CHAPTER 12

Fractional Factorial Designs at Two Levels

The number of runs required by a full 2^k factorial design increases geometrically as k is increased. It turns out, however, that when k is not small the desired information can often be obtained by performing only a fraction of the full factorial design. This chapter describes how suitable fractions can be generated and discusses their advantages and limitations.

12.1. REDUNDANCY

Consider a two-level design in seven variables. A complete factorial arrangement requires $2^7 = 128$ runs. From these runs 128 statistics can be calculated, which estimate the following effects:

		interactions					
average	main effects	2-factor	3-factor	4-factor	5-factor	6-factor	7-factor
1	7	21	35	35	21	7	1

Now the fact that all these effects can be estimated does not imply that they all are of appreciable size. There tends to be a certain hierarchy. In terms of absolute magnitude, main effects tend to be larger than two-factor interactions, which in turn tend to be larger than three-factor interactions, and so on. This fact relates directly to the properties of smoothness and similarity discussed earlier. (In particular, for quantitative variables the main effects and interactions can be associated with the terms of a Taylor series expansion of a response function. Ignoring, say, three-factor interactions corresponds to ignoring terms of third order in the Taylor expansion.)

It is often true, then, that at some point higher order interactions tend to become negligible and can properly be disregarded. Also, when a moderately large number of variables is introduced into a design, it often happens that some have no distinguishable effects at all. We can encompass both these ideas by saying that there tends to be redundancy in a 2^k design if k is not small—redundancy in terms of an excess number of interactions that can be estimated and sometimes in an excess number of variables that are studied. Fractional factorial designs exploit this redundancy. We begin by considering what effects can be estimated using only a half-fraction of a 2^5 factorial design.

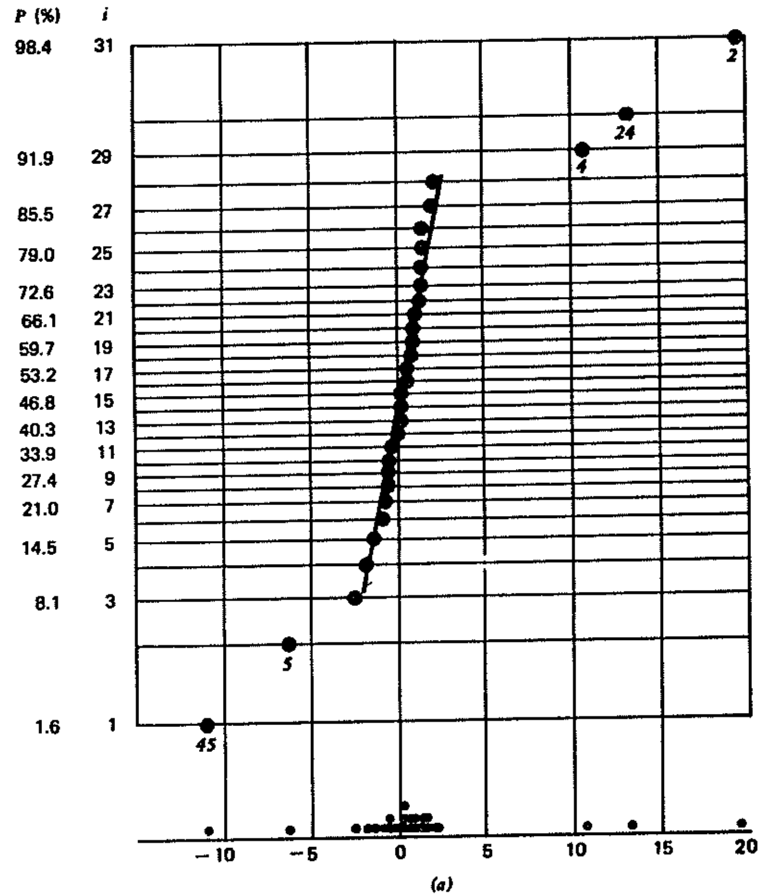


FIGURE 12.1. (*a*) Normal plot of effects from 2^5 factorial design, reactor example.

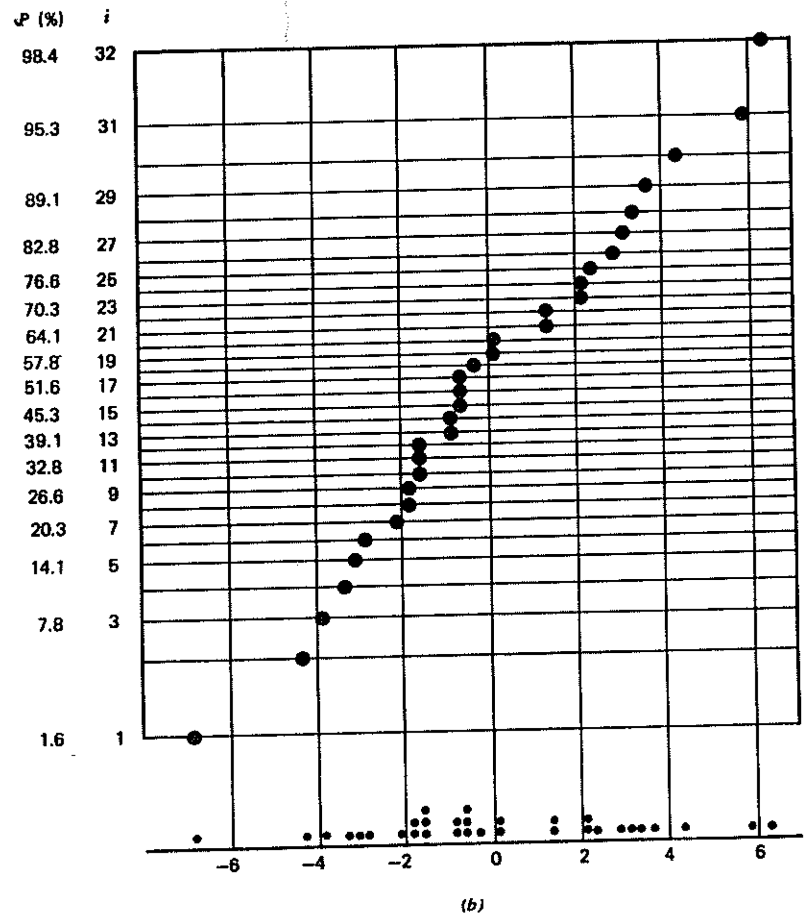


FIGURE 12.1. (*b*) Normal plot of residuals after eliminating *2*, *4*, *5*, *24*, and *45* from 2^5 factorial design, reactor example.

12.2. A HALF-FRACTION OF A 2^5 DESIGN: REACTOR EXAMPLE

Table 12.1*a* shows data from a complete 2^5 factorial design analyzed in Table 12.1*b*. Normal plots (Figure 12.1) indicate that over the ranges of the variables studied the main effects *2*, *4*, and *5* and interactions *24* and *45* are the only effects distinguishable from noise.

TABLE 12.1*a*. Results from 2^5 factorial design, reactor example

variable	−	+
1 feed rate (liters/min)	10	15
2 catalyst (%)	1	2
3 agitation rate (rpm)	100	120
4 temperature (°C)	140	180
5 concentration (%)	3	6

run	variable 1	2	3	4	5	response (% reacted) y
1	−	−	−	−	−	61
*2	+	−	−	−	−	53
*3	−	+	−	−	−	63
4	+	+	−	−	−	61
*5	−	−	+	−	−	53
6	+	−	+	−	−	56
7	−	+	+	−	−	54
*8	+	+	+	−	−	61
*9	−	−	−	+	−	69
10	+	−	−	+	−	61
11	−	+	−	+	−	94
*12	+	+	−	+	−	93
13	−	−	+	+	−	66
*14	+	−	+	+	−	60
*15	−	+	+	+	−	95
16	+	+	+	+	−	98
*17	−	−	−	−	+	56
18	+	−	−	−	+	63
19	−	+	−	−	+	70
*20	+	+	−	−	+	65
21	−	−	+	−	+	59
*22	+	−	+	−	+	55
*23	−	+	+	−	+	67
24	+	+	+	−	+	65
25	−	−	−	+	+	44
*26	+	−	−	+	+	45
*27	−	+	−	+	+	78
28	+	+	−	+	+	77
*29	−	−	+	+	+	49
30	+	−	+	+	+	42
31	−	+	+	+	+	81
*32	+	+	+	+	+	82

TABLE 12.1*b*. Analysis of 2^5 factorial design, reactor example

estimates of effects

average =	65.5		
1 =	−1.375	*123* =	1.50
2 =	19.5	*124* =	1.375
3 =	−0.625	*125* =	−1.875
4 =	10.75	*134* =	−0.75
5 =	−6.25	*135* =	−2.50
		145 =	0.625
12 =	1.375	*235* =	0.125
13 =	0.75	*234* =	1.125
14 =	0.875	*245* =	−0.250
15 =	0.125	*345* =	0.125
23 =	0.875		
24 =	13.25	*1234* =	0.0
25 =	2.0	*1245* =	0.625
34 =	2.125	*2345* =	−0.625
35 =	0.875	*1235* =	1.5
45 =	−11.0	*1345* =	1.0
		12345 =	−0.25

The full 2^5 factorial requires 32 runs. Suppose that the experimenter had chosen to make only the 16 runs marked with asterisks in Table 12.1, so that only the data of Table 12.2 were available. When the 15 main effects and two-factor interactions are calculated from the reduced set of data in Table 12.2, they produce the estimates listed there, which are not very different from those obtained from the complete factorial design. Furthermore the normal plots of Figure 12.2 call attention to precisely the same effects: *2*, *4*, *24*, *45* and *5*. Thus the essential information could have been obtained with only half the effort.

