TMA4275 LIFETIME ANALYSIS

Slides 16: Trend testing for NHPP. Brief introduction to RP

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- Trend in the ROCOF of an NHPP
- Tests for trend in single systems
  - The Laplace test
  - The Military Handbook test
- Tests for trend in multiple systems
  - The pooled Laplace test
  - The pooled Military Handbook test
- The TTT-concept for repairable systems
  - TTT-based tests for trend
  - TTT-plot for repairable systems
  - Real data examples with MINITAB
- Brief introduction to renewal processes

TREND TESTING FOR NHPP

Consider an NHPP with intensity (ROCOF) $w(t)$.

Can see "trend" of $w(t)$ by considering $W(t)$:

- if $W(t)$ is **convex**, then $w(t) \nearrow$
- if $W(t)$ is **concave**, then $w(t) \searrow$
- if $W(t)$ is linear, then $w(t)$ is constant (so we have an HPP).

Trend tests are tests for:

$$H_0 : w(t) \text{ is constant}$$

versus, alternatives such as

$$\begin{aligned} H_1 &: w(t) \nearrow \\ H_1 &: w(t) \searrow \\ H_1 &: w(t) \text{ is not constant} \end{aligned}$$

Basic result for an HPP on the interval $[0, \tau]$, with events at $S_1, S_2, \ldots, S_N$:

Given $N = n$, the n events are distributed uniformly on the interval $[0, \tau]$, i.e. they are the orderings of n independent variables, uniformly distributed on $[0, \tau]$.

These variables have mean equal to $\tau/2$ and variance $\tau^2/12$. Thus

$$\sum_{i=1}^{n} S_i \approx N(n\tau/2, n\tau^2/12)$$

Thus

$$W = \frac{\sum_{i=1}^{n} S_i - n\tau/2}{\tau\sqrt{n/12}} = \frac{\sum_{i=1}^{n} (S_i - \tau/2)}{\tau\sqrt{n/12}} \approx N(0, 1)$$

This is the test statistic of the Laplace test!

Why is this a reasonable test statistic, and for which values of W should we reject the null hypothesis?

MOTIVATION FOR THE LAPLACE TEST

Recall

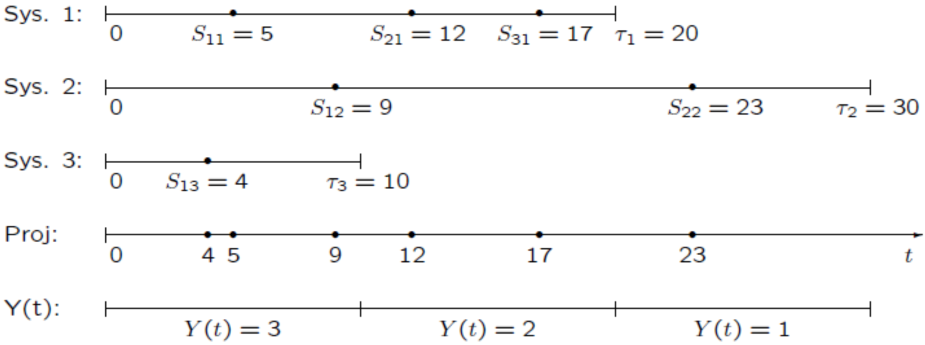
$$W = \frac{\sum_{i=1}^{n}(S_i - \tau/2)}{\tau\sqrt{n/12}}$$

Now:

- If $H_1 : w(t) \nearrow$, then most failures are above $\tau/2$, and W has a tendency to be large and positive.
- If $H_1 : w(t) \searrow$, then most failures are below $\tau/2$, and W has a tendency to be small and negative.
- If $H_1 : w(t)$ is not constant, then
  - if the alternative is that of a *monotone* $w(t)$, the W has a tendency to be either too large or too small
  - if the alternative is, e.g., a bathtub $w(t)$, then W may still be moderate, so that the test is not good in this case!

