TMA4275 LIFETIME ANALYSIS

Slides 3: Parametric families of lifetime distributions

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For a lifetime T we define

$$MTTF = E(T) = \int_0^\infty tf(t)dt = \int_0^\infty R(t)dt$$

(The last equality can be proven by partial integration, noting that $R'(t) = -f(t)$. *Try to do it! You will need that* $\lim_{t\to\infty} t\,R(t) = 0$ *which holds if* $E(T) < \infty$. *)*

$$\begin{aligned} Var(T) &= \int_0^\infty (t - E(T))^2 f(t)dt \\ &= \int_0^\infty t^2 f(t)dt - (E(T))^2 \\ &= E(T^2) - (E(T))^2 \end{aligned}$$

$$\mathrm{SD}(T) = (Var(T))^{1/2}$$

EXAMPLE: EXPONENTIAL DISTRIBUTION

Let T be exponentially distributed with density $f(t) = \lambda e^{-\lambda t}$. Then you may check the following computations:

$$
\begin{aligned}
E(T) &= \int_0^\infty t\lambda e^{-\lambda t} dt = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda} \\
\mathrm{Var}(T) &= E(T^2) - (E(T))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \\
\mathrm{SD}(T) &= \frac{1}{\lambda}
\end{aligned}
$$

Thus: For a component with exponentially distributed lifetime,

$$\mathrm{MTTF} = 1/\text{failure rate}$$

NOTE: We will mainly use the parameterization $f(t) = \frac{1}{\theta} e^{-t/\theta}$, so that

$$R(t) = e^{-t/\theta}, \;\; E(T) = \theta, \;\; \mathrm{SD} = \theta$$

The lifetime T is Weibull-distributed with *shape* parameter $\alpha > 0$ and *scale* parameter $\theta > 0$, written $T \sim \text{Weib}(\alpha, \beta)$, if

$$R(t) = e^{-(\frac{t}{\theta})^{\alpha}}$$

From this we can derive:

$$\begin{aligned} Z(t) &= \left(\frac{t}{\theta}\right)^{\alpha} \\ z(t) &= \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1} \\ f(t) &= z(t)e^{-Z(t)} = \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1} e^{-(\frac{t}{\theta})^{\alpha}} \end{aligned}$$

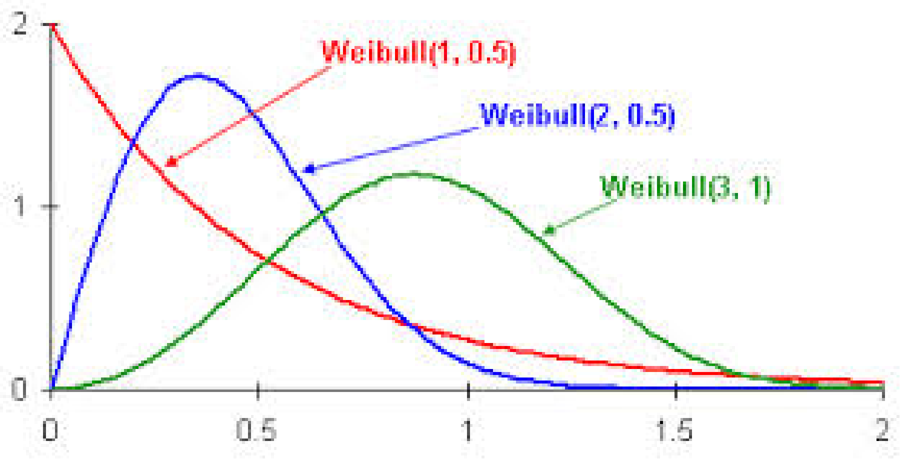
$\alpha = 1$ *corresponds to the exponential distribution;*
$\alpha < 1$ *gives a decreasing failure rate (DFR);*
$\alpha > 1$ *gives an increasing failure rate (IFR).*

$$E(T) = \int_0^\infty R(t)dt = \int_0^\infty e^{-(\frac{t}{\theta})^\alpha} dt = \cdots = \theta \cdot \Gamma\left(\frac{1}{\alpha} + 1\right)$$

where $\Gamma(\cdot)$ is the gamma-function defined by $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u} du$.

$$\begin{aligned} \mathrm{Var}(T) &= \theta^2 \left(\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma^2\left(\frac{1}{\alpha}+1\right)\right) \\ SD(T) &= \theta \left(\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma^2\left(\frac{1}{\alpha}+1\right)\right)^{1/2} \end{aligned}$$

WEIBULL DISTRIBUTION (CONT.)



