Bo Lindqvist
Department of Mathematical Sciences
Norwegian University of Science and Technology
Trondheim

http://www.math.ntnu.no/~bo/
bo@math.ntnu.no

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Trend in the ROCOF of an NHPP
Tests for trend in single systems
  - The Laplace test
  - The Military Handbook test
Tests for trend in multiple systems
  - The pooled Laplace test
  - The pooled Military Handbook test
The TTT-concept for repairable systems
  - TTT-based tests for trend
  - TTT-plot for repairable systems
  - Real data examples with MINITAB
Brief introduction to renewal processes
Consider an NHPP with intensity (ROCOF) \( w(t) \).

Can see “trend” of \( w(t) \) by considering \( W(t) \):

- If \( W(t) \) is \textbf{convex}, then \( w(t) \uparrow \)
- If \( W(t) \) is \textbf{concave}, then \( w(t) \downarrow \)
- If \( W(t) \) is linear, then \( w(t) \) is constant (so we have an HPP).

*Trend tests* are tests for:

\[ H_0 : w(t) \text{ is constant} \]

versus, alternatives such as

\[ H_1 : w(t) \uparrow \]
\[ H_1 : w(t) \downarrow \]
\[ H_1 : w(t) \text{ is not constant} \]
Basic result for an HPP on the interval \([0, \tau]\), with events at \(S_1, S_2, \ldots, S_N\):

Given \(N = n\), the \(n\) events are distributed uniformly on the interval \([0, \tau]\), i.e. they are the orderings of \(n\) independent variables, uniformly distributed on \([0, \tau]\).

These variables have mean equal to \(\tau/2\) and variance \(\tau^2/12\). Thus

\[
\sum_{i=1}^{n} S_i \approx N(n\tau/2, n\tau^2/12)
\]

Thus

\[
W = \frac{\sum_{i=1}^{n} S_i - n\tau/2}{\tau \sqrt{n/12}} = \frac{\sum_{i=1}^{n} (S_i - \tau/2)}{\tau \sqrt{n/12}} \approx N(0, 1)
\]

This is the test statistic of the Laplace test!

Why is this a reasonable test statistic, and for which values of \(W\) should we reject the null hypothesis?
Recall

$$W = \frac{\sum_{i=1}^{n} (S_i - \tau/2)}{\tau \sqrt{n/12}}$$

Now:

- If $H_1 : w(t) \nearrow$, then most failures are above $\tau/2$, and $W$ has a tendency to be large and positive.
- If $H_1 : w(t) \searrow$, then most failures are below $\tau/2$, and $W$ has a tendency to be small and negative.
- If $H_1 : w(t)$ is not constant, then
  - if the alternative is that of a monotone $w(t)$, the $W$ has a tendency to be either too large or too small
  - if the alternative is, e.g., a bathtub $w(t)$, then $W$ may still be moderate, so that the test is not good in this case!
TREND TESTING IN SIMPLE EXAMPLE WITH THREE SYSTEMS

- system 1: \[ W_1 = \frac{(5-10)+(12-10)+(17-10)}{20\sqrt{3/12}} = \frac{4}{20\sqrt{3/12}} = 0.40 \]
- system 2: \[ W_2 = \frac{(9-15)+(23-15)}{30\sqrt{2/12}} = 0.1633 \]
- system 3: \[ W_3 = \frac{4-5}{10\sqrt{1/12}} = -0.3464 \]
Basic result for an HPP stopped at the $n$th event, for a given $n$, with events at $S_1, S_2, \ldots, S_n$:

Given the value $S_n = s_n$, the $n - 1$ first events are distributed uniformly on the interval $[0, s_n]$, i.e. they are the orderings of $n - 1$ independent variables, uniformly distributed on $[0, s_n]$.