TREND TESTING IN SIMPLE EXAMPLE WITH THREE SYSTEMS

- system 1: $W_1 = \frac{(5-10)+(12-10)+(17-10)}{20\sqrt{3/12}} = \frac{4}{20\sqrt{3/12}} = 0.40$
- system 2: $W_2 = \frac{(9-15)+(23-15)}{30\sqrt{2/12}} = 0.1633$
- system 3: $W_3 = \frac{4-5}{10\sqrt{1/12}} = -0.3464$



Basic result for an HPP stopped at the nth event, for a given n, with events at $S_1, S_2, \ldots, S_n$:

Given the value $S_n = s_n$, the $n-1$ first events are distributed uniformly on the interval $[0, s_n]$, i.e. they are the orderings of $n-1$ independent variables, uniformly distributed on $[0, s_n]$.

This leads to the test statistic

$$W = \frac{\sum_{i=1}^{n-1}(S_i - S_n/2)}{S_n\sqrt{(n-1)/12}}$$

which is approximately $N(0, 1)$ under the null hypothesis of HPP.

Now use the following result:

If $S \sim U[0, \tau]$, then

$$Z = 2 \ln \frac{\tau}{S} \sim \chi^2_2$$

which is proved as follows:

$$\begin{aligned} P(Z \leq z) &= P(2 \ln \frac{\tau}{S} \leq z) \\ &= P(\ln \frac{\tau}{S} \leq \frac{z}{2}) \\ &= P(\frac{\tau}{S} \leq e^{\frac{z}{2}}) \\ &= P(S \geq \tau e^{-\frac{z}{2}}) \\ &= 1 - e^{-\frac{z}{2}} \end{aligned}$$

Thus $f_Z(z) = \frac{1}{2} e^{-\frac{z}{2}} \sim \chi^2_2$

The test statistic is

$$Z = 2\sum_{i=1}^{n} \ln \frac{\tau}{S_i} \sim \chi^2_{2n} \text{ under } H_0 \text{ (time censoring at } \tau)$$

$$Z = 2\sum_{i=1}^{n-1} \ln \frac{S_n}{S_i} \sim \chi^2_{2(n-1)} \text{ under } H_0 \text{ (failure censoring at } n\text{th failure)}$$

(the distributions under H_0 are exact distribution, not only "approximately")

Recall

$$Z = 2\sum_{i=1}^{n} \ln \frac{\tau}{S_i}$$

Now:

- If $H_1 : w(t) \nearrow$, then many of the τ/S_i are close to 1, and hence Z has a tendency to be small (note $\ln 1 = 0$).
- If $H_1 : w(t) \searrow$, then many of the S_i are small, so many of the τ/S_i are large, and hence Z has a tendency to be large.
- If $H_1 : w(t)$ is not constant, then
  - if the alternative is that of a *monotone* $w(t)$, the Z has a tendency to be either too large or too small
  - if the alternative is, e.g., a bathtub $w(t)$, then Z may still be moderate, so that the test is not good in this case!

Recall: This system has failures at 5, 12, 17; time censoring at $\tau = 20$.

$$Z = 2(\ln \frac{20}{5} + \ln \frac{20}{12} + \ln \frac{20}{17}) = 5.14$$

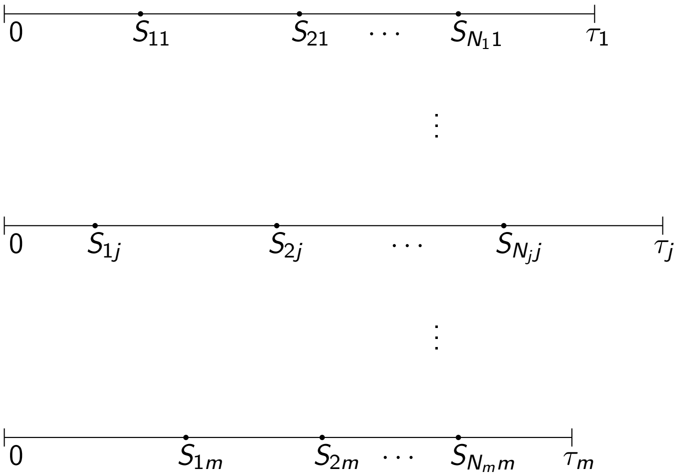
If H_0 holds (HPP), then $Z \sim \chi^2_6$ (i.e. $E(Z) = 6$), so the tendency is towards "small" value, i.e. increasing $w(t)$.