Standard normal distribution, $Z \sim N(0,1)$:

$$
\begin{aligned}
f_Z(z) &= \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \\
F_Z(z) &= \Phi(z) = \int_{-\infty}^{z} \phi(w)dw
\end{aligned}
$$

Now Let $Y \sim N(\mu, \sigma)$. Then it is well known that

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) = \Phi\left(\frac{y-\mu}{\sigma}\right) \\
M_Y(t) &= E(e^{tY}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \quad \textit{(moment generating function)}
\end{aligned}
$$

Further, if we let $Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$, then $Y = \mu + \sigma Z$.

Thus: The model $Y \sim N(\mu, \sigma)$ is a **location–scale family**, defined by the standardized random variable Z, with *location parameter* μ and *scale parameter* σ.

EXERCISE

1. Consider $Y = \mu + \sigma Z$ where $Z \sim N(0,1)$. What is the distribution of Y? Why are the names *location* parameter and *scale* parameter appropriate for, respectively, μ and σ?
2. The normal distribution is sometimes used as a lifetime distribution (in fact it is a possible choice in MINITAB). What is a possible problem with this distribution?

The lifetime T has a lognormal distribution with parameters μ and σ if $Y \equiv \ln T$ is normally distributed, $Y \sim N(\mu, \sigma)$.

We can hence write

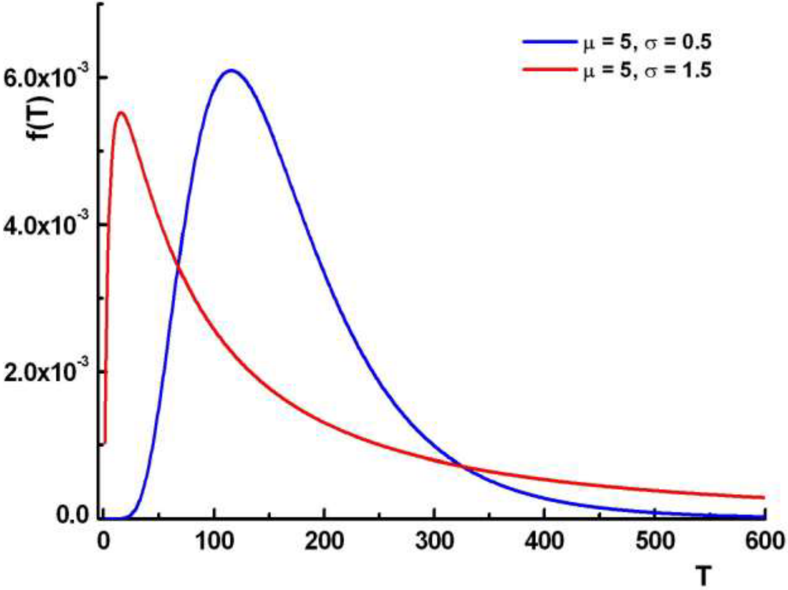
$$Y = \ln \mathrm{T} = \mu + \sigma Z \qquad (*)$$

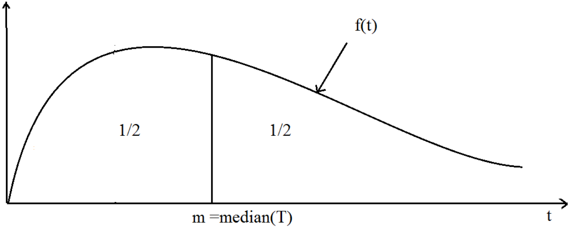
where $Z \sim N(0, 1)$.

Here μ is called the *location parameter* and σ is called the *scale parameter* of the lognormal distribution.

Because of (*) we say that the lognormal distribution is a **log-location-scale family** of distributions, meaning that the log of T defines a *location-scale* family.

LOGNORMAL DISTRIBUTION (CONT.)





$m = median(T)$ is defined by $F(m) = R(m) = 1/2$.

