**Norwegian University of Science and Technology**
**Department of Mathematical Sciences**



Corresponding teacher:
Henning Omre 90937848

# EXAM IN TMA4275 LIFETIME ANALYSIS

Wednesday June 4th 2008

Time: 09:00–13:00

*Support material:*
All printed and handwritten material.
Calculator HP30S.

Result of exam due: June 25th 2008.

ENGLISH TRANSLATION

## Problem 1

The table given below shows the tensile strength (Y in mg) for 21 wire connections. There are two types of failures (M):

M=1: rupture in the wire itself .
M=2: rupture in the wire attachment.

| Wire | Y | M |
|---|---|---|
| 1 | 550 | 2 |
| 2 | 750 | 1 |
| 3 | 950 | 2 |
| 4 | 950 | 1 |
| 5 | 1150 | 2 |
| 6 | 1150 | 2 |
| 7 | 1150 | 1 |
| 8 | 1150 | 1 |

| | | |
|---|---|---|
| 9 | 1150 | 1 |
| 10 | 1250 | 2 |
| 11 | 1250 | 2 |
| 12 | 1350 | 1 |
| 13 | 1450 | 2 |
| 14 | 1450 | 2 |
| 15 | 1450 | 1 |
| 16 | 1550 | 2 |
| 17 | 1550 | 1 |
| 18 | 1550 | 1 |
| 19 | 1850 | 1 |
| 20 | 2050 | 2 |
| 21 | 3150 | 2 |

As a model, it is assumed that for each wire connection there is a potential tensile strength $T$ for rupture in the wire itself, and a tensile strength $S$ for rupture in the wire connection. Throughout the problem it is assumed that $T$ and $S$ are independent random variables, with reliability functions $R_T(t) = P(T > t)$ and $R_S(t) = P(S > t)$ respectively, for all $t \geq 0$.

The observed tensile strength $Y$ for a wire is now given as

$$Y = \min(T, S),$$

while the type of failure is given by the corresponding $M$.

The 21 observations are assumed to be independent.

We are primarily interested in estimating the distribution of the tensile strength $T$ for rupture in the wire itself.

**a)** Explain why $R_T(t)$ under the given conditions can be estimated from the Kaplan-Meier estimator (the KM-estimator).

What type of censoring do we have in this situation, and which value of $M$ corresponds to censoring of $T$?

**b)** Perform the estimation of $R_T(t)$. Draw the Kaplan-Meier plot.

Use the plot to estimate the lower quartile and the median in the distribution of $T$.

Why cannot the KM-estimator be used to estimate the upper quartile in the distribution of $T$?

**c)** One wants to find out if an exponential distribution is an appropriate distribution for $T$.

Explain how a plot of the (estimated) cumulative hazard function $Z_T(t)$ can be used to give an idea about the appropriateness of an exponential distribution for $T$.

By using a relation between the reliability function $R_T(t)$ and the cumulative hazard function $Z_T(t)$, use the KM-estimate from point (b) to plot an estimate for $Z_T(t)$.

Does the plot indicate that $T$ is exponentially distributed?

Alternatively one decides to fit a lognormal distribution for $T$. The following is a part of a printout from MINITAB.

```
Distribution Analysis: T

Estimation Method: Maximum Likelihood

Distribution:   Lognormal


Parameter Estimates

                      Standard    95,0% Normal CI
Parameter  Estimate      Error      Lower     Upper
Location    7,41163   0,114869    7,18650   7,63677
Scale      0,413893  0,0951003   0,263819  0,649335
```

**d)** Express the density and the reliability function for the lognormal distribution by the density $\phi$ and the cumulative distribution function $\Phi$ in the standard normal distribution. (You may use that $T$ is lognormal with location $\mu$ and scale $\sigma$ if $\ln T \sim N(\mu, \sigma^2)$).

Use this to write down an expression for the likelihood function which is maximized in the MINITAB analysis. Explain shortly.

**e)** Use the MINITAB printout to calculate estimates for:

**(i)** The median in the distribution for $T$.

**(ii)** The expectation in the distribution for $T$.

**(iii)** The upper and lower quartile in the distribution for $T$.

**(iv)** A 95% confidence interval for the median of $T$.

Compare (i) and (iii) with the results in point (b). Why is it now possible to estimate the upper quartile in the distribution of $T$?

Assume in the following that both $S$ and $T$ are lognormally distributed. Fitting a lognormal distribution to the distribution of $S$ gave the following result when using MINITAB:

```
Parameter  Estimate
Location    7,39094
Scale      0,432850
```

**f)** Calculate an estimate for the probability $P(S > T)$.

What does this probability express (in words)?

Can this probability be estimated from the data if no assumptions are made about the distributions of $S$ and $T$? In that case, what estimate do you find?

## Problem 2

Consider a type of pumps where the time to failure for a new pump ($T$ in months) has the reliability function

$$R(t) = e^{-t}(t^2 + 2t + 2)/2, \quad \text{for } t > 0$$

**a)** Show that the cumulative hazard rate for $T$ is given by

$$Z(t) = t - \ln(t^2 + 2t + 2) + \ln 2, \quad \text{for } t > 0$$

where ln is the natural logarithm.

Use this to find the hazard rate for $T$.

Show that the MTTF for the pump, i.e. $E(T)$, is 3 (months).

(Here you may use that for all integers $k \geq 0$, $\int_0^\infty t^k e^{-t} dt = k!$).

The pump is repaired when it fails and is then put back into operation immediately (the repair times are assumed to be 0).

Let $N(t)$ be the number of failures for the pump in the time interval $(0, t]$, defined for all $t > 0$.

**b)** Assume in this point that after each repair, the pump is as good as new.

What kind of process is $N(t)$ in this case?

What is (approximately) the expected number of failures during the first two years?

(Here you may use a result that is valid when the time goes to infinity).

In the remaining problems, we assume that only minimal repair is performed for every failure.

**c)** Explain shortly what is meant by minimal repair, and explain why we then have a Poisson process with ROCOF given by

$$w(t) = \frac{t^2}{t^2 + 2t + 2}, \quad \text{for } t > 0$$

Write down the cumulative ROCOF $W(t)$.

Find the expected number of failures for the component during the first two years.

Compare the result with the answer in point (b) and comment.

**d)** What is the probability that the pump is functioning without failures the first 6 months?

Given that the first failure for the pump occurs after exactly 6 months, what is then the probability that the pump will not fail during the next 6 months?