The 16-run design in Table 12.2 is called a *half-fraction*. It is often designated as a 2^{5-1} fractional factorial design since

$$\tfrac{1}{2}2^5 = 2^{-1}2^5 = 2^5 2^{-1} = 2^{5-1}$$

The notation tells us that the design accommodates five variables, each at two levels, but that only $2^{5-1} = 2^4 = 16$ runs are employed.

TABLE 12.2 Analysis of a half-fraction of the full 2^5 design: a 2^{5-1} fractional factorial design, reactor example

variable	−	+
1 feed rate (liters/min)	10	15
2 catalyst (%)	1	2
3 agitation rate (rpm)	100	120
4 temperature (°C)	140	180
5 concentration (%)	3	6

run	design 1	2	3	4	5	12	13	14	15	23	24	25	34	35	45	response (% reacted) y
17	−	−	−	−	+	+	+	+	−	+	+	−	+	−	−	56
2	+	−	−	−	−	−	−	−	−	+	+	+	+	+	+	53
3	−	+	−	−	−	−	+	+	+	−	−	−	+	+	+	63
20	+	+	−	−	+	+	−	−	+	−	−	+	+	−	−	65
5	−	−	+	−	−	+	−	+	+	−	+	+	−	−	+	53
22	+	−	+	−	+	−	+	−	+	−	+	−	−	+	−	55
23	−	+	+	−	+	−	−	+	−	+	−	+	−	+	−	67
8	+	+	+	−	−	+	+	−	−	+	−	−	−	−	+	61
9	−	−	−	+	−	+	+	−	+	+	−	+	−	+	−	69
26	+	−	−	+	+	−	−	+	+	+	−	−	−	−	+	45
27	−	+	−	+	+	−	+	−	−	−	+	+	−	−	+	78
12	+	+	−	+	−	+	−	+	−	−	+	−	−	+	−	93
29	−	−	+	+	+	+	−	−	−	−	−	−	+	+	+	49
14	+	−	+	+	−	−	+	+	−	−	−	+	+	−	−	60
15	−	+	+	+	−	−	−	−	+	+	+	−	+	−	−	95
32	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	82

estimates of effects
(assuming that three-factor and higher order interactions are negligible)

average =	65.25	*12* =	1.5
1 =	−2.0	*13* =	0.5
2 =	20.5	*14* =	−0.75
3 =	0.0	*15* =	1.25
4 =	12.25	*23* =	1.50
5 =	−6.25	*24* =	10.75
		25 =	1.25
		34 =	0.25
		35 =	2.25
		45 =	−9.50

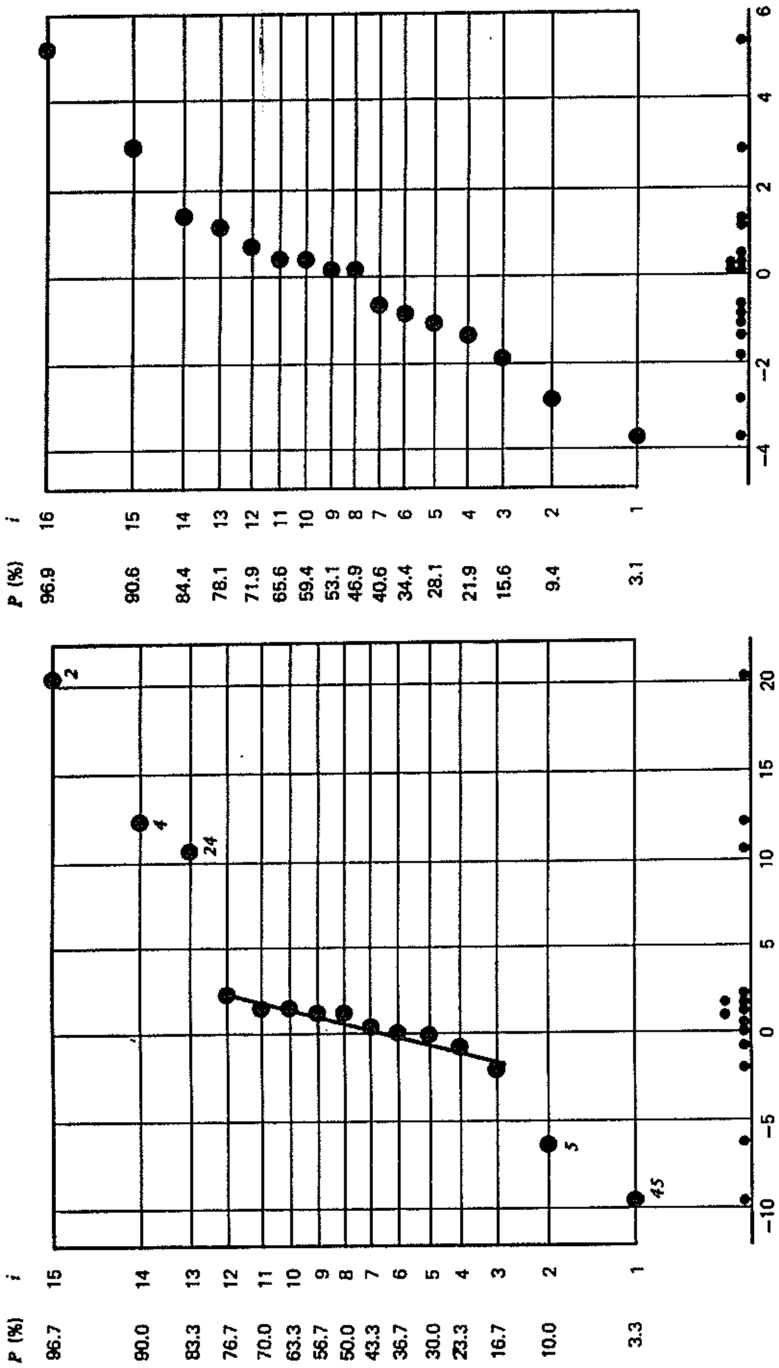


FIGURE 12.2. Normal plot of (*a*) effects and (*b*) residuals after eliminating *2*, *4*, *5*, *24*, and *45* from 2^{5-1} fractional factorial design, reactor example.

12.3. CONSTRUCTION AND ANALYSIS OF HALF-FRACTIONS: REACTOR EXAMPLE

How Were the 16 Runs Chosen?

The 2^{5-1} design in Table 12.2 was constructed as follows:

1. A full 2^4 design was written for the four variables **1**, **2**, **3**, and **4**.
2. The column of signs for the **1234** interaction was written, and these were used to define the levels of variable **5**. Thus we made **5** = **1234**.

Exercise 12.1. By using this procedure, verify that the design obtained is the one given in Table 12.2.

The Anatomy of the Half-Fraction

At this point we seem to have gained something for nothing. It is natural to ask, Have we lost anything? Look again at the fractional factorial design of Table 12.2. We have made 16 runs and estimated 16 quantities: the mean, the 5 main effects, and the 10 two-factor interactions. But what happened to the remaining 16 effects we were able to estimate with the full factorial design—the 10 three-factor interactions, the 5 four-factor interactions, and the 1 five-factor interaction?

Let us try to estimate the value of the three-factor interaction *123*. Multiplying the signs in columns **1**, **2**, and **3**, we obtain the sequence (which, to save space, we write as a row rather than a column)

$$\mathbf{123} = - + + - + - - + - + + - + - - +$$

We notice that this is identical to

$$\mathbf{45} = - + + - + - - + - + + - + - - +$$

Thus **123** = **45**, and as a consequence the *123* and *45* interactions are confounded. Equivalently, in the fractional design the individual interactions *123* and *45* are said to be *aliases* of each other. Now suppose that we use the symbol l_{45} to denote the linear function of the observations which we used to estimate the *45* interaction:

$$l_{45} = \tfrac{1}{8}(-56 + 53 + 63 - 65 + 53 - 55 - 67 + 61 - 69 + 45 + 78$$
$$-93 + 49 - 60 - 95 + 82) = -9.50 \qquad (12.1)$$

We can call this the l_{45} *contrast* since it is the *difference* between two averages of eight results. Properly speaking, contrast l_{45} estimates the *sum* of the mean values of effects *45* and *123*. We indicate this by the notation $l_{45} \to 45 + 123$. If the columns of signs corresponding to all the other three-factor, four-factor, and five-factor interactions are obtained by multiplying signs, we get the results shown in Table 12.3.