Compute the median m when T is

1. Exponentially distributed with parameter θ, i.e. $T \sim \text{Expon}(\theta)$
2. $T \sim \text{Weib}(\alpha, \theta)$
3. $T \sim \text{lognormal}(\mu, \sigma)$

Recall:

$$T \sim \text{lognormal}(\mu, \sigma) \Longleftrightarrow \ln T \sim N(\mu, \sigma)$$

Thus

$$\begin{aligned} R(t) &= P(T > t) = P(\ln T > \ln t) \\ &= 1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right) \end{aligned}$$

and

$$\begin{aligned} f(t) &= -R'(t) = \phi\left(\frac{\ln t - \mu}{\sigma}\right) \cdot \frac{1}{t\sigma} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \frac{1}{t} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}} \quad \text{for } t > 0 \end{aligned}$$

HAZARD FUNCTION OF THE LOGNORMAL DISTRIBUTION

$$z(t) = \frac{f(t)}{R(t)} = \frac{\phi\left(\frac{\ln t - \mu}{\sigma}\right)/(t\sigma)}{1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right)}$$

Let $T \sim \text{lognormal}(\mu, \sigma)$. Then $Y = \ln T \sim N(\mu, \sigma)$, and hence

$$M_Y(t) = E(e^{tY}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Thus:

$$\begin{aligned}
E(T) &= E(e^Y) = M_Y(1) = e^{\mu + \frac{1}{2}\sigma^2} \\
E(T^2) &= E(e^{2Y}) = M_Y(2) = e^{2\mu + 2\sigma^2} \\
\text{Var}(T) &= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)
\end{aligned}$$

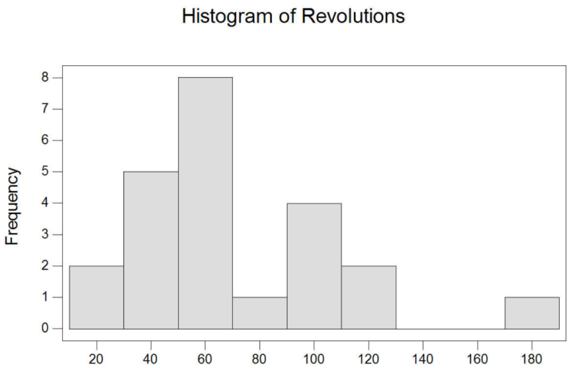
On the other hand,

$$\textit{median}(T) = e^{\mu}$$

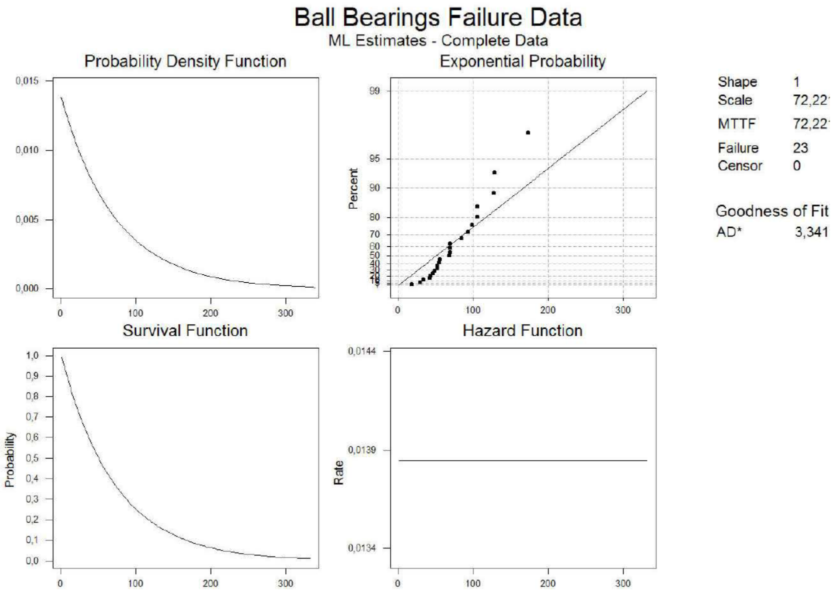
since $P(T \leq e^{\mu}) = P(\ln T \leq \mu) = P(Y \leq \mu) = 1/2$.

