TMA4275 LIFETIME ANALYSIS

Slides 14: Recurrent events; Repairable systems

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NTNU, Spring 2020

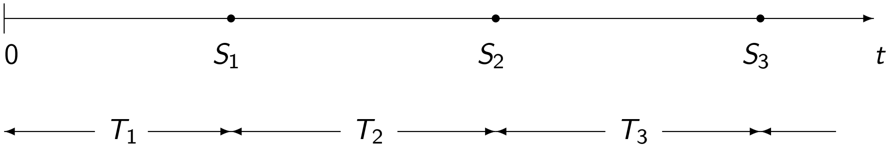
- Repairable systems
  - Definition
  - Notation
  - Real life examples
- Probabilistic modeling of repairable systems
  - Rate of Occurrence of Failures (ROCOF)
- Basic models for repairable systems
  - Perfect repair: Renewal Process (RP)
  - Minimal repair: Nonhomogneous Poisson Process (NHPP)
- Nonparametric estimation of ROCOF
  - Theory
  - MINITAB

REPAIRABLE SYSTEMS/RECURRENT EVENTS/ COUNTING PROCESSES

Definition of repairable system (Ascher and Feingold 1984):

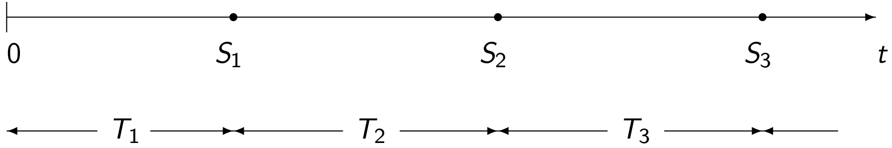
"A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, other than replacement of the entire system".

TYPICAL EXAMPLES



1. System is repaired and put into use again.
2. Machine part is replaced.
3. Relapse from disease (epileptic seizures, recurrence of tumors)

NOTATION



Modeling as a *counting process;* i.e. counting events on a time axis.

$N(t) = \#$ events in (0,t].

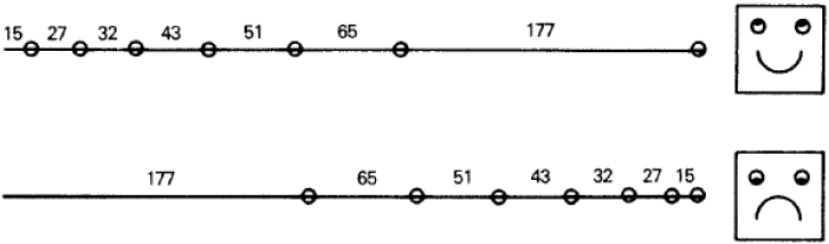
$N(s,t) = \#$ events in $(s,t] = N(s) - N(t)$.

$S_1, S_2, \cdots$ are event times.

$T_1, T_2, \cdots$ are times *between* events; also called "sojourn times".

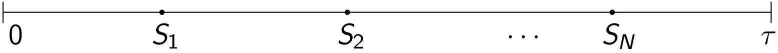
NOTE: It is common to disregard *repair times*, but one could have situations where "up times" alternate with "down times" of a system.

Ascher and Feingold presented the following example of a "happy" and "sad" system:



- *Their claim:* Reliability engineers do not recognize the difference between these cases since they always treat times between failures as i.i.d. and fit probability models like Weibull.
- *Their conclusion:* Use point process models to analyze repairable systems data!

APPLICATIONS

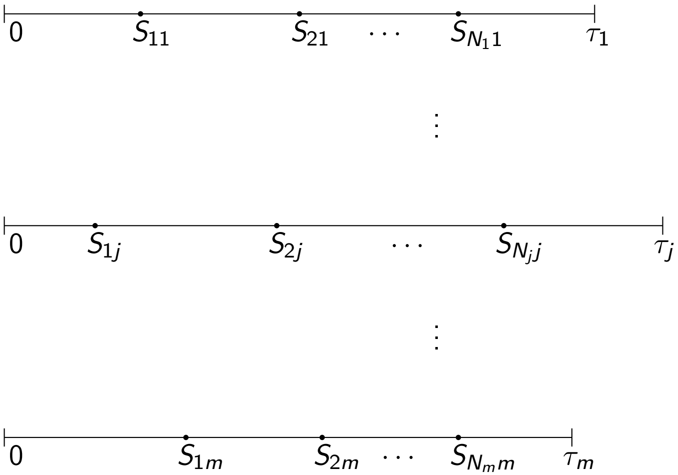


- Applications: engineering and reliability studies, public health, clinical trials, politics, finance, insurance, sociology, etc.

