TMA4275 LIFETIME ANALYSIS

Slides 15: Parametric estimation in NHPPs

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NTNU, Spring 2015

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Power law NHPP (also called Weibull-process)

$w(t) = \lambda\beta t^{\beta-1}$ for $\lambda, \beta > 0$

$\searrow$ if $\beta < 1$

$\nearrow$ if $\beta > 1$

HPP if $\beta = 1$

$W(t) = \int_0^t w(u)du = \int_0^t \lambda\beta u^{\beta-1}du = \lambda t^\beta$

Similar to Weibull hazard $z(t) = \frac{\alpha}{\theta}(\frac{t}{\theta})^{\alpha-1}$

Log linear NHPP

$w(t) = e^{\alpha+\beta t}$ for $-\infty < \lambda, \beta < \infty$

$\searrow$ if $\beta < 0$

$\nearrow$ if $\beta > 0$

HPP if $\beta = 0$

$W(t) = \frac{e^\alpha}{\beta}(e^{\beta t} - 1)$

Suppose first that we observe a *single* NHPP with ROCOF $w(t;\theta)$ on the time interval $[0,\tau]$, where τ is fixed.

Then we know that if N is the number of events, then $N \equiv N(\tau) \sim Poisson(W(\tau;\theta))$, where $W(\tau;\theta)) = \int_0^\tau w(u;\theta)du$.

Let the times of events be: $(0 \leq)\ s_1 \leq s_2 \leq \cdots \leq s_N\ (\leq \tau)$.

Define also

$$W(s,t;\theta) = \int_s^t w(u;\theta)du = E[N(s,t)]$$

On the next slides we derive the likelihood function for these observations.

Divide time axis at $h_0 = 0 \le h_1 < h_2 < \cdots < h_r \equiv \tau$.

Let $D_i = \#$events in $(h_{i-1}, h_i]$. Then $D_i \sim Poisson(W(h_{i-1}, h_i; \theta))$, and $D_1, D_2, \cdots, D_r$ are independent (properties of NHPP), so the likelihood is

$$
\begin{aligned}
L(\theta) &= P(D_1 = d_1, D_2 = d_2, \cdots, D_r = d_r) \\
&= \prod_{i=1}^{r} P(D_i = d_i) = \prod_{i=1}^{r} \frac{W(h_{i-1}, h_i; \theta)^{d_i}}{d_i!} e^{-W(h_{i-1}, h_i; \theta)} \\
&= \left\{ \prod_{i=1}^{r} \frac{W(h_{i-1}, h_i; \theta)^{d_i}}{d_i!} \right\} \cdot e^{-\sum_{i=1}^{r} W(h_{i-1}, h_i; \theta)} \\
&= \left\{ \prod_{i=1}^{r} \frac{W(h_{i-1}, h_i; \theta)^{d_i}}{d_i!} \right\} \cdot e^{-W(\tau; \theta)}
\end{aligned}
$$

since
$\sum_{i=1}^{r} W(h_{i-1}, h_i; \theta) = \sum_{i=1}^{r} \int_{h_{i-1}}^{h_i} w(u; \theta) du = \int_0^{\tau} w(u; \theta) du = W(\tau; \theta)$

Recall $L(\theta) = \left\{ \prod_{i=1}^{r} \frac{W(h_{i-1},h_i;\theta)^{d_i}}{d_i!} \right\} \cdot e^{-W(\tau;\theta)}$.

If data are given by counts D_i in intervals $(h_{i-1}, h_i]$, then we maximize this $L(\theta)$ to find MLE.

If times are instead given by exact times $s_1, s_2, \cdots, s_N$, then we let the grid of h_i be more and more dense and get in the limit 0 or 1 event in each interval $(h_{i-1}, h_i]$.

Now, when $d_i = 0$, the contribution to the product is $\frac{W(h_{i-1},h_i;\theta)}{0!} = 1$, which can be ignored.

When $d_i = 1$, the contribution is

$$W(h_{i-1}, h_i; \theta) = \int_{h_{i-1}}^{h_i} w(u;\theta)du \approx w(s_k;\theta)(h_i - h_{i-1})$$

Since the $h_i - h_{i-1}$ are fixed by us, we let the contribution to the likelihood be $w(s_k, \theta)$ for $k = 1, 2, \ldots, N$.