BUT: If $H_1 : w(t) \nearrow$, then we should reject at 5% level if $Z < \chi^2_6(0.05) = 1.64$, so we are frar from rejecting the HPP!

Suppose we have m processes which each are NHPPs

H_0: the m processes are all HPP.
H_1: the m processes have increasing trend (at least one of them); or
H_1: the m processes have decreasing trend (at least one of them); or
H_1: not all of the m processes are HPP.

TYPICAL DATA



THE POOLED LAPLACE TEST

Under the null hypothesis of m HPP, we have for each j:

Given $N_j = n_j$, the $S_{1j}, \cdots, S_{n_j j}$ are orderings of n_j uniforms on $(0, \tau_j)$.
Thus, $E(S_{ij}) = \frac{\tau_j}{2}$, $Var(S_{ij}) = \frac{\tau_j^2}{12}$.

The pooled Laplace test is defined by the test statistic

$$W_{pooled} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} S_{ij} - E\left[\sum_{j=1}^m \sum_{i=1}^{n_j} S_{ij}\right]}{\sqrt{Var\left[\sum_{j=1}^m \sum_{i=1}^{n_j} S_{ij}\right]}}$$

Here
$E[\sum_{j=1}^m \sum_{i=1}^{n_j} S_{ij}] = \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\tau_j}{2} = \sum_{j=1}^m \frac{n_j \tau_j}{2}$
$Var[\sum_{j=1}^m \sum_{i=1}^{n_j} S_{ij}] = \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{\tau_j^2}{12} = \sum_{j=1}^m \frac{n_j \tau_j^2}{12}$

$$\Rightarrow W_{pooled} = \frac{\sum_{j=1}^m \sum_{i=1}^{n_j} S_{ij} - \sum_{j=1}^m \frac{n_j \tau_j}{2}}{\sqrt{\sum_{j=1}^m \frac{n_j \tau_j^2}{12}}} \approx N(0,1) \quad \text{under } H_0$$

$$W_{pooled} = \frac{5+12+17+9+23+4-\frac{1}{2}(3\cdot 20+2\cdot 30+1\cdot 10)}{\sqrt{\frac{1}{12}[3\cdot 20^2+2\cdot 30^2+1\cdot 10^2]}}$$
$$= \frac{70-65}{\sqrt{\frac{1}{12}[3100]}}$$
$$= \frac{5}{\sqrt{\frac{3100}{12}}}$$
$$= 0.3111$$

p-value for a test of “all HPP” vs “not all HPP” is
$2 \cdot P(W \geq 0.3111) = 2 \cdot 0.378 = 0.756.$

Note: *This test is in fact a test of the null hypothesis that processes are all HPP's but possibly with different individual hazards.*

Under the null hypothesis of m HPP, we have for each j:
$Z_{ij} = 2\sum_{i=1}^{n_j} \ln \frac{\tau_j}{S_{ij}} \sim \chi^2_{2n_j}$. This suggests to define

$$Z_{pooled} = \sum_{j=1}^{m} \underbrace{\sum_{i=1}^{n_j} 2\ln \frac{\tau_j}{S_{ij}}}_{\chi^2_{2n_j}} \sim \chi^2_{2n}$$

where $n = \sum_{j=1}^{m} n_j$.

Can write simply $Z_{pooled} = \sum_{j=1}^{m} Z_{ij}$.

