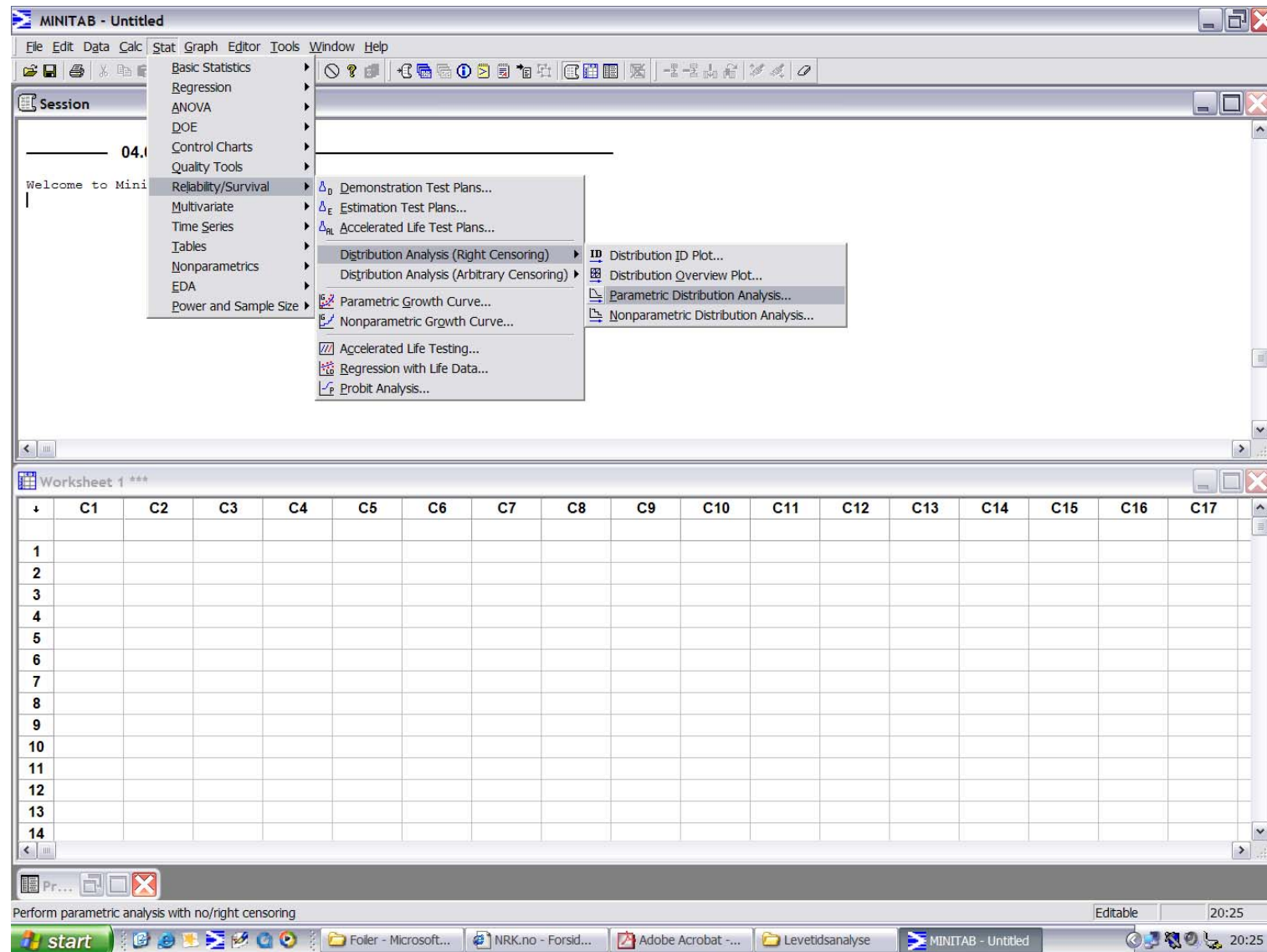


## Why parametric models?

- Complements nonparametric techniques.
- Parametric models can be described concisely with just a few parameters, instead of having to report an entire curve.
- It is possible to use a parametric model to extrapolate (in time) to the lower or upper tail of a distribution.
- Parametric models provide smooth estimates of failure-time distributions. In practice it is often useful to compare various parametric and nonparametric analyses of a data set.

# PARAMETRIC LIFETIME ANALYSIS IN MINITAB



## DATA OPTIONS

RIGHT CENSORING:

| $Y_i$         | $\delta_i$                                  |
|---------------|---|
| Observed time | Cens. status<br>1: Lifetime<br>0: Censoring |

ARBITRARY CENSORING:

| Start variable<br>$A_i$ | End variable<br>$B_i$ |  |
|-------------------------|-----------------------|--|
| 1.7                     | 1.7                   | Exact lifetime 1.7                                       |
| 2.0                     | *                     | Right censoring at time 2.0,<br>i.e. lifetime is $> 2.0$ |
| *                       | 0.5                   | Left censoring at time 0.5,<br>i.e. lifetime is $< 0.5$  |
| 1.0                     | 1.5                   | Interval censoring:<br>Lifetime between 1.0 and 1.5      |

## LIKELIHOOD CONTRIBUTION

| Obs. type      | Start variable<br>$A_i$ | End variable<br>$B_i$ | Likelihood contribution           |
|----------------|-------------------------|-----------------------|-----------------------------------|
| Exact lifetime | 1.7                     | 1.7                   | $f(1.7; \theta)$                  |
| Right cens.    | 2.0                     | *                     | $1 - F(2.0; \theta)$              |
| Left cens.     | *                       | 0.5                   | $F(0.5; \theta)$                  |
| Interval cens. | 1.0                     | 1.5                   | $F(1.5; \theta) - F(1.0; \theta)$ |

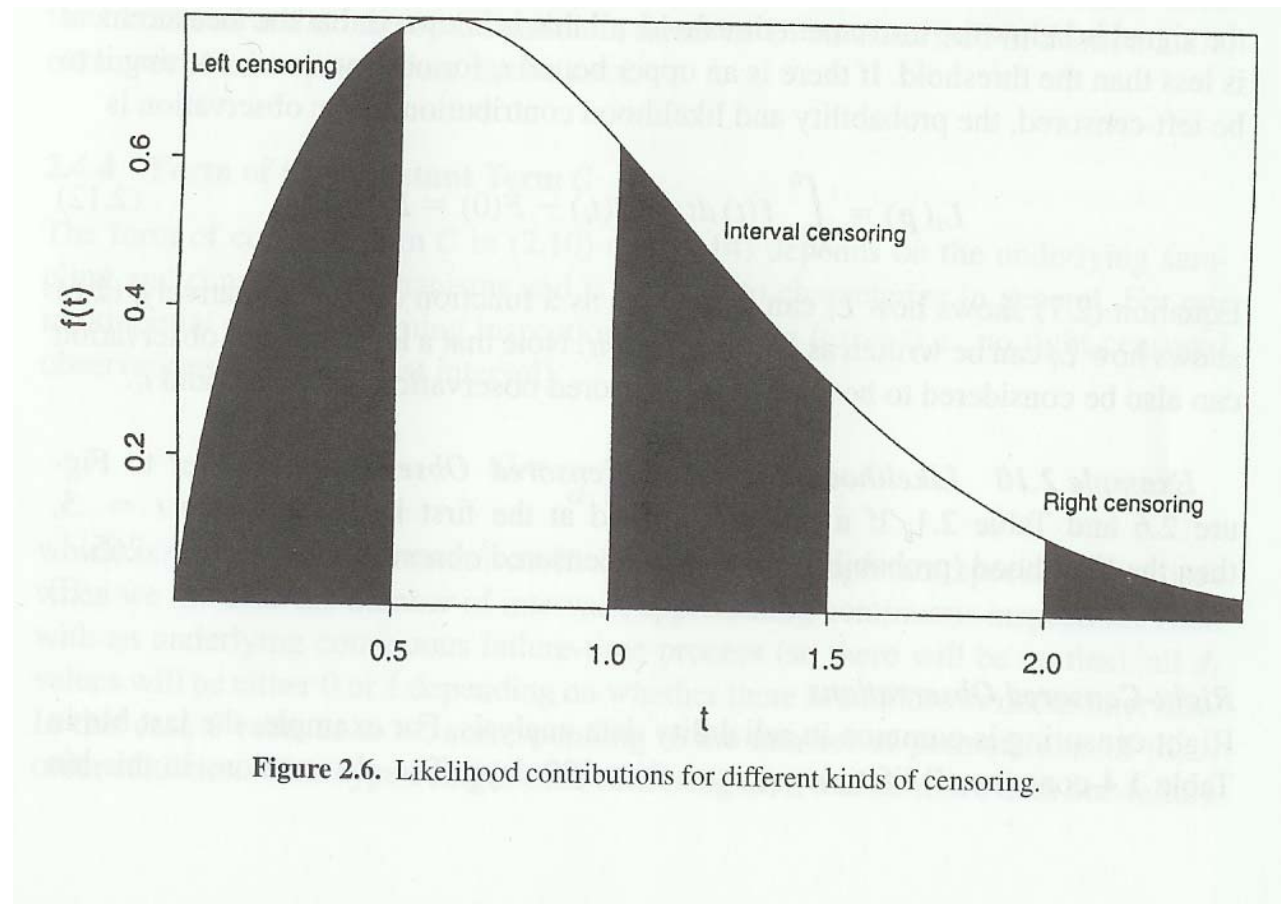
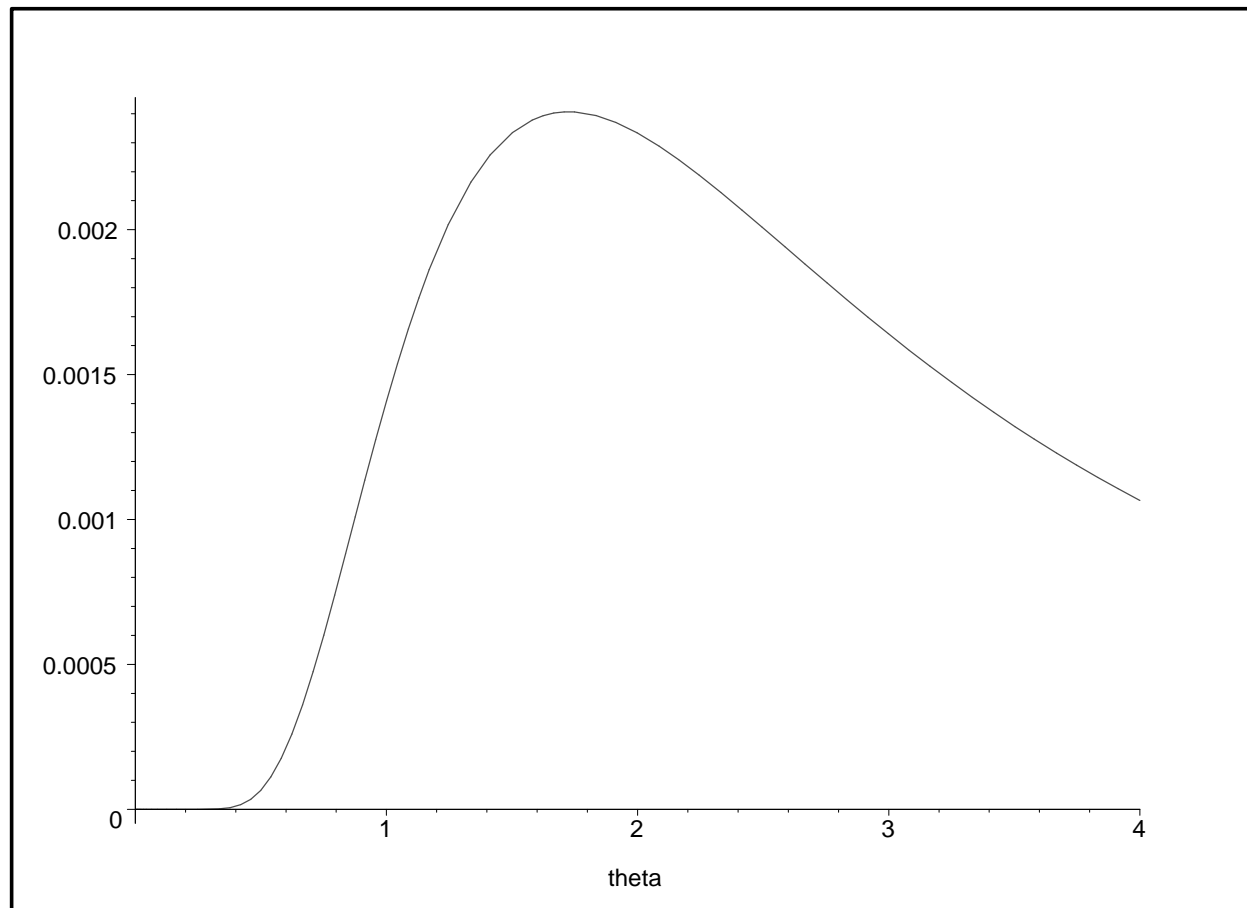