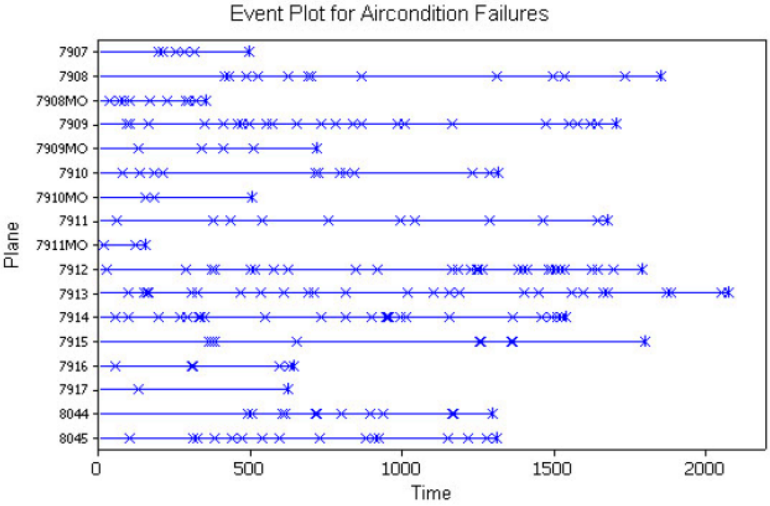
Reliability applications:

- breakdown or failure of a mechanical or electronic system
- discovery of a bug in an operating system software
- the occurrence of a crack in concrete structures
- the breakdown of a fiber in fibrous composites
- Warranty claims of manufactured products

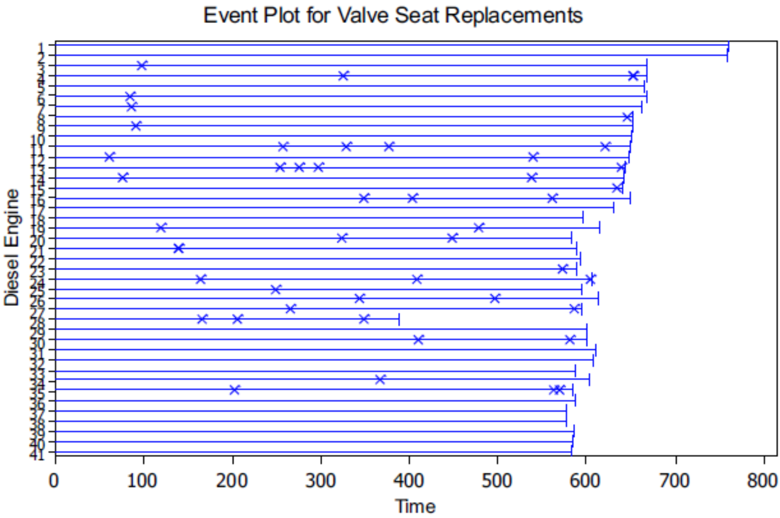
TYPICAL DATA FORMAT; EVENT PLOT



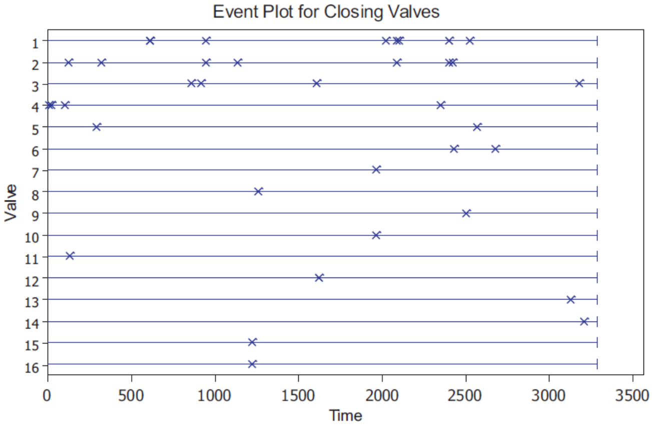
Times of failures of aircondition system in a fleet of Boeing 720 airplanes



Times of valve-seat replacements in a fleet of 41 diesel engines



- Failure data for closing valves in safety systems at two nuclear reactor plants in Finland. Failures type: *External Leakage*, follow-up 9 years for 104 valves. 88 valves had no failures



Aalen and Husebye (1991): Migratory motor complex (MMC) periods in 19 patients, 1-9 events per individual.