IN SIMPLE EXAMPLE:

$$Z_{pooled} = 2(\ln\frac{20}{5} + \ln\frac{20}{12} + \ln\frac{20}{17} \ln\frac{30}{9} + \ln\frac{30}{23} + \ln\frac{10}{4}) = 8.89$$

UnderH_0 : $Z_{pooled} \sim \chi^2_{12}$, so P-value is $2P(\chi^2_{12} \leq 8.89) = 0.5754$

(Use parametric repairable systems analysis).

Trend Tests

	MIL-Hdbk-189		Laplace's		
	TTT-based	Pooled	TTT-based	Pooled	Anderson-Darling
Test Statistic	9,59	8,89	0,12	0,31	0,24
P-Value	0,697	0,576	0,906	0,756	0,977
DF	12	12			

MINITAB also reports a **TTT-based test**, based on the article:

ELSEVIER

PII: S0951-8320(97)00099-9

Reliability Engineering and System Safety **60** (1998) 13–28

0951–8320/98/$19.00

TTT-based tests for trend in repairable systems data

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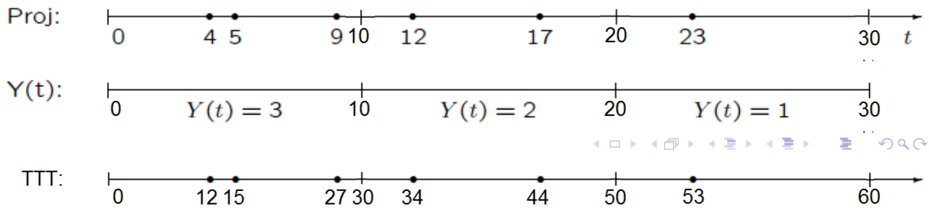
This is for the for the null hypothesis that *all* the m processes are HPP with the *same* intensity.

H_0: *the individual processes are HPP(λ), with the same λ.*

This is done by constructing a single process based on all data, which is HPP(λ) if H_0 holds.

This construction is based on the Total Time on Test (TTT) idea and can be illustrated by the simple example with 3 systems:



The idea is that - on the lower axis - time is running proportional to the number of processes under observation. The “failure” times on that axis forms an HPP(λ) under H_0.

TTT-based Laplace-tests and Mil Hbk tests are obtained by first constructing the TTT-process and then applying the corresponding tests for single systems.

For the simple example we get for the TTT-based Laplace test:

$$W_{TTT} = \frac{12 + 15 + 27 + 34 + 44 + 53 - 6 \cdot 30}{60\sqrt{6/12}} = 0.1179$$

so a two-sided p-value is $2 \cdot P(N(0,1) > .1179) = 0.906$ (check also earlier MINITAB-output!), so there is no reason to reject the null hypothesis.

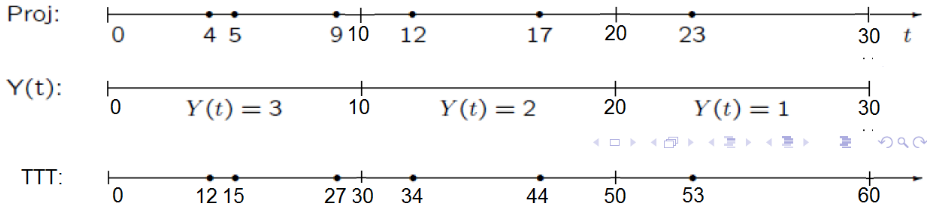
The corresponding Mil Hbk test gives:

$$Z_{TTT} = 2(6 \ln 60 - (\ln 12 + \ln 15 + \ln 27 + \ln 34 + \ln 44 + \ln 53)) = 9.59$$

Hence the two-sided p-value is $2 \cdot P(\chi^2_{12} < 9.59) = 0.696$ (check MINITAB). Again there is no reason to reject the null hypothesis.

Consider m systems observed over possibly different time lengths τ_j, and let $Y(t)$ be the number of systems observed at time t.

The Total Time on Test at time t is defined by $\mathcal{T}(t) = \int_0^t Y(u)du$. The figure below shows how $\mathcal{T}(t)$ develops in the simple example.