Figure 2.6. Likelihood contributions for different kinds of censoring.

LIKELIHOOD FOR MODEL  $f(t; \theta) = (1/\theta)e^{-t/\theta}$

$$L(\theta) = \left(\frac{1}{\theta}e^{-1.7/\theta}\right) \cdot (e^{-2.0/\theta}) \cdot (1 - e^{-0.5/\theta}) \cdot (e^{-1.0/\theta} - e^{-1.5/\theta})$$



Maximum likelihood estimate:  $\hat{\theta} = 1.725$

# ARBITRARY CENSORED DATA: MINITAB OUTPUT

## Distribution Analysis, Start = A and End = B

Variable Start: A End: B

| Censoring Information   | Count |
|-------------------------|-------|
| Uncensored value        | 1     |
| Right censored value    | 1     |
| Interval censored value | 1     |
| Left censored value     | 1     |

Estimation Method: Maximum Likelihood

Distribution: Exponential

### Parameter Estimates

| Parameter | Estimate | Standard Error | 95,0% Normal CI |         |
|-----------|----------|----------------|-----------------|---------|
|           |          |                | Lower           | Upper   |
| Mean      | 1,72529  | 0,998421       | 0,554978        | 5,36353 |

Log-Likelihood = -6,029

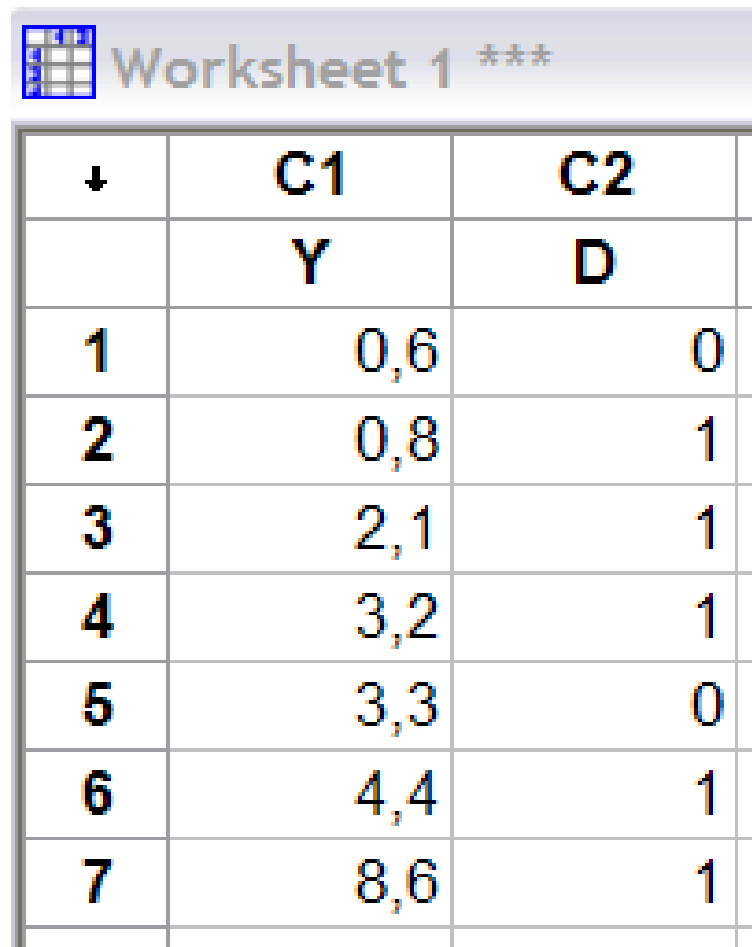
Goodness-of-Fit

Anderson-Darling (adjusted) = 4,933

### Characteristics of Distribution

|                          | Estimate | Standard Error | 95,0% Normal CI |         |
|--------------------------|----------|----------------|-----------------|---------|
|                          |          |                | Lower           | Upper   |
| Mean(MTTF)               | 1,72529  | 0,998421       | 0,554978        | 5,36353 |
| Standard Deviation       | 1,72529  | 0,998421       | 0,554978        | 5,36353 |
| Median                   | 1,19588  | 0,692053       | 0,384682        | 3,71771 |
| First Quartile(Q1)       | 0,496336 | 0,287228       | 0,159657        | 1,54299 |
| Third Quartile(Q3)       | 2,39177  | 1,38411        | 0,769363        | 7,43543 |
| Interquartile Range(IQR) | 1,89543  | 1,09688        | 0,609706        | 5,89244 |

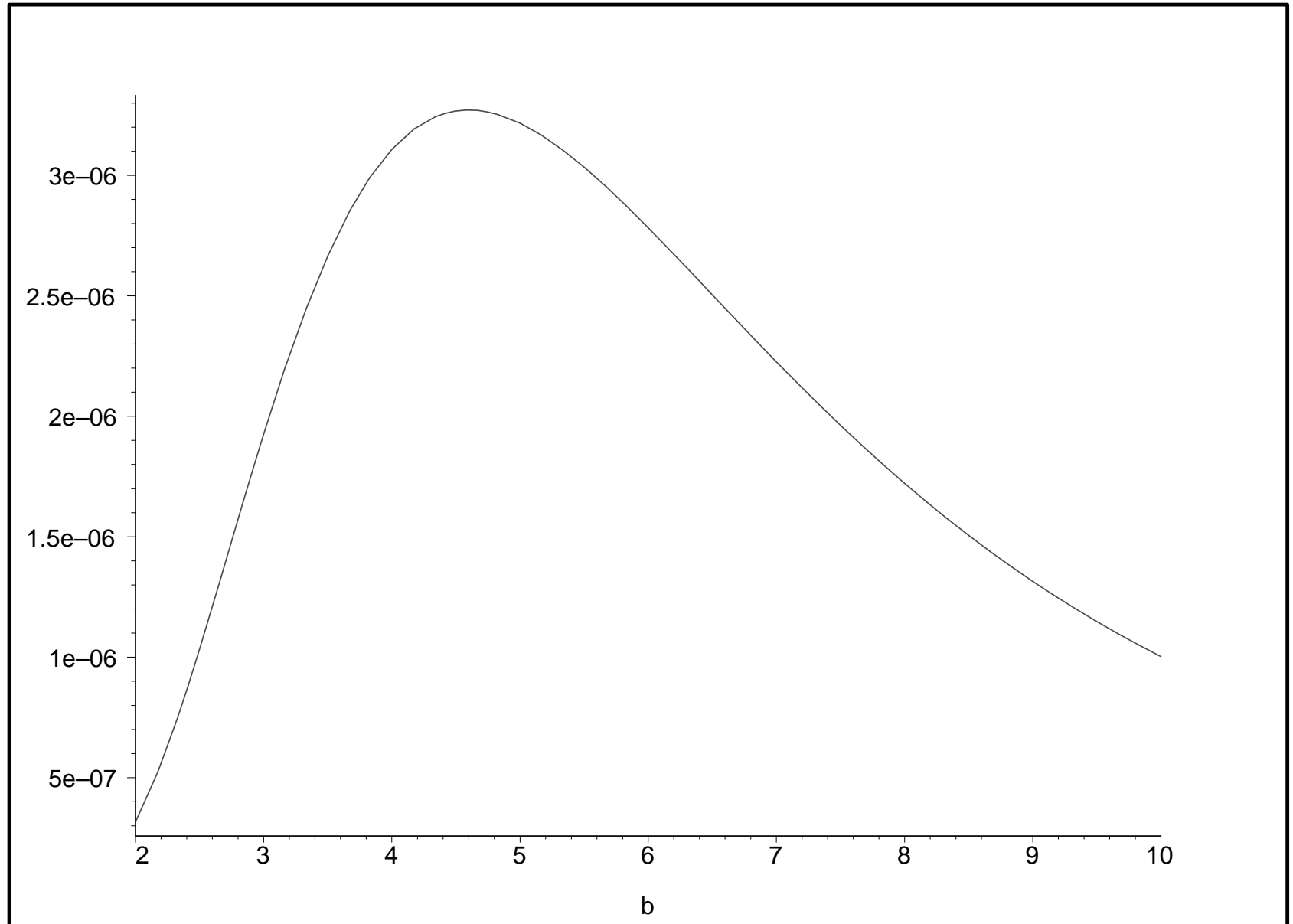
## EXAMPLE: RIGHT CENSORED DATA



The image shows a screenshot of a spreadsheet application window titled "Worksheet 1 \*\*\*". The spreadsheet contains a table with 7 rows of data. The first row is a header with columns labeled "C1" and "C2". The second row is a sub-header with columns labeled "Y" and "D". The subsequent rows contain numerical values for "Y" and "D".

