One alternative (equivalent) representation of $\hat{S}(t)$ is given by:

$$\hat{S}(t) = \prod_{j\,:\,v_j \le t} \left(\frac{Y(v_j) - d_j}{Y(v_j)} \right) \qquad \text{for } t \le max(v_i), \qquad (5.3)$$

where $v_1 < v_2 < \cdots$ are the distinct observed failure times.

It is instructive to think about how the Kaplan Meier estimator places mass at the observed failure times. One way of gaining insight into this is by a construction of $\hat{S}(t)$ due to Efron (1967). This is known as the 'Redistribution of Mass' algorithm (for another algorithm, see Dinse 1985).

Step 1 Arrange data in increasing order, with censored observations to the right of uncensored observations in the case of ties.

Step 2 Put mass $1/n$ at each observation.

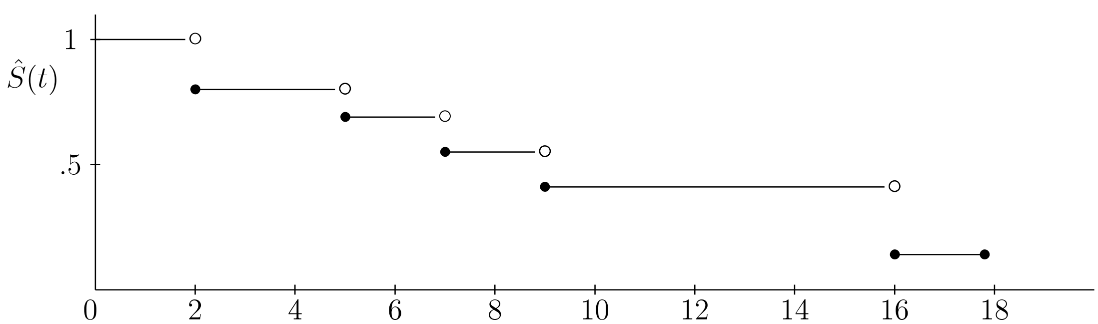
Step 3 Start from the smallest observation and move 'right'. Each time a censored observation is reached, redistribute its mass evenly to all observations to the right.

Step 4 Repeat Step 3 until all censored observations (except largest observations) have no mass. If largest (v_g) is censored, regard this mass as $> v_g$.

Let's illustrate this with the previous Example with $n = 10$.

Step 1	2	2	3^+	5	5^+	7	9	16	16	18^+
Step 2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
Step 3	$\downarrow$	$\downarrow$	$\hookrightarrow$	$\frac{1}{70}$	$\frac{1}{70}$	$\frac{1}{70}$	$\frac{1}{70}$	$\frac{1}{70}$	$\frac{1}{70}$	$\frac{1}{70}$
	$\downarrow$	$\downarrow$		$\downarrow$	$\hookrightarrow$	$\frac{1}{5}(\frac{8}{70})$	$\frac{1}{5}(\frac{8}{70})$	$\frac{1}{5}(\frac{8}{70})$	$\frac{1}{5}(\frac{8}{70})$	$\frac{1}{5}(\frac{8}{70})$
Total Mass	$\frac{2}{10}$		0	$\frac{8}{70}$	0	$\frac{24}{175}$	$\frac{24}{175}$	$\frac{48}{175}$		Assume this is somewhere > 18

$$\therefore \; \hat{S}(t) \; = \begin{cases} 1 & 0 \le t < 2 \\ .8 & 2 \le t < 5 \\ .69 & 5 \le t < 7 \\ .55 & 7 \le t < 9 \\ .41 & 9 \le t < 16 \\ .14 & 16 \le t \le 18 \\ \text{not defined} & 18 < t \end{cases}$$



Self-Consistency The Kaplan-Meier Estimator can also be viewed as the self-consistency estimator. To be specific, if there is no censoring, the survival function can be estimated as

$$\hat{S}(t) = n^{-1} \sum_{i=1}^{n} I(T_i > t).$$

In the presence of censoring, however the observed data consists of $\{(U_i, \delta_i), i = 1, \cdots, n\}$, the survival function can be estimated as

$$\hat{S}(t) = n^{-1} \sum_{i=1}^{n} E(I(T_i > t)|U_i, \delta_i)$$

where

- $E(I(T_i > t)|U_i, \delta_i = 1) = I(U_i > t)$,
- $E(I(T_i > t)|U_i, \delta_i = 0) = S(t)/S(U_i)I(t \geq U_i) + I(t < U_i)$.

However, since $S(\cdot)$ is unknown, $\hat{S}(t)$ can only be obtained by iteratively calculating

$$\hat{S}_{new}(t) = n^{-1} \sum_{i=1}^{n} \left\{ I(U_i > t) + (1 - \delta_i) \frac{\hat{S}_{old}(t)}{\hat{S}_{old}(Y_i)} I(t \geq U_i) \right\}.$$

The limit solves the self-consistency equation:

$$\hat{S}(t) = n^{-1} \sum_{i=1}^{n} \left\{ I(U_i > t) + (1 - \delta_i) \frac{\hat{S}(t)}{\hat{S}(Y_i)} I(t \geq U_i) \right\}$$

and is the same as the Kaplan-Meier estimator. The self-consistency principle can be used to construct estimator under other type of censoring such as interval censoring.