TABLE 12.3. Confounding pattern and estimates from 2^{5-1} design of Table 12.2

relationship between column pairs	confounding pattern	estimate
1 = 2345	$l_1 \to 1 + 2345$	$l_1 = -2.0$
2 = 1345	$l_2 \to 2 + 1345$	$l_2 = 20.5$
3 = 1245	$l_3 \to 3 + 1245$	$l_3 = 0.0$
4 = 1235	$l_4 \to 4 + 1235$	$l_4 = 12.25$
5 = 1234	$l_5 \to 5 + 1234$	$l_5 = -6.25$
12 = 345	$l_{12} \to 12 + 345$	$l_{12} = 1.5$
13 = 245	$l_{13} \to 13 + 245$	$l_{13} = 0.5$
14 = 235	$l_{14} \to 14 + 235$	$l_{14} = -0.75$
15 = 234	$l_{15} \to 15 + 234$	$l_{15} = 1.25$
23 = 145	$l_{23} \to 23 + 145$	$l_{23} = 1.5$
24 = 135	$l_{24} \to 24 + 135$	$l_{24} = 10.75$
25 = 134	$l_{25} \to 25 + 134$	$l_{25} = 1.25$
34 = 125	$l_{34} \to 34 + 125$	$l_{34} = 0.25$
35 = 124	$l_{35} \to 35 + 124$	$l_{35} = 2.25$
45 = 123	$l_{45} \to 45 + 123$	$l_{45} = -9.50$
(I = 12345)	$[l_I \to \text{average} + \frac{1}{2}(12345)]$	$(l_I = 65.25)$

Exercise 12.2. As was done for columns **45** and **123**, verify that columns **24** and **135** are identical. Verify the identity of the other column pairs in Table 12.3.

A Justification for the Analysis

Evidently our earlier analysis would be justified if it could be assumed that effects of third and fourth order (represented by three-factor and four-factor interactions) could be ignored. In the reactor example the assumption was apparently justified. We shall see later that the analysis could also be justified on different and somewhat more subtle grounds (see the subsection entitled "An Alternative Rationale for the Half-Fraction Design in the Reactor Experiment").

How to Find the Confounding Patterns

In manipulating fractional factorials it is important to be able to obtain the confounding pattern for any given design. The method of associating like sign sequences is extremely tedious. Fortunately a much more expeditious route is available. To understand it remember the following four points:

1. Boldface numerals (e.g., **3** and **12**) refer to *columns* of plus and minus signs.
2. A product column is obtained by multiplication of the individual elements in the columns that make up that product. (The product column **124**, for instance, is obtained by multiplication of the individual elements in the corresponding columns, **1**, **2**, and **4**.)
3. Multiplying the elements in any column by a column of identical elements gives a column of plus signs, which is designated by the letter **I**, that is, $\mathbf{1} \times \mathbf{1} = \mathbf{1}^2 = \mathbf{I}$, $\mathbf{2}^2 = \mathbf{I}$, $\mathbf{3}^2 = \mathbf{I}$, $\mathbf{4}^2 = \mathbf{I}$, and so forth.
4. A contrast like l_{45} in Equation 12.1 is obtained by multiplying the observations by the appropriate plus and minus signs in column **45** and dividing by $N/2 = 8$ where N is the number of observations (16 in this case). Each quantity l is thus a contrast between two averages, each of $N/2$ observations. The single exception is $l_I = \bar{y}$, which is obtained by multiplying the observations by the column **I** of plus signs (i.e., summing the observations) and dividing the result by N (in this example $N = 16$).

Generator and Defining Relation

The 2^{5-1} design in Table 12.2 was constructed by setting

$$\mathbf{5} = \mathbf{1234} \tag{12.2}$$

This relation is called the *generator* of the design. Multiplying both sides by **5**, we obtain

$$\mathbf{5} \times \mathbf{5} = \mathbf{1234} \times \mathbf{5} \tag{12.3}$$

or

$$\mathbf{5}^2 = \mathbf{12345} \tag{12.4}$$

Thus the generator for the design can equivalently (and more conveniently) be written as

$$\mathbf{I} = \mathbf{12345} \tag{12.5}$$

This version of the identity is readily confirmed by multiplying together the elements in columns **1**, **2**, **3**, **4**, and **5**, and noting that a column of plus signs,

l_1 is actually obtained. The half-fraction is defined* by a single generator, so that the relation **I** = **12345** also provides the *defining relation* of the design.

This defining relation is the key to the confounding pattern. For example, multiplying the defining relation on both sides by **1** yields

$$\mathbf{1} = \mathbf{2345}$$

In a similar way multiplying by **2** gives **2** = **1345** and so on to produce all the identities in the first column of Table 12.3.

The Complementary Half-Fraction

In the above example the generator **5** = **1234**, or, equivalently, **I** = **12345**, produced the defining relation for the design. In other words, by generating a new column **5** = **1234** we obtained the half-fraction corresponding to the runs marked with asterisks in Table 12.1. The defining relation **I** = **12345** provided by this generator immediately yields the confounding pattern of Table 12.3. The complementary half-fraction is generated by putting **5** = **−1234**. We then obtain the half-fraction corresponding to the runs of the original 2^5 that are *not* marked with asterisks in Table 12.1. The defining relation for this design may be written as

$$\mathbf{I} = -\mathbf{12345}$$

In practice either half-fraction can equally well be used. For the data of Table 12.1 the complementary half-fraction would have given, for example,

$$l'_1 = -0.75 \rightarrow 1 - 2345$$

$$l'_2 = 18.50 \rightarrow 2 - 1345$$

Exercise 12.3. For the 16 runs in Table 12.1 that do *not* have asterisks, calculate the average and the 15 contrasts $l'_1, l'_2, \ldots, l'_{45}$. Show by making a normal plot that the conclusions that would result from this fraction would be similar to those obtained from the other one.

Answer: (average, *1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45*) = (65.75, −0.75, 18.5, −1.25, 9.25, −6.25, 1.25, 1.0, −1.0, −1.0, 0.25, 15.75, 2.75, 4.0, −0.5, −12.5).

Combining the Two Half-Fractions

Suppose that after completing one of the half-fractions the other was subsequently added, so that the whole factorial was available. Unconfounded estimates of all effects could then be obtained by analyzing the 32 runs as a full 2^5 factorial design run in two blocks of 16. The same result would be obtained by suitably adding and subtracting estimates from the two individual fractions. For example, we have

first fraction	second fraction
$l_2 = 20.5 \rightarrow 2 + 1345$	$l'_2 = 18.5 \rightarrow 2 - 1345$

whence

$$\tfrac{1}{2}(l_2 + l'_2) = \tfrac{1}{2}(20.5 + 18.5) = 19.5 \rightarrow 2$$
$$\tfrac{1}{2}(l_2 - l'_2) = \tfrac{1}{2}(20.5 - 18.5) = 1.0 \rightarrow 1345 \qquad (12.6)$$

These values for *2* and *1345* agree with those given in Table 12.1 for the complete 2^5 design.

* When higher fractions are employed, there is more than one generator. For example, a quarter-fraction is defined by two generators. For more complicated fractions see Appendix 12A.

12.4. THE CONCEPT OF DESIGN RESOLUTION: REACTOR EXAMPLE

The 2^{5-1} fraction is called a *resolution V* design. Looking at the confounding pattern in Table 12.3, we see, for example, that $l_1 \rightarrow 1 + 2345$ and $l_{12} \rightarrow 12 + 345$. Thus main effects are confounded with four-factor interactions, and two-factor interactions with three-factor interactions.

In general, a design of resolution R is one in which no p-factor effect is confounded with any other effect containing less than $R - p$ factors. The resolution of a design is denoted by the appropriate Roman letter appended as a subscript. Thus we could refer to the design of Table 12.2 as a 2^{5-1}_{V} design. To illustrate:

1. A design of resolution R = III does not confound main effects with one another but does confound main effects with two-factor interactions.
2. A design of resolution R = IV does not confound main effects and two-factor interactions but does confound two-factor interactions with other two-factor interactions.
3. A design of resolution R = V does not confound main effects and two-factor interactions with each other, but does confound two-factor interactions with three-factor interactions, and so on.

