

where T_i indicates the observed time (in hours) and δ_i is the censoring indicator ($\delta_i = 0$ indicates a censored observation)

a) Let $R(t)$ be the survival function for T. Compute the Kaplan-Meier estimate for $R(t)$ in the above case and draw the estimated curve.

State the assumptions behind the Kaplan-Meier estimator?

If possible, find an estimate for the first and third quartile and for the median of T. If it is not possible to estimate such quantities give a reason.

b) What kind of error would you commit by ignoring the censoring information and assuming all recorded times are observed failures?

(*Hint:* Compute an estimate of $R(t)$ by assuming $\delta_i = 1$ for all data points, compute an estimate of the first and third quartile and and of the median and compare with those obtained in a)

One wants to fit a Weibull distribution for T. The pdf is as follows:

$$f(t; \alpha, \theta) = \frac{\alpha}{\theta^\alpha} t^{\alpha-1} \exp\left(-\frac{t}{\theta}\right)^\alpha$$

c) Explain how to compute the the profile log-likelihood $\tilde{l}(\alpha)$ of α. (You do not need to do the computations).

A graph of $\tilde{l}(\alpha)$ is given in Figure 1.

d) Use Figure 1 to find an (approximate) maximum likelihood estimate for α. Find also an (approximate) 95% confidence interval for α.

e) One wants to find out whether an exponential distribution is appropriate for T.

Write down the appropriate null and alternative hypotheses and perform the test at 5% level withthe help of Figure (1).

f) Show that the Weibull distribution is a member of the log-location-scale family, i.e. that $Y = \log T$ has a cumulative distribution of the form

$$F_Y(y) = P(T \leq y) = \Phi_0\left(\frac{y-\mu}{\sigma}\right)$$

for some cdf Φ_0 and constants μ (location) and σ (scale). Write down the expression for Φ_0 and the parameters μ and σ as functions of α and θ.

Explain how this can be used to define a parametric regression model for lifetimes T with corresponding covariate vector $\mathbf{x} = (x_1, \ldots, x_k)$

