TMA4275 LIFETIME ANALYSIS

Slides 15: Recurrent events; Repairable systems

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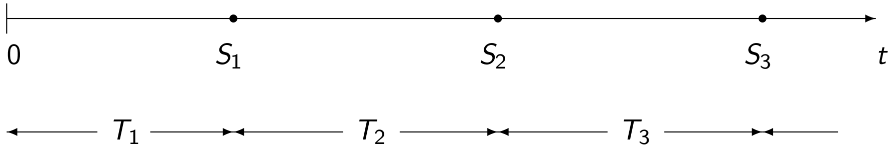
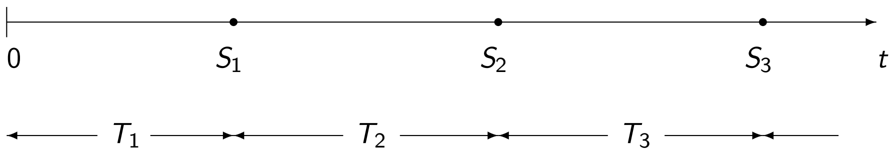
REPAIRABLE SYSTEMS/RECURRENT EVENTS/ COUNTING PROCESSES

Definition of repairable system (Ascher and Feingold 1984):

"A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, other than replacement of the entire system".



1. System is repaired and put into use again.
2. Machine part is replaced.
3. Relapse from disease (epileptic seizures, recurrence of tumors)



Modeling as a *counting process;* i.e. counting events on a time axis.

$N(t) = \#$ events in (0,t].

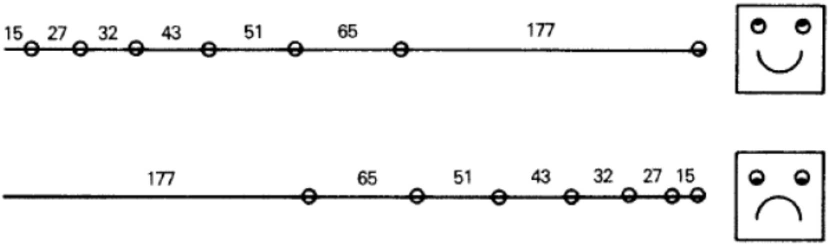
$N(s,t) = \#$ events in $(s,t] = N(s) - N(t)$.

$S_1, S_2, \cdots$ are event times.

$T_1, T_2, \cdots$ are times *between* events; also called "sojourn times".

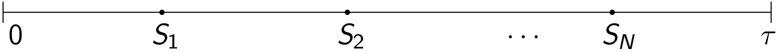
NOTE: It is common to disregard *repair times*, but one could have situations where "up times" alternate with "down times" of a system.

Ascher and Feingold presented the following example of a "happy" and "sad" system:



- *Their claim:* Reliability engineers do not recognize the difference between these cases since they always treat times between failures as i.i.d. and fit probability models like Weibull.
- *Their conclusion:* Use point process models to analyze repairable systems data!

