TMA4275 LIFETIME ANALYSIS

Slides 17: Trend testing for NHPP

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NTNU, Spring 2014

Consider an NHPP with intensity (ROCOF) $w(t)$.

Can see "trend" of $w(t)$ by considering $W(t)$:

- if $W(t)$ is **convex**, then $w(t) \nearrow$
- if $W(t)$ is **concave**, then $w(t) \searrow$
- if $W(t)$ is linear, then $w(t)$ is constant (so we have an HPP).

Trend tests are tests for:

$$H_0 : w(t) \text{ is constant}$$

versus, alternatives such as

$$\begin{aligned} H_1 &: w(t) \nearrow \\ H_1 &: w(t) \searrow \\ H_1 &: w(t) \text{ is not constant} \end{aligned}$$

Basic result for an HPP on the interval $[0, \tau]$, with events at $S_1, S_2, \ldots, S_N$:

Given $N = n$, the n events are distributed uniformly on the interval $[0, \tau]$, i.e. they are the orderings of n independent variables, uniformly distributed on $[0, \tau]$.

These variables have mean equal to $\tau/2$ and variance $\tau^2/12$. Thus

$$\sum_{i=1}^{n} S_i \approx N(n\tau/2, n\tau^2/12)$$

Thus

$$W = \frac{\sum_{i=1}^{n} S_i - n\tau/2}{\tau\sqrt{n/12}} = \frac{\sum_{i=1}^{n}(S_i - \tau/2)}{\tau\sqrt{n/12}} \approx N(0,1)$$

This is the test statistic of the Laplace test!

Why is this a reasonable test statistic, and for which values of W should we reject the null hypothesis?

Recall

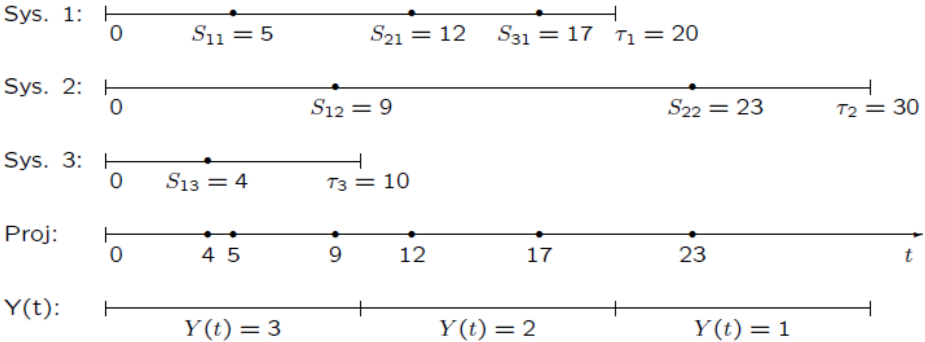
$$W = \frac{\sum_{i=1}^{n}(S_i - \tau/2)}{\tau\sqrt{n/12}}$$

Now:

- If $H_1 : w(t) \nearrow$, then most failures are above $\tau/2$, and W has a tendency to be large and positive.
- If $H_1 : w(t) \searrow$, then most failures are below $\tau/2$, and W has a tendency to be small and negative.
- If $H_1 : w(t)$ is not constant, then
  - if the alternative is that of a *monotone* $w(t)$, the W has a tendency to be either too large or too small
  - if the alternative is, e.g., a bathtub $w(t)$, then W may still be moderate, so that the test is not good in this case!

TREND TESTING IN SIMPLE EXAMPLE WITH THREE SYSTEMS

- system 1: $W_1 = \frac{(5-10)+(12-10)+(17-10)}{20\sqrt{3/12}} = \frac{4}{20\sqrt{3/12}} = 0.40$
- system 2: $W_2 = \frac{(9-15)+(23-15)}{30\sqrt{2/12}} = 0.1633$
- system 3: $W_3 = \frac{4-5}{10\sqrt{1/12}} = -0.3464$



Basic result for an HPP stopped at the nth event, for a given n, with events at $S_1, S_2, \ldots, S_n$:

Given the value $S_n = s_n$, the $n-1$ first events are distributed uniformly on the interval $[0, s_n]$, i.e. they are the orderings of $n-1$ independent variables, uniformly distributed on $[0, s_n]$.

