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Statistical Methods for Reliability Data

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2.4 LIKELIHOOD

2.4.1 Likelihood-Based Statistical Methods

The general idea of likelihood inference is to fit models to data by entertaining model–parameter combinations for which the probability of the data is large. Model–parameter combinations with relatively high probabilities are more plausible than combinations with low probability. Likelihood methods provide general and versatile tools for fitting models to data. The methods can be applied with a wide variety of parametric and nonparametric models with censored, interval, and truncated data. It is also possible to fit models with explanatory variables (i.e., regression analysis).

There is a well-developed large-sample likelihood theory for regular models that provides straightforward methods for fitting models to data. The theory guarantees that these methods are, in large samples, statistically efficient (i.e., yield the most accurate estimates). These properties are approximate in moderate and small sample sizes, and various studies have shown that likelihood methods generally perform as well as other available methods. With censored data, "large sample" really means "large number of failures" and a typical guideline for large is 20 or more, but this really depends on the problem and the questions to be answered.

Likelihood theory can be extended to more complicated *nonregular* models and the basic concepts are similar. Also, much current statistical research is focused on the development of more refined, but computationally intensive, methods that will work better for smaller sample sizes.

2.4.2 Specifying the Likelihood Function

The likelihood function is either equal to or approximately proportional to the probability of the data. This section describes a general method of computing the probability of a given data set. Then, for a given set of data and specified model, the likelihood is viewed as a function of the unknown model parameters (where we can use either the π_i values or the p_i values in the multinomial model introduced in Section 2.2). The form of the likelihood function will depend on factors like:

- The assumed probability model.
- The form of available data (censored, interval censored, etc.).
- The question or focus of the study. This includes issues relating to identifiability of parameters (i.e., the data's ability or inability to estimate certain features of a statistical model).

The total likelihood can be written as the joint probability of the data. Assuming n independent observations, the sample likelihood is

$$L(\boldsymbol{p}) = L(\boldsymbol{p}; \text{DATA}) = C \prod_{i=1}^{n} L_i(\boldsymbol{p}; \text{data}_i), \tag{2.10}$$

where $L_i(\boldsymbol{p}; \text{data}_i)$ is the probability of the observation i, data_i is the data for observation i, and $\boldsymbol{p}$ is the vector of parameters to be estimated. To estimate $\boldsymbol{p}$ from the available DATA, we find the values of $\boldsymbol{p}$ that maximize $L(\boldsymbol{p})$. In the usual situations where the constant term C in (2.10) does not depend on $\boldsymbol{p}$, one can simply take $C = 1$ for purposes of estimating $\boldsymbol{p}$ (see Section 2.4.4 for more information on C). The likelihood in (2.10) can also be written as a function of the multinomial cell probabilities $\boldsymbol{\pi}$. Similarly, if there is a specified parametric form for $F(t; \boldsymbol{\theta})$ the likelihood can be written as a function of the parameters $\boldsymbol{\theta}$. We use $\boldsymbol{p}$ here because Chapter 3 illustrates the direct estimation of $\boldsymbol{p}$.

2.4.3 Contributions to the Likelihood Function

Figure 2.6 illustrates the intervals of uncertainty for examples of left-censored, interval-censored, and right-censored observations. The likelihood contributions for each of these cases, shown in Table 2.2, is simply the probability of failing in the corresponding interval of uncertainty.

Interval-Censored Observations

If a unit's failure time is known to have occurred between times t_{i-1} and t_i, the probability of this event is

$$L_i(\boldsymbol{p}) = \int_{t_{i-1}}^{t_i} f(t)\, dt = F(t_i) - F(t_{i-1}). \tag{2.11}$$

The three middle rows in Table 1.4 are examples of interval-censored observations.