The TTT-process transfers the projected failure times $S_1, S_2, \cdots S_n$ on $[0, \tau_{max}]$ on the upper axis to $\mathcal{T}(S_1), \mathcal{T}(S_2), \cdots, \mathcal{T}(S_n)$ on the interval $[0, \mathcal{T}(\tau_{max})]$ on the lower axis.

TTT-PLOT FOR REPAIRABLE SYSTEMS

Plot

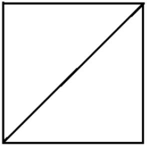
$$\left(\frac{i}{n}, \frac{\mathcal{T}(S_i)}{\mathcal{T}(\tau_{max})}\right) \text{ for } i = 1, 2, \ldots, n.$$

This is similar to the TTT-plot for survival data treated earlier.

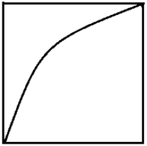
Straight line expected for HPP, i.e., if there is no trend.

Concave shape indicates increasing trend (more large intervals in the beginning)

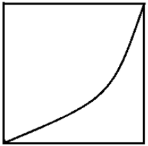
Convex shape indicates decreasing trend (more short intervals in the beginning)



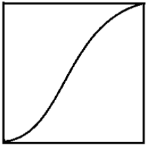
HPP



Increasing trend (sad system)

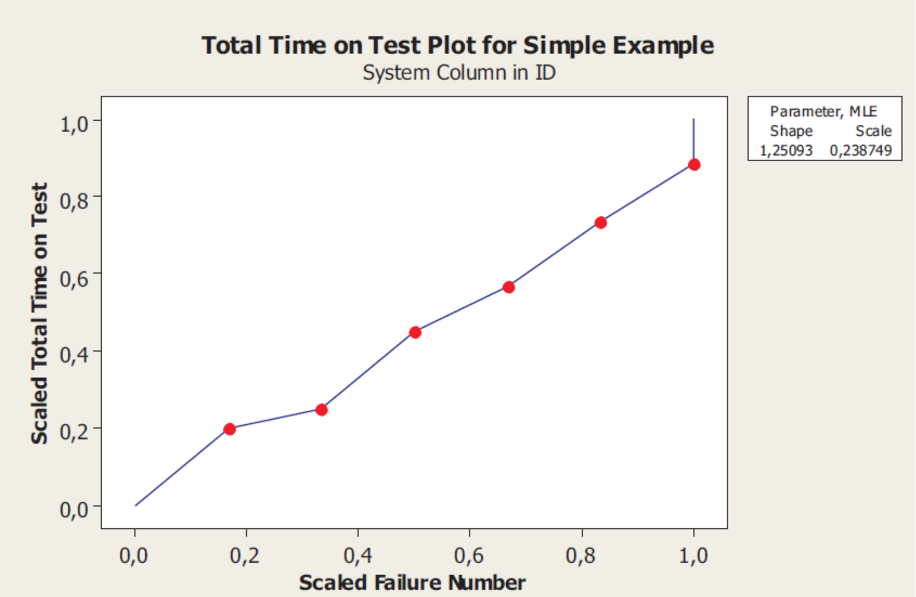


Decreasing trend (happy system)