This leads to the test statistic

$$W = \sum_{i=1}^{n-1}(S_i - S_n/2) \quad S_n \sqrt{(n - 1)/12}$$

which is approximately $N(0, 1)$ under the null hypothesis of HPP.
Now use the following result:

If $S \sim U[0, \tau]$, then

$$Z = 2 \ln \frac{\tau}{S} \sim \chi^2_2$$

which is proved as follows:

$$P(Z \leq z) = P(2 \ln \frac{\tau}{S} \leq z)$$

$$= P(\ln \frac{\tau}{S} \leq \frac{z}{2})$$

$$= P(\frac{\tau}{S} \leq e^{\frac{z}{2}})$$

$$= P(S \geq \tau e^{-\frac{z}{2}})$$

$$= 1 - e^{-\frac{z}{2}}$$

Thus $f_Z(z) = \frac{1}{2} e^{-\frac{z}{2}} \sim \chi^2_2$
The test statistic is

\[ Z = 2 \sum_{i=1}^{n} \ln \frac{\tau}{S_i} \sim \chi_{2n}^2 \text{ under } H_0 \text{ (time censoring at } \tau) \]

\[ Z = 2 \sum_{i=1}^{n-1} \ln \frac{S_n}{S_i} \sim \chi_{2(n-1)}^2 \text{ under } H_0 \text{ (failure censoring at } n\text{th failure)} \]

(the distributions under \( H_0 \) are exact distribution, not only ”approximately”)

Bo Lindqvist Slides 16
TMA4275 LIFETIME ANALYSIS
Recall

\[ Z = 2 \sum_{i=1}^{n} \ln \frac{\tau}{S_i} \]

Now:

- If \( H_1 : w(t) \uparrow \), then many of the \( \tau/S_i \) are close to 1, and hence \( Z \) has a tendency to be small (note \( \ln 1 = 0 \)).
- If \( H_1 : w(t) \downarrow \), then many of the \( S_i \) are small, so many of the \( \tau/S_i \) are large, and hence \( Z \) has a tendency to be large.
- If \( H_1 : w(t) \) is not constant, then
  - if the alternative is that of a \emph{monotone} \( w(t) \), the \( Z \) has a tendency to be either too large or too small
  - if the alternative is, e.g., a bathtub \( w(t) \), then \( Z \) may still be moderate, so that the test is not good in this case!
Recall: This system has failures at 5, 12, 17; time censoring at $\tau = 20$. 

\[
Z = 2\left(\ln \frac{20}{5} + \ln \frac{20}{12} + \ln \frac{20}{17}\right) = 5.14
\]

If $H_0$ holds (HPP), then $Z \sim \chi^2_6$ (i.e. $E(Z) = 6$), so the tendency is towards “small” value, i.e. increasing $w(t)$.

BUT: If $H_1: w(t) \nearrow$, then we should reject at 5% level if $Z < \chi^2_6(0.05) = 1.64$, so we are far from rejecting the HPP!
Suppose we have $m$ processes which each are NHPPs

$H_0$: the $m$ processes are all HPP.

$H_1$: the $m$ processes have increasing trend (at least one of them); or

$H_1$: the $m$ processes have decreasing trend (at least one of them); or

$H_1$: not all of the $m$ processes are HPP.
TYPICAL DATA

\[
\begin{aligned}
0 & \quad S_{11} \quad S_{21} \quad \cdots \quad S_{N_1 1} \quad \tau_1 \\
& \quad \vdots \\
0 & \quad S_{1j} \quad S_{2j} \quad \cdots \quad S_{N_j j} \quad \tau_j \\
& \quad \vdots \\
0 & \quad S_{1m} \quad S_{2m} \quad \cdots \quad S_{N_m m} \quad \tau_m 
\end{aligned}
\]
THE POOLED LAPLACE TEST

Under the null hypothesis of $m$ HPP, we have for each $j$:

Given $N_j = n_j$, the $S_{1j}, \ldots, S_{nj}$ are orderings of $n_j$ uniforms on $(0, \tau_j)$.

Thus, $E(S_{ij}) = \frac{\tau_j}{2}$, $\text{Var}(S_{ij}) = \frac{\tau_j^2}{12}$.