*In general, the *resolution* of a two-level fractional design is *the length of the shortest word in the defining relation.*

Resolutions of Some Half-Fractions

For any half-fraction the number of symbols on the right of the defining relation denotes the resolution of the design. Thus a 2^{5-1} half-fraction with defining relation $\mathbf{I} = \pm\mathbf{12345}$ has resolution V. In Table 12.4 the 2^{3-1} half-fractions with defining relations $\mathbf{I} = \pm\mathbf{123}$ have resolution III, and the 2^{4-1} fractions with defining relations $\mathbf{I} = \pm\mathbf{1234}$ have resolution IV.

Half-Fractions of Highest Resolution

At the beginning of Section 12.3 we gave a procedure for constructing a 2^{5-1} design. In fact, it would have been possible to use *any* interaction or main effect column to accommodate the fifth variable. The choice we made yields a half-fraction with *highest possible resolution.* In general, to construct a 2^{k-1} fractional factorial design of highest possible resolution:

1. Write a full factorial design for the first $k-1$ variables.
2. Associate the kth variable with plus or minus the interaction column **123** . . . **(k − 1)**.

Table 12.4 gives examples of 2_{III}^{3-1}, 2_{IV}^{4-1}, and 2_{V}^{5-1} half-fractions of this kind. The two 2^{3-1} half-fractions obtained by the above rule are shown geometrically in Table 12.4.

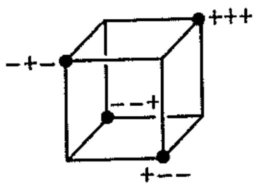
Exercise 12.4. Obtain the confounding pattern for a 2^{5-1} design generated by setting **5 = 123**. Discuss its properties. What is its resolution? Can you imagine circumstances in which it might be preferred to the resolution V design?

Partial answer: $l_1 \to 1 + 235$, $l_2 \to 2 + 135$, $R = \text{IV}$.

An Alternative Rationale for the Half-Fraction Design in the Reactor Experiment

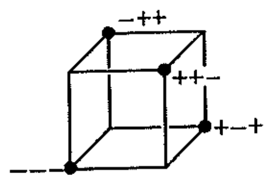
Consider the 2_{V}^{5-1} half-fraction with $\mathbf{I} = \mathbf{12345}$ given in Tables 12.2 and 12.4. Obviously (from its mode of construction), if we omit the fifth column of plus and minus signs from this design, we have a complete factorial in variables **1**, **2**, **3**, and **4**. But try omitting column **1** instead. There is now a complete factorial in variables **2**, **3**, **4**, and **5**! Indeed, a *complete* factorial in the remaining variables is obtained *whichever column is omitted*. We have already seen that the experimenter could justify the 2^{5-1} half-fraction on the assumption that three-factor, four-factor, and five-factor interactions could be ignored. An alternative justifying assumption is that at most only four of the five variables will produce detectable effects and the other will be essentially

TABLE 12.4. Best half-fractions for $k = 3$, $k = 4$, and $k = 5$



2_{III}^{3-1}, **3 = 12** (**I = 123**)

1	2	3
−	−	+
+	−	−
−	+	−
+	+	+



2_{III}^{3-1}, **3 = −12** (**I = −123**)

1	2	3
−	−	−
+	−	+
−	+	+
+	+	−

2_{IV}^{4-1}, **4 = 123** (**I = 1234**)

1	2	3	4
−	−	−	−
+	−	−	+
−	+	−	+
+	+	−	−
−	−	+	+
+	−	+	−
−	+	+	−
+	+	+	+

2_{IV}^{4-1}, **4 = −123** (**I = −1234**)

1	2	3	4
−	−	−	+
+	−	−	−
−	+	−	−
+	+	−	+
−	−	+	−
+	−	+	+
−	+	+	+
+	+	+	−

2_{V}^{5-1}, **5 = 1234** (**I = 12345**)

1	2	3	4	5
−	−	−	−	+
+	−	−	−	−
−	+	−	−	−
+	+	−	−	+
−	−	+	−	−
+	−	+	−	+
−	+	+	−	+
+	+	+	−	−
−	−	−	+	−
+	−	−	+	+
−	+	−	+	+
+	+	−	+	−
−	−	+	+	+
+	−	+	+	−
−	+	+	+	−
+	+	+	+	+

2_{V}^{5-1}, **5 = −1234** (**I = −12345**)

1	2	3	4	5
−	−	−	−	−
+	−	−	−	+
−	+	−	−	+
+	+	−	−	−
−	−	+	−	+
+	−	+	−	−
−	+	+	−	−
+	+	+	−	+
−	−	−	+	+
+	−	−	+	−
−	+	−	+	−
+	+	−	+	+
−	−	+	+	−
+	−	+	+	+
−	+	+	+	+
+	+	+	+	−

inert—it will have no detectable main effect or interaction with any other variable. On the assumption of one or more inert variables, the 2_V^{5-1} design will generate complete factorials in the remaining variables, *no matter which variables these are.*

In fact, our analysis for the reactor example suggests that only three of the variables had detectable effects: **2**, **4**, and **5** (catalyst, temperature, and con- centration). Since variables **1** and **3** were effectively inert, we had a *replicated* 2^3 factorial in variables **2**, **4**, and **5**, and the results can be assembled as in Figure 12.3.

Factorials Embedded in Fractions:
The General Importance of the Concept of Resolution

In general, it can be shown that a fractional factorial design of resolution R contains complete factorials (possibly replicated) in every set of $R - 1$ variables. Suppose, then, that the experimenter has a number of candidate variables but believes that all but $R - 1$ of them (specific identity unknown) may have no detectable effects. Then, if he employs a design of resolution R and his conjecture is justified, he will have a complete factorial design in the effective variables. This idea is illustrated with the 2_{III}^{3-1} design in Figure 12.4, which projects a 2^2 pattern in every subspace of two dimensions.

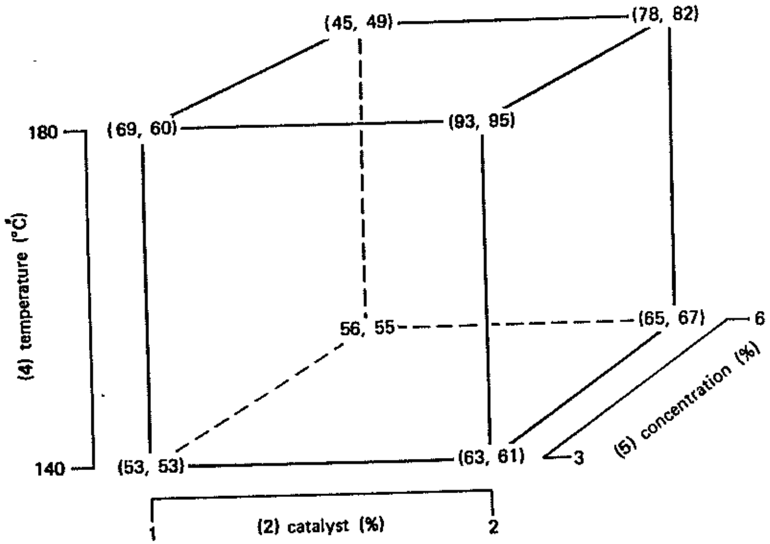


FIGURE 12.3. Data (% reacted) from a 2^{5-1} fraction, shown as replicated 2^3 factorial in variables *2*, *4*, and *5*, reactor example.

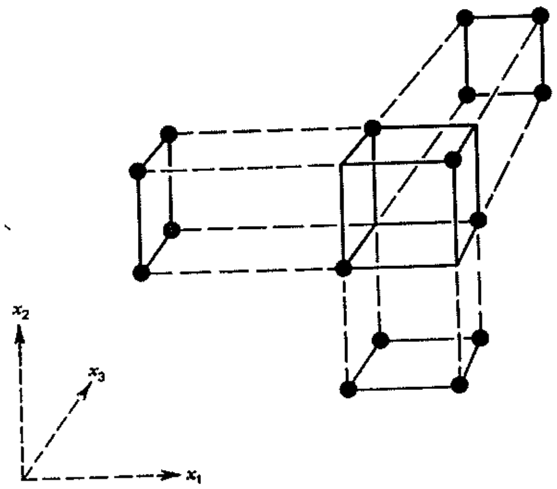


FIGURE 12.4. A 2_{III}^{3-1} design, showing projections into three 2^2 factorials.

Exercise 12.5. If a resolution R design gives a full factorial in every set of $R - 1$ variables, is it necessarily true that a full factorial is obtained in every subset containing fewer than $R - 1$ variables? *Answer*: Yes.

Exercise 12.6. A 2_V^{5-1} design gives full factorials in every subset of q variables. What is the value of q? *Answer*: 4, 3, 2, or 1 (for an example of $q = 3$ see Figure 12.3).