Individual	Observed periods (minutes)					
1	112	145	39	52	21	34
	33	51	(54)			
2	206	147	(30)			
3	284	59	186	(4)		
4	94	98	84	(87)		
5	67	(131)				
6	124	34	87	75	43	38
	58	142	75	(23)		
7	116	71	83	68	125	(111)
8	111	59	47	95	(110)	
9	98	161	154	55	(44)	
10	166	56	(122)			
11	63	90	63	103	51	(85)
12	47	86	68	144	(72)	
⋮			⋮			

Definition: $W(t) =_{def} E[N(t)] =$ expected # events (failures) in (0,t].

$w(t) =_{def} W'(t) =$ Rate of Occurrence of Failures (ROCOF).

$$\begin{aligned} w(t) &= \lim_{h\to 0} \frac{W(t+h) - W(t)}{h} \\ &= \lim_{h\to 0} \frac{E[N(t+h)] - E[N(t)]}{h} \\ &= \lim_{h\to 0} \frac{E[N(t+h) - N(t)]}{h} \\ &= \lim_{h\to 0} \frac{E[N(t,t+h)]}{h} \\ &= \lim_{h\to 0} \frac{\text{expected\#events in}(t,t+h)}{h} \end{aligned}$$

So expected number of events in $(t, t+h) \approx w(t)h$

Definition: Counting process is *regular* if

$$P(N(t, t+h) \geq 2) = o(h)$$

i.e. *small*, even compared to h, meaning that $\frac{o(h)}{h} \to 0$, as $h \to 0$
In practice *regular* means *"No simultaneous events"*. So:

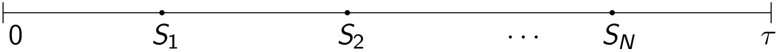
$$\begin{aligned} E[N(t, t+h)] &= 0 \cdot P(N(t, t+h) = 0) + 1 \cdot P(N(t, t+h) = 1) \\ &+ 2 \cdot P(N(t, t+h) = 2) + \cdots \end{aligned}$$

Hence

$$\frac{E[N(t, t+h)]}{h} \approx \frac{P(N(t, t+h) = 1)}{h} + \frac{o(h)}{h},$$

so $w(t) = \lim_{h \to 0} \frac{P(N(t,t+h)=1)}{h}$ or $P(N(t, t+h) = 1) \approx w(t) \cdot h$
(for a regular process).
This is analogous to $P(t < T \leq t + h | T > t) \approx z(t)h$
for hazard rates (which sometimes are called FOM = Force of Mortality)

BASIC MODELS FOR REPAIRABLE SYSTEMS



- RP(F): Renewal process with interarrival distribution F.

  Defining property:

  - Times between events are i.i.d. with distribution F

- NHPP($w(\cdot)$): Nonhomogeneous Poisson process with intensity $w(t)$.

  Defining property:

  1. Number of events in $(0, t]$ is Poisson-distributed with expectation $\int_0^t w(u)du = W(t)$
  2. Number of events in disjoint time intervals are stochastically independent

The NHPP is given by

- Specifying the ROCOF (intensity) $w(t)$,
- which has the basic property that $P(N(t, t+h) = 1) \approx w(t)h$
- assuming regularity of point process
- assuming independence of number of events in disjoint intervals

Properties of NHPP:

- $N(s, t) = \#$ events in (s,t] is Poisson($\int_s^t w(u)du$)
- $N(t) = \#$ in (0,t] is Poisson($\int_0^t w(u)du$), i.e. Poisson($W(t)$).
- $P(N(t) = j) = \frac{W(t)^j}{j!} e^{-W(t)}$, for $j = 0, 1, \cdots$ and $E(N(t)) = W(t)$ so $w(t) = W'(t)$ is really the ROCOF.
- $E[N(s, t)] = \int_s^t w(u)du = W(t) - W(s)$

Advantage of NHPP and reason for its extensive use:

Can model a *trend* in the rate of failures, because
$P(\text{failure in } (t, t+h)) \approx w(t)h$.

- $w(t) \nearrow$ deteriorating system ("sad system") e.g. aging of a mechanical system
- $w(t) \searrow$ improving system ("happy system") e.g. software reliability.
- $w(t) = \lambda$ (constant): Homogeneous Poisson process (HPP)

Let S_1 is the time to first failure. For HPP, this is expon(λ).

For NHPP, $P(T_1 > t) = P(N(t) = 0)$,

so since $N(t) \sim Poisson(W(t))$,

$R_{T_1}(t) = P(T_1 > t) = \frac{W(t)^0}{0!} e^{-W(t)} = e^{-W(t)}$

Thus, $Z_{T_1}(t) = W(t)$, so

$z_{T_1}(t) = w(t)$,

i.e. the ROCOF $w(t)$ for an NHPP equals the hazard rate for the time to first failure.