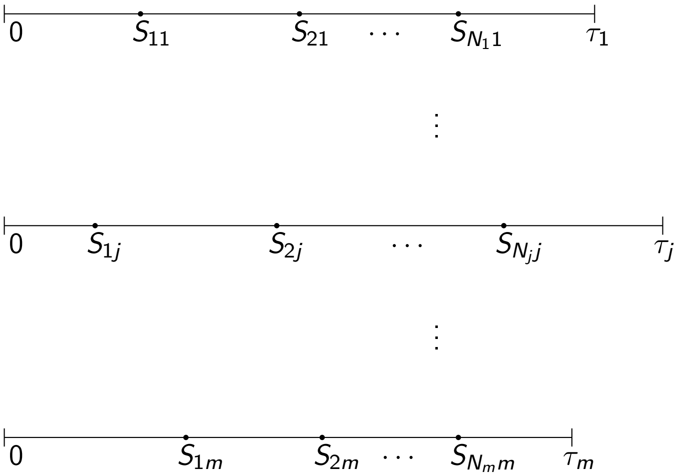
APPLICATIONS



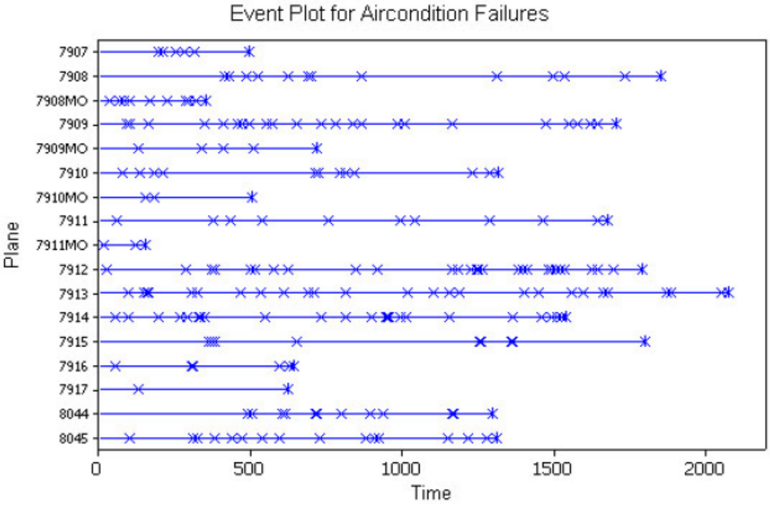
- Applications: engineering and reliability studies, public health, clinical trials, politics, finance, insurance, sociology, etc.

Reliability applications:

- breakdown or failure of a mechanical or electronic system
- discovery of a bug in an operating system software
- the occurrence of a crack in concrete structures
- the breakdown of a fiber in fibrous composites
- Warranty claims of manufactured products

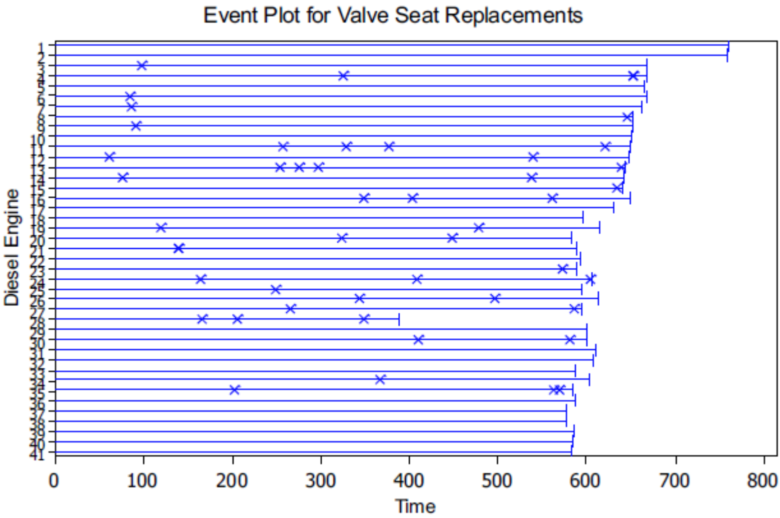

0
S_{11}
S_{21}
$\cdots$
$S_{N_1 1}$
τ_1
$\vdots$
0
S_{1j}
S_{2j}
$\cdots$
$S_{N_j j}$
τ_j
$\vdots$
0
S_{1m}
S_{2m}
$\cdots$
$S_{N_m m}$
τ_m

Times of failures of aircondition system in a fleet of Boeing 720 airplanes



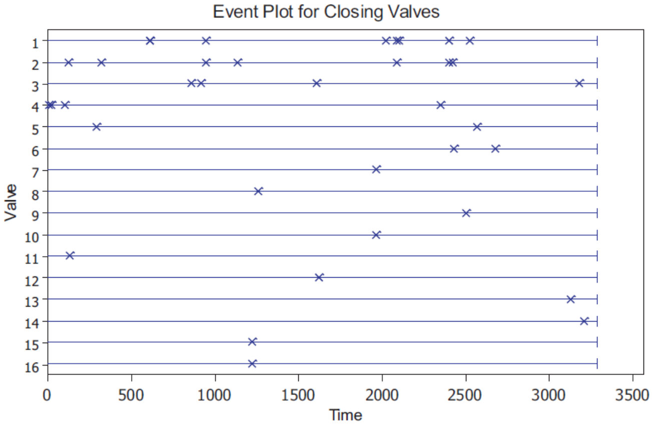
NELSON (1995) VALVESEAT DATA

Times of valve-seat replacements in a fleet of 41 diesel engines



BHATTACHARJEE ET AL. (2003) NUCLEAR PLANT FAILURE DATA

- Failure data for closing valves in safety systems at two nuclear reactor plants in Finland. Failures type: *External Leakage*, follow-up 9 years for 104 valves. 88 valves had no failures



Aalen and Husebye (1991): Migratory motor complex (MMC) periods in 19 patients, 1-9 events per individual.