Economy in Experimentation Arising from the
Sequential Use of Fractional Designs

Suppose that an experimenter who can make his runs sequentially wishes to investigate five factors, each at two levels, and is contemplating a 2^5 design involving 32 runs. It is almost always better for him to run a half-fraction containing 16 runs first, analyze the results, and think about them. If necessary, he can always run the second fraction later to complete the full design. Frequently, however, the first half-fraction itself will allow him to proceed to the next stage of experimental iteration, which may involve, for example, the introduction of new variables or different levels of the old ones. Use of this sequential approach can thus greatly accelerate progress. It is worth noting that:

1. The experimenter should randomize within each fraction.
2. If eventually it is decided to run both fractions, these fractions will be randomized orthogonal blocks of the complete design.

3. No information will be "lost" except that concerning the interaction which is actually confounded with the block contrast.
4. The design run as two randomized fractions can give greater precision than the whole design run in random order because the block difference is eliminated.

Recapitulation

We began the chapter by discussing redundancy. It was pointed out that, for moderate k, a full factorial design frequently makes possible the estimation of many more effects than are detectably different from the noise. Sometimes these nondetectable effects are high-order interactions and sometimes they are all the effects associated with some inert variable or variables.

The fractional factorials discussed in this chapter are ideally suited to exploiting the probable existence of redundancy of one or both of these kinds for the following reason:

1. It can be arranged so that the confounding that occurs is between effects of high and low order,
2. A complete factorial design is available for whichever subset of $R - 1$ variables turns out to have appreciable effects.

In sequential experimentation, unless the total number of runs for a full or replicated factorial is needed to achieve sufficient precision, it is usually better to run fractional factorial designs. The fractions, used as building blocks, can build up to the full factorial design if this is necessary.

We now illustrate these ideas for designs of resolution III.

12.5. RESOLUTION III DESIGNS: BICYCLE EXAMPLE*

Suppose that the hypothetical data of Table 12.5 are times in seconds for a particular person to complete eight trial bicycle runs up a hill between fixed marks. These runs were performed in random order on eight successive days. The design is of resolution III and is a $\frac{8}{128} = \frac{1}{16}$ fraction of the full 2^7 factorial. Thus it is a 2_{III}^{7-4} design. (Note that $2^{7-4} = 2^7 2^{-4} = 2^{-4} 2^7 = \frac{1}{16} 2^7$.)

Table 12.6 gives the calculated contrasts. For example,

$$l_1 = \tfrac{1}{4}(-69 + 52 - 60 + 83 - 71 + 50 - 59 + 88) \qquad (12.7)$$

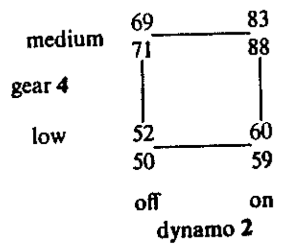
* This hypothetical example is an extension of the real one in Appendix 11A, but it is assumed now that both the rider and the bicycle are different.

TABLE 12.5. An eight-run experimental design for studying how time to cycle up a hill is affected by seven variables (I = 124, I = 135, I = 236, I = 1237).

run	seat up/down 1	dynamo off/on 2	handlebars up/down 3	gear low/medium 4 12	raincoat on/off 5 13	breakfast yes/no 6 23	tires hard/soft 7 123	time to climb hill (sec) y
1	−	−	−	+	+	+	−	69
2	+	−	−	−	−	+	+	52
3	−	+	−	−	+	−	+	60
4	+	+	−	+	−	−	−	83
5	−	−	+	+	−	−	+	71
6	+	−	+	−	+	−	−	50
7	−	+	+	−	−	+	−	59
8	+	+	+	+	+	+	+	88

TABLE 12.6. Calculated contrasts and abbreviated confounding pattern for data and design in Table 12.5

seat	$l_1 =$ 3.5	$\rightarrow$ *1 + 24 + 35 + 67*
dynamo	$l_2 =$ (12.0)	$\rightarrow$ *2 + 14 + 36 + 57*
handlebars	$l_3 =$ 1.0	$\rightarrow$ *3 + 15 + 26 + 47*
gear	$l_4 =$ (22.5)	$\rightarrow$ *4 + 12 + 56 + 37*
raincoat	$l_5 =$ 0.5	$\rightarrow$ *5 + 13 + 46 + 27*
breakfast	$l_6 =$ 1.0	$\rightarrow$ *6 + 23 + 45 + 17*
tires	$l_7 =$ 2.5	$\rightarrow$ *7 + 34 + 25 + 16*
	($l_I =$ 66.5	$\rightarrow$ average)



The table also gives an abbreviated* confounding pattern in which interactions between three or more factors have been ignored. Suppose that previous experience suggested that the standard deviation for repeated runs up the hill under the same conditions is about 3 seconds. Thus the calculated effects $l_1, l_2, \ldots, l_7$ have a standard error of about

$$\sqrt{\frac{3^2}{4} + \frac{3^2}{4}} = 2.1$$

Evidently only two contrasts, l_2 and l_4, are distinguishable from noise. Their values are circled in Table 12.6. The simplest interpretation of the results is that only two of the seven factors, the dynamo (**2**) and gear (**4**), exert a detectable influence, and they do so by way of their main effects. Having the dynamo on adds about 12 seconds to the time, and using medium gear instead of low gear adds about 22 seconds. On this interpretation we have in effect

* The method by which the confounding pattern has been obtained is given in Appendix 12A.

a replicated 2^2 design in the variables **2** and **4**, as indicated at the bottom of Table 12.6. There is, of course, some ambiguity in these conclusions. It is possible, for example, that l_4 is large, not because of a large main effect *4*, but because one or more of the interactions *12, 56, 37* are large. We see in Appendix 12B how sequential addition of further runs can resolve such ambiguities. However, for this example we suppose that the experimenter's knowledge of the nature of his bicycle suggests that the simpler explanation is likely to be right. The experimenter might well decide to proceed to the next stage of the investigation at this point.

Because one use of resolution III designs is to determine the main effects of each of the factors, assuming that they do not interact, these arrangements have sometimes been called "main effect plans."

Embedded 2^2 Factorials in Resolution III designs

A resolution R design has a complete factorial (possibly replicated) in every subset of $R - 1$ variables. For the resolution III design of Table 12.5, for example, whichever two columns of the design are chosen, they form a complete 2^2 factorial replicated twice. Also notice what happens to the confounding pattern in Table 12.6 supposing that two variables, say **2** and **4**, are effective, and the rest, that is, **1**, **3**, **5**, **6**, and **7**, are essentially inert. Then all interactions and main effects containing these numbers vanish, $l_2 \rightarrow 2$, $l_4 \rightarrow 4$, and $l_I \rightarrow 24$, and the remaining l's measure experimental error only.

Exercise 12.7. For the examples in Table 12.4, verify that any subset of $R - 1$ variables from a design of resolution R produces a full factorial design.

Construction of 2_{III}^{7-4} Design

The 2^{7-4} design in Table 12.5 can be constructed as follows:

1. Write a full factorial design for the three variables, **1**, **2**, and **3**.
2. Associate additional variables **4**, **5**, **6**, and **7** with all the interaction columns **12**, **13**, **23**, and **123**, respectively.

The design is obtained by associating every available contrast with a variable and is therefore sometimes called a *saturated design*.*

* It is actually possible to construct supersaturated designs, but we do not recommend them in ordinary circumstances.

Other 2^{7-4} Fractions

In Table 12.5 a one-sixteenth fraction of a full 2^7 factorial design is shown. How can the other one-sixteenth fractions that make up the full factorial design be generated? The first design was generated by setting

$$\mathbf{4} = +\mathbf{12} \quad \mathbf{5} = +\mathbf{13} \quad \mathbf{6} = +\mathbf{23} \quad \mathbf{7} = +\mathbf{123} \tag{12.8}$$

but, for example, we could equally well have used

$$\mathbf{4} = -\mathbf{12} \quad \mathbf{5} = +\mathbf{13} \quad \mathbf{6} = +\mathbf{23} \quad \mathbf{7} = +\mathbf{123} \tag{12.9}$$

This gives a different one-sixteenth fraction, which is shown in Table 12.7 with further hypothetical data on times to cycle up the hill. Note that none of the runs in this new design is the same as any of those in the preceding design. Calculated contrasts for this design are shown in Table 12.8.

TABLE 12.7. A second 2^{7-4}_{III} fractional factorial design with times to cycle up a hill ($\mathbf{I} = -\mathbf{124}, \mathbf{I} = \mathbf{135}, \mathbf{I} = \mathbf{236}, \mathbf{I} = \mathbf{1{,}237}$).

run	seat **1**	dynamo **2**	handlebars **3**	gear **4** −**12**	raincoat **5** **13**	breakfast **6** **23**	tires **7** **123**	time to climb hill (sec)
9	−	−	−	−	+	+	−	47
10	+	−	−	+	−	+	+	74
11	−	+	−	+	+	−	+	84
12	+	+	−	−	−	−	−	62
13	−	−	+	−	−	−	+	53
14	+	−	+	+	+	−	−	78
15	−	+	+	+	−	+	−	87
16	+	+	+	−	+	+	+	60

What is the confounding pattern for the new fraction? Notice that the new fraction was obtained by switching signs for variable **4** in the first design (variable **4** was associated with −**12** instead of +**12**). The abbreviated confounding pattern for this new fraction may be obtained, therefore, by switching signs in the confounding pattern of Table 12.6. This gives the confounding pattern in Table 12.8.