Bathtub-shaped ROCOF

TTT-PLOT FOR SIMPLE EXAMPLE WITH 3 SYSTEMS

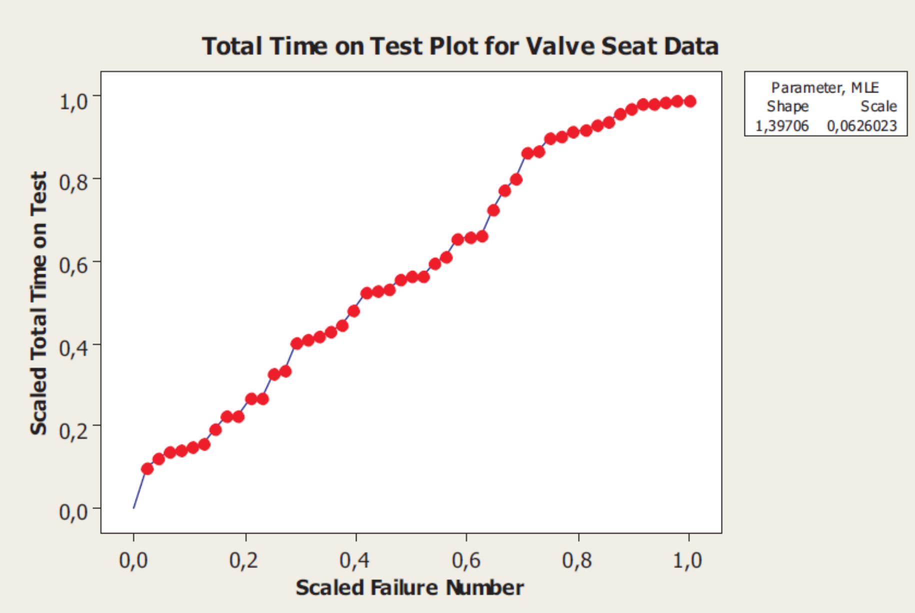


Results for: TMA4275NelsonValveseat.MTW

Parametric Growth Curve: T

```
System:   ID

Model: Power-Law Process
Estimation Method: Maximum Likelihood


Parameter Estimates

                         Standard     95% Normal CI
Parameter   Estimate        Error     Lower     Upper
Shape        1,39958        0,201   1,05695   1,85327
Scale        553,643       57,864   451,094   679,505


Test for Equal Shape Parameters
Bartlett's Modified Likelihood Ratio Chi-Square

* NOTE * Test skipped - There must be at least one other failure which is not
at the time the
         system is retired: Check ID = 251.


Trend Tests

                    MIL-Hdbk-189          Laplace's
                TTT-based  Pooled  TTT-based  Pooled  Anderson-Darling
Test Statistic      68,72   66,15       2,03    2,38              3,17
P-Value             0,032   0,017      0,043   0,017             0,022
DF                     96      96
```

TTT-PLOT FOR VALVESEAT DATA



Results for: TMA4275ProschanAircondition.MTW

Parametric Growth Curve: T

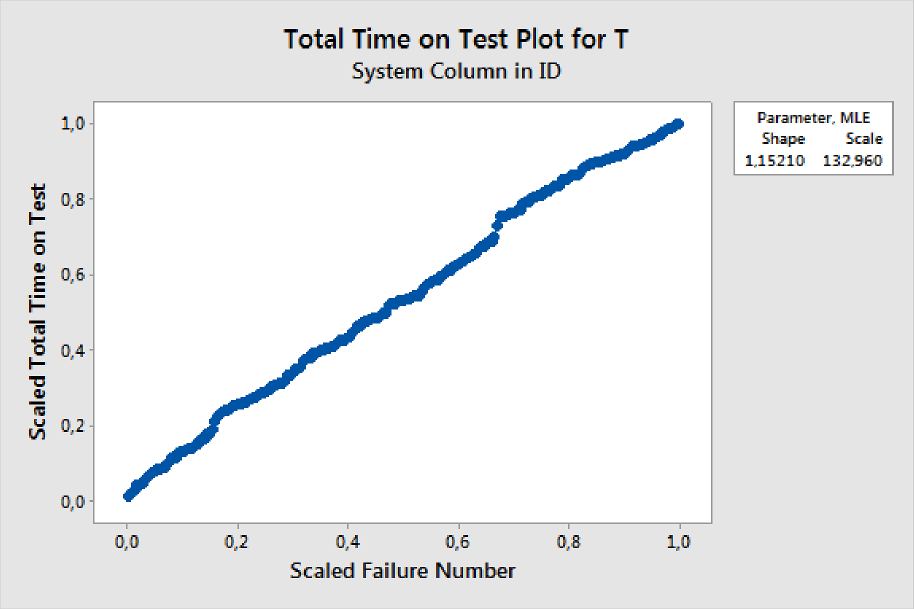
```
System:  ID

Model: Power-Law Process
Estimation Method: Maximum Likelihood


Parameter Estimates

                         Standard     95% Normal CI
Parameter  Estimate      Error      Lower     Upper
Shape       1,15210      0,066   1,02906   1,28985
Scale       132,960     20,216   98,6964   179,119


Test for Equal Shape Parameters
Bartlett's Modified Likelihood Ratio Chi-Square

Test Statistic    6,70
P-Value          0,979
DF                  16


Trend Tests

                      MIL-Hdbk-189         Laplace's
                   TTT-based  Pooled   TTT-based  Pooled  Anderson-Darling
Test Statistic       363,67   350,40        1,88    0,79              2,25
P-Value               0,031    0,129       0,061   0,428             0,068
DF                      424      392
```