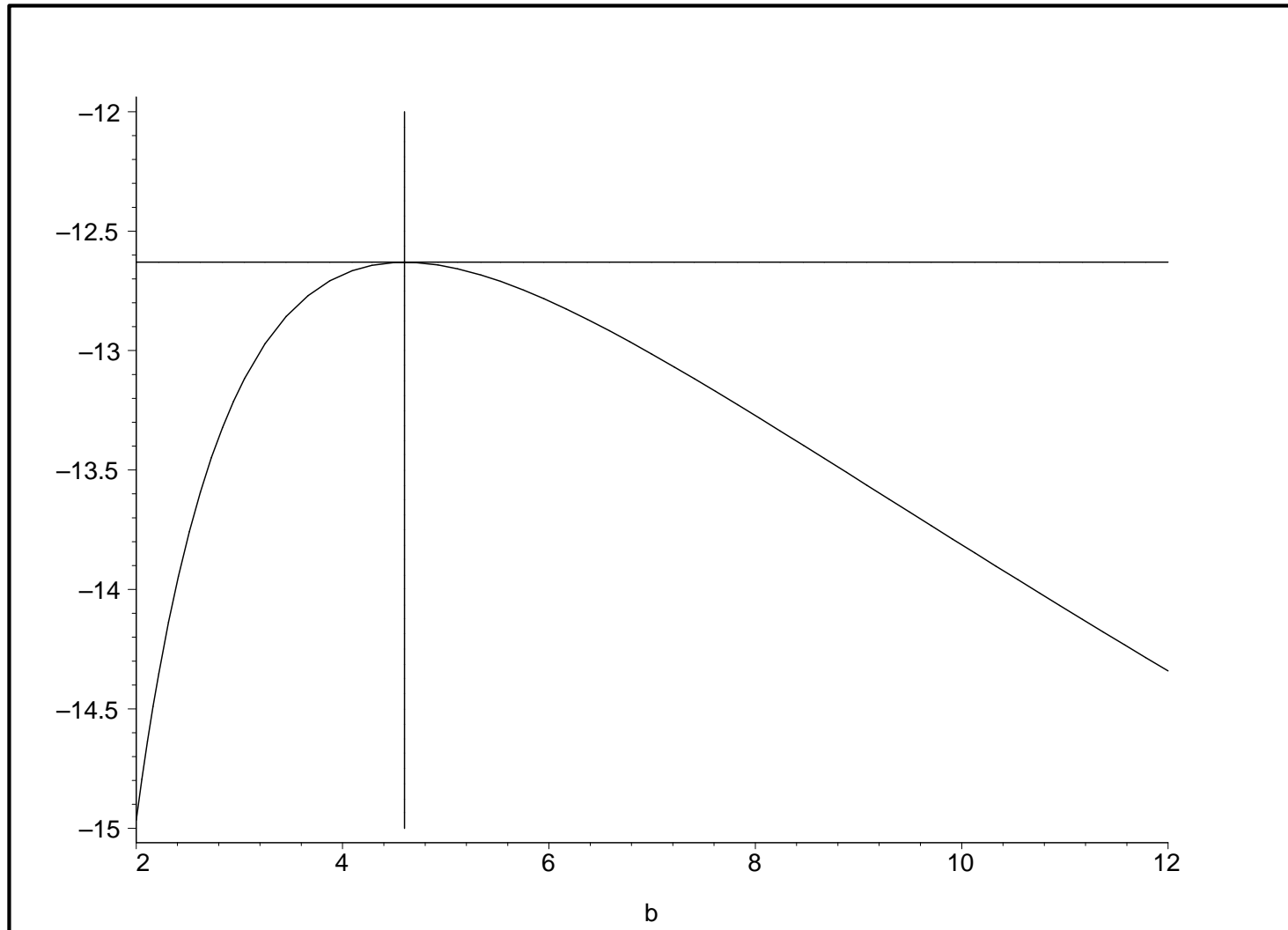
|          | <b>C1</b> | <b>C2</b> |
|----------|-----------|-----------|
|          | <b>Y</b>  | <b>D</b>  |
| <b>1</b> | 0,6       | 0         |
| <b>2</b> | 0,8       | 1         |
| <b>3</b> | 2,1       | 1         |
| <b>4</b> | 3,2       | 1         |
| <b>5</b> | 3,3       | 0         |
| <b>6</b> | 4,4       | 1         |
| <b>7</b> | 8,6       | 1         |

LIKELIHOOD FUNCTION FOR MODEL  $f(t; \theta) = (1/\theta)e^{-t/\theta}$





# LOG-LIKELIHOOD FUNCTION



Maximum likelihood estimate:  $\hat{\theta} = 4.6$

## Distribution Analysis: Y

Variable: Y

Censoring Information    Count  
Uncensored value            5  
Right censored value        2

Censoring value: D = 0

Estimation Method: Maximum Likelihood

Distribution:    Exponential

### Parameter Estimates

| Parameter | Estimate | Standard Error | 95,0% Normal CI |         |
|-----------|----------|----------------|-----------------|---------|
|           |          |                | Lower           | Upper   |
| Mean      | 4,6      | 2,05718        | 1,91465         | 11,0516 |

Log-Likelihood = -12,630

Goodness-of-Fit

Anderson-Darling (adjusted) = 3,767

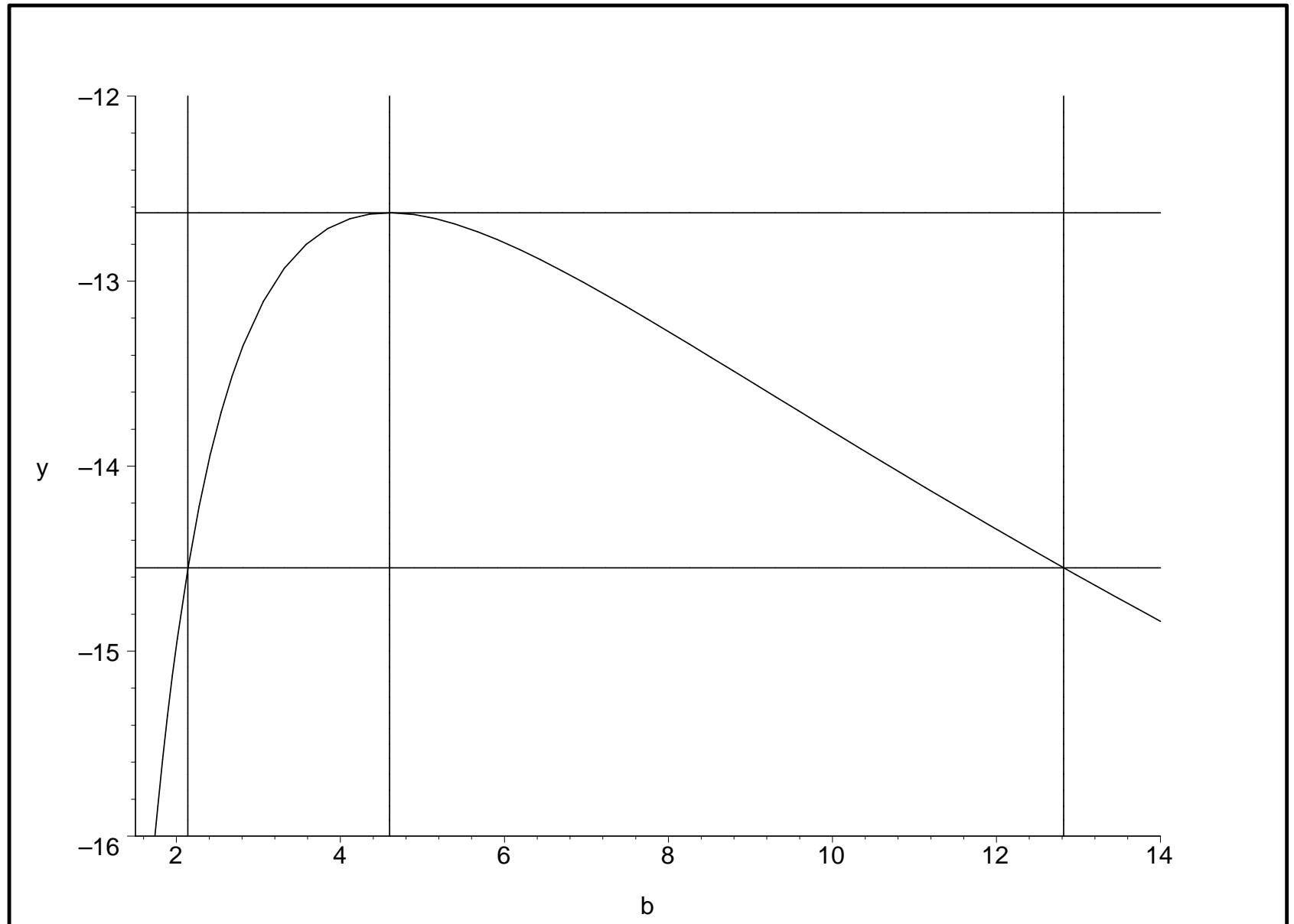
### Characteristics of Distribution

|                           | Estimate | Standard Error | 95,0% Normal CI |         |
|---------------------------|----------|----------------|-----------------|---------|
|                           |          |                | Lower           | Upper   |
| Mean (MTTF)               | 4,6      | 2,05718        | 1,91465         | 11,0516 |
| Standard Deviation        | 4,6      | 2,05718        | 1,91465         | 11,0516 |
| Median                    | 3,18848  | 1,42593        | 1,32713         | 7,66041 |
| First Quartile (Q1)       | 1,32334  | 0,591815       | 0,550810        | 3,17936 |
| Third Quartile (Q3)       | 6,37695  | 2,85186        | 2,65427         | 15,3208 |
| Interquartile Range (IQR) | 5,05362  | 2,26005        | 2,10346         | 12,1415 |