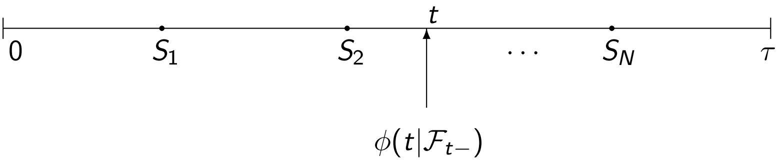
EXAMPLE

Suppose $S_1, S_2, \ldots$ is and NHPP with $w(t) = 2t$ and $W(t) = t^2$.

What is the expected # of failures in the time intervals $[0, 1], [1, 2], [2, 3]$, all having length 1?

$$E[N(0, 1)] = W(1) - W(0) = 1$$
$$E[N(1, 2)] = W(2) - W(1) = 2^2 - 1^2 = 3$$
$$E[N(2, 3)] = W(3) - W(2) = 9 - 4 = 5$$

Time to the first failure:
$R_{T_1}(t) = P(T_1 > t) = P(N(0, t) = 0) = \frac{W(t)^0}{0!} e^{-W(t)} = e^{-t^2}$
$\Rightarrow f_{T_1}(t) = -R'_{T_1}(t) = 2te^{-t^2} = w(t)e^{-W(t)}$, which is a Weibull distribution.



- $\mathcal{F}_{t-}$ = history of events until time t.
- Conditional intensity at t given history until time t,

$$\phi(t|\mathcal{F}_{t-}) = \lim_{h \downarrow 0} \frac{Pr(\text{failure in } [t, t+h)|\mathcal{F}_{t-})}{h}$$

- NHPP($w(\cdot)$):

$$\phi(t|\mathcal{F}_{t-}) = w(t)$$

so conditional intensity is independent of history.
Interpreted as "minimal repair" at failures

- RP(F) (where F has hazard rate $z(\cdot)$):

$$\phi(t|\mathcal{F}_{t-}) = z(t - S_{N(t-)})$$

so conditional intensity depends (only) on time since last event.
Interpreted as "perfect repair" at failures

- Between minimal and perfect repair? So called *imperfect repair* models.

Assume that we have a component or system with lifetime T, and corresponding hazard rate $z(t)$.

Perfect repair: Assume that the component at each failure is repaired to as good as new (or, possibly, is replaced). Then we can consider the inter-failure times $T_1, T_2, \ldots$ as independent realizations of T, hence $S_1, S_2, \ldots$ is a renewal process.

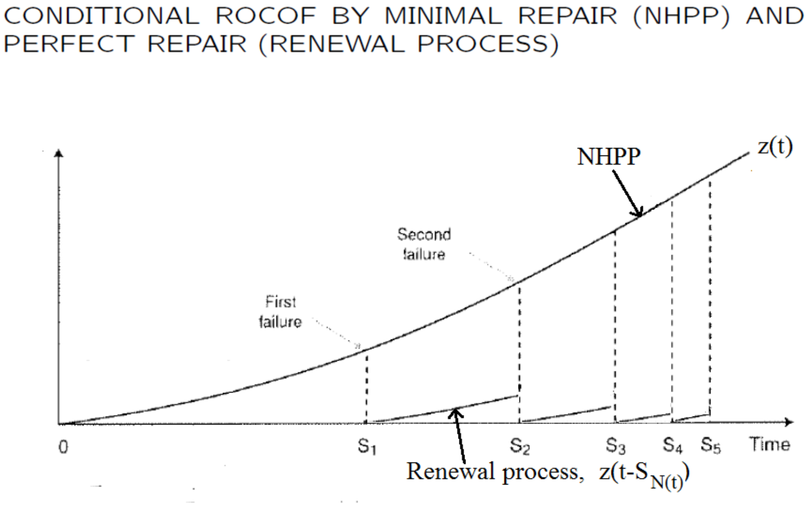
Thus, conditional ROCOF at t is $z(\text{time since last failure}) = z(t - S_{N(t)})$

Minimal repair: Assume that the system at each failure is repaired only to the same state as immediately before the failure. Then the probability of failing in $(t, t+h)$ will always be the same as for a system starting at time 0 which never has failed, namely $\approx z(t)h$. Thus rate of occurence of failures is independent of the history.

Can be shown that minimal repair as defined above, corresponds to the property of an NHPP with ROCOF $w(t) = z(t)$.