Example 2.9 Likelihood for an Interval Censored Observation. Refer to Figure 2.6 and Table 2.1. If a unit is still operating at the $t = 1.0$ inspection but a

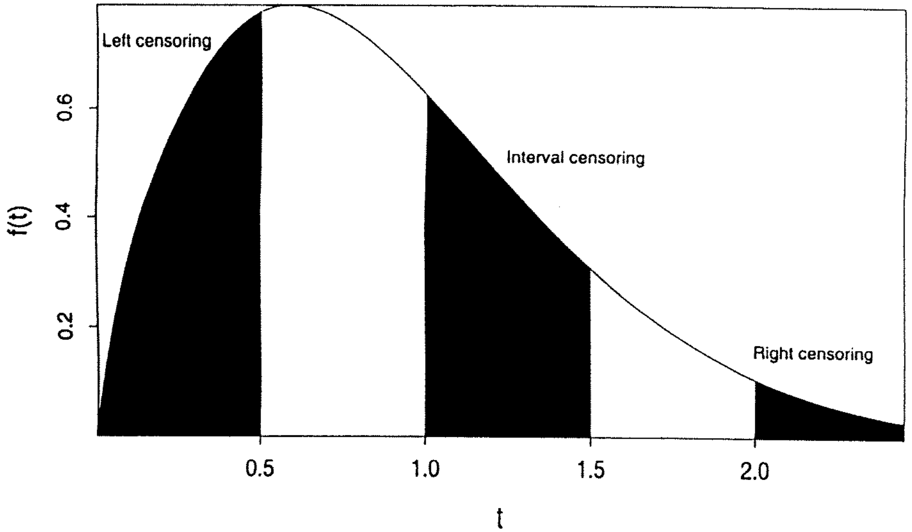

Figure 2.6. Likelihood contributions for different kinds of censoring.

Table 2.2. Contributions to Likelihood for Life Table Data

Censoring Type	Range	Likelihood
d_i observations interval-censored in t_{i-1} and t_i	$t_{i-1} < T \le t_i$	$[F(t_i) - F(t_{i-1})]^{d_i}$
ℓ_i observations left-censored at t_i	$T \le t_i$	$[F(t_i)]^{\ell_i}$
r_i observations right-censored at t_i	$T > t_i$	$[1 - F(t_i)]^{r_i}$

failure is found at the $t = 1.5$ inspection, then the likelihood (probability) for the interval-censored observation is $\pi_i = F(1.5) - F(1.0) = .231$. □

Although most data arising from observation of a continuous-time process can be thought of as having occurred in intervals similar to (t_{i-1}, t_i), the following important special cases warrant separate consideration.

Left-Censored Observations

Left-censored observations occur in life test applications when a unit has failed at the time of its first inspection; all that is known is that the unit failed before the inspection time (e.g., the first row of Table 1.4). In other situations, left-censored observations arise when the exact value of a response has not been observed and we have, instead, an upper bound on that response. Consider, for example, a measuring instrument that lacks the sensitivity needed to measure observations below a known threshold (e.g., a noise floor in an ultrasonic measuring system). When the measurement is taken, if the signal is below the instrument threshold, all that is known is that the measurement is less than the threshold. If there is an upper bound t_i for observation i, causing it to be left-censored, the probability and likelihood contribution of the observation is

$$L_i(\mathbf{p}) = \int_0^{t_i} f(t)\,dt = F(t_i) - F(0) = F(t_i). \tag{2.12}$$

Equation (2.7) shows how L_i can be written as a function of $\mathbf{p}$. Alternatively, (2.6) shows how L_i can be written as a function of $\boldsymbol{\pi}$. Note that a left-censored observation can also be considered to be an interval-censored observation between 0 and t_i.

Example 2.10 Likelihood of a Left-Censored Observation. Refer to Figure 2.6 and Table 2.1. If a failure is found at the first inspection time $t = .5$, then the likelihood (probability) for the left-censored observation is $F(.5) = .265$. □

Right-Censored Observations

Right censoring is common in reliability data analysis. For example, the last bin in Table 1.4 contains all lifetimes greater than 100 days. The observations in this bin

are right-censored because all that is known about the failure times in this bin is that they were greater than 100 days.