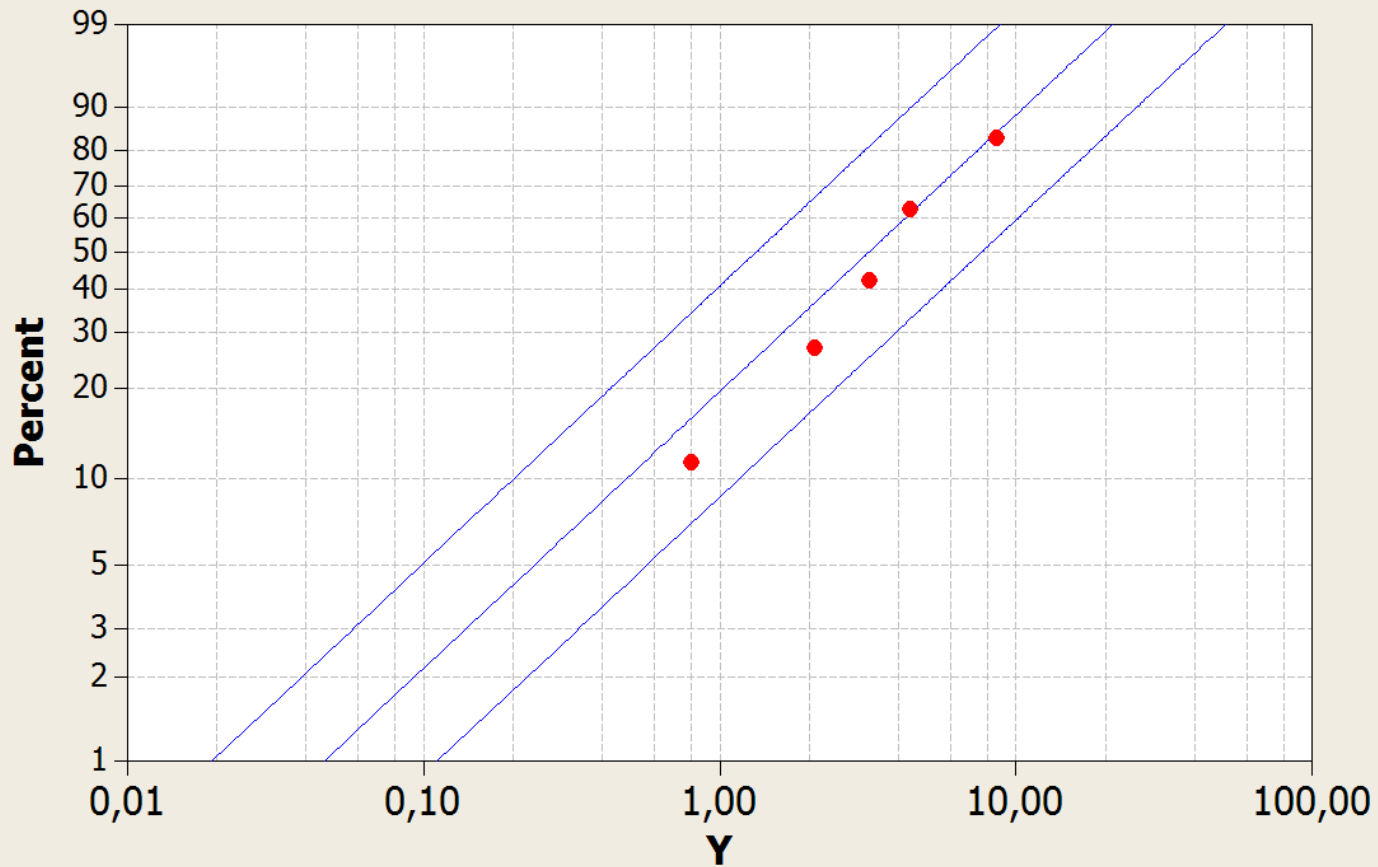
Table of Percentiles

| Percent | Percentile | Standard<br>Error | 95,0% Normal CI |          |
|---------|------------|-------------------|-----------------|----------|
|         |            |                   | Lower           | Upper    |
| 1       | 0,0462315  | 0,0206754         | 0,0192429       | 0,111073 |
| 2       | 0,0929325  | 0,0415607         | 0,0386811       | 0,223273 |
| 3       | 0,140112   | 0,0626601         | 0,0583187       | 0,336624 |
| 4       | 0,187781   | 0,0839783         | 0,0781597       | 0,451150 |
| 5       | 0,235949   | 0,105520          | 0,0982086       | 0,566875 |
| 6       | 0,284627   | 0,127289          | 0,118470        | 0,683825 |
| 7       | 0,333825   | 0,149291          | 0,138947        | 0,802025 |
| 8       | 0,383555   | 0,171531          | 0,159646        | 0,921504 |
| 9       | 0,433829   | 0,194014          | 0,180572        | 1,04229  |
| 10      | 0,484658   | 0,216746          | 0,201728        | 1,16441  |
| 20      | 1,02646    | 0,459047          | 0,427241        | 2,46610  |
| 30      | 1,64070    | 0,733745          | 0,682907        | 3,94184  |
| 40      | 2,34980    | 1,05086           | 0,978051        | 5,64546  |
| 50      | 3,18848    | 1,42593           | 1,32713         | 7,66041  |
| 60      | 4,21494    | 1,88498           | 1,75437         | 10,1265  |
| 70      | 5,53827    | 2,47679           | 2,30518         | 13,3059  |
| 80      | 7,40341    | 3,31091           | 3,08151         | 17,7869  |
| 90      | 10,5919    | 4,73684           | 4,40864         | 25,4473  |
| 91      | 11,0765    | 4,95358           | 4,61037         | 26,6117  |
| 92      | 11,6184    | 5,19588           | 4,83588         | 27,9134  |
| 93      | 12,2326    | 5,47058           | 5,09155         | 29,3892  |
| 94      | 12,9417    | 5,78770           | 5,38669         | 31,0928  |
| 95      | 13,7804    | 6,16277           | 5,73577         | 33,1078  |
| 96      | 14,8068    | 6,62182           | 6,16301         | 35,5739  |
| 97      | 16,1302    | 7,21363           | 6,71382         | 38,7532  |
| 98      | 17,9953    | 8,04775           | 7,49015         | 43,2343  |
| 99      | 21,1838    | 9,47368           | 8,81728         | 50,8947  |

# LOG-LIKELIHOOD FUNCTION



**Probability Plot for Y**  
Exponential - 95% CI  
Censoring Column in D - ML Estimates

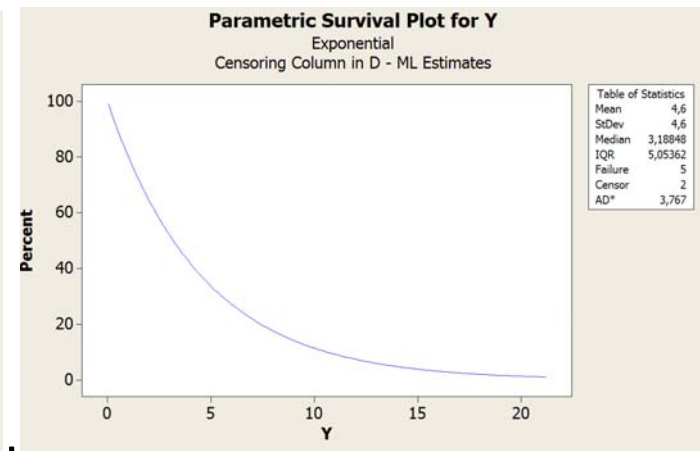
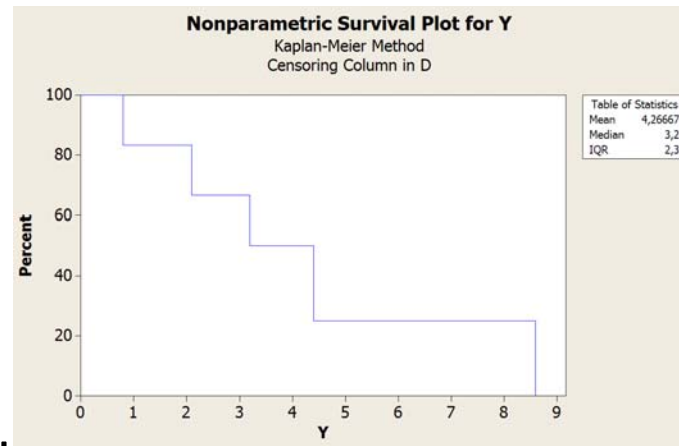


| Table of Statistics |         |
|---------------------|---------|
| Mean                | 4,6     |
| StDev               | 4,6     |
| Median              | 3,18848 |
| IQR                 | 5,05362 |
| Failure             | 5       |
| Censor              | 2       |
| AD*                 | 3,767   |

# Nonparametric and parametric survival plots for data from exponential example

Worksheet 1 \*\*\*

| ↓ | C1  | C2 |
|---|-----|----|
|   | Y   | D  |
| 1 | 0,6 | 0  |
| 2 | 0,8 | 1  |
| 3 | 2,1 | 1  |
| 4 | 3,2 | 1  |
| 5 | 3,3 | 0  |
| 6 | 4,4 | 1  |
| 7 | 8,6 | 1  |



RIGHT CENSORED DATA  
 EXPONENTIAL AND WEIBULL MODELS

| Row | C1   | C2 |
|-----|------|----|
| 1   | 0,35 | 1  |
| 2   | 0,50 | 0  |
| 3   | 0,75 | 0  |
| 4   | 1,00 | 1  |
| 5   | 1,30 | 1  |
| 6   | 1,80 | 1  |
| 7   | 3,00 | 0  |
| 8   | 3,15 | 0  |
| 9   | 4,85 | 0  |
| 10  | 5,50 | 1  |
| 11  | 5,50 | 0  |
| 12  | 6,25 | 0  |

Variable: C1  
 Censoring Information  
 Count  
 Uncensored value 5  
 Right censored value 7  
 Censoring value: C2 = 0