The pooled Laplace test is defined by the test statistic

$$W_{\text{pooled}} = \frac{\sum_{j=1}^{m} \sum_{i=1}^{n_j} S_{ij} - E \left[ \sum_{j=1}^{m} \sum_{i=1}^{n_j} S_{ij} \right]}{\sqrt{\text{Var} \left[ \sum_{j=1}^{m} \sum_{i=1}^{n_j} S_{ij} \right]}}$$

Here

$$E \left[ \sum_{j=1}^{m} \sum_{i=1}^{n_j} S_{ij} \right] = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \frac{\tau_j}{2} = \sum_{j=1}^{m} \frac{n_j \cdot \tau_j}{2}$$

$$\text{Var} \left[ \sum_{j=1}^{m} \sum_{i=1}^{n_j} S_{ij} \right] = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \frac{\tau_j^2}{12} = \sum_{j=1}^{m} \frac{n_j \cdot \tau_j^2}{12}$$

$$\Rightarrow W_{\text{pooled}} = \frac{\sum_{j=1}^{m} \sum_{i=1}^{n_j} S_{ij} - \sum_{j=1}^{m} \frac{n_j \cdot \tau_j}{2}}{\sqrt{\sum_{j=1}^{m} \frac{n_j \cdot \tau_j^2}{12}}} \approx N(0, 1) \quad \text{under } H_0$$
POOLED LAPLACE TEST FOR SIMPLE SYSTEMS

\[ W_{\text{pooled}} = \frac{5 + 12 + 17 + 9 + 23 + 4 - \frac{1}{2}(3 \cdot 20 + 2 \cdot 30 + 1 \cdot 10)}{\sqrt{\frac{1}{12}[3 \cdot 20^2 + 2 \cdot 30^2 + 1 \cdot 10^2]}} \]

\[ = \frac{70 - 65}{\sqrt{\frac{1}{12}[3100]}} \]

\[ = \frac{5}{\sqrt{\frac{3100}{12}}} \]

\[ = 0.3111 \]

\( p \)-value for a test of “all HPP” vs “not all HPP” is

\[ 2 \cdot P(W \geq 0.3111) = 2 \cdot 0.378 = 0.756. \]

**Note:** This test is in fact a test of the null hypothesis that processes are all HPP’s but possibly with different individual hazards.
Under the null hypothesis of \( m \) HPP, we have for each \( j \):
\[
Z_{ij} = 2 \sum_{i=1}^{n_j} \ln \frac{\tau_j}{S_{ij}} \sim \chi^2_{2n_j}.
\]
This suggests to define
\[
Z_{pooled} = \sum_{j=1}^{m} \sum_{i=1}^{n_j} 2 \ln \frac{\tau_j}{S_{ij}} \sim \chi^2_{2n}
\]
where \( n = \sum_{j=1}^{m} n_j \).

Can write simply \( Z_{pooled} = \sum_{j=1}^{m} Z_{ij} \).

**IN SIMPLE EXAMPLE:**
\[
Z_{pooled} = 2(\ln \frac{20}{5} + \ln \frac{20}{12} + \ln \frac{20}{17} \ln \frac{30}{9} + \ln \frac{30}{23} + \ln \frac{10}{4}) = 8.89
\]

Under \( H_0 \): \( Z_{pooled} \sim \chi^2_{12} \), so P-value is \( 2P(\chi^2_{12} \leq 8.89) = 0.5754 \)
(Use parametric repairable systems analysis).

### Trend Tests

<table>
<thead>
<tr>
<th></th>
<th>MIL-Hdbk-189</th>
<th></th>
<th>Laplace’s</th>
<th></th>
<th>Anderson-Darling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TTT-based</td>
<td>Pooled</td>
<td>TTT-based</td>
<td>Pooled</td>
<td></td>
</tr>
<tr>
<td>Test Statistic</td>
<td>9.59</td>
<td>8.89</td>
<td>0.12</td>
<td>0.31</td>
<td>0.24</td>
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<tr>
<td>P-Value</td>
<td>0.697</td>
<td>0.576</td>
<td>0.906</td>
<td>0.756</td>
<td>0.977</td>
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<tr>
<td>DF</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>
TTT-BASED TREND TESTS

MINITAB also reports a **TTT-based test**, based on the article:

**TTT-based tests for trend in repairable systems data**

Jan Terje Kvaløy & Bo Henry Lindqvist

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7034 Trondheim, Norway

(Received 25 September 1996; revised 24 January 1997; accepted 15 July 1997)

This is for the null hypothesis that *all* the *m* processes are HPP with the *same* intensity.
$H_0$: the individual processes are $HPP(\lambda)$, with the same $\lambda$.