If there is a lower bound t_i for the ith failure time, the failure time is somewhere in the interval (t_i, ∞). Then the probability and likelihood contribution for this right-censored observation is

$$L_i(\mathbf{p}) = \int_{t_i}^{\infty} f(t)\,dt = F(\infty) - F(t_i) = 1 - F(t_i). \tag{2.13}$$

Example 2.11 Likelihood of a Right-Censored Observation. Refer to Figure 2.6 and Table 2.1. If a unit has not failed by the last inspection at $t = 2$, then the likelihood (probability) for the right-censored observation is $1 - F(2) = .0388$. □

Total Likelihood

The total likelihood, or joint probability of the DATA, for n independent observations is

$$L(\mathbf{p};\text{DATA}) = \mathcal{C}\prod_{i=1}^{n} L_i(\mathbf{p};\text{data}_i) \tag{2.14}$$

$$= \mathcal{C}\prod_{i=1}^{m+1} [F(t_i)]^{\ell_i}\,[F(t_i) - F(t_{i-1})]^{d_i}\,[1 - F(t_i)]^{r_i},$$

where $n = \sum_{j=1}^{m+1}(d_j + r_j + \ell_j)$ and $\mathcal{C}$ is a constant depending on the sampling inspection scheme but not on the parameters $\mathbf{p}$. So we can take $\mathcal{C} = 1$. We want to find $\mathbf{p}$ so that $L(\mathbf{p})$ is large. The $\mathbf{p}$ that maximizes $L(\mathbf{p})$ provides a maximum likelihood estimate of $F(t)$. For some problems, it will be more convenient to write the likelihood and do the optimization in terms of $\boldsymbol{\pi}$. As described in Section 2.2.1, either set of basic parameters can be used.

2.4.4 Form of the Constant Term $\mathcal{C}$

The form of constant term $\mathcal{C}$ in (2.10) and (2.14) depends on the underlying sampling and censoring mechanisms and is difficult to characterize in general. For our multinomial model, assuming inspection data and *no losses* (i.e., no right-censored observations before the last interval),

$$\mathcal{C} = \frac{n!}{d_1!\cdots d_{m+1}!},$$

which is the usual multinomial coefficient. Another important special case arises when we increase the number of intervals, approaching continuous inspection. Then with an underlying continuous failure-time process (so there will be no ties), all d_i values will be either 0 or 1 depending on whether there is a failure or not in interval i. In this case $\mathcal{C}$ reduces to $n!$, corresponding to the number of permutations of the n order statistics. With Type I single-time censoring at t_m and no more than one failure in any of the intervals before t_m, $\mathcal{C} = n!/r_{m+1}!$, where $r_{m+1} = d_{m+1}$ is the number of right-censored observations, all of which are beyond t_m.

Because, for most models, $\mathcal{C}$ is a constant that does not depend on the model parameters, it is common practice to take $\mathcal{C} = 1$ and suppress $\mathcal{C}$ from likelihood expressions and computations.

2.4.5 Likelihood Terms for General Reliability Data

Although some reliability data sets are reported in life table form (e.g., Table 1.4), other data sets report only the times or the intervals in which failures actually occurred or observations were censored. For such data sets there is an alternative, more general form for writing the likelihood. This form of the likelihood is commonly used as input for computer software for analyzing failure-time data. In general, observation i consists of an interval $(t_i^L, t_i]$, $i = 1, \dots, n$, that contains the failure time T for unit i in the sample. The intervals $(t_i^L, t_i]$ may overlap and their union may not cover the entire time line $(0, \infty)$. In general, $t_i^L \neq t_{i-1}$. Assuming that the censoring is at t_i the likelihood for individual observations can be computed as shown in Table 2.3; the joint likelihood for the DATA with n independent observations is

$$L(\boldsymbol{p}; \text{DATA}) = \prod_{i=1}^{n} L_i(\boldsymbol{p}; \text{data}_i).$$

