TMA 4275 Lifetime analysis
Exercise 1 - solution

Problem 1

a) The reliability, or survivor, function of an item is defined by(p. 17, eq. 2.2): $R(t) = 1 - F(t) = Pr(T > t)$ for $t > 0$. Two months without a failure means that $t = 24 * 60$(hours*days). Note that we need to calculate the time t in hours, since $z(t)$ is defined in hours for this exercise. Note also that $R(t) = e^{-\int_0^t z(u)du}$ (p. 19, eq 2.9). Therefore:

$$\begin{aligned} Pr(T > 24 \cdot 60) = R(1440) = & \\ & = e^{-\int_0^t z(u)du}|_{t=1440} = \\ & = e^{-(\lambda * u)|_{u=0}^{u=t}}|_{t=1440} = \\ & = e^{-2.5 \cdot 10^{-5} \cdot 24 \cdot 60} = 0.9646 \end{aligned} \tag{1}$$

b) The mean time to failure of an object is given by $MTTF = \int_0^\infty R(t)dt$ (p.22 eq. 2.12). Therefore:

$$\begin{aligned} MTTF = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t} &= -\frac{1}{\lambda}(e^{-\lambda\infty} - e^{-\lambda 0}) = \\ &= \frac{1}{\lambda} = 40000 \text{ hours} . \end{aligned} \tag{2}$$

Because $e^{-\infty} \to 0$.

c) In other words we need to calculate $R(t)$ for $t =$MTTF. That is:

$$\begin{aligned} Pr(T > MTTF) = R(MTTF) = & \\ & = e^{-\lambda MTTF} = e^{-\lambda MTTF} = \\ & = e^{-\lambda \frac{1}{\lambda}} = e^{-1} = 0.9646 \end{aligned} \tag{3}$$

And e^{-1} does not depend on λ.

Problem 2

a) Constant failure rate means exponential failure function: $z(t) = \lambda$ (p.27 eq. 2.30). We also know that $Pr(T > 100) = 0.5 = R(100)$. Therefore, as in problem 1, we get: $R(100) = e^{-\int_0^t z(u)du}|_{t=100} = e^{-\lambda \cdot 100} = 0.5$. So that: $-\lambda 100 = log(0.5) \Rightarrow \lambda = \frac{-\log(0.5)}{100}$hours^{-1}

b) Like in problem 1a, we need to find the survivor function for $t = 500$. Therefore:
$R(500) = Pr(T > 500) = e^{-\lambda \cdot 500} = e^{5\log(0.5)} = 0.5^5 = 0.03$

c) Here we use the rule of conditional probabilities $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$. So that:

$$\begin{aligned}
Pr(T < 1000|T > 500) &= 1 - Pr(T > 1000|T > 500) \\
&= 1 - \frac{Pr(T > 1000 \cap T > 500)}{Pr(T > 500)} \\
&= 1 - \frac{Pr(T > 1000)}{Pr(T > 500)} \\
&= 1 - \frac{0.5^{10}}{0.5^5} = 1 - 0.5^5 = 0.97
\end{aligned}$$

Analogously:

$$\begin{aligned}
Pr(T < 1000|T > 100) &= 1 - \frac{Pr(T > 1000)}{Pr(T > 100)} \\
&= 1 - \frac{0.5^{10}}{0.5} = 1 - 0.5^9 = 0.99
\end{aligned}$$

Where $Pr(T > 1000)$ and $Pr(T > 100)$ were calculated as in 2b.

Problem 3

a) First we need to find the survivor function:
$R(t) = e^{-\int_0^t z(u)du} = e^{-kt|_{u=0}^{u=t}} = e^{-\frac{1}{2}kt^2}$.
Then the probability that the component survives 200 hours is:
$Pr(T > 200) = R(200) = e^{-\frac{1}{2}2*10^{-6}*(200)^2} = 0.9608$.

b) As before, $MTTF = \int_0^\infty R(t)dt = \int_0^\infty e^{-\frac{1}{2}kt^2}dt = \frac{1}{2}\sqrt{\frac{2\pi}{k}} = 886$ hours.