Individual	Observed periods (minutes)					
1	112	145	39	52	21	34
	33	51	(54)			
2	206	147	(30)			
3	284	59	186	(4)		
4	94	98	84	(87)		
5	67	(131)				
6	124	34	87	75	43	38
	58	142	75	(23)		
7	116	71	83	68	125	(111)
8	111	59	47	95	(110)	
9	98	161	154	55	(44)	
10	166	56	(122)			
11	63	90	63	103	51	(85)
12	47	86	68	144	(72)	
⋮			⋮			

Definition: $W(t) =_{def} E[N(t)]$ = expected # events (failures) in (0,t].

$w(t) =_{def} W'(t)$ = Rate of Occurrence of Failures (ROCOF).

$$
\begin{aligned}
w(t) &= \lim_{h\to 0} \frac{W(t+h) - W(t)}{h} \\
&= \lim_{h\to 0} \frac{E[N(t+h)] - E[N(t)]}{h} \\
&= \lim_{h\to 0} \frac{E[N(t+h) - N(t)]}{h} \\
&= \lim_{h\to 0} \frac{E[N(t,t+h)]}{h} \\
&= \lim_{h\to 0} \frac{\text{expected\#events in}(t,t+h)}{h}
\end{aligned}
$$

So expected number of events in $(t, t+h) \approx w(t)h$

Definition: Counting process is *regular* if

$$P(N(t, t+h) \geq 2) = o(h)$$

i.e. *small*, even compared to h, meaning that $\frac{o(h)}{h} \to 0$, as $h \to 0$
In practice *regular* means "*No simultaneous events*". So:

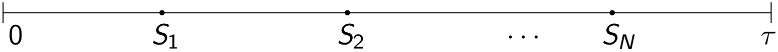
$$\begin{aligned} E[N(t, t+h)] &= 0 \cdot P(N(t, t+h) = 0) + 1 \cdot P(N(t, t+h) = 1) \\ &+ 2 \cdot P(N(t, t+h) = 2) + \cdots \end{aligned}$$

Hence

$$\frac{E[N(t, t+h)]}{h} \approx \frac{P(N(t, t+h) = 1)}{h} + \frac{o(h)}{h},$$

so $w(t) = \lim_{h \to 0} \frac{P(N(t,t+h)=1)}{h}$ or $P(N(t, t+h) = 1) \approx w(t) \cdot h$
(for a regular process).
This is analogous to $P(t < T \leq t + h | T > t) \approx z(t)h$
for hazard rates (which sometimes are called FOM = Force of Mortality)

BASIC MODELS FOR REPAIRABLE SYSTEMS



- RP(F): Renewal process with interarrival distribution F.

  Defining property:

  - Times between events are i.i.d. with distribution F

- NHPP($w(\cdot)$): Nonhomogeneous Poisson process with intensity $w(t)$.

  Defining property:

  1. Number of events in $(0, t]$ is Poisson-distributed with expectation $\int_0^t w(u)du = W(t)$
  2. Number of events in disjoint time intervals are stochastically independent

The NHPP is given by

- Specifying the ROCOF (intensity) $w(t)$,
- which has the basic property that $P(N(t, t+h) = 1) \approx w(t)h$
- assuming regularity of point process
- assuming independence of number of events in disjoint intervals

Properties of NHPP:

- $N(s, t) = \#$ events in (s,t] is Poisson($\int_s^t w(u)du$)
- $N(t) = \#$ in (0,t] is Poisson($\int_0^t w(u)du$), i.e. Poisson($W(t)$).
- $P(N(t) = j) = \frac{W(t)^j}{j!} e^{-W(t)}$, for $j = 0, 1, \cdots$ and $E(N(t)) = W(t)$ so $w(t) = W'(t)$ is really the ROCOF.
- $E[N(s, t)] = \int_s^t w(u)du = W(t) - W(s)$

Advantage of NHPP and reason for its extensive use:

Can model a *trend* in the rate of failures, because
$P(\text{failure in } (t, t+h)) \approx w(t)h$.