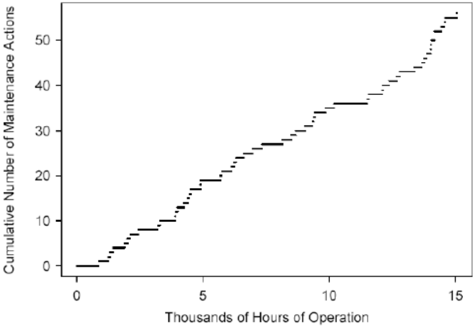
CONDITIONAL INTENSITY FOR REPAIRED COMPONENT

Consider a component with hazard rate $z(t)$, which is repaired at failures.



First: Assume that we have data for *one* system only. Then since $W(t) = E[N(t)]$, we estimate $W(t)$ by $\hat{W}(t) = N(t)$.

Cumulative Number of Unscheduled Maintenance
Actions Versus Operating Hours
for a USS Grampus Diesel Engine
Lee (1980)



Assume now, more generally:

- m processes are observed, assumed to have the same $W(t)$
- processes are not necessarily NHPPs
- first process is observed on time interval $(0, \tau_1]$
  second process on $(0, \tau_2]$
  $\vdots$
  Let τ_{max} = largest τ_j
- $Y(t) = \#$ processes under observation at time t.

We want to estimate $W(t)$

Divide the time axis at $h_0 = 0, h_1, h_2, \ldots$ up to τ_{max}.
Assume for simplicity that all the τ_j are among the h_j.

Let $D_i = \#$ events in $(h_{i-1}, h_i]$ *(total for all systems)*
and $y_i =$ value of $Y(t)$ in $(h_{i-1}, h_i]$

For each process:
$E[N(h_{i-1}, h_i)] = E[N(h_i)] - E[N(h_{i-1})] = W(h_i) - W(h_{i-1})$

Thus when all processes are considered:
$E(D_i) = y_i(W(h_i) - W(h_{i-1}))$,
and $E(\frac{D_i}{y_i}) = W(h_i) - W(h_{i-1})$ for $i = 1, 2, \ldots$

But then
$E[\frac{D_1}{y_1}] + E[\frac{D_2}{y_2}] + \ldots + E[\frac{D_k}{y_k}]$

$= W(h_1) - W(h_0) + W(h_2) - W(h_1) + \ldots + W(h_k) - W(h_{k-1})$

$= W(h_k) - W(h_0) = W(h_k)$

Recall $E[\frac{D_1}{y_1}] + E[\frac{D_2}{y_2}] + \ldots + E[\frac{D_k}{y_k}] = W(h_k)$

This suggests the estimator

$$\hat{W}(h_k) = \sum_{i=1}^{k} \frac{D_i}{y_i} \text{ for } k = 1, 2 \ldots$$

Suppose the failure times, when joined for all the m processes, are ordered as $t_1 < t_2 < \ldots < t_n$

Then by letting the h_i be more and more dense, we get contributions for at most *one* failure time in each interval (h_{i-1}, h_i).

Then we get, letting $d(t_i) = \#$events at t_i (so $d(t_i) = 1$ if regular process) $Y(t_i) = \#$processes observes at t_i

$$\hat{W}(t) = \sum_{t_i \leq t} \frac{d(t_i)}{Y(t_i)}$$

Going back to the first estimator $\hat{W}(h_k) = \sum_{i=1}^{k} \frac{D_i}{y_i}$ for $k = 1, 2 \ldots$, if the processes are *NHPPs* with CROCOF $W(t)$, then

1. $D_i \sim Poisson(y_i(W(h_i) - W(h_{i-1})))$
2. The D_i are *independent* (very important implication of NHPP)

Now $Var(\frac{D_i}{y_i}) = \frac{1}{y_i^2} Var(D_i) = \frac{E(D_i)}{y_i^2}$
and hence

$$Var(\hat{W}(h_k)) = \sum_{i=1}^{k} Var(\frac{D_i}{y_i}) = \sum_{i=1}^{k} \frac{Var(D_i)}{y_i^2} = \sum_{i=1}^{k} \frac{E(D_i)}{y_i^2}$$