This leads to the test statistic

$$W = \frac{\sum_{i=1}^{n-1}(S_i - S_n/2)}{S_n\sqrt{(n-1)/12}}$$

which is approximately $N(0,1)$ under the null hypothesis of HPP.

A PRELIMINARY RESULT

Now use the following result:

If $S \sim U[0, \tau]$, then

$$Z = 2 \ln \frac{\tau}{S} \sim \chi_2^2$$

which is proved as follows:

$$\begin{aligned} P(Z \leq z) &= P(2 \ln \frac{\tau}{S} \leq z) \\ &= P(\ln \frac{\tau}{S} \leq \frac{z}{2}) \\ &= P(\frac{\tau}{S} \leq e^{\frac{z}{2}}) \\ &= P(S \geq \tau e^{-\frac{z}{2}}) \\ &= 1 - e^{-\frac{z}{2}} \end{aligned}$$

Thus $f_Z(z) = \frac{1}{2} e^{-\frac{z}{2}} \sim \chi_2^2$

The test statistic is

$$Z = 2\sum_{i=1}^{n} \ln \frac{\tau}{S_i} \sim \chi^2_{2n} \text{ under } H_0 \text{ (time censoring at } \tau)$$

$$Z = 2\sum_{i=1}^{n-1} \ln \frac{S_n}{S_i} \sim \chi^2_{2(n-1)} \text{ under } H_0 \text{ (failure censoring at } n\text{th failure)}$$

(the distributions under H_0 are exact distribution, not only "approximately")

Recall

$$Z = 2 \sum_{i=1}^{n} \ln \frac{\tau}{S_i}$$

Now:

- If $H_1 : w(t) \nearrow$, then many of the τ / S_i are close to 1, and hence Z has a tendency to be small (note $\ln 1 = 0$).
- If $H_1 : w(t) \searrow$, then many of the S_i are small, so many of the τ / S_i are large, and hence Z has a tendency to be large.
- If $H_1 : w(t)$ is not constant, then
  - if the alternative is that of a *monotone* $w(t)$, the Z has a tendency to be either too large or too small
  - if the alternative is, e.g., a bathtub $w(t)$, then Z may still be moderate, so that the test is not good in this case!

COMPUTATION FOR SIMPLE SYSTEM NO 1

Recall: This system has failures at 5, 12, 17; time censoring at $\tau = 20$.

$$Z = 2(\ln \frac{20}{5} + \ln \frac{20}{12} + \ln \frac{20}{17}) = 5.14$$

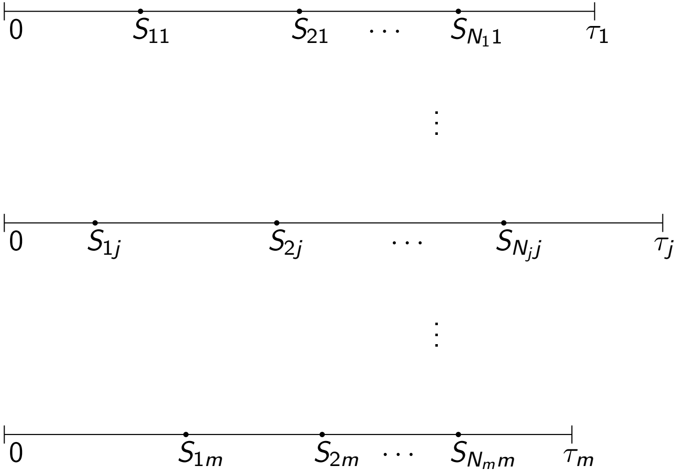
If H_0 holds (HPP), then $Z \sim \chi^2_6$ (i.e. $E(Z) = 6$), so the tendency is towards "small" value, i.e. increasing $w(t)$.

BUT: If $H_1 : w(t) \nearrow$, then we should reject at 5% level if $Z < \chi^2_6(0.05) = 1.64$, so we are frar from rejecting the HPP!

Suppose we have m processes which each are NHPPs

H_0: the m processes are all HPP.
H_1: the m processes have increasing trend (at least one of them); or
H_1: the m processes have decreasing trend (at least one of them); or
H_1: not all of the m processes are HPP.