- $w(t) \nearrow$ deteriorating system ("sad system") e.g. aging of a mechanical system
- $w(t) \searrow$ improving system ("happy system") e.g. software reliability.
- $w(t) = \lambda$ (constant): Homogeneous Poisson process (HPP)

Let S_1 is the time to first failure. For HPP, this is expon(λ).

For NHPP, $P(T_1 > t) = P(N(t) = 0)$,

so since $N(t) \sim Poisson(W(t))$,

$R_{T_1}(t) = P(T_1 > t) = \frac{W(t)^0}{0!}e^{-W(t)} = e^{-W(t)}$

Thus, $Z_{T_1}(t) = W(t)$, so

$z_{T_1}(t) = w(t)$,

i.e. the ROCOF $w(t)$ for an NHPP equals the hazard rate for the time to first failure.

EXAMPLE

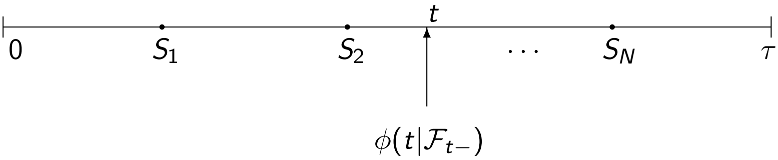
Suppose $S_1, S_2, \ldots$ is and NHPP with $w(t) = 2t$ and $W(t) = t^2$.

What is the expected # of failures in the time intervals $[0, 1], [1, 2], [2, 3]$, all having length 1?

$$E[N(0,1)] = W(1) - W(0) = 1$$
$$E[N(1,2)] = W(2) - W(1) = 2^2 - 1^2 = 3$$
$$E[N(2,3)] = W(3) - W(2) = 9 - 4 = 5$$

Time to the first failure:
$R_{T_1}(t) = P(T_1 > t) = P(N(0,t) = 0) = \frac{W(t)^0}{0!} e^{-W(t)} = e^{-t^2}$
$\Rightarrow f_{T_1}(t) = -R'_{T_1}(t) = 2te^{-t^2} = w(t)e^{-W(t)}$, which is a Weibull distribution.

POINT PROCESS MODELING OF RECURRENT EVENTS



- $\mathcal{F}_{t-}$ = history of events until time t.
- Conditional intensity at t given history until time t,

$$\phi(t|\mathcal{F}_{t-}) = \lim_{h \downarrow 0} \frac{Pr(\text{failure in } [t, t+h)|\mathcal{F}_{t-})}{h}$$

SPECIAL CASES: THE BASIC MODELS

- NHPP($w(\cdot)$):

$$\phi(t|\mathcal{F}_{t-}) = w(t)$$

so conditional intensity is independent of history.
Interpreted as "minimal repair" at failures

- RP(F) (where F has hazard rate $z(\cdot)$):

$$\phi(t|\mathcal{F}_{t-}) = z(t - S_{N(t-)})$$

so conditional intensity depends (only) on time since last event.
Interpreted as "perfect repair" at failures

- Between minimal and perfect repair? So called *imperfect repair* models.

PERFECT AND MINIMAL REPAIR

Assume that we have a component or system with lifetime T, and corresponding hazard rate $z(t)$.

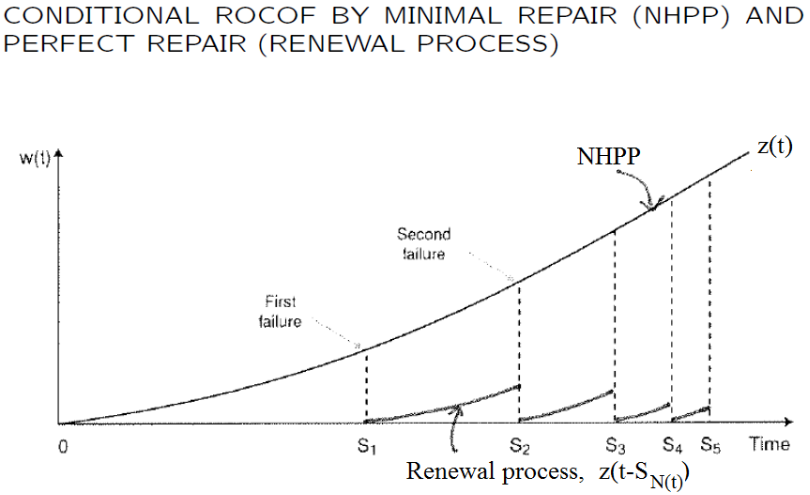
Perfect repair: Assume that the component at each failure is repaired to as good as new (or, possibly, is replaced). Then we can consider the inter-failure times $T_1, T_2, \ldots$ as independent realizations of T, hence $S_1, S_2, \ldots$ is a renewal process.