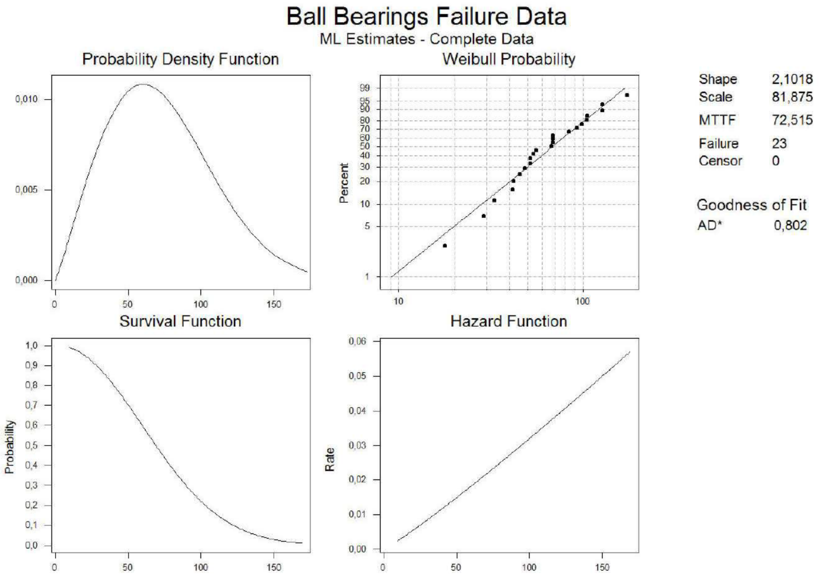
17,88	28,92	33,00	41,52	42,12	45,60	48,40	51,84
51,96	54,12	55,56	67,80	68,64	68,64	68,88	84,12
93,12	98,64	105,12	105,84	127,92	128,04	173,40	

Question: *How can we fit a parametric lifetime model to these data?*

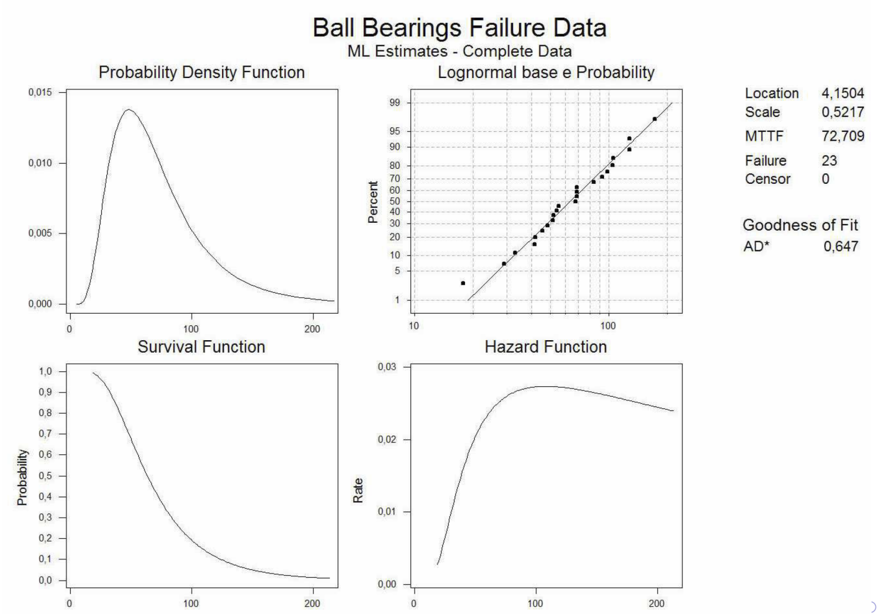


BB-DATA: EXPONENTIAL DISTRIBUTION (MINITAB)