Some of the failure times or intervals may appear more than once in a data set. Then w_j is used to denote the frequency (weight or multiplicity) of such identical observations and

$$L(\boldsymbol{p}; \text{DATA}) = \prod_{j=1}^{k} \left[L_j(\boldsymbol{p}; \text{data}_j)\right]^{w_j}. \tag{2.15}$$

Chapter 3 shows how to compute the maximum likelihood estimate of $F(t)$ without having to make any assumption about the underlying distribution of T. Starting in Chapter 7 we show how to estimate a small number of unknown parameters from a more highly structured parametric model for $F(t)$.

Table 2.3. Contributions to the Likelihood for General Failure Time Data

Type of Observation	Characteristic	Likelihood of a Single Response $L_i(\boldsymbol{p}; \text{data}_i)$
Interval-censored	$t_i^L < T \le t_i$	$F(t_i) - F(t_i^L)$
Left-censored at t_i	$T \le t_i$	$F(t_i)$
Right-censored at t_i	$T > t_i$	$1 - F(t_i)$

2.4.6 Other Likelihood Terms

The likelihood contributions used in (2.14) and (2.15) will cover the vast majority of reliability data analysis problems that arise in practice. There are, however, other kinds of observations and corresponding likelihood contributions that can arise and these can be handled with only a slight extension of this framework.

Random Censoring in the Intervals

Until now, it has been assumed that right censoring occurs at the end of the inspection intervals. If C is a random censoring time, an observation is censored *in* the interval $(t_{i-1}, t_i]$ if $t_{i-1} < C \le t_i$ and $C \le T$. Similarly, an observation is a failure in that interval if $t_{i-1} < T \le t_i$ and $T \le C$. To account for right-censored observations that occur at unknown random points in the intervals, one usually assumes that the censoring is determined by a random variable C with pdf $f_C(t)$ and cdf $F_C(t)$ and that the failure time T and censoring time C are statistically independent. (But it is important to recognize that making such an assumption does not make it so!) Then for continuous T, the joint probability (likelihood) for r_i right-censored observations in $(t_{i-1}, t_i]$ and d_i failures in $(t_{i-1}, t_i]$ is

$$\begin{aligned} L_i(\boldsymbol{p}; \text{data}_i) &= \{\Pr[(T \le C) \cap (t_{i-1} < T \le t_i)]\}^{d_i}\{\Pr[(C \le T) \cap (t_{i-1} < C \le t_i)]\}^{r_i} \\ &= \left\{\int_{t_{i-1}}^{t_i} f_T(t)\,[1 - F_C(t)]\,dt\right\}^{d_i} \times \left\{\int_{t_{i-1}}^{t_i} f_C(t)\,[1 - F_T(t)]\,dt\right\}^{r_i}. \end{aligned} \tag{2.16}$$

Example 2.12 Battery Failure Data with Multiple Failure Modes. Morgan (1980) presents data from a study conducted on 68 battery cells. The purpose of the test was to determine early causes of failure, to determine which causes reduce product life the most, and to estimate failure-time distributions. Each test cell was subjected to automatic cycling (charging and discharging) at normal operating conditions. Some survived until the end of the test and others were removed before failure for physical examination. The original data giving precise times of failure or removal were not available. Instead, the data in Appendix Table C.6 provide a useful summary. By the nature of this summary, however, the removals (censoring times) do not occur at the ends of the intervals (as in the examples in Chapter 1). □

Truncated Data

In some reliability studies, observations may be *truncated*. Truncation, which is similar to but different from censoring, arises when observations are actually observed only when they take on values in a particular range. For observations that fall outside the certain range, the *existence* is not known (and this is what distinguishes truncation from censoring). Equivalently, sampling from a truncated distribution leads to truncated data. Examples and appropriate likelihood-based methods for handling truncated data, based on conditional probabilities, will be given in Section 11.6.