This is done by constructing a single process based on all data, which is $HPP(\lambda)$ if $H_0$ holds.

This construction is based on the Total Time on Test (TTT) idea and can be illustrated by the simple example with 3 systems:

The idea is that - on the lower axis - time is running proportional to the number of processes under observation. The “failure” times on that axis forms an $HPP(\lambda)$ under $H_0$. 
TTT-based Laplace-tests and Mil Hbk tests are obtained by first constructing the TTT-process and then applying the corresponding tests for single systems.

For the simple example we get for the TTT-based Laplace test:

$$W_{TTT} = \frac{12 + 15 + 27 + 34 + 44 + 53 - 6 \cdot 30}{60 \sqrt{6/12}} = 0.1179$$

so a two-sided p-value is $2 \cdot P(N(0, 1) > .1179) = 0.906$ (check also earlier MINITAB-output!), so there is no reason to reject the null hypothesis.

The corresponding Mil Hbk test gives:

$$Z_{TTT} = 2(6 \ln 60 - (\ln 12 + \ln 15 + \ln 27 + \ln 34 + \ln 44 + \ln 53)) = 9.59$$

Hence the two-sided p-value is $2 \cdot P(\chi^2_{12} < 9.59) = 0.696$ (check MINITAB). Again there is no reason to reject the null hypothesis.
Consider $m$ systems observed over possibly different time lengths $\tau_j$, and let $Y(t)$ be the number of systems observed at time $t$.

The Total Time on Test at time $t$ is defined by $\mathcal{T}(t) = \int_0^t Y(u)du$. The figure below shows how $\mathcal{T}(t)$ develops in the simple example.

The TTT-process transfers the projected failure times $S_1, S_2, \ldots, S_n$ on $[0, \tau_{max}]$ on the upper axis to $\mathcal{T}(S_1), \mathcal{T}(S_2), \ldots, \mathcal{T}(S_n)$ on the interval $[0, \mathcal{T}(\tau_{max})]$ on the lower axis.
TTT-PLOT FOR REPAIRABLE SYSTEMS

Plot
\[
\left( \frac{i}{n}, \frac{T(S_i)}{T(T_{max})} \right) \quad \text{for } i = 1, 2, \ldots, n.
\]

This is similar to the TTT-plot for survival data treated earlier.

*Straight line* expected for HPP, i.e., if there is no trend.

*Concave shape* indicates increasing trend (more large intervals in the beginning)

*Convex shape* indicates decreasing trend (more short intervals in the beginning)
Results for: TMA4275NelsonValveseat.MTW

Parametric Growth Curve: T

System: ID

Model: Power-Law Process
Estimation Method: Maximum Likelihood

Parameter Estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>95% Normal CI Lower</th>
<th>95% Normal CI Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape</td>
<td>1,39958</td>
<td>0,201</td>
<td>1,05695</td>
<td>1,85327</td>
</tr>
<tr>
<td>Scale</td>
<td>553,643</td>
<td>57,864</td>
<td>451,094</td>
<td>679,505</td>
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Test for Equal Shape Parameters
Bartlett’s Modified Likelihood Ratio Chi-Square

* NOTE * Test skipped – There must be at least one other failure which is not at the time the system is retired: Check ID = 251.