So an estimator is

$$\widehat{Var(\hat{W}(h_k))} = \sum_{i=1}^{k} \frac{D_i}{y_i^2}$$

which in the limit gives

$$\widehat{Var(\hat{W}(t))} = \sum_{t_i \leq t} \frac{d(t_i)}{Y(t_i)^2}$$

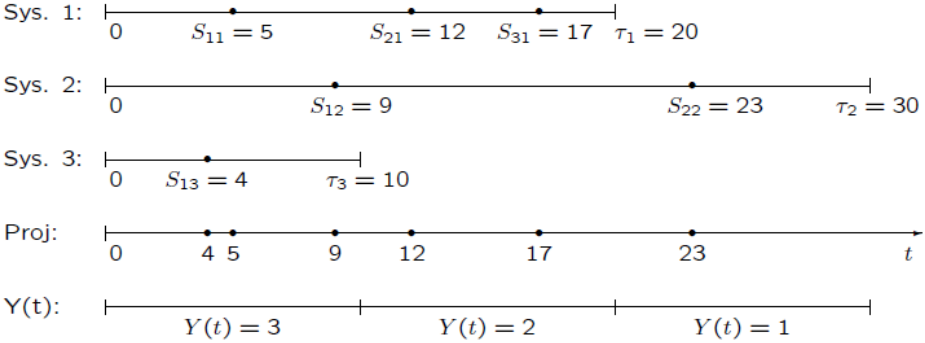
1. Order all failure times as $t_1 < t_2 < \dots t_n$.
2. Let $d_j(t_i) = \#$ events in system j at t_i.
3. Let $d(t_i) = \sum_{j=1}^{m} d_j(t_i) = \#$ events in all systems at t_i.
4. Let $Y_j(t) = \begin{cases} 1 & \text{if system } j \text{ is under observation at time } t \\ 0 & \text{otherwise} \end{cases}$
5. Let $Y(t) = \sum_{j=1}^{m} Y_j(t) = \#$ systems under observation at time t .

Then

$$\text{Under general assumptions: } \widehat{W}(t) = \sum_{t_i \le t} \frac{d(t_i)}{Y(t_i)}.$$

$$\text{Assuming NHPP: } \widehat{\text{Var}\, \widehat{W}(t)} = \sum_{t_i \le t} \frac{d(t_i)}{\{Y(t_i)\}^2}$$

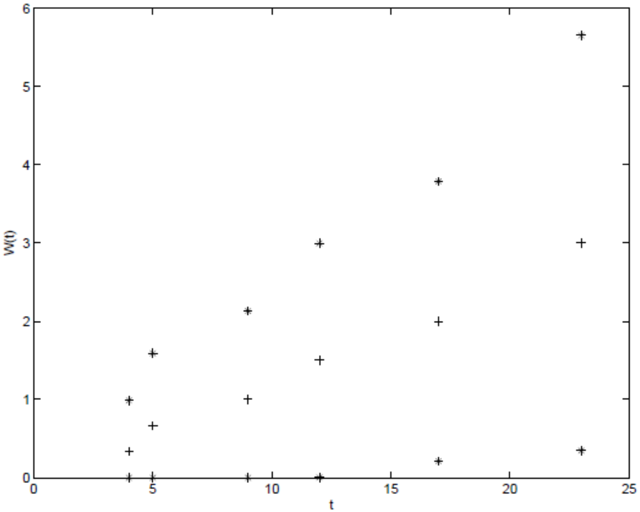
$$\text{Under general assumptions (MINITAB): } \widehat{\text{Var}\, \widehat{W}(t)} = \sum_{j=1}^{m} \left\{ \sum_{t_i \le t} \frac{Y_j(t_i)}{Y(t_i)} \left[d_j(t_i) - \frac{d(t_i)}{Y(t_i)} \right] \right\}^2$$


Sys. 1:
0
$S_{11} = 5$
$S_{21} = 12$
$S_{31} = 17$
$\tau_1 = 20$
Sys. 2:
0
$S_{12} = 9$
$S_{22} = 23$
$\tau_2 = 30$
Sys. 3:
0
$S_{13} = 4$
$\tau_3 = 10$
Proj:
0
4 5
9
12
17
23
t
Y(t):
$Y(t) = 3$
$Y(t) = 2$
$Y(t) = 1$