Thus, conditional ROCOF at t is $z(\text{time since last failure}) = z(t - S_{N(t)})$

Minimal repair: Assume that the system at each failure is repaired only to the same state as immediately before the failure. Then the probability of failing in $(t, t+h)$ will always be the same as for a system starting at time 0 which never has failed, namely $\approx z(t)h$. Thus rate of occurence of failures is independent of the history.

Can be shown that minimal repair as defined above, corresponds to the property of an NHPP with ROCOF $w(t) = z(t)$.

Consider a component with hazard rate $z(t)$, which is repaired at failures.



First: If we have data for *one* system only, Then since $W(t) = E[N(t)]$, we estimate $W(t)$ by $\hat{W}(t) = N(t)$ (also valid outside the class of NHPPs).

Assume more generally:

- m processes are observed, assumed to have the same $W(t)$
- processes are not necessarily NHPPs
- first process is observed on time interval $(0, \tau_1]$
  second process on $(0, \tau_2]$
  $\vdots$
  Let τ_{max} = largest τ_j
- $Y(t) = \#$ processes under observation at time t.

We want to estimate $W(t)$

TOWARDS THE NELSON-AALEN ESTIMATOR FOR $W(t)$

Divide the time axis at $h_0 = 0, h_1, h_2, \dots$ up to τ_{max}.
Assume for simplicity that all the τ_j are among the h_j.

Let $D_i = \#$ events in $(h_{i-1}, h_i]$ *(total for all systems)*
and $y_i =$ value of $Y(t)$ in $(h_{i-1}, h_i]$

For each process:
$E[N(h_{i-1}, h_i)] = E[N(h_i)] - E[N(h_{i-1})] = W(h_i) - W(h_{i-1})$

Thus when all processes are considered:
$E(D_i) = y_i(W(h_i) - W(h_{i-1}))$,
and $E(\frac{D_i}{y_i}) = W(h_i) - W(h_{i-1})$ for $i = 1, 2, \dots$

But then
$E[\frac{D_1}{y_1}] + E[\frac{D_2}{y_2}] + \dots + E[\frac{D_k}{y_k}]$

$= W(h_1) - W(h_0) + W(h_2) - W(h_1) + \dots + W(h_k) - W(h_{k-1})$

$= W(h_k) - W(h_0) = W(h_k)$

Recall $E[\frac{D_1}{y_1}] + E[\frac{D_2}{y_2}] + \ldots + E[\frac{D_k}{y_k}] = W(h_k)$

This suggests the estimator

$$\hat{W}(h_k) = \sum_{i=1}^{k} \frac{D_i}{y_i} \quad \text{for } k = 1, 2 \ldots$$

Suppose the failure times, when joined for all the m processes, are ordered as $t_1 < t_2 < \ldots < t_n$

Then by letting the h_i be more and more dense, we get contributions for at most *one* failure time in each interval (h_{i-1}, h_i).

Then we get, letting $d(t_i) = \#$events at t_i (so $d(t_i) = 1$ if regular process)
$Y(t_i) = \#$processes observes at t_i

$$\hat{W}(t) = \sum_{t_i \leq t} \frac{d(t_i)}{Y(t_i)}$$

Going back to the first estimator $\hat{W}(h_k) = \sum_{i=1}^{k} \frac{D_i}{y_i}$ for $k = 1, 2 \ldots$, if the processes are *NHPPs* with CROCOF $W(t)$, then

1. $D_i \sim Poisson(y_i(W(h_i) - W(h_{i-1})))$
2. The D_i are *independent* (very important implication of NHPP)

Now $Var(\frac{D_i}{y_i}) = \frac{1}{y_i^2} Var(D_i) = \frac{E(D_i)}{y_i^2}$
and hence

$$Var(\hat{W}(h_k)) = \sum_{i=1}^{k} Var(\frac{D_i}{y_i}) = \sum_{i=1}^{k} \frac{Var(D_i)}{y_i^2} = \sum_{i=1}^{k} \frac{E(D_i)}{y_i^2}$$