Estimation Method: Maximum Likelihood Distribution: Exponential

Parameter Estimates

| Parameter | Estimate | Standard Error | 95,0% Normal CI |        |
|-----------|----------|----------------|-----------------|--------|
|           |          |                | Lower           | Upper  |
| Shape     | 1,00000  |                |                 |        |
| Scale     | 6,790    | 3,037          | 2,826           | 16,313 |

Log-Likelihood = -14,577

# WEIBULL MODEL

Variable: C1

Censoring Information

Count

Uncensored value 5

Right censored value 7

Censoring value: C2 = 0

Estimation Method: Maximum Likelihood Distribution: Weibull

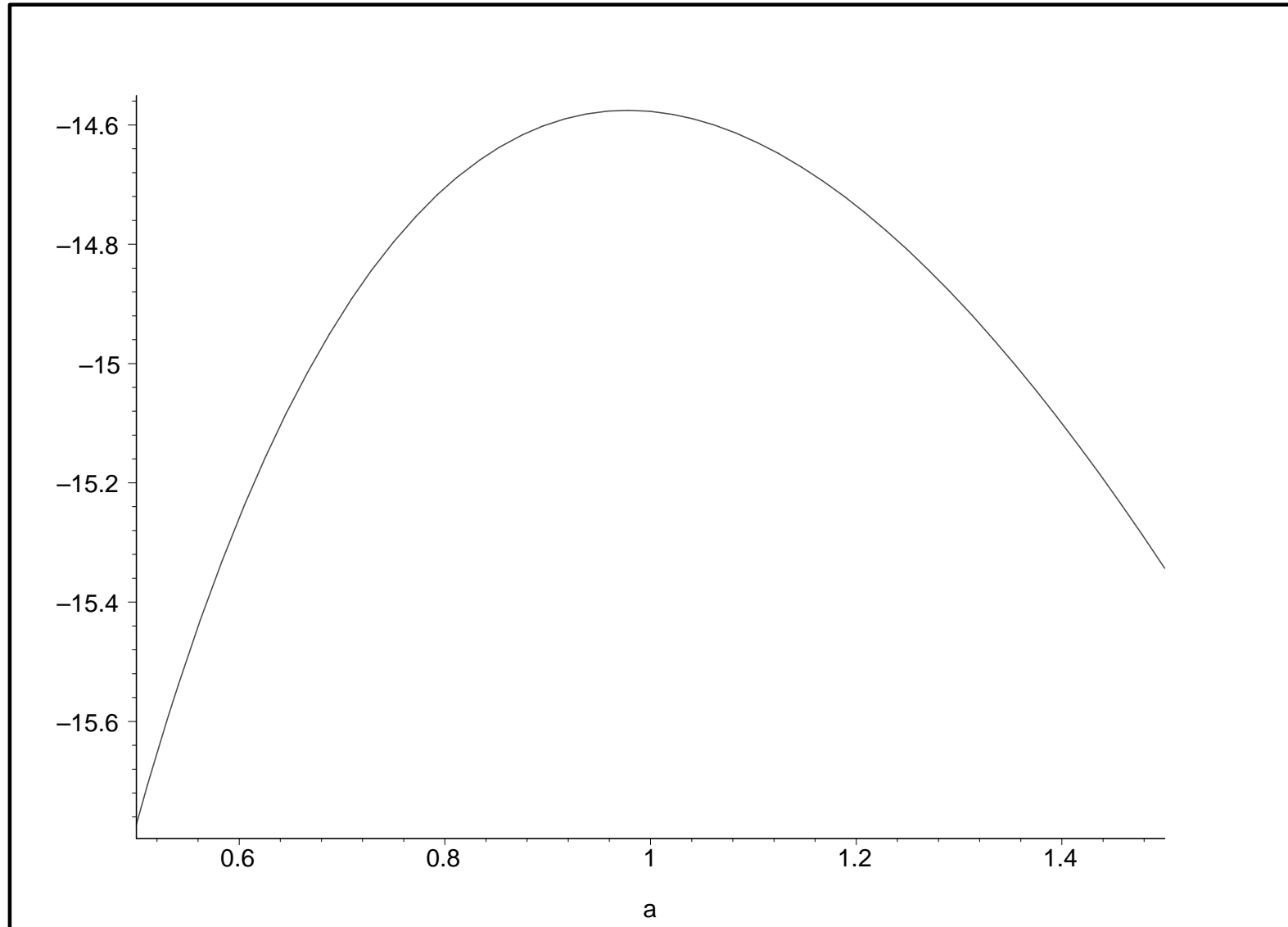
## Parameter Estimates

| Parameter | Estimate | Standard<br>Error | 95,0% Normal CI |        |
|-----------|----------|-------------------|-----------------|--------|
|           |          |                   | Lower           | Upper  |
| Shape     | 0,9780   | 0,3694            | 0,4665          | 2,0504 |
| Scale     | 6,880    | 3,517             | 2,526           | 18,740 |

Log-Likelihood = -14,576



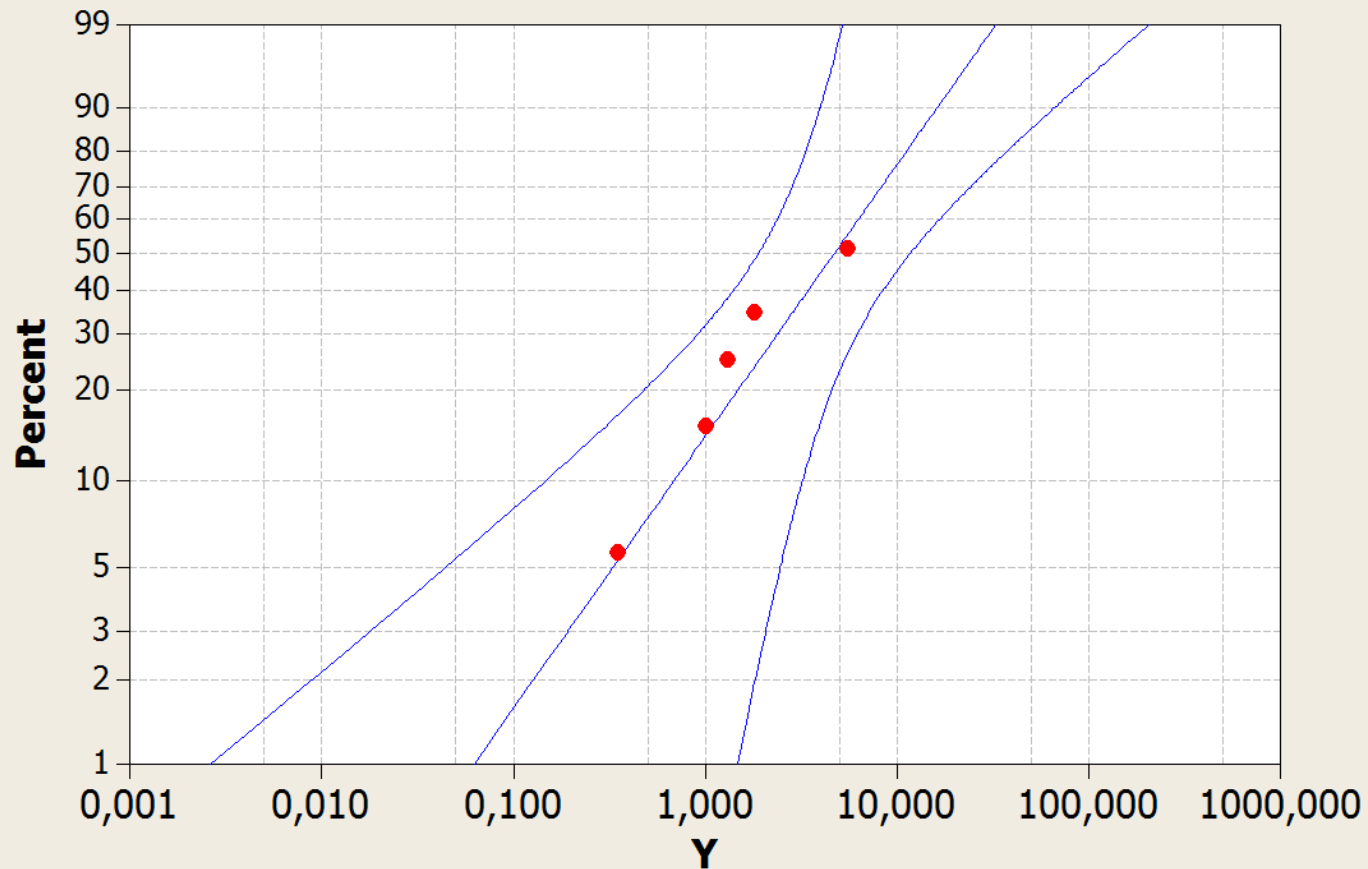
# PROFILE LOG LIKELIHOOD FOR SHAPE PARAMETER $\alpha$ IN WEIBULL DISTRIBUTION