TYPICAL DATA



THE POOLED LAPLACE TEST

Under the null hypothesis of m HPP, we have for each j:

Given $N_j = n_j$, the $S_{1j}, \cdots, S_{N_j j}$ are orderings of n_j uniforms on $(0, \tau_j)$.
Thus, $E(S_{ij}) = \frac{\tau_j}{2}$, $Var(S_{ij}) = \frac{\tau_j^2}{12}$.

The pooled Laplace test is defined by the test statistic

$$W_{pooled} = \frac{\sum_{j=1}^{m} \sum_{i=1}^{N_j} S_{ij} - E\left[\sum_{j=1}^{m} \sum_{i=1}^{N_j} S_{ij}\right]}{\sqrt{Var\left[\sum_{j=1}^{m} \sum_{i=1}^{N_j} S_{ij}\right]}}$$

Here
$E[\sum_{j=1}^{m} \sum_{i=1}^{N_j} S_{ij}] = \sum_{j=1}^{m} \sum_{i=1}^{N_j} \frac{\tau_j}{2} = \sum_{j=1}^{m} \frac{n_j \tau_j}{2}$
$Var[\sum_{j=1}^{m} \sum_{i=1}^{N_j} S_{ij}] = \sum_{j=1}^{m} \sum_{i=1}^{N_j} \frac{\tau_j^2}{12} = \sum_{j=1}^{m} \frac{n_j \tau_j^2}{12}$

$$\Rightarrow W_{pooled} = \frac{\sum_{j=1}^{m} \sum_{i=1}^{N_j} S_{ij} - \sum_{j=1}^{m} \frac{n_j \tau_j}{2}}{\sqrt{\sum_{j=1}^{m} \frac{n_j \tau_j^2}{12}}} \approx N(0,1) \quad \text{under } H_0$$

$$
\begin{aligned}
W_{pooled} &= \frac{5+12+17+9+23+4-\frac{1}{2}(3\cdot 20+2\cdot 30+1\cdot 10)}{\sqrt{\frac{1}{12}[3\cdot 20^2+2\cdot 30^2+1\cdot 10^2]}} \\
&= \frac{70-65}{\sqrt{\frac{1}{12}[3100]}} \\
&= \frac{5}{\sqrt{\frac{3100}{12}}} \\
&= 0.3111
\end{aligned}
$$

p-value for a test of "all HPP" vs "not all HPP" is $2\cdot P(W\geq 0.3111)=2\cdot 0.378=0.756$.

Note: *This test is in fact a test of the null hypothesis that processes are all HPP's but possibly with different individual hazards.*

Under the null hypothesis of m HPP, we have for each j:
$Z_{ij} = 2\sum_{i=1}^{n_j} \ln \frac{\tau_j}{S_{ij}} \sim \chi^2_{2n_j}$. This suggests to define

$$Z_{pooled} = \sum_{j=1}^{m} \underbrace{\sum_{i=1}^{n_j} 2\ln \frac{\tau_j}{S_{ij}}}_{\chi^2_{2n_j}} \sim \chi^2_{2n}$$

where $n = \sum_{j=1}^{m} n_j$.

Can write simply $Z_{pooled} = \sum_{j=1}^{m} Z_{ij}$.

IN SIMPLE EXAMPLE:

$$Z_{pooled} = 2(\ln \frac{20}{5} + \ln \frac{20}{12} + \ln \frac{20}{17} \ln \frac{30}{9} + \ln \frac{30}{23} + \ln \frac{10}{4}) = 8.89$$

UnderH_0 : $Z_{pooled} \sim \chi^2_{12}$, so P-value is $2P(\chi^2_{12} \leq 8.89) = 0.5754$

(Use parametric repairable systems analysis).

Trend Tests

	MIL-Hdbk-189		Laplace's		
	TTT-based	Pooled	TTT-based	Pooled	Anderson-Darling
Test Statistic	9,59	8,89	0,12	0,31	0,24
P-Value	0,697	0,576	0,906	0,756	0,977
DF	12	12			

MINITAB also reports a **TTT-based test**, based on the article:

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PII: S0951-8320(97)00099-9

Reliability Engineering and System Safety **60** (1998) 13–28
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0951–8320/98/$19.00

TTT-based tests for trend in repairable systems data

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(Received 25 September 1996; revised 24 January 1997; accepted 15 July 1997)

This is for the for the null hypothesis that *all* the *m* processes are HPP with the *same* intensity.