So an estimator is

$$\widehat{Var(\hat{W}(h_k))} = \sum_{i=1}^{k} \frac{D_i}{y_i^2}$$

which in the limit gives

$$\widehat{Var(\hat{W}(t))} = \sum_{t_i \leq t} \frac{d(t_i)}{Y(t_i)^2}$$

NELSON-AALEN ESTIMATOR FOR CUMULATIVE ROCOF $W(t)$

1. Order all failure times as $t_1 < t_2 < \ldots t_n$.
2. Let $d_j(t_i) = \#$ events in system j at t_i.
3. Let $d(t_i) = \sum_{j=1}^{m} d_j(t_i) = \#$ events in all systems at t_i.
4. Let $Y_j(t) = \begin{cases} 1 & \text{if system } j \text{ is under observation at time } t \\ 0 & \text{otherwise} \end{cases}$
5. Let $Y(t) = \sum_{j=1}^{m} Y_j(t) = \#$ systems under observation at time t .

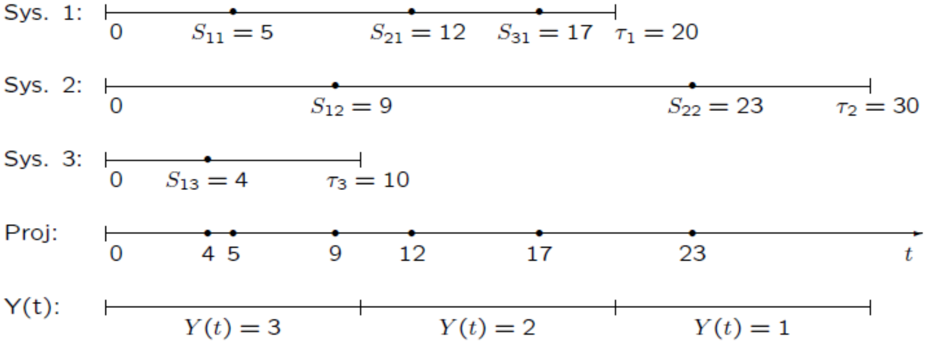
Then

$$\text{Under general assumptions: } \widehat{W}(t) = \sum_{t_i \leq t} \frac{d(t_i)}{Y(t_i)}.$$

$$\text{Assuming NHPP: } \widehat{\text{Var }\widehat{W}(t)} = \sum_{t_i \leq t} \frac{d(t_i)}{\{Y(t_i)\}^2}$$

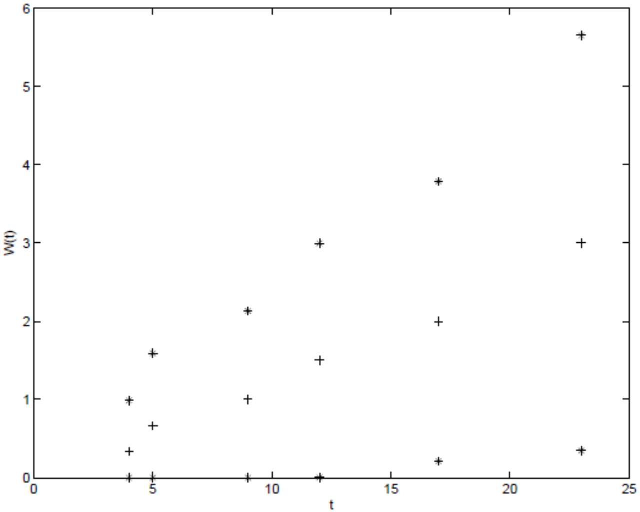
$$\text{Under general assumptions (MINITAB): } \widehat{\text{Var }\widehat{W}(t)} = \sum_{j=1}^{m} \left\{ \sum_{t_i \leq t} \frac{Y_j(t_i)}{Y(t_i)} \left[d_j(t_i) - \frac{d(t_i)}{Y(t_i)} \right] \right\}^2$$