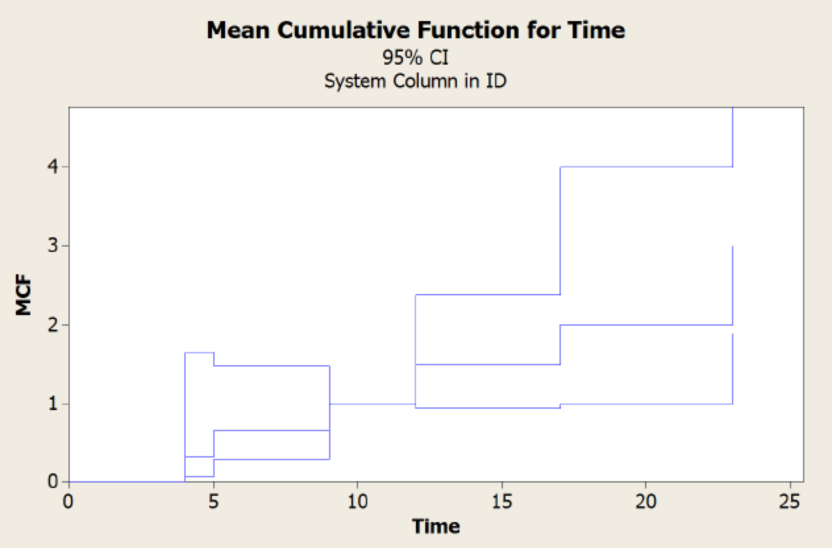
t	$1/Y(t)$	$1/Y(t)^2$	$\hat{W}(t)$	$\widehat{Var\hat{W}(t)}$	$\widehat{SD\hat{W}(t)}$
4	1/3	1/9	1/3	1/9	0.3333
5	1/3	1/9	2/3	2/9	0.4714
9	1/3	1/9	1	1/3	0.5774
12	1/2	1/4	3/2	7/12	0.7638
17	1/2	1/4	2	5/6	0.9129
23	1	1	3	11/6	1.3540

ESTIMATED $W(t)$ WITH CONFIDENCE LIMITS (NHPP)

ESTIMATED $W(t)$ with 95% confidence limits (Nelson-Aalen)



ESTIMATED $W(t)$ WITH CONFIDENCE LIMITS (GENERAL)



Compare with MINITAB Output:

$$\begin{aligned}\widehat{\text{Var}\ \widehat{W}(4)} &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2+\left\{\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2+\left\{\frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\ &= \frac{6}{81}=0.2722^2\end{aligned}$$

$$\begin{aligned}\widehat{\text{Var}\ \widehat{W}(5)} &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\ &= \frac{6}{81}=0.2722^2\end{aligned}$$

$$\begin{aligned}\widehat{\text{Var}\ \widehat{W}(9)} &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\ &= 0\end{aligned}$$

$$\begin{aligned}\widehat{\text{Var}\ \widehat{W}(12)} &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{2}\left[1-\frac{1}{2}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{2}\left[0-\frac{1}{2}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\ &= \frac{1}{8} = 0.3536^2\end{aligned}$$

Valve Seat Replacement Times
(Nelson and Doganaksoy 1989)

Data collected from valve seats from a fleet of 41 diesel engines (days of operation)

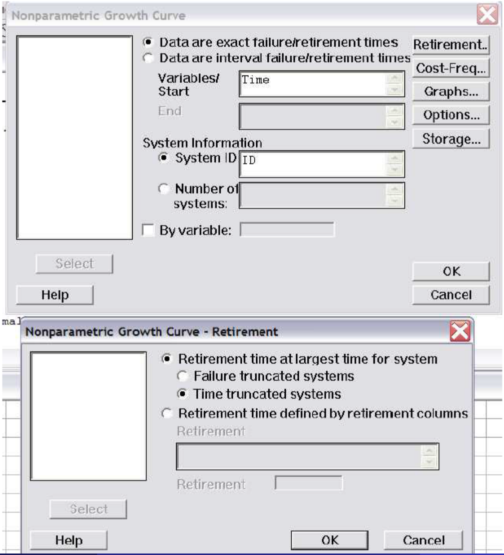
- Each engine has 16 valves
- Does the replacement rate increase with age?
- How many replacement valves will be needed in the future?
- Can valve life in these systems be modeled as a renewal process?