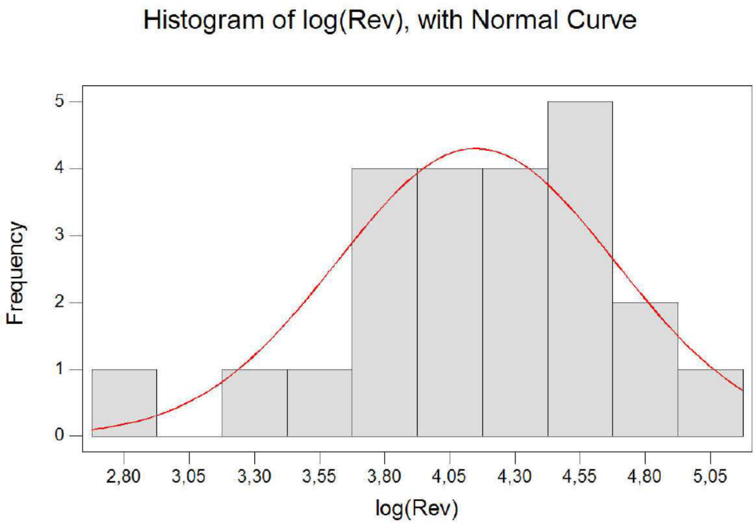




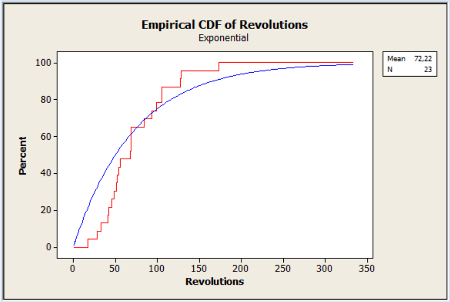
Ball Bearings Failure Data
ML Estimates - Complete Data
Probability Density Function
Weibull Probability
Survival Function
Hazard Function
Percent
Probability
Rate
Shape 2,1018
Scale 81,875
MTTF 72,515
Failure 23
Censor 0
Goodness of Fit
AD* 0,802

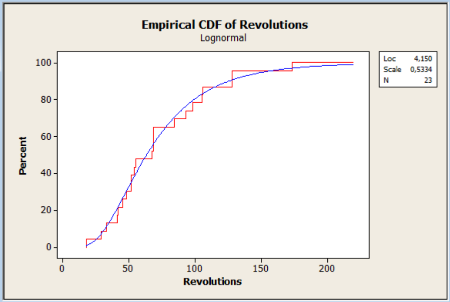

Ball Bearings Failure Data
ML Estimates - Complete Data
Probability Density Function
0,015
0,010
0,005
0,000
0
100
200
Lognormal base e Probability
Percent
99
95
90
80
70
60
50
40
30
20
10
5
1
10
100
Location 4,1504
Scale 0,5217
MTTF 72,709
Failure 23
Censor 0
Goodness of Fit
AD* 0,647
Survival Function
Probability
1,0
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0,0
0
100
200
Hazard Function
Rate
0,03
0,02
0,01
0,00
0
100
200

BB-DATA: HISTOGRAM OF LOG-LIFETIMES



BB-DATA: EMPIRICAL DISTRIBUTION COMPARED TO PARAMETRIC FITS





BB-DATA: SUMMARY OF ESTIMATION RESULTS

Model	$\widehat{\text{MTTF}}$	$\widehat{\text{STD}(T)}$	$\widehat{\text{med}(T)}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\sigma}$
Exp	72.221	72.221	50.060		72.221		
Weib	72.515	36.250	68.773	2.102	81.875		
Logn	72.710	40.664	63.458			4.150	0.522
Norm	72.221	36.667	72.221			72.221	36.667

Method: Maximum likelihood.

MOTIVATION FOR EXPONENTIAL DISTRIBUTION

- Simplest distribution used in the analysis of reliability data.
- Has the important characteristic that its hazard function is constant (does not depend on time t).
- Popular distribution for some kinds of electronic components (e.g., capacitors or robust, high-quality integrated circuits).
- Might be useful to describe failure times for components that exhibit physical wearout only after expected technological life of the system, in which the component would be replaced.

- The theory of extreme values shows that the Weibull distribution can be used to model the minimum of a large number of independent positive random variables from a certain class of distributions.
    - Failure of the weakest link in a chain with many links with failure mechanisms (e.g. fatigue) in each link acting approximately independently.
    - Failure of a system with a large number of components in series and with approximately independent failure mechanisms in each component.
- The more common justification for its use is empirical: the Weibull distribution can be used to model failure-time data with a decreasing or an increasing hazard function.

MOTIVATION FOR LOGNORMAL DISTRIBUTION

- The lognormal distribution is a common model for failure times.
- It can be justified for a random variable that arises from the product of a number of identically distributed independent positive random quantities (remember central limit theorem for sum of normals).
- It has been suggested as an appropriate model for failure time caused by a degradation process with combinations of random rates that combine multiplicatively.
- Widely used to describe time to fracture from fatigue crack growth in metals.

- Definition of
  - MTTF$= E(T)$
  - $\text{Var}(T)$
  - $\text{SD}(T) = \sqrt{\text{Var}\, T}$
- Presentation of distributions:
  - *Weibull-distribution*, $\text{Weib}(\alpha, \theta)$
  - *Normal distribution*, $N(\mu, \sigma)$
  - *Lognormal distribution*, $\text{lognormal}(\mu, \sigma)$
- Definition of
  - *location-scale family*, e.g. $N(\mu, \sigma)$
  - *log-location-scale family*, e.g. $\text{lognormal}(\mu, \sigma)$