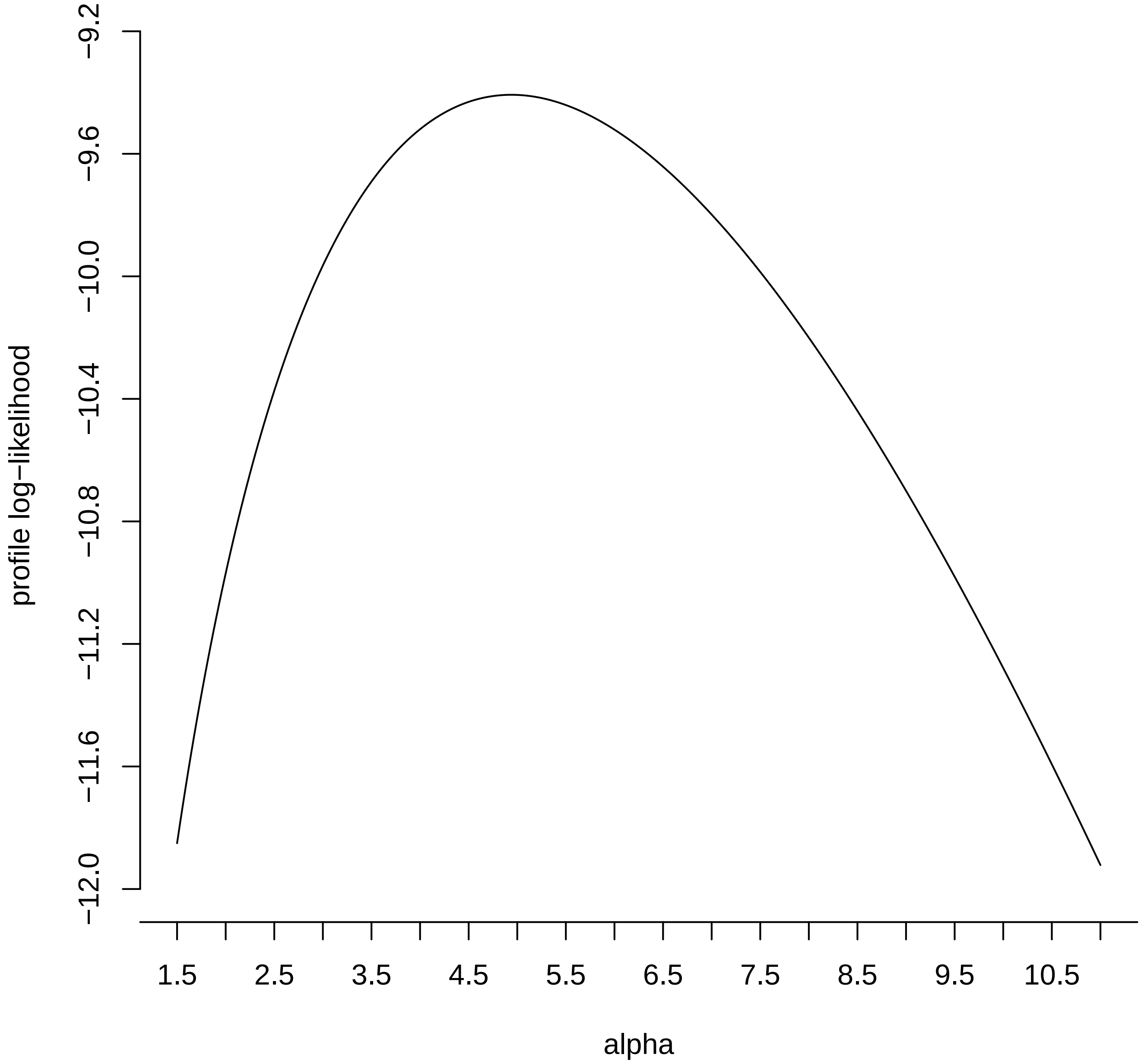

Figure 1: Profile log-likelihood

Problem 2

Let the lifetime T have the cumulative distribution function (cdf)

$$F(t;\alpha,\mu) = \Pr(T \leq t) = \exp\left\{-\left(\frac{\alpha}{t-\mu}\right)\right\} \qquad t > \mu, \alpha > 0 \tag{1}$$

a) Check that the function in (1) satisfies the requirements of a cumulative distribution.

Derive also an expression for the hazard rate $z(t)$ of T.

b) Derive an expression for the p percentile t_p of the distribution of T, $0 \leq p \leq 1$.

c) Write down the log-likelihood for a sample of n observations from $F(t;\alpha,\mu)$ which are all reported as exact failures (i.e. no censoring).

d) Use the log-likelihood in (c) to derive the maximum likelihood estimator $\hat{\alpha}$ for α assuming that the value of μ is given. Find also an estimate for the variance of $\hat{\alpha}$.

e) Compute $(-\log(F(t)))^{-1}$.

Explain how this can be used to construct a plot for checking whether a right censored data-set can be assumed to come from distribution (1).

Problem 3

Consider a component where the time to failure T (in days) has the hazard function

$$z(t) = \begin{cases} 2 - \frac{1}{10}t & \text{for} \quad 0 \leq t \leq 10 \\ 1 & \text{for} \quad 10 \leq t \leq 20 \\ t - 19 & \text{for} \quad t \geq 20 \end{cases}$$

It is assumed that repairs are minimal and take negligible time. Let $N(t)$ be the numbers of failures for the component in the time interval $(0, t]$, for $t > 0$.

a) What kind of process is $N(t)$ in this case?

What is the ROCOF of such process?

Explain briefly the main characteristics of a non-homogeneous Poisson process (NHPP).

b) Make a sketch of the ROCOF $w(t)$, and of the corresponding cumulative ROCOF $W(t)$ as functions of t.

Use the shape of $w(t)$ and $W(t)$ to describe the behaviour of the system.

Find the expected number of failures during the first 12 days.

c) What is the probability that there will be exactly 1 failure in the first 2 days?

d) Find the survival function for the time to the first event T_1, $R_{T_1}(t)$.

What is the probability that the first failure does not happen in day 1?

Given that the first failure happens on day 1, what is the probability that the system will not fail during the next 2 days?