# Probability Plot for Y

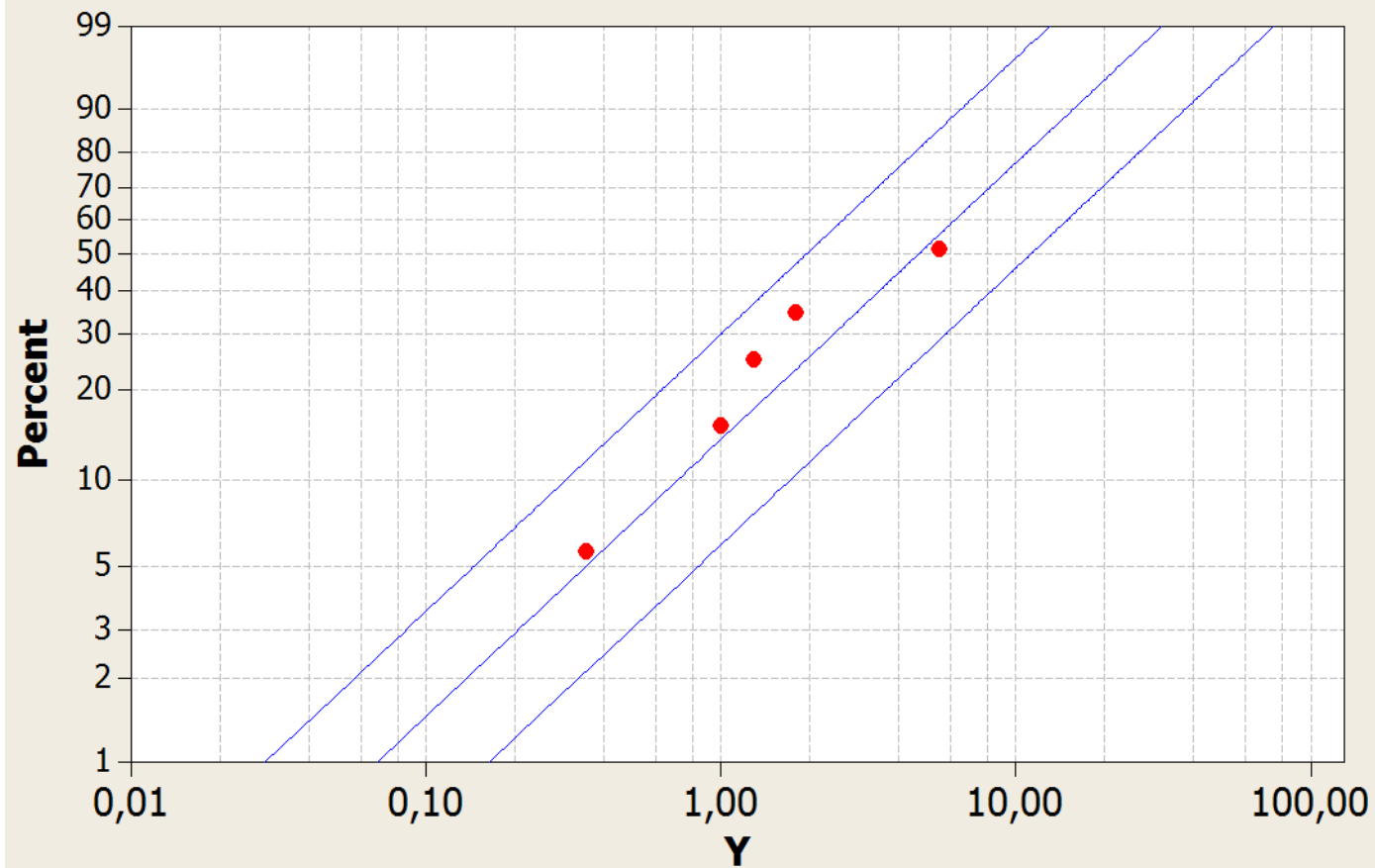
Weibull - 95% CI

Censoring Column in D - ML Estimates



| Table of Statistics |          |
|---------------------|----------|
| Shape               | 0,977997 |
| Scale               | 6,88032  |
| Mean                | 6,94720  |
| StDev               | 7,10402  |
| Median              | 4,72991  |
| IQR                 | 7,68386  |
| Failure             | 5        |
| Censor              | 7        |
| AD*                 | 30,049   |

**Probability Plot for Y**  
Exponential - 95% CI  
Censoring Column in D - ML Estimates



| Table of Statistics |         |
|---------------------|---------|
| Mean                | 6,79    |
| StDev               | 6,79    |
| Median              | 4,70647 |
| IQR                 | 7,45958 |
| Failure             | 5       |
| Censor              | 7       |
| AD*                 | 30,051  |

# Shock absorber data

Y = kilometers to failure, F = failure mode (0 is censoring)

| Row | Y     | F |    |       |   |
|-----|-------|---|----|-------|---|
|     |       |   | 19 | 14300 | 1 |
| 1   | 6700  | 1 | 20 | 17520 | 1 |
| 2   | 6950  | 0 | 21 | 17540 | 0 |
| 3   | 7820  | 0 | 22 | 17890 | 0 |
| 4   | 8790  | 0 | 23 | 18450 | 0 |
| 5   | 9120  | 2 | 24 | 18960 | 0 |
| 6   | 9660  | 0 | 25 | 18980 | 0 |
| 7   | 9820  | 0 | 26 | 19410 | 0 |
| 8   | 11310 | 0 | 27 | 20100 | 2 |
| 9   | 11690 | 0 | 28 | 20100 | 0 |
| 10  | 11850 | 0 | 29 | 20150 | 0 |
| 11  | 11880 | 0 | 30 | 20320 | 0 |
| 12  | 12140 | 0 | 31 | 20900 | 2 |
| 13  | 12200 | 1 | 32 | 22700 | 1 |
| 14  | 12870 | 0 | 33 | 23490 | 0 |
| 15  | 13150 | 2 | 34 | 26510 | 1 |
| 16  | 13330 | 0 | 35 | 27410 | 0 |
| 17  | 13470 | 0 | 36 | 27490 | 1 |
| 18  | 14040 | 0 | 37 | 27890 | 0 |
|     |       |   | 38 | 28100 | 0 |

## Shock Absorber Failure Data

First reported in O'Connor (1985).

- Failure times, in number of kilometers of use, of vehicle shock absorbers.
- Two failure modes, denoted by M1 and M2.
- One might be interested in the distribution of time to failure for mode M1, mode M2, or in the overall failure-time distribution of the part.

Here we do not differentiate between modes M1 and M2. We will estimate the distribution of time to failure by either mode M1 or M2.

## Shock absorber data

Estimation Method: Maximum Likelihood Distribution: Lognormal

### Parameter Estimates

| Parameter | Estimate | Standard Error | 95,0% Normal CI |          |
|-----------|----------|----------------|-----------------|----------|
|           |          |                | Lower           | Upper    |
| Location  | 10,1448  | 0,144175       | 9,86219         | 10,4273  |
| Scale     | 0,530068 | 0,112683       | 0,349447        | 0,804047 |