For this set of data the contrasts calculated from the second fraction confirm the conclusions from the first fraction.

TABLE 12.8. Calculated contrasts and abbreviated confounding pattern for second design in bicycle experiment

$l'_1 = 0.8 \rightarrow 1 - 24 + 35 + 67$
$l'_2 = 10.2 \rightarrow 2 - 14 + 36 + 57$
$l'_3 = 2.7 \rightarrow 3 + 15 + 26 - 47$
$l'_4 = 25.2 \rightarrow 4 - 12 - 56 - 37$ (i.e., $l'_{-4} = -25.2 \rightarrow -4 + 12 + 56 + 37$)
$l'_5 = -1.7 \rightarrow 5 + 13 - 46 + 27$
$l'_6 = 2.2 \rightarrow 6 + 23 - 45 + 17$
$l'_7 = -0.7 \rightarrow 7 - 34 + 25 + 16$

The Sixteen Different Fractions

In all there are 16 different ways of allocating signs to the four generators:

$$\mathbf{4} = \pm\mathbf{12}, \quad \mathbf{5} = \pm\mathbf{13}, \quad \mathbf{6} = \pm\mathbf{23}, \quad \mathbf{7} = \pm\mathbf{123} \tag{12.10}$$

Thus appropriate sign switching in columns* **4**, **5**, **6**, and **7** of Table 12.5 produces 16 fractional factorial designs which together make up the complete 2^7 factorial design. Corresponding sign switching in Table 12.6 produces the 16 different confounding patterns.

Designing Two Fractions

Consider again the bicycle example. Suppose that the 16 results from the two 2^{7-4}_{III} fractionals were considered together. What conclusions could be drawn? Combining the results from Tables 12.6 and 12.8, we obtain Table 12.9.

Conclusions would now be somewhat more certain. In particular, the large main effect of factor **4** (gear) is now estimated free of bias from two-factor interactions, and has a value close to that conjectured earlier. The joint effect of the string of interactions *12 + 56 + 37* can now be estimated separately from the main effect *4*, and it is shown to be small. Most interestingly, all the two-factor interactions involving the important variable 4 are now *free of aliases*. (Of course we continue to assume all three-factor and higher order interactions to be zero.) For this particular set of data, however, none of these two-factor interactions is distinguishable from noise. Factor **2** (dynamo), somewhat less aliased than before, is showing an effect similar to that previously conjectured.

* The reader can confirm by experimentation that switching signs in other columns of the design only produces one or another of these basic 16 fractions. However, the *order* in which the runs appear can be different.

TABLE 12.9. Analysis of complete set of 16 runs, combining the results of the two fractions, bicycle example

seat	$\frac{1}{2}(l_1 + l'_1) = \frac{1}{2}(3.5 + 0.8)$	$= 2.2$	$\rightarrow 1 + 35 + 67$
dynamo	$\frac{1}{2}(l_2 + l'_2) = \frac{1}{2}(12.0 + 10.2)$	$= 11.1$	$\rightarrow 2 + 36 + 57$
handlebars	$\frac{1}{2}(l_3 + l'_3) = \frac{1}{2}(1.0 + 2.7)$	$= 1.9$	$\rightarrow 3 + 15 + 26$
gear	$\frac{1}{2}(l_4 + l'_4) = \frac{1}{2}(22.5 + 25.2)$	$= 23.9$	$\rightarrow 4$
raincoat	$\frac{1}{2}(l_5 + l'_5) = \frac{1}{2}(0.5 - 1.7)$	$= -0.6$	$\rightarrow 5 + 13 + 27$
breakfast	$\frac{1}{2}(l_6 + l'_6) = \frac{1}{2}(1.0 + 2.2)$	$= 1.8$	$\rightarrow 6 + 23 + 17$
tires	$\frac{1}{2}(l_7 + l'_7) = \frac{1}{2}(2.5 - 0.7)$	$= 0.9$	$\rightarrow 7 + 25 + 16$
	$\frac{1}{2}(l_1 - l'_1) = \frac{1}{2}(3.5 - 0.8)$	$= 1.3$	$\rightarrow 24$
	$\frac{1}{2}(l_2 - l'_2) = \frac{1}{2}(12.0 - 10.2)$	$= 0.9$	$\rightarrow 14$
	$\frac{1}{2}(l_3 - l'_3) = \frac{1}{2}(1.0 - 2.7)$	$= -0.9$	$\rightarrow 47$
	$\frac{1}{2}(l_4 - l'_4) = \frac{1}{2}(22.5 - 25.2)$	$= -1.4$	$\rightarrow 12 + 56 + 37$
	$\frac{1}{2}(l_5 - l'_5) = \frac{1}{2}(0.5 + 1.7)$	$= 1.1$	$\rightarrow 46$
	$\frac{1}{2}(l_6 - l'_6) = \frac{1}{2}(1.0 - 2.2)$	$= -0.6$	$\rightarrow 45$
	$\frac{1}{2}(l_7 - l'_7) = \frac{1}{2}(2.5 + 0.7)$	$= 1.6$	$\rightarrow 34$

Sequential Use of Highly Fractionated Designs

The preceding example illustrates a useful application of highly fractionated designs as sequential building blocks. Additional fractions may be selected to resolve ambiguities, which knowledge of the variables and data available so far suggest may be of importance. We explore two important applications of this idea. The reader can devise others to suit particular circumstances.

Addition of a Second Fraction to De-alias Any One Main Effect and All Its Associated Two-Factor Interactions

Consider the two fractions used in the bicycle experiment. The largest effect obtained from the first set of eight runs was associated with the choice of gear (variable **4**). It might have been argued, therefore, that if further runs were to be made, they could best be employed to de-alias **4** and all the interactions of other variables with **4**.

Table 12.9 shows that by adding a second fraction in which the sign of variable **4** has been switched, a design of 16 runs possessing the desired property is obtained. This ability to de-alias one effect and all its two-factor interactions by adding a second fraction with the appropriate column of signs switched is a handy device for the sequential use of these designs.

Adding a Second Fraction to De-alias All Main Effects

Consider Table 12.5 again, and suppose that a different second fraction is added in which signs are switched in *all* the columns. Then for the new fraction

the first two rows in the confounding pattern (obtained by switching signs in Table 12.6) are

$$l'_1 \rightarrow 1 - 24 - 35 - 67 \qquad (l'_{-1} \rightarrow -1 + 24 + 35 + 67)$$
$$l'_2 \rightarrow 2 - 14 - 36 - 57 \qquad (l'_{-2} \rightarrow -2 + 14 + 36 + 57) \qquad (12.11)$$

By combining this second fraction with the original fraction, we obtain

$$\tfrac{1}{2}(l_1 + l'_1) \rightarrow 1, \qquad \tfrac{1}{2}(l_1 - l'_1) \rightarrow 24 + 35 + 67$$
$$\tfrac{1}{2}(l_2 + l'_2) \rightarrow 2, \qquad \tfrac{1}{2}(l_2 - l'_2) \rightarrow 14 + 36 + 57 \qquad (12.12)$$

and so on.

This way of augmenting the design yields *all* main effects clear of all two-factor interactions, but the two-factor interactions themselves are still confounded in groups of three. An example of the use of this sequence is given in Section 13.3.

Exercise 12.8. Show that the second fraction obtained above by switching all signs may also be obtained (with runs in a different order) by switching signs in columns **4**, **5**, **6**, and **7** only. Can you find other ways to reproduce the second fraction? Explain the equivalences you find.

General Construction of Resolution III Designs

Resolution III designs for $2^k - 1$ variables may be obtained by saturating a 2^k factorial with additional variables. For example, to construct a saturated 16-run design in 15 variables first write a full factorial design for four variables and then associate the extra variables **5**, **6**, ..., **15** with the 11 interaction columns **12**, **13**, **14**, **23**, **24**, **34**, **123**, **124**, **134**, **234**, and **1234**, respectively. The resulting design is a 2_{III}^{15-11} fractional factorial design for 15 variables in 16 runs.

Exercise 12.9. Construct a two-level fractional factorial design for 31 variables in 32 runs. This is a 2^{k-p} design; what values do k and p have? *Answer*: $k = 31$, $p = 26$.

Exercise 12.10. Indicate how you could construct a 2^{63-57} fractional factorial design. Is this a saturated design? *Answer*: Yes.