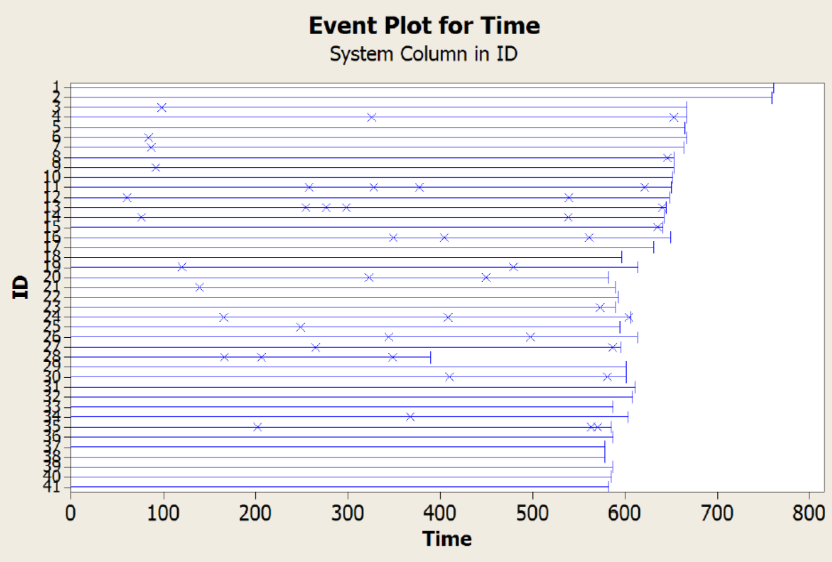
VALVESEAT DATA

TMA4275valveseat.MTW ***

↓	C1	C2	C3	C4	C5	C6	C7	C8	C
	ID	Time							
1	1	761							
2	2	759							
3	3	98							
4	3	667							
5	4	326							
6	4	653							
7	4	653							
8	4	667							
9	5	665							
10	6	84							
11	6	667							
12	7	87							
13	7	663							
14	8	646							
15	8	653							
16	9	92							
17	9	653							
18	10	651							
19	11	258							
20	11	328							
21	11	377							
22	11	621							
23	11	650							
24	12	61							
25	12	539							
26	12	648							

Reliability/Survival > Repairable System Analysis
> Nonparametric Growth Curve




Event Plot for Time
System Column in ID
ID
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
0
100
200
300
400
500
600
700
800
Time

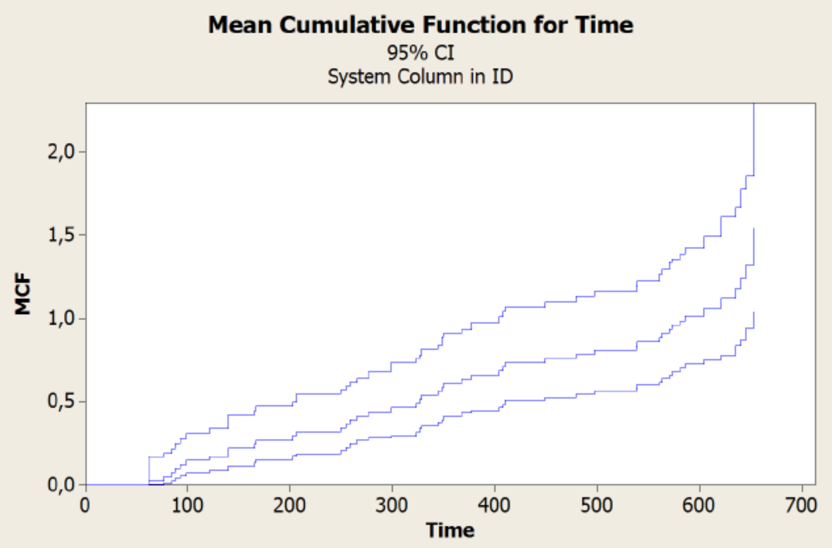
Nonparametric Growth Curve: Time

System: ID

Nonparametric Estimates

Table of Mean Cumulative Function

Time	Mean Cumulative Function	Standard Error	95% Normal CI Lower	95% Normal CI Upper	System
61	0,02439	0,024091	0,00352	0,16903	12
76	0,04878	0,033641	0,01262	0,18848	14
84	0,07317	0,040670	0,02462	0,21750	6
87	0,09756	0,046340	0,03846	0,24750	7
92	0,12195	0,051105	0,05364	0,27726	9
98	0,14634	0,055199	0,06987	0,30650	3
120	0,17073	0,058764	0,08696	0,33519	19
139	0,19512	0,061891	0,10479	0,36333	21
139	0,21951	0,073270	0,11411	0,42226	21
165	0,24390	0,075417	0,13305	0,44711	24
166	0,26829	0,077317	0,15251	0,47196	28
202	0,29268	0,078988	0,17246	0,49672	35
206	0,31707	0,087527	0,18458	0,54467	28
249	0,34146	0,088680	0,20525	0,56807	25
254	0,36585	0,089656	0,22631	0,59143	13
258	0,39024	0,090461	0,24775	0,61468	11
265	0,41463	0,091101	0,26955	0,63780	27
276	0,43902	0,097858	0,28363	0,67955	13
298	0,46341	0,109607	0,29150	0,73671	13
323	0,48780	0,109740	0,31387	0,75812	20
326	0,51220	0,109740	0,33656	0,77949	4
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]


Mean Cumulative Function for Time
95% CI
System Column in ID
MCF
2,0
1,5
1,0
0,5
0,0
0
100
200
300
400
500
600
700
Time