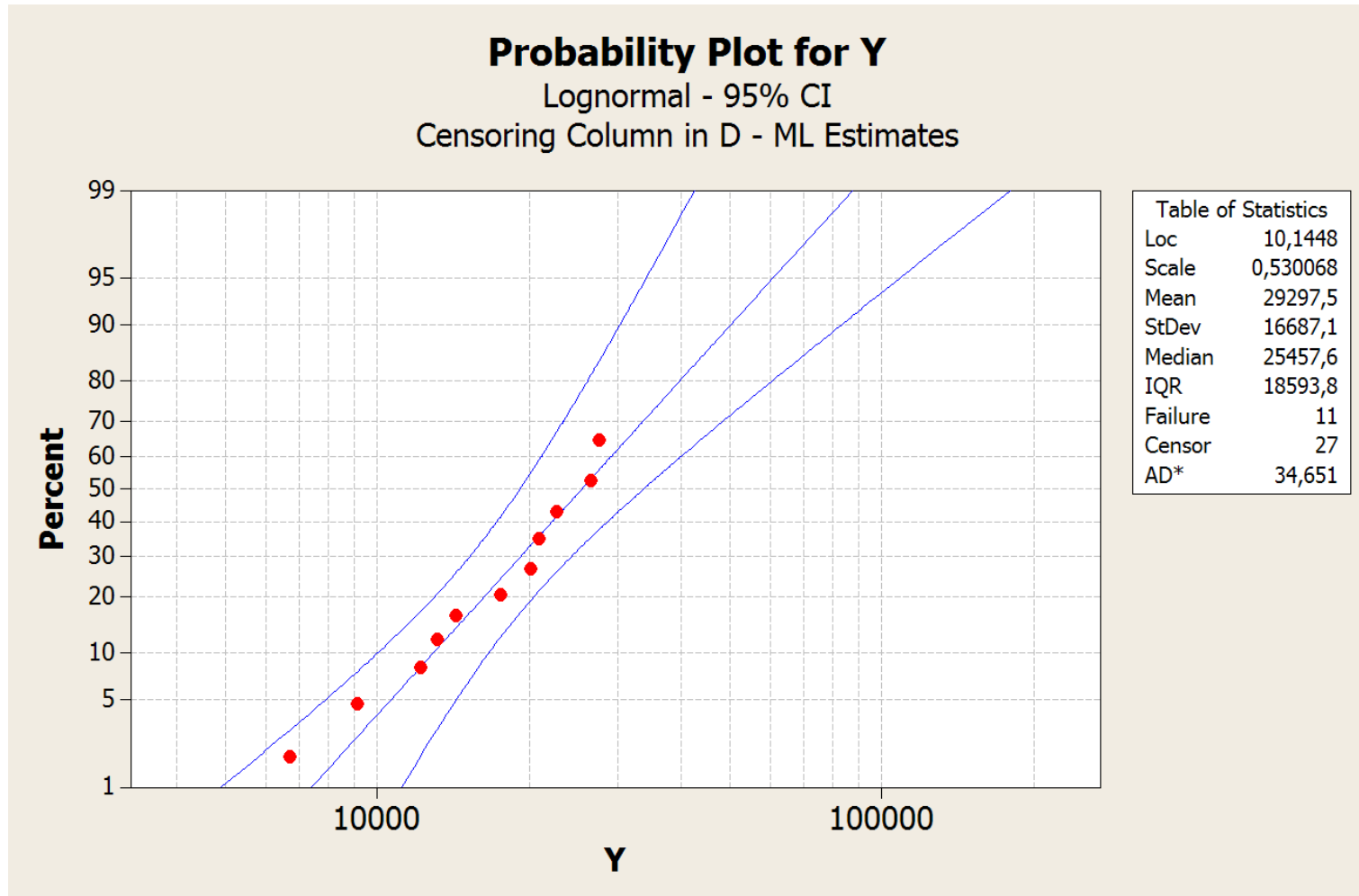
Log-Likelihood = -124,609

Goodness-of-Fit Anderson-Darling (adjusted) = 34,651

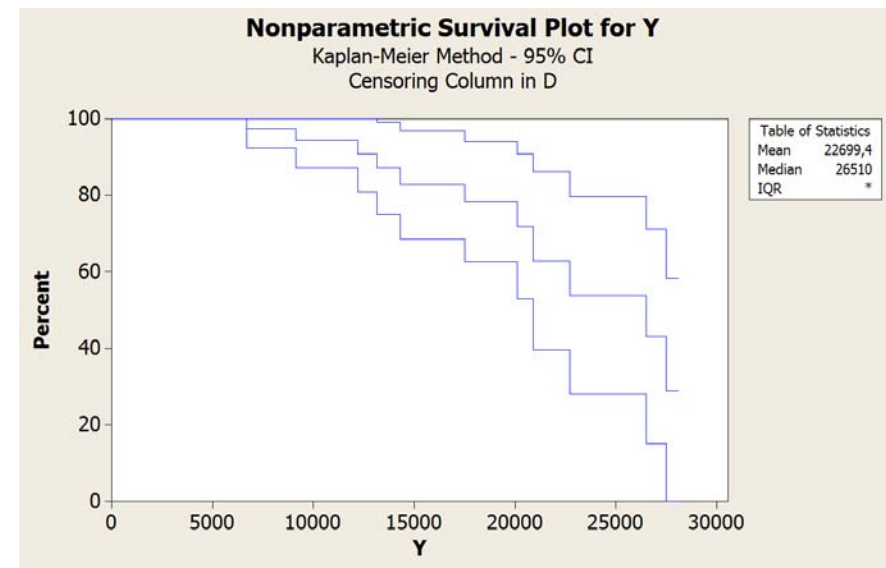
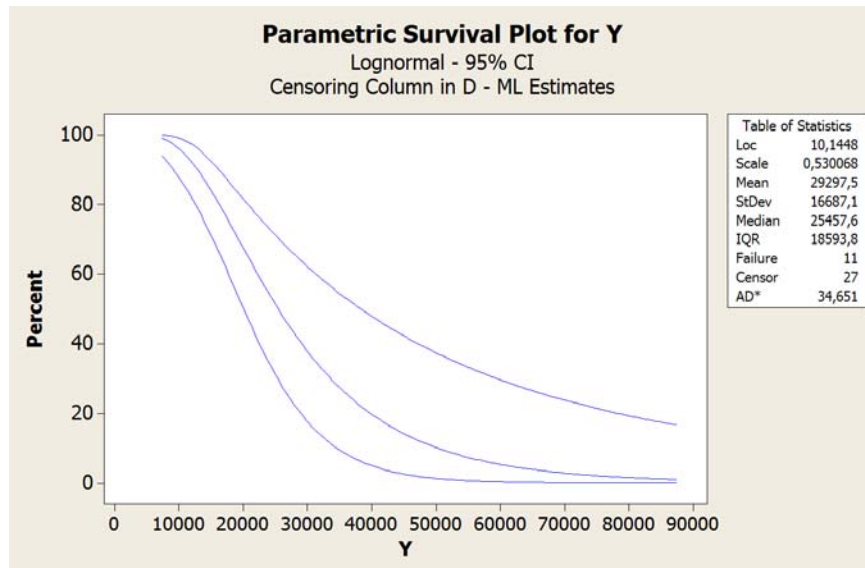
### Characteristics of Distribution

|                          | Estimate | Standard Error | 95,0% Normal CI |         |
|--------------------------|----------|----------------|-----------------|---------|
|                          |          |                | Lower           | Upper   |
| Mean(MTTF)               | 29297,5  | 5455,91        | 20338,3         | 42203,2 |
| Standard Deviation       | 16687,1  | 6787,01        | 7519,35         | 37032,5 |
| Median                   | 25457,6  | 3670,36        | 19190,9         | 33770,7 |
| First Quartile(Q1)       | 17805,2  | 2062,96        | 14188,1         | 22344,4 |
| Third Quartile(Q3)       | 36399,0  | 7252,61        | 24631,2         | 53789,0 |
| Interquartile Range(IQR) | 18593,8  | 6115,60        | 9758,96         | 35426,9 |

# Shock absorber data

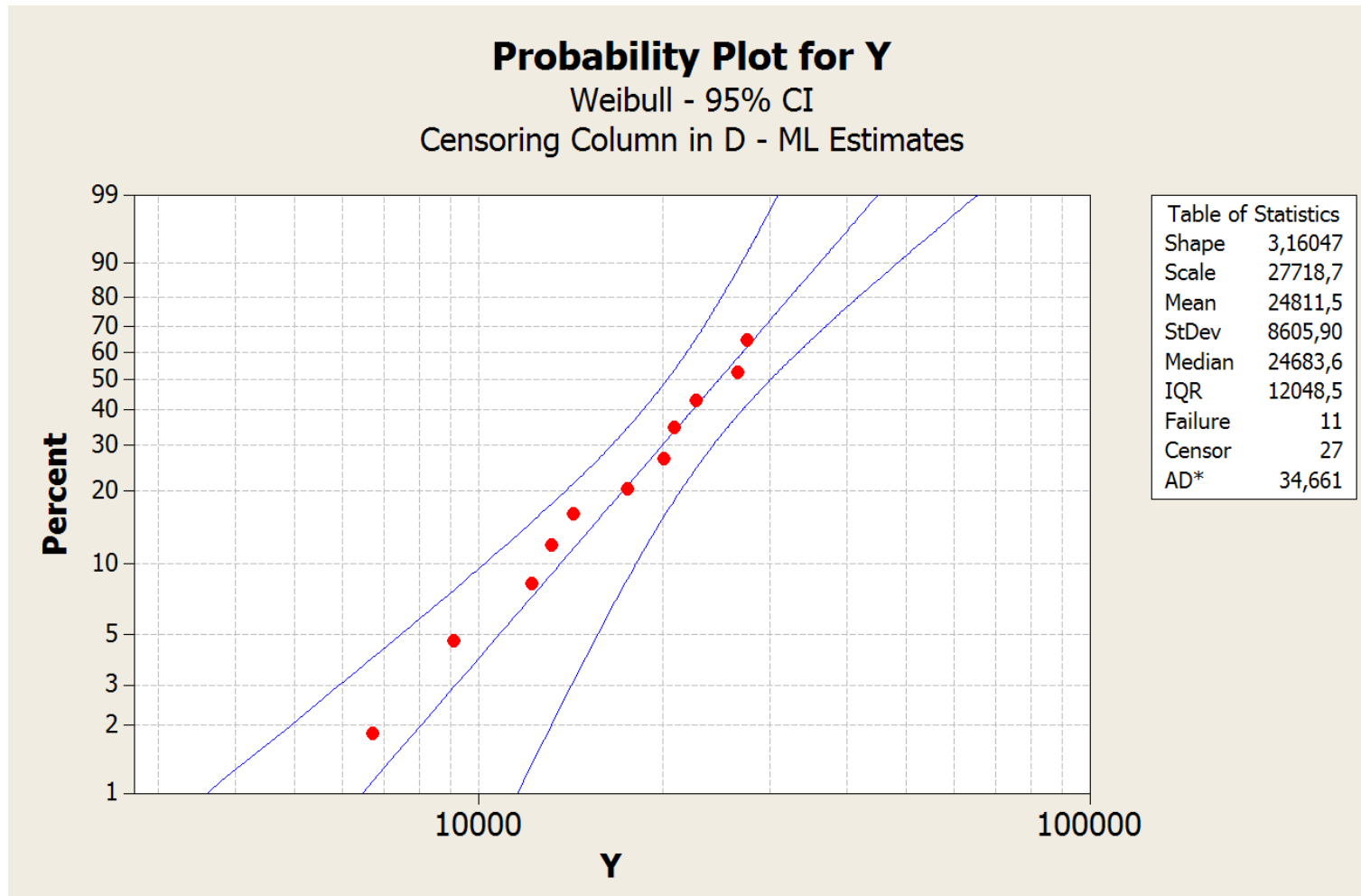


# Shock absorber data

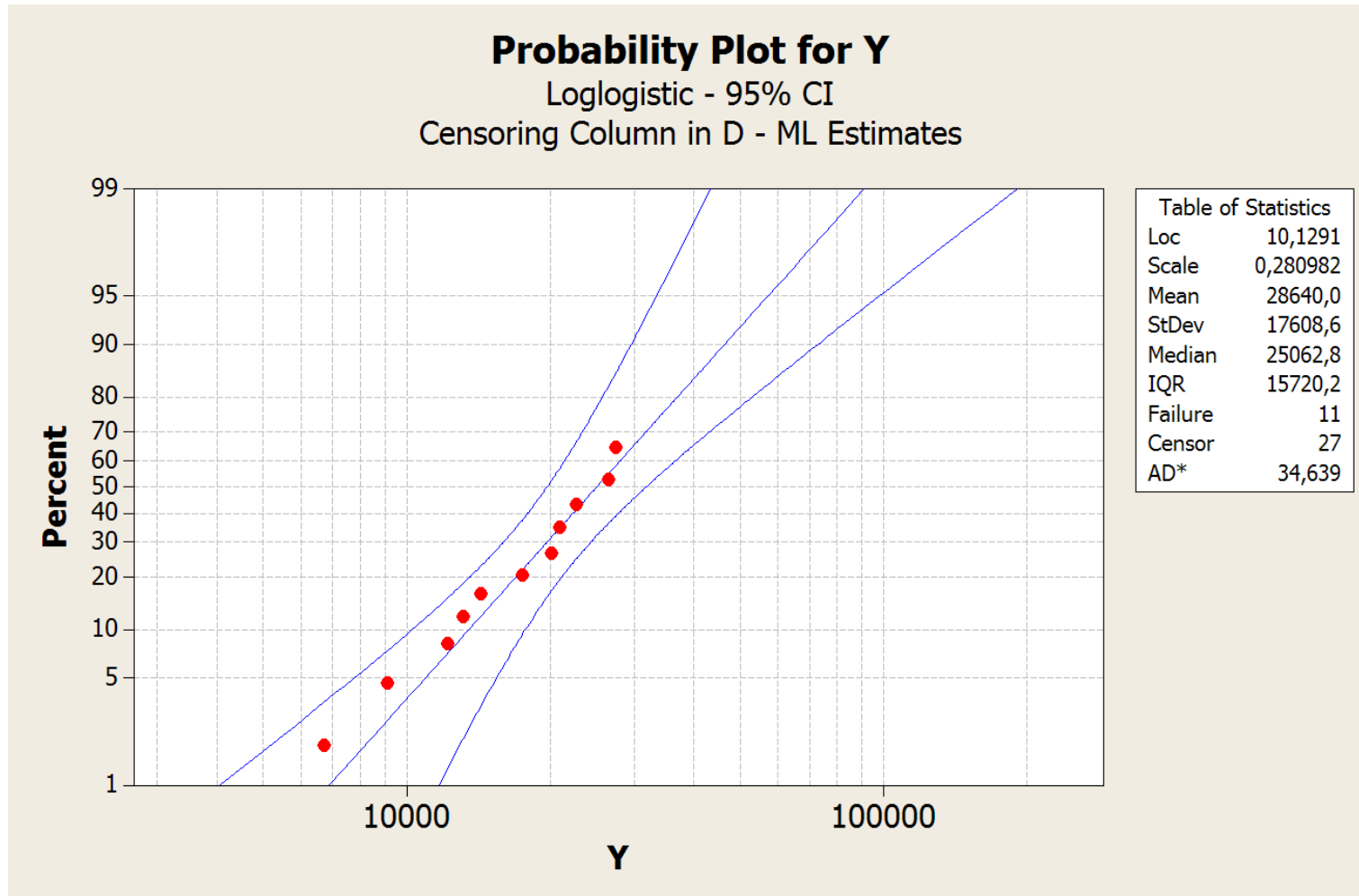




# Shock absorber data



# Shock absorber data



Shock absorber data:

Results for loglogistic (left), lognormal (middle), Weibull (right)

| Table of Statistics |          |
|---------------------|----------|
| Loc                 | 10,1291  |
| Scale               | 0,280982 |
| Mean                | 28640,0  |
| StDev               | 17608,6  |
| Median              | 25062,8  |
| IQR                 | 15720,2  |
| Failure             | 11       |
| Censor              | 27       |
| AD*                 | 34,639   |

| Table of Statistics |          |
|---------------------|----------|
| Loc                 | 10,1448  |
| Scale               | 0,530068 |
| Mean                | 29297,5  |
| StDev               | 16687,1  |
| Median              | 25457,6  |
| IQR                 | 18593,8  |
| Failure             | 11       |
| Censor              | 27       |
| AD*                 | 34,651   |

| Table of Statistics |         |
|---------------------|---------|
| Shape               | 3,16047 |
| Scale               | 27718,7 |
| Mean                | 24811,5 |
| StDev               | 8605,90 |
| Median              | 24683,6 |
| IQR                 | 12048,5 |
| Failure             | 11      |
| Censor              | 27      |
| AD*                 | 34,661  |

## Confidence Interval for the Mean Life of a New Insulating Material

- A life test for a new insulating material used 25 specimens which were tested simultaneously at a high voltage of 30 kV.
- The test was run until 15 of the specimens failed.
- The 15 failure times (hours) were recorded as:

1.08, 12.20, 17.80, 19.10, 26.00, 27.90, 28.20, 32.20, 35.90, 43.50, 44.00, 45.20, 45.70, 46.30, 47.80

Then  $TTT = 1.08 + \dots + 47.80 + 10 \times 47.80 = 950.88$  hours.

- The ML estimate of  $\theta$  and a 95% confidence interval are:

$$\begin{aligned}\hat{\theta} &= 950.88/15 = 63.392 \text{ hours} \\ \left[ \underline{\theta}, \tilde{\theta} \right] &= \left[ \frac{2(950.88)}{\chi_{(.975;30)}^2}, \frac{2(950.88)}{\chi_{(.025;30)}^2} \right] = \left[ \frac{1901.76}{46.98}, \frac{1901.76}{16.79} \right] \\ &= [40.48, 113.26].\end{aligned}$$

# Pike (1966) cancer data for rats

| Row | Y   | D |
|-----|-----|---|
| 1   | 143 | 1 |
| 2   | 164 | 1 |
| 3   | 188 | 1 |
| 4   | 188 | 1 |
| 5   | 190 | 1 |
| 6   | 192 | 1 |
| 7   | 206 | 1 |
| 8   | 209 | 1 |
| 9   | 213 | 1 |
| 10  | 216 | 1 |
| 11  | 220 | 1 |
| 12  | 227 | 1 |
| 13  | 230 | 1 |
| 14  | 234 | 1 |
| 15  | 246 | 1 |
| 16  | 265 | 1 |
| 17  | 304 | 1 |
| 18  | 216 | 0 |
| 19  | 244 | 0 |

# Pike (1966) cancer data for rats: 3-parameter Weibull

## Distribution Analysis: C1

Variable: C1

| Censoring Information | Count |
|-----------------------|-------|
| Uncensored value      | 17    |
| Right censored value  | 2     |

Censoring value: C2 = 0

Estimation Method: Maximum Likelihood

Distribution: 3-Parameter Weibull

### Parameter Estimates

| Parameter | Estimate | Standard Error | 95,0% Normal CI |         |
|-----------|----------|----------------|-----------------|---------|
|           |          |                | Lower           | Upper   |
| Shape     | 2,71148  | 1,05876        | 1,26135         | 5,82878 |
| Scale     | 108,383  | 32,5734        | 60,1367         | 195,335 |
| Threshold | 122,026  | 28,6924        | 65,7898         | 178,262 |

Log-Likelihood = -87,324

Goodness-of-Fit

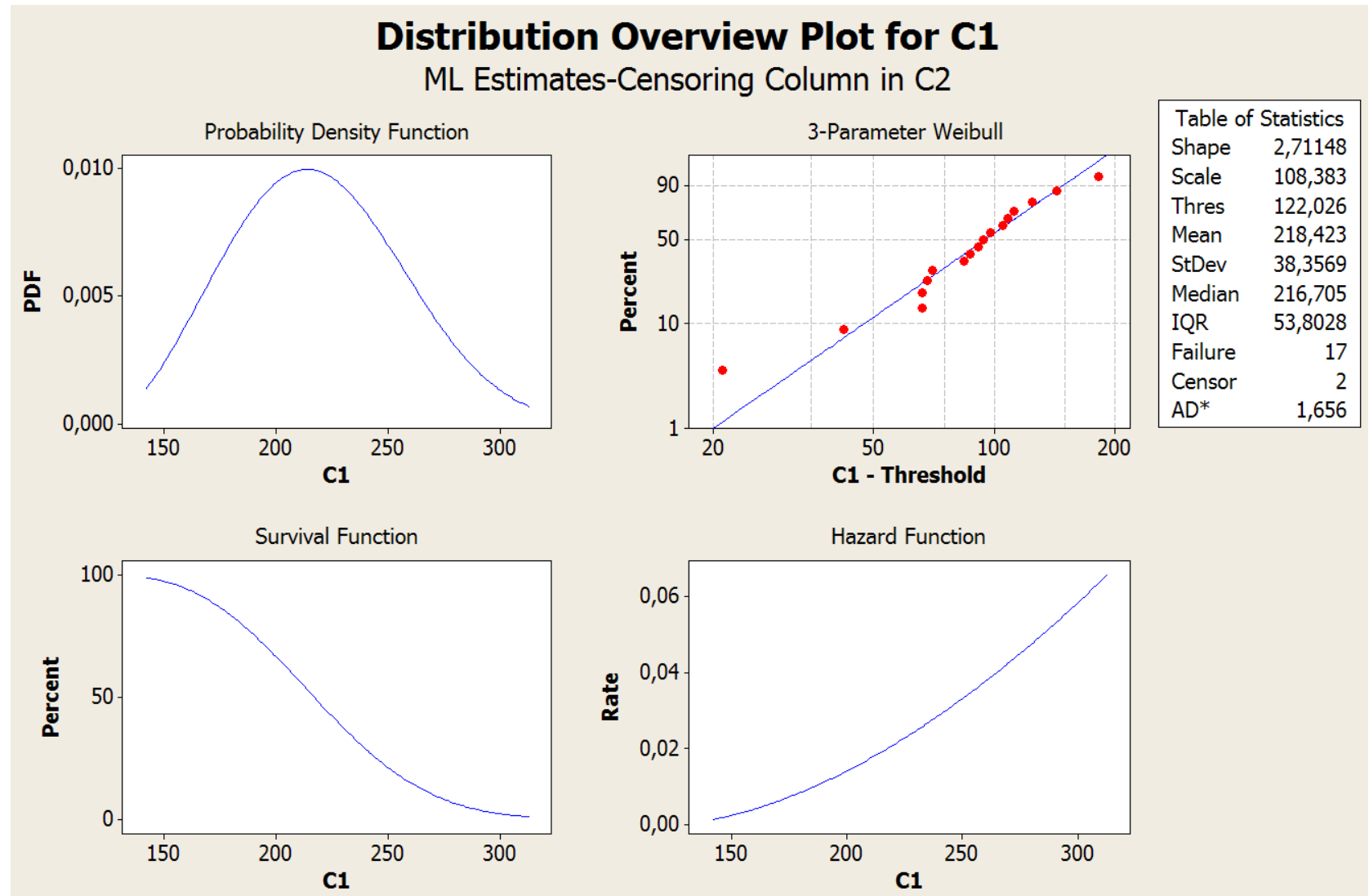
Anderson-Darling (adjusted) = 1,656

### Characteristics of Distribution

|                           | Estimate | Standard Error | 95,0% Normal CI |         |
|---------------------------|----------|----------------|-----------------|---------|
|                           |          |                | Lower           | Upper   |
| Mean (MTTF)               | 218,423  | 8,99156        | 201,492         | 236,777 |
| Standard Deviation        | 38,3569  | 6,41597        | 27,6352         | 53,2383 |
| Median                    | 216,705  | 9,89384        | 198,156         | 236,991 |
| First Quartile (Q1)       | 190,481  | 9,63934        | 172,495         | 210,342 |
| Third Quartile (Q3)       | 244,284  | 11,0118        | 223,627         | 266,849 |
| Interquartile Range (IQR) | 53,8028  | 8,97770        | 38,7945         | 74,6172 |

# Pike (1966) cancer data for rats

## 3-parameter Weibull



Pike 3-parameter Weibull: Profile log likelihood for  $\gamma$

| $\gamma$ | $\hat{\theta}(\gamma)$ | $\hat{\alpha}(\gamma)$ | $\tilde{l}(\gamma)$ |
|----------|------------------------|------------------------|---------------------|
| 0        | 234.3                  | 6.08                   | -88.233             |
| 60       | 173.2                  | 4.49                   | -87.831             |
| 100      | 131.8                  | 3.38                   | -87.467             |
| 110      | 121.2                  | 3.08                   | -87.381             |
| 120      | 110.6                  | 2.78                   | -87.327             |
| 122      | 108.4                  | 2.71                   | -87.324             |
| 125      | 105.2                  | 2.61                   | -87.330             |
| 130      | 99.7                   | 2.44                   | -87.382             |
| 135      | 94.0                   | 2.24                   | -87.542             |
| 140      | 88.0                   | 1.99                   | -88.064             |
| 142      | 85.2                   | 1.80                   | -88.773             |
| 143      | 81.1                   | 1.00                   | -91.718             |