Hence we get the likelihood for exactly observed failure times from a single repairable system:

$$L(\theta) = \left\{ \prod_{i=1}^{N} w(s_i, \theta) \right\} e^{-W(\tau;\theta)}$$

If there are data from m systems with the same $w(t;\theta)$, then the likelihood is the product of the likelihoods for each system:

$$L(\theta) = \prod_{j=1}^{m} \left(\left\{ \prod_{i=1}^{N_j} w(s_{ij};\theta) \right\} e^{-W(\tau_j;\theta)} \right)$$

Here system j ($j = 1, \ldots, m$) is observed on the time interval $(0, \tau_j]$, with N_j observations $s_{1j}, s_{2j}, \ldots, s_{N_j j}$.

The log-likelihood function for m systems is hence

$$\ell(\theta) = \sum_{j=1}^{m} \sum_{i=1}^{N_j} \ln w(s_{ij}; \theta) - \sum_{j=1}^{m} W(\tau_j; \theta)$$

Example - power law NHPP: $w(t; \lambda, \beta) = \lambda \beta t^{\beta - 1}$, $W(t; \lambda, \beta) = \lambda t^{\beta}$

$$\begin{aligned}\ell(\lambda, \beta) &= \sum_{j=1}^{m} \sum_{i=1}^{N_j} (\ln \lambda + \ln \beta + (\beta - 1) \ln s_{ij}) - \sum_{j=1}^{m} \lambda \tau_j^{\beta})) \\ &= N \ln \lambda + N \ln \beta + (\beta - 1) S - \lambda \sum_{j=1}^{m} \tau_j^{\beta}\end{aligned}$$

where $N = \sum_{j=1}^{m} N_j$ = total number of observations,
and $S = \sum_{j=1}^{m} \sum_{i=1}^{N_j} \ln s_{ij}$ = sum of logs of all observations

Recall $\ell(\lambda, \beta)) = N \ln \lambda + N \ln \beta + (\beta - 1)S - \lambda \sum_{j=1}^{m} \tau_j^{\beta}$

$$\frac{\partial \ell}{\partial \lambda} = \frac{N}{\lambda} - \sum_{j=1}^{m} \tau_j^{\beta} = 0 \qquad (1)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{N}{\beta} + S - \lambda \sum_{j=1}^{m} \tau_j^{\beta} \ln \tau_j \qquad (2)$$

(1) gives $\lambda = \frac{N}{\sum_{j=1}^{m} \tau_j^{\beta}}$
put this into (2)

$$\frac{N}{\beta} + S - \frac{N \sum_{j=1}^{m} \tau_j^{\beta} \ln \tau_j}{\sum_{j=1}^{m} \tau_j^{\beta}} = 0$$

Can solve this numerically for β to get $\hat{\beta}$ and then put $\hat{\lambda} = \frac{N}{\sum_{j=1}^{m} \tau_j^{\hat{\beta}}}$

Special case: If all $\tau_j \equiv \tau$ are equal (including the case m=1)

$$\frac{N}{\beta} + S - \frac{Nm\tau^\beta \ln \tau}{m\tau^\beta} = 0$$

$$\frac{N}{\beta} + S - N \ln \tau = 0$$

$$\beta = \frac{N}{N \ln \tau - S}$$

So in this case:

$$\hat{\beta} = \frac{N}{N \ln \tau - S}$$

FINDING MLE BY PROFILE LOG LIKELIHOOD

If β is *known*, then we found by solving $\partial\ell/\partial\lambda = 0$,

$$\hat{\lambda}(\beta) = \frac{N}{\sum_{j=1}^{m} \tau_j^{\beta}}$$

The profile log likelihood for β is therefore

$$\begin{aligned}
\tilde{\ell}(\beta) &= \ell(\hat{\lambda}(\beta), \beta) \\
&= N \ln \hat{\lambda}(\beta) + N \ln \beta + (\beta - 1)S - \hat{\lambda}(\beta) \cdot \sum_{j=1}^{m} \tau_j^{\beta} \\
&= N \ln N - N \ln(\sum_{j=1}^{m} \tau_j^{\beta}) + N \ln \beta + (\beta - 1)S - N
\end{aligned}$$

Then to find the MLE of λ, β we can

1. Maximize $\tilde{\ell}(\beta)$ to find $\hat{\beta}$
2. Put $\hat{\lambda} = \hat{\lambda}(\hat{\beta})$