The null hypothesis of "the processes are all HPP with the **same** *ROCOF λ" is rejected (by the TTT-based tests, at least the one based on Mil Hbk).*

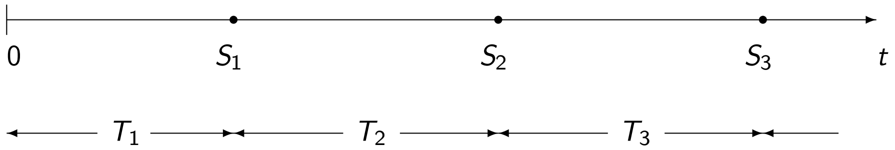
On the other hand, the null hypothesis of "the processes are all HPP, possibly with **different** *ROCOFs" is not rejected by any of the tests.*

This is in accordance with Proschan's conclusion in 1963, namely that the processes are HPPs with different ROCOFs.

TTT-PLOT FOR AIRCONDITIONER DATA



RENEWAL PROCESSES - A SHORT INTRODUCTION



Definition of **renewal process**:

$T_1, T_2, \ldots$ are i.i.d. with common cdf $F(t)$.

1. Time to rth event: $S_r = T_1 + \ldots + T_r$ (with distribution called the rth *convolution* of T).
2. $N(t) =$ number of events in $(0, t] = \max\{r : S_r \leq t\}$
3. $W(t) = E(N(t))$ is called the **renewal function** ($=$ CROCOF))
4. $w(t) = W'(t)$ is called the **renewal density** ($=$ ROCOF))

Usually difficult to find nice expressions for $W(t)$ and $w(t)$ (except for HPP). Thus approximations are needed (see next slide).

Let

$$\begin{aligned} \mu &= E(T_i) \;\; (= MTBF) \\ \sigma^2 &= Var(T_i) \end{aligned}$$

The HPP is an RP with $T_i \sim \text{expon}(\lambda)$. Hence

$$\begin{aligned} W(t) &= \lambda t = \frac{1}{\mu} t, \quad \text{so} \\ \frac{W(t)}{t} &= \frac{1}{\mu} \end{aligned}$$

For general renewal processes we have
The Elementary Renewal Theorem:

$$\lim_{t \to \infty} \frac{W(t)}{t} = \frac{1}{\mu}$$

The Elementary Renewal Theorem implies (R&H, p.253):

$$W(t) \approx \frac{t}{\mu} \quad \text{for large } t \ .$$

A better approximation is

$$W(t) \approx \frac{t}{\mu} + \frac{1}{2}\left(\frac{\sigma^2}{\mu^2} - 1\right)$$

Finally, for an HPP(λ) we clearly have exactly

$$W(t+\alpha) - W(t) = \frac{1}{\mu}(t+\alpha) - \frac{1}{\mu}t = \frac{\alpha}{\mu}$$

For general RP we have *Blackwell's theorem:*

$$\lim_{t\to\infty} \left(W(t+\alpha) - W(t)\right) = \frac{\alpha}{\mu}$$