Trend Tests

<table>
<thead>
<tr>
<th>MIL-Hdbk-189</th>
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</tr>
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<tbody>
<tr>
<td>TTT-based</td>
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</tr>
<tr>
<td>Pooled</td>
<td>Pooled</td>
</tr>
<tr>
<td>Anderson-Darling</td>
<td></td>
</tr>
<tr>
<td>Test Statistic</td>
<td>68,72</td>
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<tr>
<td>P-Value</td>
<td>0,032</td>
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<tr>
<td>DF</td>
<td>96</td>
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</tbody>
</table>
Total Time on Test Plot for Valve Seat Data

Scaled Total Time on Test vs. Scaled Failure Number

Parameter, MLE
Shape: 1.39706
Scale: 0.0626023
Results for: TMA4275ProschanAircondition.MTW

Parametric Growth Curve: T

System: ID

Model: Power-Law Process
Estimation Method: Maximum Likelihood

Parameter Estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>95% Normal CI Lower</th>
<th>95% Normal CI Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape</td>
<td>1,15210</td>
<td>0,066</td>
<td>1,02906</td>
<td>1,28985</td>
</tr>
<tr>
<td>Scale</td>
<td>132,960</td>
<td>20,216</td>
<td>98,6964</td>
<td>179,119</td>
</tr>
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</table>

Test for Equal Shape Parameters
Bartlett’s Modified Likelihood Ratio Chi-Square

Test Statistic  6,70
P-Value  0,979
DF  16

Trend Tests

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<tbody>
<tr>
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<td>TTT-based</td>
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<tr>
<td>Test Statistic</td>
<td>363,67</td>
<td>350,40</td>
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<tr>
<td>P-Value</td>
<td>0,031</td>
<td>0,129</td>
</tr>
<tr>
<td>DF</td>
<td>424</td>
<td>392</td>
</tr>
</tbody>
</table>
The null hypothesis of “the processes are all HPP with the same ROCOF $\lambda$” is rejected (by the TTT-based tests, at least the one based on Mil Hbk).

On the other hand, the null hypothesis of “the processes are all HPP, possibly with different ROCOFs” is not rejected by any of the tests.

This is in accordance with Proschan’s conclusion in 1963, namely that the processes are HPPs with different ROCOFs.
Total Time on Test Plot for T
System Column in ID

Parameter, MLE
Shape  Scale
1.15210  132.960
RENEWAL PROCESSES - A SHORT INTRODUCTION

Definition of renewal process:

\( T_1, T_2, \ldots \) are i.i.d. with common cdf \( F(t) \).

1. Time to \( r \)th event: \( S_r = T_1 + \ldots + T_r \)
   (with distribution called the \( r \)th convolution of \( T \)).
2. \( N(t) = \) number of events in \((0, t] = \max \{ r : S_r \leq t \} \)
3. \( W(t) = E(N(t)) \) is called the renewal function (= CROCOF)
4. \( w(t) = W'(t) \) is called the renewal density (= ROCOF)

Usually difficult to find nice expressions for \( W(t) \) and \( w(t) \) (except for HPP). Thus approximations are needed (see next slide).
Let

\[ \mu = E(T_i) \quad (= \text{MTBF}) \]
\[ \sigma^2 = \text{Var}(T_i) \]

The HPP is an RP with \( T_i \sim \text{expon}(\lambda) \). Hence

\[ W(t) = \lambda t = \frac{1}{\mu} t, \quad \text{so} \]
\[ \frac{W(t)}{t} = \frac{1}{\mu} \]

For general renewal processes we have

*The Elementary Renewal Theorem:*

\[ \lim_{t \to \infty} \frac{W(t)}{t} = \frac{1}{\mu} \]
The Elementary Renewal Theorem implies (R&H, p.253):

\[ W(t) \approx \frac{t}{\mu} \quad \text{for large } t. \]

A better approximation is

\[ W(t) \approx \frac{t}{\mu} + \frac{1}{2} \left( \frac{\sigma^2}{\mu^2} - 1 \right) \]

Finally, for an HPP(\(\lambda\)) we clearly have exactly

\[ W(t + \alpha) - W(t) = \frac{1}{\mu} (t + \alpha) - \frac{1}{\mu} t = \frac{\alpha}{\mu} \]

For general RP we have *Blackwell’s theorem*:

\[ \lim_{t \to \infty} (W(t + \alpha) - W(t)) = \frac{\alpha}{\mu} \]