Useful designs may be obtained by appropriately deleting columns from the saturated designs. For example, dropping columns **4** and **7** from the design matrix for a 2^{7-4} design yields a 2^{5-2} design, the defining relation for which can be obtained from that for the 2^{7-4} design by deleting all words containing **4** and **7**. The variables to be dropped are selected so as to obtain the most satisfactory alias arrangement.

Plackett and Burman Saturated Designs

The saturated fractional factorial designs have the following orthogonal* property: if we take any two columns, then, corresponding to the $N/2$ plus signs in the first column, there will be $N/4$ plus and $N/4$ minus signs in the second column, and similarly for the minus signs in the first column. Provided that all interactions are negligible, designs with this property allow unbiased estimation of all main effects of $N - 1$ variables with smallest possible variance. The fractional factorials so far discussed are available only if N is a power of 2. Plackett and Burman (1946) have obtained arrangements with this same orthogonal property when N is a multiple of 4. For example, their design for $k = 11$ factors in $N = 12$ runs is shown in Table 12.10. The fashion in which two-factor interactions confound main effects for most Plackett and Burman designs is complicated. However, fold-over pairs of any such orthogonal design are of resolution IV (see Box and Wilson, 1951).

TABLE 12.10. Plackett and Burman design for study of 11 factors in 12 runs

	variable										
run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	$\overline{\mathbf{10}}$	$\overline{\mathbf{11}}$
1	+	−	+	−	−	−	+	+	+	−	+
2	+	+	−	+	−	−	−	+	+	+	−
3	−	+	+	−	+	−	−	−	+	+	+
4	+	−	+	+	−	+	−	−	−	+	+
5	+	+	−	+	+	−	+	−	−	−	+
6	+	+	+	−	+	+	−	+	−	−	−
7	−	+	+	+	−	+	+	−	+	−	−
8	−	−	+	+	+	−	+	+	−	+	−
9	−	−	−	+	+	+	−	+	+	−	+
10	+	−	−	−	+	+	+	−	+	+	−
11	−	+	−	−	−	+	+	+	−	+	+
12	−	−	−	−	−	−	−	−	−	−	−

12.6. RESOLUTION IV DESIGNS: INJECTION MOLDING EXAMPLE

We have seen that for designs of resolution V main effects are confounded only with four-factor interactions, and two-factor interactions only with three-factor interactions. Full factorial designs are generated by every subset

* If the level of the ith variable is represented by $x_i = \pm 1$ and that of the jth variable by $x_j = \pm 1$, then $\Sigma x_i = 0$, $\Sigma x_j = 0$, and $\Sigma x_i x_j = 0$ for every i and j.

of four variables. Designs of resolution III introduce much more serious confounding, with main effects having two-factor interactions as aliases. For these designs full factorial designs exist for every subset of two variables. Designs of resolution IV occupy an intermediate position. No main effect is confounded with any two-factor interaction, but two-factor interactions *are* confounded with each other. For these designs full factorial designs exist for every subset of three variables.

An Experiment on Injection Molding

In an injection molding experiment (Table 12.11) eight variables were studied in a 2_{IV}^{8-4} (a $\frac{1}{16}$ replicate of a 2^8 factorial of resolution IV). The normal plots, shown in Figure 12.5, suggest that the linear contrasts l_3, l_{15}, and l_5 are distinguishable from the noise. The largest remaining effect is l_8. The confounding pattern, assuming negligible interaction between three or more factors, is shown in Table 12.12. It seems likely that main effects associated with holding pressure (**3**) and booster pressure (**5**) exist. Also, the interactions most likely to explain the large size of l_{15} are perhaps *15* and *38*, since these involve factors **3** and **5**, which have large main effects. It is, however, possible that interactions exist between factors that have no main effects. Without further information the situation is uncertain. One way to proceed is to choose a further fraction of eight or 16 runs designed to resolve the ambiguity. However, in this particular example the large size of l_{15} suggested that the problem might be resolved with even fewer than eight runs. We show in Appendix 12B how four additional runs were chosen and used to discover and estimate the responsible interaction.

Construction of the Resolution IV Design by "Folding Over" a Resolution III Design

The sixteen-run 2_{IV}^{8-4} design in Table 12.11 was constructed as follows. The eight-run 2_{III}^{7-4} design was first written as in Table 12.5 for the seven variables **1**, **2**, **3**, ..., **7**. A further column labeled **8** and consisting entirely of plus signs was then added. The remaining eight runs were obtained by switching all signs in the first set of eight runs. Thus run 9 was obtained by switching all signs in run 1 and so on.

The Alias Pattern

The alias pattern for the folded-over design given in Table 12.11 can be obtained from that of the resolution III design (Table 12.6) by the following argument. Suppose that we compute for the first set of eight runs

$$l_1' = \tfrac{1}{4}(-y_1 + y_2 \ldots + y_8)$$

and for the second set of eight runs

$$-l'_1 = \tfrac{1}{8}(-y_9 + y_{10} \cdots + y_{16})$$

Then using Table 12.6

$l_1 \to 1 + 18 + 24 + 35 + 67$ and $-l'_1 = -1 + 18 + 24 + 35 + 67$

Now the contrast l_1 for the complete set of 16 runs is

$$l_1 = \tfrac{1}{8}(-y_1 + y_2 + \cdots + y_8 + y_9 - y_{10} \cdots - y_{16}) = \tfrac{1}{2}(l_1 + l'_1)$$

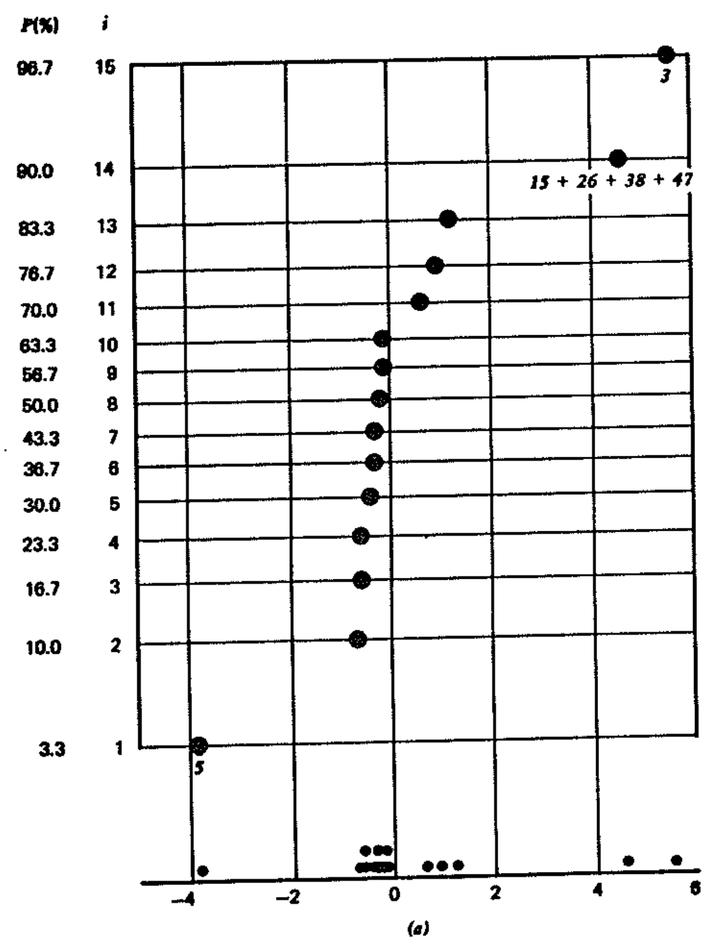


FIGURE 12.5. (*a*) Normal plot of contrasts, injection molding example.

Similarly for the contrast associated with the interaction *18* it is

$$l_{18} = \tfrac{1}{8}(-y_1 + y_2 + \cdots - y_8 - y_9 + y_{10} \cdots + y_{16}) = \tfrac{1}{2}(l_1 - l'_1).$$

Thus $l_1 \to 1$ and $l_{18} \to 18 + 24 + 35 + 67$. The same argument applied to the remaining contrasts yields the confounding pattern of Table 12.12. A more complete discussion is given in Appendix 12A.