Since $\int\limits_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ for $a > 0$.

c) This is the same as in problem 2c. So by using the same rule we get:

$$Pr(T > 400|T > 200) = \frac{Pr(T > 400 \cap T > 200)}{Pr(T > 200)} = \frac{Pr(T > 400)}{Pr(T > 200)} = \frac{R(400)}{R(200)} = 0.8869$$

Analogously

$$Pr(T > 300|T > 100) = \frac{R(300)}{R(100)} = 0.9231$$

Where $R(t)$ can be calculated by 3a.

d) The survivor function is of the form: $R(t) = e^{-(\lambda t)^a}$ for $\lambda = \sqrt{\frac{k}{2}}$ and $\alpha = 2$. So this is Weibull distribution with shape parameter $\alpha = 2$ and scale parameter $\lambda = \sqrt{\frac{k}{2}}$ (p/ 38, eq 2.52).

Problem 4

a) Figure 1 shows the sketch of the failure function.

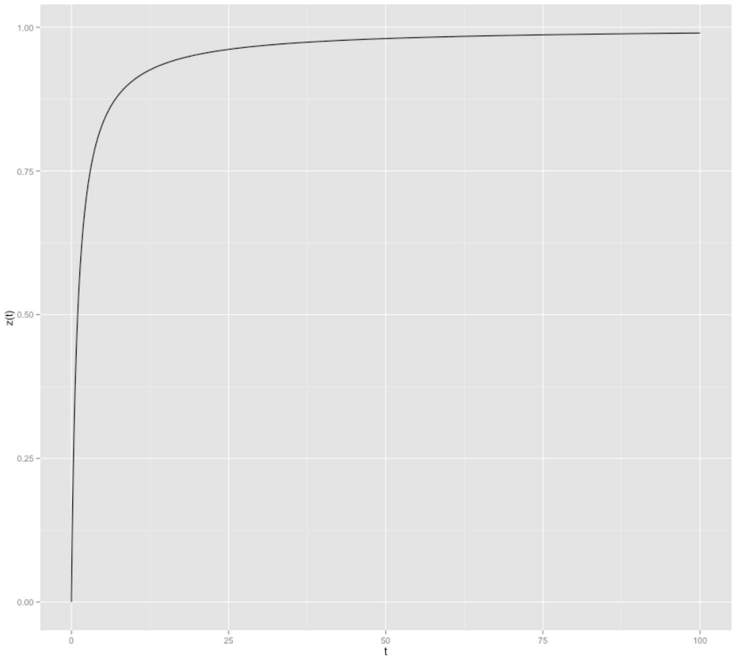

Figure 1: The failure rate function $z(t)$

b) We know that $f(t) = z(t)e^{-\int_0^t z(u)du}$ for $t > 0$ (p.19 eq. 2.10). So that:

$$
\begin{aligned}
f(t) = z(t)e^{-\int_0^t z(u)du} &= \\
&= \frac{t}{1+t}e^{-\int_0^t \frac{u}{1+u}du} \\
&= \frac{t}{1+t}e^{-\int_0^t (1-\frac{1}{1+u})du} \\
&= \frac{t}{1+t}e^{-(u-log(1+u))|_0^t} \\
&= \frac{t}{1+t}e^{-t+log(1+t)} = \frac{t(1+t)}{1+t}e^{-t} = te^{-t}
\end{aligned}
$$

c) We know that $R(t) = \int_t^\infty f(u)du$ (p. 18, eq 2.3). Such that:

$$
\begin{aligned}
R(t) = \int_t^\infty f(u)du = \\
&= \int_t^\infty ue^{-u}du \\
&= -ue^{-u}|_t^\infty + \int_t^\infty e^{-u}du \\
&= -0 + te^{-t} + e^{-u}|_t^\infty = +te^{-t} + 0 - e^{-t} = (1+t)e^{-t}
\end{aligned}
$$

Since $-ue^{-u} \to 0$ when $u \to \infty$ as e^{-u} goes faster to zero than $-u$ goes to $-\infty$. Accordingly:

$$
\begin{aligned}
MTTF = \int_0^\infty R(t)dt = \\
&= \int_0^\infty (1+t)e^{-t}dt = \int_0^\infty e^{-t}dt + \int_0^\infty te^{-t}dt = \\
&= -e^{-t}|_0^\infty - te^{-t}|_0^\infty + \int_0^\infty e^{-t}dt = \\
&= -0 + 1 - 0 + 0 - e^{-t}|_0^\infty = 1 + 0 + 1 = 2
\end{aligned}
$$

d) Note that $MTTF$ is of the form $MTTF = \frac{k}{\lambda}$, where $\lambda = 1$ and $k = 2$. This MTTF belongs to the Gamma with parameters $k = 2$ and $\lambda = 1$. Note also that this can be seen by the survivor function, although it is a little bit more difficult (p.34 eq. 2.43,2.45)