SIMPLE EXAMPLE WITH THREE SYSTEMS

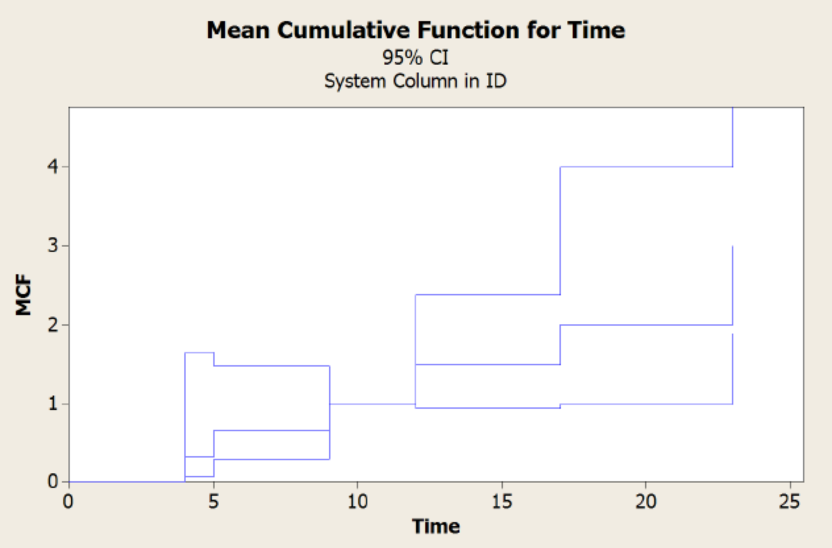


t	$1/Y(t)$	$1/Y(t)^2$	$\hat{W}(t)$	$\widehat{Var\hat{W}(t)}$	$\widehat{SD\hat{W}(t)}$
4	1/3	1/9	1/3	1/9	0.3333
5	1/3	1/9	2/3	2/9	0.4714
9	1/3	1/9	1	1/3	0.5774
12	1/2	1/4	3/2	7/12	0.7638
17	1/2	1/4	2	5/6	0.9129
23	1	1	3	11/6	1.3540

ESTIMATED $W(t)$ with 95% confidence limits (Nelson-Aalen)



ESTIMATED $W(t)$ WITH CONFIDENCE LIMITS (GENERAL)



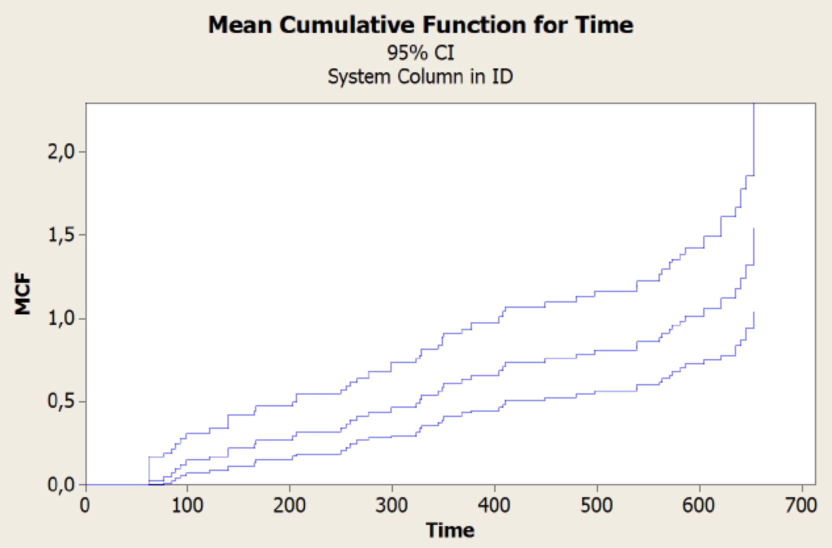
Compare with MINITAB Output:

$$\widehat{\text{Var}\ \widehat{W}(4)} = \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 + \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 + \left\{\frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 = \frac{6}{81} = 0.2722^2$$

$$\begin{aligned}\widehat{\text{Var}\ \widehat{W}(5)} &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\ &+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\ &= \frac{6}{81} = 0.2722^2\end{aligned}$$

$$
\begin{aligned}
\widehat{\text{Var}\ \widehat{W}(9)} &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\text{Var}\ \widehat{W}(12)} &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{2}\left[1-\frac{1}{2}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{2}\left[0-\frac{1}{2}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]+\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&= \frac{1}{8} = 0.3536^2
\end{aligned}
$$


Mean Cumulative Function for Time
95% CI
System Column in ID
MCF
2,0
1,5
1,0
0,5
0,0
0
100
200
300
400
500
600
700
Time

GRAMPUS DATA: ESTIMATION OF $W(t)$

Cumulative Number of Unscheduled Maintenance Actions Versus Operating Hours for a USS Grampus Diesel Engine Lee (1980)