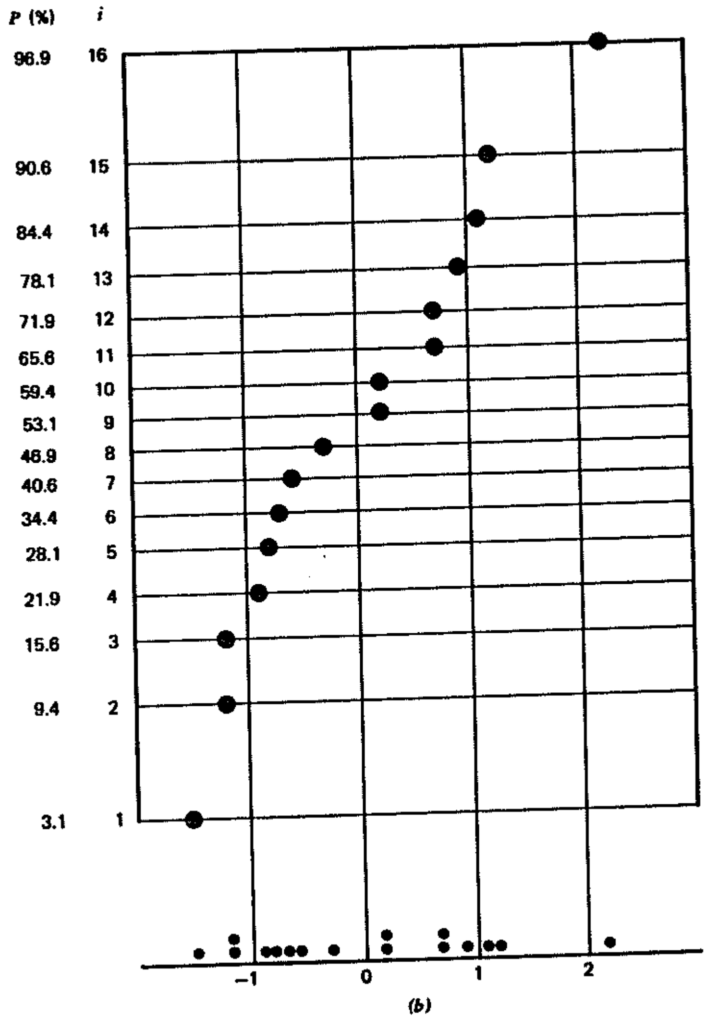


FIGURE 12.5. (*b*) Normal plot of residuals 2^{8-4}_{IV} design, injection molding example.

TABLE 12.11. A 2^{8-4} resolution IV design, molding example (I = 1248, I = 1358, I = 2368, I = 1237).

	mold temperature	moisture content	holding pressure	cavity thickness	booster pressure	cycle time	gate size	screw speed	shrinkage
run	1	2	3	4	5	6	7	8	y
1	−	−	−	+	+	+	−	+	14.0
2	+	−	−	−	−	+	+	+	16.8
3	−	+	−	−	+	−	+	+	15.0
4	+	+	−	+	−	−	−	+	15.4
5	−	−	+	+	−	−	+	+	27.6
6	+	−	+	−	+	−	−	+	24.0
7	−	+	+	−	−	+	−	+	27.4
8	+	+	+	+	+	+	+	+	22.6
9	+	+	+	−	−	−	+	−	22.3
10	−	+	+	+	+	−	−	−	17.1
11	+	−	+	+	−	+	−	−	21.5
12	−	−	+	−	+	+	+	−	17.5
13	+	+	−	−	+	+	−	−	15.9
14	−	+	−	+	−	+	+	−	21.9
15	+	−	−	+	+	−	+	−	16.7
16	−	−	−	−	−	−	−	−	20.3

TABLE 12.12. Calculated contrasts with their expected values: interactions between three or more factors ignored, molding example

$l_1 = -0.7 \rightarrow 1$
$l_2 = -0.1 \rightarrow 2$
$l_3 = 5.5 \rightarrow 3$
$l_4 = -0.3 \rightarrow 4$
$l_5 = -3.8 \rightarrow 5$
$l_6 = -0.1 \rightarrow 6$
$l_7 = 0.6 \rightarrow 7$
$l_8 = 1.2 \rightarrow 8$
$l_{12} = -0.6 \rightarrow 12 + 37 + 48 + 56$
$l_{13} = 0.9 \rightarrow 13 + 27 + 46 + 58$
$l_{14} = -0.4 \rightarrow 14 + 28 + 36 + 57$
$l_{15} = 4.6 \rightarrow 15 + 26 + 38 + 47$
$l_{16} = -0.3 \rightarrow 16 + 25 + 34 + 78$
$l_{17} = -0.2 \rightarrow 17 + 23 + 68 + 45$
$l_{18} = -0.6 \rightarrow 18 + 24 + 35 + 67$
average = 19.75

Alternative 2^{8-4} Fractions

Sixteen different 2_{IV}^{8-4} fractions are members of the family making up the complete 2^8 design. Individual members of the family may be generated by sign switching. Exactly as with the resolution III designs, the switching of signs in one or more columns will always yield a member of the family, and the associated confounding pattern is obtained by making the corresponding sign changes in the alias patterns of Table 12.12.

Building Blocks

Resolution IV designs may be used sequentially as were the resolution III designs. As before, sign switching may be used to eliminate particular confounding links.

General Construction of Resolution IV Designs

The construction of a resolution IV design containing 2^k variables follows exactly the pattern given for the 2_{IV}^{8-4} design:

1. Write a complete 2^k factorial with added columns for all interaction terms.
2. Generate a resolution III design for $2^k - 1$ variables by saturating this design with variables.
3. Add a further variable as a column of plus signs.
4. Repeat the design with all signs reversed to give a resolution IV design for 2^k variables in 2^{k+1} runs.

An alternative general method is given in Appendix 12A.

12.7. ELIMINATION OF BLOCK EFFECTS IN FRACTIONAL DESIGNS

Fractional designs may be run in blocks, with suitable contrasts used as "block variables." A design in 2^q blocks is defined by q independent contrasts. All effects (including aliases) associated with these basic contrasts *and all their interactions* are confounded with blocks.

Example 2_V^{5-1} Design in Two Blocks of Eight

Consider again the 2^{5-1} design of Table 12.2. Suppose the investigator decided that interaction between feed rate and catalyst concentration was likely to be negligible. This interaction *13* could then be used for blocking. The eight runs 2, 20, 5, . . . , 15, having a minus sign in the **13** column, would be run in one block, and the eight runs 17, 3, 22, . . . , 32 in the other. Notice that in this design the alias *245* (here assumed negligible) of *13* is also confounded with blocks.

Example: 2_V^{5-1} Design in Four Blocks of Four Runs

Suppose that, in the 2^{5-1} design of Table 12.2, columns **13** and **23** are confounded with blocks. Then the interaction between these columns **13** × **23** = **123**^{2} = **12** is also confounded. The design would thus be appropriate if we were prepared to confound with blocks all two-factor interactions between variables **1**, **2**, and **3** and their aliases. To achieve this arrangement, runs 20, 5, 12, and 29, for which the **13** and **23** columns have signs (− −), could be put in the first block, runs 2, 23, 26, and 15, for which columns **13** and **23** have signs (− +) in the second block, and so on. Thus in terms of a two-way table the arrangement would be as follows:

13		**23** −		**23** +	
+		III		IV	
		+	−	+	+
−		I		II	
		−	−	−	+

The Resolution IV Designs as Main Effect Plans in Blocks of Two

It occasionally happens that we must work with very small block sizes. A remarkable class of such designs based on the resolution **IV** arrangement provides economical main effect plans with a block size of only two. In one investigation the subject of study was an effluent impurity that tended to vary slowly with time. Runs made consecutively were thus much more comparable than those made further apart. It was possible to run the design in blocks of 2-hour periods, one experimental condition being run in the first hour and one in the second. At one stage of the investigation a 16-run main effect plan was used to study the main effects of eight variables based on a blocked 2_{IV}^{8-4} design. The plan is shown in Table 12.13. To see how this is derived, consider the original design given in Table 12.11 and the aliasing strings in Table 12.12. For the blocking scheme suppose that we use any two-factor interaction contrast, say l_{12}, to accommodate $\mathbf{B}_1$, and a second, say l_{13}, to accommodate $\mathbf{B}_2$; then l_{17} cannot be used for $\mathbf{B}_3$ since it can be obtained by multiplying the signs of **12** and **13**. Suppose, therefore, we use l_{14} for $\mathbf{B}_3$. (The reader may confirm that any other remaining two-factor interaction contrast can equally well be employed.) Then the seven columns of signs obtained for $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_1\mathbf{B}_2$, $\mathbf{B}_1\mathbf{B}_3$, $\mathbf{B}_2\mathbf{B}_3$, $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3$ exactly correspond to the contrasts $l_{12}, l_{13}, l_{14}, l_{15}, l_{16}, l_{17}, l_{18}$, in some order. They thus involve only the strings of interactions and not the main effects. When the design is rearranged in the eight blocks as on the right of Table 12.13, it is seen that the second run in each block is the mirror image or "fold-over" of the first run, that is, the signs in one run are exactly reversed in the other.

In designs of this kind, both the ordering within pairs and the sequence in which the pairs (blocks) are run should be random.

Rather than regard all between-block information as lost, the design can be analyzed on the basis that there are two different error variances. The within-block variance is appropriate for inferences about main effects, and the between-block variance for inferences about the strings of two-factor